On the lattice point covering problem in dimension 2

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Abstract

In this paper we study the lattice point covering property of some regular polygons in dimension 2.

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1 Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the n-dimensional Euclidean space $\mathbb{R}^n$ with non-empty interior. We denote by $\mathcal{K}^n_{(o)} \subset \mathcal{K}^n$ the set of all convex bodies, having the origin as an interior point, i.e., $\mathbf{0} \in \operatorname{int}(K)$, and by $\mathcal{K}^n_{(s)} \subset \mathcal{K}^n_{(o)}$ those bodies which are symmetric with respect to $\mathbf{0}$, i.e., $K = -K$.

We say that a convex body $K \in \mathcal{K}^n$ has the lattice point covering property, if K contains a lattice point of $\mathbb{Z}^n$ in any position, i.e., in any translation and rotation of K.

There are several beautiful results about the lattice point covering property.

Theorem 1 (Niven & Zuckerman,[4])**.** *A triangle with sides of lengths a, b, c, with $a \leqslant b \leqslant c$, has the lattice point covering property if and only if $2\Delta(c-1) \geqslant c^2$ where Δ is the area of the triangle.*

Theorem 2 (Niven & Zuckerman,[4])**.** *Let a and b be the distances between the pairs of opposite sides, say with $a \leqslant b$, of a parallelogram $ABCD$ with an interior angle $\gamma \leqslant \pi/2$. The parallelogram has the lattice point covering property if and only if $a \geqslant 1$ and one of the following conditions hold:*

(i) $b \geqslant \sqrt{2}$;

(ii) $b \leqslant \sqrt{2}$ *and* $\alpha + \beta + \gamma \leqslant \pi/2$, *where* $\alpha = \arccos(a/\sqrt{2})$ *and* $\beta = \arccos(b/\sqrt{2})$.

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Theorem 3 (Henk & Tsintsifas,[3]). *Let $\mathcal{E} \subset \mathbb{R}^n$ be an ellipsoid with semi-axes α_i, $1 \leqslant i \leqslant n$. The following statements are equivalent:*

(i) $\mathcal{E}$ *contains a lattice point of* $\mathbb{Z}^n$ *in any position,*

(ii) $\sum_{i=1}^n \frac{1}{\alpha_i^2} \leqslant 4$,

(iii) $\mathcal{E}$ *contains a cube of edge length* 1.

Let $K \in \mathcal{K}^n$. Denote by $Z(K)$ the lattice point covering radius of K, i.e., the smallest positive number r, such that rK has the lattice point covering property. We are here concerned with the lattice point covering properties of regular polygons. Let us denote by

$$H_n = \operatorname{conv}\left\{\left(\cos\left(\frac{2k\pi}{n}\right), \sin\left(\frac{2k\pi}{n}\right)\right) : k = 0, 1, \cdots, n-1\right\}$$

the regular n-gon.

Our main result is:

Theorem 4. *Let $t_i > 0$, $i \in \mathbb{N}$.*

(1) The following statements are equivalent:

(i) $t_{4n} \cdot H_{4n}$ *contains a lattice point of* $\mathbb{Z}^2$ *in any position,*

(ii) $t_{4n} \cdot H_{4n}$ *contains a ball with radius* $\frac{\sqrt{2}}{2}$,

(iii) $t_{4n} \geqslant \frac{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{2n}}$.

(2) The following statements are equivalent for $n = 1, 2$:

(i) $t_{4n+2} \cdot H_{4n+2}$ *contains a lattice point of* $\mathbb{Z}^2$ *in any position,*

(ii) $t_{4n+2} \cdot H_{4n+2}$ *contains* $[-\frac{1}{2}, \frac{1}{2}]^2$,

(iii) $t_6 \geqslant \frac{1}{3-\sqrt{3}} \approx 0.788675\ldots$; $t_{10} \geqslant \frac{\cos\frac{\pi}{5} - \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}} \approx 0.734342\ldots$.

Corollary 5.

