Uglov bipartitions and extended Young diagrams

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Abstract

We study the class of Uglov bipartitions and prove a generalization of a conjecture by Dipper, James and Murphy. We give two consequences concerning the computation of canonical bases in affine type A and the description of decomposition matrices for Hecke algebras of type B_n in arbitrary characteristic.

Mathematics Subject Classifications: 20C08, 05E10, 17B37

1 Introduction

Uglov bipartitions are a class of combinatorial objects which were first defined in the context of the representation theory of quantum groups. More precisely, in the case of the affine special linear group $\widehat{\mathfrak{sl}}_e$, these pairs of integer partitions are known to naturally label the crystal graph of the irreducible highest weight modules (of level two) and thus their canonical bases. Since their introduction by Uglov in [17], they have appeared in various (but connected) situations:

- the representation theory of Cherednik algebras [4, 5] (as the bipartitions indexing the standard modules which are not killed by the KZ functor in type B_n),

- the representation theory of Hecke algebras [8] (as the bipartitions labelling the so called canonical basic sets in characteristic 0),

- the Harish-Chandra theory for unitary groups [9] (as the bipartitions labelling certain weak Harish-Chandra series).

The definition of these bipartitions depends on the choice of a pair of integers $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. If $\mathbf{s} \in \mathbb{Z}^2$ and $\mathbf{s}' \in \mathbb{Z}^2$ are in the same orbit modulo an action of an extended affine symmetric group, the associated classes of bipartitions are in bijection. For a certain choice of $\mathbf{s} \in \mathbb{Z}^2$, the associated bipartitions are known as FLOTW bipartitions (a special case of "cylindric bipartitions"), for another choice (called asymptotic), the associated bipartitions are known as Kleshchev bipartitions. Both types of bipartitions have been extensively studied in recent years. In the general case, even if these bipartitions have a nice and relatively easy recursive definition (see §3.2), it can be difficult to characterize them explicitly or to study their properties. In [13], a new combinatorial (but still recursive) definition has been given (it concerns in fact the more general class of Uglov multipartitions). As a consequence, an old conjecture by Dipper, James and Murphy has been deduced but only for the class of Kleshchev multipartitions (the papers [10, 11] consider special cases of this conjecture). This result shows that the Kleshchev multipartitions may be easily obtained as the maximal elements with respect to the lexicographic order on bipartitions in certain combinatorial expressions. Note also that the tableaux theoretic version in section 4.4 provide lower bounds for the dimensions of simple modules for the Hecke algebra[1].

The aim of this note is to continue the work of [13] and to obtain a general proof of the Dipper-James-Murphy conjecture for the whole class of Uglov bipartitions (see Theorem 15 or its reformulation in Corollary 24). To do this, we use the fact that this conjecture may be easily proved for the class of FLOTW bipartitions and we use the bijections between classes of Uglov bipartitions that we have already studied and defined in a number of papers. We in particular obtain a different proof of the conjecture for the class of Kleshchev bipartitions than the one presented in [2]. We note that, even if some of the results of [13] are used in this paper, our result is essentially independent of [13]. In particular, it does not use the notion of staggered sequence which is used to prove the conjecture for the Kleshchev multipartitions in this paper. One of the main interest of this conjecture lies in its application. We here give two consequences of this result. The first one (which has been already mentioned in [13]) concerns the computation of canonical bases for irreducible highest weight modules fo $\widehat{\mathfrak{sl}}_e$. The second one concerns the form of the decomposition matrices for Hecke algebras of type B_n in arbitrary characteristic. Thanks to our main result, we give an elementary proof for the existence of canonical basic sets for these algebras. Recently, such a result has been also obtained by C. Bowman using the theory of Cherednik algebras [4] (even in the wider context of Ariki-Koike algebras). In general, this was previously only known assuming the validity of certain Lusztig's conjectures on Hecke algebras with unequal parameters (see [7]).

2 Several definitions

In this part, we recall several combinatorial notions concerning Young diagrams, most of them can be already found in [13].

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[1]we thank the referee for pointing this to us

2.1 Extended Nodes of a bipartition

A *partition* λ of rank $n \in \mathbb{Z}_{>0}$ is a sequence of non negative and non increasing integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that we have $|\lambda| := \sum_{1 \leqslant i \leqslant r} \lambda_i = n$. The unique partition of rank 0, the empty partition, is denoted by $\varnothing$. We denote by $\ell(\lambda)$ the minimal integer such that $\lambda_{\ell(\lambda)+1} = 0$ with the convention that $\ell(\varnothing) = 0$. We will be interested here on the set of bipartitions of rank n:

$$\mathcal{P}^2(n) := \{(\lambda^1, \lambda^2) \mid |\lambda^1| + |\lambda^2| = n\}.$$

The *nodes* of the bipartition $\boldsymbol{\lambda}$ are the elements (a, b, c) where $c \in \{1, 2\}$, $a \in \{1, \ldots, \ell(\lambda^c)\}$, $b \in \{1, \ldots, \lambda_a^c\}$. The set of all nodes of $\boldsymbol{\lambda}$ is denoted by $\mathcal{Y}(\boldsymbol{\lambda})$. It is called the *Young diagram* of $\boldsymbol{\lambda}$. The *extended nodes* of the bipartition $\boldsymbol{\lambda}$ are the following elements of $\mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \times \{1, 2\}$:

1. the elements of $\mathcal{Y}(\boldsymbol{\lambda})$,

2. the elements of the form $(0, b, c)$ where $b > \lambda_1^c$ and $c \in \{1, 2\}$,

3. the elements of the form $(a, 0, c)$ where $a > \ell(\lambda^c)$ and $c \in \{1, 2\}$.