(i) $Z(H_{4n}) = \frac{\frac{\sqrt{2}}{2}}{\cos\frac{\pi}{2n}}$, *for* $n \in \mathbb{N}$.

(ii) $Z(H_6) = \frac{1}{3-\sqrt{3}}$.

(iii) $Z(H_{10}) = \frac{\cos\frac{\pi}{5} - \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}$.

This paper is organized as follows: in the Section 2 we will introduce the covering radius and the gauge function; then we will give a necessary and a sufficient condition of the lattice point covering property in the Section 3 and the Section 4, respectively; the content of the Section 5 will be the proof of the main theorem.

2 Covering Radius

The covering radius of $K \in \mathcal{K}^n_{(o)}$ with respect to $\mathbb{Z}^n$ is denoted by

$$c(K) = c(K, \mathbb{Z}^n) = \min\{\lambda > 0 : \lambda K + \mathbb{Z}^n = \mathbb{R}^n\}.$$

The gauge function $\|\cdot\|$ associated to a $K \in \mathcal{K}^n_{(o)}$ is the function

$$\|\cdot\|_K : \mathbb{R}^n \to [0, \infty)$$

defined by $\|\boldsymbol{v}\|_K = \min\{t > 0 : \boldsymbol{v} \in tK\}$.

Theorem 6. *Let $K \in \mathcal{K}^n_{(o)}$. Then K contains a lattice point of $\mathbb{Z}^n$ in any position if and only if, for any $o(K)$ rotation of K, $c(o(K)) \leqslant 1$.*

Proof. If $c(o(K)) > 1$ for some rotation $o(K)$, then there exists a point $\boldsymbol{x} \in \mathbb{R}^n$, such that for every $\boldsymbol{u} \in \mathbb{Z}^2$,

$$\|\boldsymbol{x} - \boldsymbol{u}\|_{o(K)} > 1 \Longleftrightarrow \|\boldsymbol{u} - \boldsymbol{x}\|_{-o(K)} > 1.$$

Therefore, $-o(K) + \boldsymbol{x}$ does not contain a lattice point of $\mathbb{Z}^2$.

If $c(o(K)) \leqslant 1$ for any rotation $o(K)$, then for any point $\boldsymbol{x} \in \mathbb{R}^n$, since $o(K) + \mathbb{Z}^n = \mathbb{R}^n$, there exists a lattice point $\boldsymbol{u} \in \mathbb{Z}^2$, such that

$$\|\boldsymbol{x} - \boldsymbol{u}\|_{o(K)} \leqslant 1 \Longleftrightarrow \|\boldsymbol{u} - \boldsymbol{x}\|_{-o(K)} \leqslant 1.$$

So, $-o(K) + \boldsymbol{x}$ contain a lattice point u. ☐

Therefore, the lattice point covering property of a convex body depends on the covering radius of all rotations of this convex body.

3 Necessary Condition

According to the knowledge of the lattice covering for a centrally symmetric convex body, we have:

Theorem 7 (I.Fáry,[1]). *Let $K \in \mathcal{K}^2_{(s)}$, such that $K + \mathbb{Z}^2$ is a lattice covering. Then K contains a spacefiller L, i.e., a parallelogram or a centrally symmetric hexagon, such that $L + \mathbb{Z}^2$ is a lattice tiling.*

Since the lattice point covering property depends on the lattice covering of all rotations, we have:

Theorem 8. *Let $K \in \mathcal{K}^2_{(s)}$. Then K contains a lattice point of $\mathbb{Z}^2$ in any position, if and only if*

(1) $o(K) + \mathbb{Z}^2 = \mathbb{R}^2$,

(2) $o(K)$ contains a spacefiller L, i.e., parallelogram or a centrally symmetric hexagon, such that $L + \mathbb{Z}^2$ is a lattice tiling,

for any rotation $o(K)$.

Proof. Apply Theorem 6 and 7. □

We here call it a necessary condition of the lattice point covering property, because this condition has to hold, but is uneasy to check.

4 Sufficient Condition

For a planar convex body, it is possible to check some inscribed parallelograms, i.e., by checking the Steiner symmetrization of the convex body. We will give a sufficient condition of the lattice point covering property in this way.

The Steiner symmetrization of $K \in \mathcal{K}^2$ with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, denoted by $\mathrm{St}_1(K)$, is a convex body symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, such that for each line l vertical to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$,

$$\mathrm{vol}_1(K \cap l) = \mathrm{vol}_1(\mathrm{St}_1(K) \cap l),$$

where $\mathrm{vol}_1(L)$ denotes the length of a line segment L. For more information on the Steiner symmetrization, we refer to [2, Section 9.1]. It is obvious that $\mathrm{St}_1(K) \subset \mathrm{St}_1(L)$ for two convex bodies $K \subset L$.

Theorem 9. *Let $K \in \mathcal{K}^2_{(s)}$. If for each rotation $o(K)$ of K, $\mathrm{St}_1(o(K))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then K contains a lattice point of $\mathbb{Z}^2$ in any position.*

Proof. Notice that if $\mathrm{St}_1(o(K))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then $o(K)$ contains a parallelogram in the form of $L = \mathrm{conv}\,\{(\frac{1}{2}, a), (\frac{1}{2}, a+1), (-\frac{1}{2}, -a), (-\frac{1}{2}, -a-1)\}$, which is a spacefiller with respect to $\mathbb{Z}^2$. Therefore K has the lattice point covering property (cf. Theorem 6). □

We also have the following proposition of lattice covering for sets symmetric with respect to the axis ox and oy.

Proposition 10. *Let $K \in \mathcal{K}^2_{(s)}$. If K is symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$ and $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, then $K + \mathbb{Z}^2 = \mathbb{R}^2$ if and only if K contains $[-\frac{1}{2}, \frac{1}{2}]^2$.*

Proof. If K contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then $K + \mathbb{Z}^2$ is a lattice covering. Otherwise, if K does not contain $[-\frac{1}{2}, \frac{1}{2}]^2$, i.e., $(\frac{1}{2}, \frac{1}{2}) \notin K$, then since K is symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$ and $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$, K does not contain any point of $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$, thus $K + \mathbb{Z}^2$ does not contain $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. □

5 Proof of the Main Theorem

In this section we discuss the lattice point covering property of some regular polygons. The proofs have the following steps:

1. Prove that the Steiner symmetrizations of all rotations of the convex body contain $[-\frac{1}{2}, \frac{1}{2}]^2$ (using Theorem 9).

2. Prove that a smaller copy of the convex body does not have the lattice point covering property (using Theorem 8(1) and Proposition 10).

We first look at the regular $4n$-gon.

Proof of Theorem 4(1). If $t \cdot H_{4n}$ contains $B_2(\frac{\sqrt{2}}{2})$, where $B_2(r)$ denotes the Euclidean disk of radius r centered at $\mathbf{0}$, then each rotation $o(t \cdot H_{4n})$ also contains $B_2(\frac{\sqrt{2}}{2})$. Notice that $B_2(\frac{\sqrt{2}}{2})$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, therefore $o(t \cdot H_{4n}) + \mathbb{Z}^2$ is always a lattice covering, thus $t \cdot H_{4n}$ has the lattice point covering property (cf. Theorem 6).

If $t \cdot H_{4n}$ does not contain $B_2(\frac{\sqrt{2}}{2})$, then $o(t \cdot H'_{4n}, \frac{\pi}{4}) + (\frac{1}{2}, \frac{1}{2})$, does not contain any lattice point of $\mathbb{Z}^2$, where $o(t \cdot H'_{4n}, \frac{\pi}{4})$ is the rotation of $t \cdot H'_{4n}$ by angle $\frac{\pi}{4}$. $\square$

Denote by $o(K, \theta)$ the counterclockwise rotation of K by angle θ, i.e.,

$$o(H_n, \theta) = \text{conv}\left\{\left(\cos\left(\frac{2k\pi}{n} + \theta\right), \sin\left(\frac{2k\pi}{n} + \theta\right)\right) : k = 0, 1, \cdots, n-1\right\}.$$

Then we look at the regular hexagon.

Proof of Theorem 4(2), $n = 1$. For the symmetric reason between H_6 and $\mathbb{Z}^2$, the case $\frac{\pi}{12} \leqslant \theta \leqslant \frac{\pi}{6}$ is actually symmetric to the case $0 \leqslant \theta \leqslant \frac{\pi}{12}$ with respect to the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$. We are going to prove that $\text{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$ for $0 \leqslant \theta \leqslant \frac{\pi}{12}$ (cf. Theorem 9).

By calculation,

$$\begin{aligned}\text{St}_1(o(H_6, \theta)) = \text{conv}\Bigg\{&(\pm\cos\theta, 0), \left(\pm\cos(\theta - \frac{\pi}{3}), \pm\frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}\right), \\ &\left(\pm\cos(\theta + \frac{\pi}{3}), \pm\frac{\sqrt{3}}{2\cos\theta}\right)\Bigg\},\end{aligned}$$

which is also symmetric with respect to $\{\boldsymbol{x} \in \mathbb{R}^2 : x_1 = 0\}$. In order to check whether $\text{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$ for $0 \leqslant \theta \leqslant \frac{\pi}{12}$, notice that when $0 \leqslant \theta \leqslant \frac{\pi}{12}$, it holds

$$\cos(\theta + \frac{\pi}{3}) \leqslant \frac{\sqrt{3}}{2\cos\theta} \tag{1}$$

(cf. Proposition 12), therefore the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ may intersect the boundary of $\text{St}_1(o(H_6, \theta))$ with the edge

$$\text{conv}\left\{(\cos\theta, 0), \left(\cos(\theta - \frac{\pi}{3}), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}\right)\right\}$$

or the edge

$$\text{conv}\left\{\left(\cos\left(\theta - \frac{\pi}{3}\right), \frac{\sin\frac{\pi}{3}}{2\sin\left(\theta + \frac{\pi}{6}\right)}\right), \left(\cos(\theta + \frac{\pi}{3}), \frac{\sqrt{3}}{2\cos\theta}\right)\right\}.$$

Case 1: $\cos(\theta - \frac{\pi}{3}) \leqslant \frac{\sin\frac{\pi}{3}}{2\sin(\theta+\frac{\pi}{6})}$, i.e., $0 \leqslant \theta \leqslant \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}$.

In this case, the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ intersects the edge

$$\operatorname{conv}\left\{(\cos\theta, 0), \left(\cos\left(\theta - \frac{\pi}{3}\right), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}\right)\right\}$$

with $(s(\theta), s(\theta))$, where

$$\frac{s(\theta)}{s(\theta) - \cos\theta} = \frac{\frac{\sin\frac{\pi}{3}}{2\sin(\theta+\frac{\pi}{6})}}{\cos(\theta - \frac{\pi}{3}) - \cos\theta},$$

thus

$$s(\theta) = \frac{\sin\frac{\pi}{3}\cos\theta}{\sin\frac{\pi}{3} - 2\sin^2(\theta + \frac{\pi}{6}) + 2\cos\theta\sin(\theta + \frac{\pi}{6})}. \tag{2}$$

This function $s(\theta)$ is increasing in $\left[0, \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}\right]$ (cf. Proposition 13), therefore,

$$s(\theta) \geqslant s(0) = \frac{\sqrt{3}}{\sqrt{3}+1},$$

and

$$\frac{1}{3-\sqrt{3}}s(\theta) \geqslant \frac{1}{2}.$$

So $\mathrm{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta))$ always contains $[-\frac{1}{2}, \frac{1}{2}]^2$ when $\theta \in [0, \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}]$.

Case 2: $\cos(\theta - \frac{\pi}{3}) \geqslant \frac{\sin\frac{\pi}{3}}{2\sin(\theta+\frac{\pi}{6})}$, i.e., $\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{12}$.

In this case, the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ intersects the edge

$$\operatorname{conv}\left\{\left(\cos\left(\theta - \frac{\pi}{3}\right), \frac{\sin\frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})}\right), \left(\cos\left(\theta + \frac{\pi}{3}\right), \frac{\sqrt{3}}{2\cos\theta}\right)\right\}.$$

with $(t(\theta), t(\theta))$, where

$$\frac{2t(\theta) - \frac{\sqrt{3}}{\cos\theta}}{t(\theta) - \cos\left(\theta + \frac{\pi}{3}\right)} = \frac{\frac{\sin\frac{\pi}{3}}{\sin\left(\theta+\frac{\pi}{6}\right)} - \frac{\sqrt{3}}{\cos\theta}}{\cos\left(\theta - \frac{\pi}{3}\right) - \cos\left(\theta + \frac{\pi}{3}\right)},$$

i.e.,

$$t(\theta) = \frac{2\sqrt{3}\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(\theta + \frac{\pi}{3}\right)}{4\cos\theta\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}}. \tag{3}$$

This function $t(\theta)$ is decreasing in $\left[\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}, \frac{\pi}{12}\right]$, and in fact it is decreasing in $\left[\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}, \frac{\pi}{6}\right]$ (cf. Proposition 14), therefore

$$t(\theta) \geqslant t\left(\frac{\pi}{12}\right) > t\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{\sqrt{3}+1},$$

and

$$\frac{1}{3-\sqrt{3}}t(\theta) > \frac{1}{2}.$$

Thus $\mathrm{St}_1(o(\frac{1}{3-\sqrt{3}}H_6,\theta))$ always contains $[-\frac{1}{2},\frac{1}{2}]^2$ when $\theta \in \left[\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}, \frac{\pi}{12}\right]$.

To see that $\frac{1}{3-\sqrt{3}}$ is the minimum number, we refer to Proposition 10 applied to ρH_6 for any $\rho < \frac{1}{3-\sqrt{3}}$. □

Now we look at the regular 10-gon.

Proof of Theorem 4(2), $n = 2$. For the symmetric reason between H_{10} and $\mathbb{Z}^2$, the case $\frac{\pi}{20} \leqslant \theta \leqslant \frac{\pi}{10}$ is actually symmetric to the case $0 \leqslant \theta \leqslant \frac{\pi}{20}$ with respect to the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$. We are going to prove that $\mathrm{St}_1(o(\frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}H_{10},\theta))$ contains $[-\frac{1}{2},\frac{1}{2}]^2$ for $0 \leqslant \theta \leqslant \frac{\pi}{20}$ (cf. Theorem 9).

By calculation,

$$\begin{aligned}\mathrm{St}_1(o(H_{10},\theta)) = \mathrm{conv}\Bigg\{ &\pm(\cos\theta, 0),\\ &\left(\pm\cos(\theta-\frac{\pi}{5}), \pm\frac{\sin\frac{\pi}{10}\sin\frac{\pi}{5}}{\sin(\frac{\pi}{10}+\theta)}\right), \left(\pm\cos(\theta+\frac{\pi}{5}), \pm\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}\right),\\ &\left(\pm\cos(\theta-\frac{2\pi}{5}), \pm\frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}\right), \left(\pm\cos(\theta+\frac{2\pi}{5}), \pm\frac{\sin\frac{2\pi}{5}}{\cos\theta}\right)\Bigg\}.\end{aligned}$$

While $\theta \in [0, \frac{\pi}{20}]$, it holds

$$\cos\left(\theta - \frac{\pi}{5}\right) > \frac{\sin\frac{\pi}{10}\sin\frac{\pi}{5}}{\sin\left(\frac{\pi}{10}+\theta\right)} \qquad \text{(cf. Proposition 16)},$$

$$\cos\left(\theta + \frac{\pi}{5}\right) > \frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin\left(\frac{3\pi}{10}-\theta\right)} \qquad \text{(cf. Proposition 17)},$$

$$\cos\left(\theta - \frac{2\pi}{5}\right) < \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin\left(\frac{3\pi}{10}+\theta\right)} \qquad \text{(cf. Proposition 18)},$$

$$\cos\left(\theta + \frac{2\pi}{5}\right) < \frac{\sin\frac{2\pi}{5}}{\cos\theta} \qquad \text{(cf. Proposition 19)}.$$

Therefore, the line $\{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = x_1\}$ intersects $\mathrm{St}_1(o(H_{10},\theta))$ with the edge

$$\mathrm{conv}\left\{\left(\cos(\theta+\frac{\pi}{5}), \frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}\right), \left(\cos(\theta-\frac{2\pi}{5}), \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}\right)\right\}$$

at the point $(t(\theta), t(\theta))$, where

$$\frac{t(\theta) - \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{t(\theta) - \cos(\theta - \frac{2\pi}{5})} = \frac{\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)} - \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{\cos(\theta+\frac{\pi}{5}) - \cos(\theta-\frac{2\pi}{5})}. \tag{4}$$

The function $t(\theta)$ is increasing in $[0, \frac{\pi}{20}]$ (cf. Proposition 15), therefore

$$t(\theta) \geqslant t(0) = \frac{\sin\frac{\pi}{5}}{\cos\frac{\pi}{5} - \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}},$$

and

$$\frac{\cos\frac{\pi}{5} - \sin\frac{\pi}{5} + \sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}} t(\theta) \geqslant \frac{1}{2}.$$

So $\mathrm{St}_1(o(\frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}} H_{10}, \theta))$ always contains $[-\frac{1}{2}, \frac{1}{2}]^2$ when $\theta \in [0, \frac{\pi}{20}]$.

To see that $\frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}$ is the minimum number, we refer to Proposition 10 applied to ρH_{10} for any $\rho < \frac{\cos\frac{\pi}{5}-\sin\frac{\pi}{5}+\sin\frac{2\pi}{5}-\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}$. □

Remark 11. $\mathrm{St}_1(o(H_{2n}, \theta))$, $\theta \in [0, \frac{\pi}{4n}]$, has the vertices

$$\begin{aligned}&(\cos\theta, 0),\\ &\left(\cos(\theta - \frac{2\pi}{n}), \frac{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}}{\sin(\frac{\pi}{n}+\theta)}\right), \left(\cos(\theta + \frac{2\pi}{n}), \frac{\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}}{\sin(\frac{3\pi}{n}-\theta)}\right),\\ &\left(\cos(\theta - \frac{4\pi}{n}), \frac{\sin\frac{3\pi}{n}\sin\frac{4\pi}{n}}{\sin(\frac{3\pi}{n}+\theta)}\right), \left(\cos(\theta + \frac{4\pi}{n}), \frac{\sin\frac{4\pi}{n}\sin\frac{5\pi}{n}}{\sin(\frac{5\pi}{n}-\theta)}\right) \dots\end{aligned}$$

One can deal with all regular $(4n+2)$-gons similarly.

A Some Inequalities

Proposition 12.

$$\cos(\theta + \frac{\pi}{3}) \leqslant \frac{\sqrt{3}}{2\cos\theta}$$

for $0 \leqslant \theta \leqslant \frac{\pi}{12}$.

Proof. It is equivalent to

$$\begin{aligned}&2\cos\theta\cos(\theta+\frac{\pi}{3}) \leqslant \sqrt{3}\\ \Longleftrightarrow &\cos\frac{\pi}{3} + \cos(2\theta + \frac{\pi}{3}) \leqslant \sqrt{3}\\ \Longleftrightarrow &\cos(2\theta+\frac{\pi}{3}) \leqslant \sqrt{3} - \frac{1}{2},\end{aligned}$$

which holds true since $\cos(2\theta + \frac{\pi}{3})$ is decreasing in $\theta \in [0, \frac{\pi}{12}]$ and $\cos\frac{\pi}{3} < \sqrt{3} - \frac{1}{2}$. □

Proposition 13.

$$s(\theta) = \frac{\sin\frac{\pi}{3}\cos\theta}{\sin\frac{\pi}{3} - 2\sin^2(\theta+\frac{\pi}{6}) + 2\cos\theta\sin(\theta+\frac{\pi}{6})} \geqslant s(0)$$

for $0 \leqslant \theta \leqslant \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}$.

Proof. Notice that
$$s(\theta) = \frac{\sqrt{3}\cos\theta}{-3+\sqrt{3}+4\cos^2\theta}.$$
Since $\frac{x}{-3+\sqrt{3}+4x^2}$ is decreasing in $x \in [\cos\left(\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}\right), 1]$ and $\cos\theta$ is decreasing in $\theta \in [0, \arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}]$ then $s(\theta)$ is increasing. □

Proposition 14.
$$t(\theta) = \frac{2\sqrt{3}\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(\theta + \frac{\pi}{3}\right)}{4\cos\theta\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}} \geqslant t(\frac{\pi}{6})$$

for $\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{6}$.

Proof. Notice that
$$t(\theta) = \frac{3\sin(\theta + \frac{\pi}{3})}{4\sin^2(\theta + \frac{\pi}{3}) + \sqrt{3} - 1}.$$
Since $\frac{x}{4x^2+\sqrt{3}-1}$ is decreasing in $x \in [\sin\left(\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}\right), \frac{1}{2}]$ and $\sin(\theta + \frac{\pi}{3})$ is increasing in $\theta \in [\arcsin\left(\frac{\sqrt[4]{3}}{2}\right) - \frac{\pi}{6}, \frac{\pi}{6}]$ then $t(\theta)$ is decreasing. □

Proposition 15.
$$t(\theta) \geqslant t(0)$$
for $0 \leqslant \theta \leqslant \frac{\pi}{20}$ *where*
$$\frac{t(\theta) - \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{t(\theta) - \cos(\theta - \frac{2\pi}{5})} = \frac{\frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)} - \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}}{\cos(\theta + \frac{\pi}{5}) - \cos(\theta - \frac{2\pi}{5})}.$$

Proof. Notice that
$$t(\theta) = \frac{2\sin\frac{3\pi}{10}\cos\frac{\pi}{10}\cos\theta}{2\cos^2\theta + \sin\frac{3\pi}{5} - \cos\frac{3\pi}{5} - 1}.$$
Since $\frac{x}{2x^2+\sin\frac{3\pi}{5}-\cos\frac{3\pi}{5}-1}$ is decreasing in $x \in [\cos\frac{\pi}{20}, 1]$ and $\cos\theta$ is decreasing in $\theta \in [0, \frac{\pi}{20}]$ thus $t(\theta)$ is increasing. □

Proposition 16.
$$\cos(\theta - \frac{\pi}{5}) > \frac{\sin\frac{\pi}{10}\sin\frac{\pi}{5}}{\sin(\frac{\pi}{10} + \theta)}$$
for $0 \leqslant \theta \leqslant \frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned}
&\sin(\frac{\pi}{10}+\theta)\cos(\theta-\frac{\pi}{5}) > \sin\frac{\pi}{10}\sin\frac{\pi}{5}\\
\Leftrightarrow\, &\sin(2\theta-\frac{\pi}{10})+\sin\frac{3\pi}{10} > 2\sin\frac{\pi}{10}\sin\frac{\pi}{5}\\
\Leftrightarrow\, &\sin(2\theta-\frac{\pi}{10}) > 2\sin\frac{\pi}{10}\sin\frac{\pi}{5}-\sin\frac{3\pi}{10}.
\end{aligned}$$

Since $\sin(2\theta-\frac{\pi}{10})$ is increasing for $0\leqslant\theta\leqslant\frac{\pi}{20}$, and

$$\sin\frac{3\pi}{10}-\sin\frac{\pi}{10}-2\sin\frac{\pi}{10}\sin\frac{\pi}{5}>0,$$

the inequality holds for $0\leqslant\theta\leqslant\frac{\pi}{20}$. □

Proposition 17.

$$\cos(\theta+\frac{\pi}{5}) > \frac{\sin\frac{\pi}{5}\sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10}-\theta)}$$

for $0\leqslant\theta\leqslant\frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned}
&\sin(\frac{3\pi}{10}-\theta)\cos(\theta+\frac{\pi}{5}) > \sin\frac{\pi}{5}\sin\frac{3\pi}{10}\\
\Leftrightarrow\, &\sin\frac{\pi}{2}+\sin(\frac{\pi}{10}-2\theta) > 2\sin\frac{\pi}{5}\sin\frac{3\pi}{10}\\
\Leftrightarrow\, &\sin(\frac{\pi}{10}-2\theta) > 2\sin\frac{\pi}{5}\sin\frac{3\pi}{10}-\sin\frac{\pi}{2}.
\end{aligned}$$

Since $\sin(\frac{\pi}{10}-2\theta)$ is decreasing for $0\leqslant\theta\leqslant\frac{\pi}{20}$, and

$$0 > 2\sin\frac{\pi}{5}\sin\frac{3\pi}{10}-\sin\frac{\pi}{2},$$

the inequality holds for $0\leqslant\theta\leqslant\frac{\pi}{20}$. □

Proposition 18.

$$\cos(\theta-\frac{2\pi}{5}) < \frac{\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10}+\theta)}$$

for $0\leqslant\theta\leqslant\frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned}
&\sin(\frac{3\pi}{10}+\theta)\cos(\theta-\frac{2\pi}{5}) < \sin\frac{3\pi}{10}\sin\frac{2\pi}{5}\\
\Leftrightarrow\, &\sin(2\theta-\frac{\pi}{10})+\sin\frac{7\pi}{10} < 2\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}\\
\Leftrightarrow\, &\sin(2\theta-\frac{\pi}{10}) < 2\sin\frac{3\pi}{10}\sin\frac{2\pi}{5}-\sin\frac{7\pi}{10}
\end{aligned}$$

Since $\sin(2\theta - \frac{\pi}{10})$ is increasing for $0 \leqslant \theta \leqslant \frac{\pi}{20}$, and

$$0 < 2\sin\frac{3\pi}{10}\sin\frac{2\pi}{5} - \sin\frac{7\pi}{10},$$

the inequality holds for $0 \leqslant \theta \leqslant \frac{\pi}{20}$. □

Proposition 19.

$$\cos(\theta + \frac{2\pi}{5}) < \frac{\sin\frac{2\pi}{5}}{\cos\theta}$$

for $0 \leqslant \theta \leqslant \frac{\pi}{20}$.

Proof. The statement is equivalent to

$$\begin{aligned} &\cos\theta\cos(\theta + \frac{2\pi}{5}) < \sin\frac{2\pi}{5} \\ \Leftrightarrow\ &\cos(2\theta + \frac{2\pi}{5}) + \cos\frac{2\pi}{5} < 2\sin\frac{2\pi}{5} \\ \Leftrightarrow\ &\cos(2\theta + \frac{2\pi}{5}) < 2\sin\frac{2\pi}{5} - \cos\frac{2\pi}{5}. \end{aligned}$$

Since $\cos(2\theta + \frac{2\pi}{5})$ is decreasing for $0 \leqslant \theta \leqslant \frac{\pi}{20}$, and

$$\cos\frac{2\pi}{5} < 2\sin\frac{2\pi}{5} - \cos\frac{2\pi}{5},$$

the inequality holds for $0 \leqslant \theta \leqslant \frac{\pi}{20}$. □

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