The set of all extended nodes of $\boldsymbol{\lambda}$ is called the *extended Young diagram* of $\boldsymbol{\lambda}$ and it is denoted by $\mathcal{Y}^{\text{ext}}(\boldsymbol{\lambda})$. It thus contains the Young diagram of $\boldsymbol{\lambda}$. One can represent it as a collection of boxes. Each box then corresponds to an extended node of the bipartition as in the following example.

Example 1. We consider the bipartition $(3.3.1, 2.1)$ of $n = 10$. The extended Young diagram is given as follows.
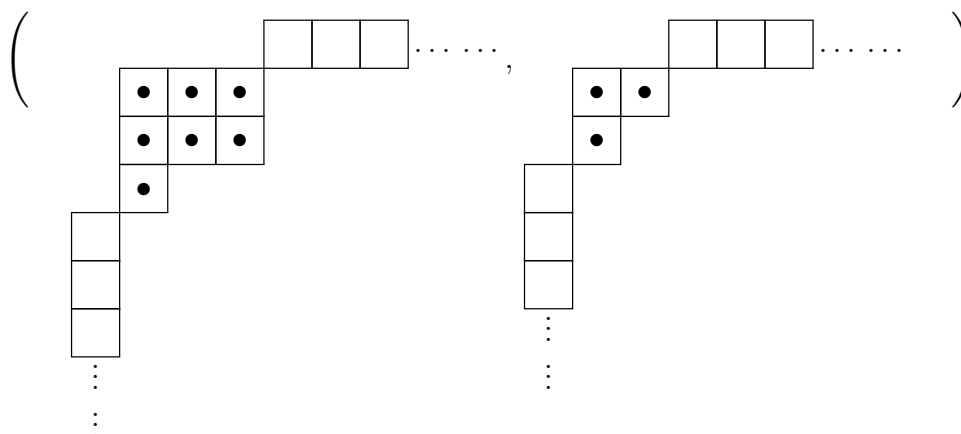
The boxes containing a bullet correspond to the boxes of the usual Young diagram.

Let $e \in \mathbb{Z}_{>1} \sqcup \{\infty\}$. We now fix $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. One can attach to each extended node $\gamma = (a, b, c)$ of the extended Young diagram its *content* (depending on the choice of $\mathbf{s}$):

$$\text{cont}(\gamma) = b - a + s_c \in \mathbb{Z}.$$

By definition, the *residue* (depending on the choice of $\mathbf{s}$ and e) $\operatorname{res}(\gamma)$ of the extended node γ is the content modulo e if e is finite and the content if $e = \infty$. Throughout the paper, we set $I := \mathbb{Z}/e\mathbb{Z}$ (which will be identified with $\{0, \ldots, e-1\}$) if e is finite and $I := \mathbb{Z}$ if $e = \infty$. If $\operatorname{res}(\gamma) = \mathfrak{j}$ then we say that γ is a *(extended) $\mathfrak{j}$-node.* In this paper, we will use "roman" letters for contents and "mathfrak" letters for residues to avoid confusion.

The *boundary* of the extended Young diagram is by definition given by the extended nodes $(a, b, c) \in \mathcal{Y}^{\mathrm{ext}}(\boldsymbol{\lambda})$ such that:

- $(a, b+1, c)$ is not in $\mathcal{Y}^{\mathrm{ext}}(\boldsymbol{\lambda})$. Such nodes constitute the *vertical boundary* of the extended Young diagram.

- $(a+1, b, c)$ is not in $\mathcal{Y}^{\mathrm{ext}}(\boldsymbol{\lambda})$. Such nodes constitute the *horizontal boundary* of the extended Young diagram.

The extended nodes which are in the vertical or horizontal boundary without being in the Young diagram are called *virtual*.

A node $\gamma = (a, b, c)$ of $\mathcal{Y}(\boldsymbol{\lambda})$ is said to be *removable* for $\boldsymbol{\lambda}$ if $\mathcal{Y}(\boldsymbol{\lambda}) \setminus \{\gamma\}$ is the Young tableau of a bipartition $\boldsymbol{\mu}$. If $\gamma = (a, b, c) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \{1, \ldots, l\}$ is such that $\mathcal{Y}(\boldsymbol{\lambda}) \sqcup \{\gamma\}$ is the Young tableau of a bipartition $\boldsymbol{\mu}$ then it is said to be *addable* for $\boldsymbol{\lambda}$.

The intersection between the vertical and the horizontal boundary is given by the set of removable nodes. We see also that there always exists one unique extended node in a fixed component with a given content which is either addable, or in the boundary of $\boldsymbol{\lambda}$. For $\mathfrak{j} \in I$, we will denote by $\mathcal{E}_{\mathfrak{j}}(\boldsymbol{\lambda})$ the set consisting of:

- addable $\mathfrak{j}$-nodes of $\boldsymbol{\lambda}$, we say that these nodes are of *nature A*.

- extended $\mathfrak{j}$-nodes of the boundary of $\boldsymbol{\lambda}$. Such node may be either removable (we then say that they are of nature R), or in the vertical boundary without being removable (we say that they are of nature B_v) or in the horizontal boundary without being removable (of nature B_h)

Given the nature of a node of content j in a component $c \in \{1, 2\}$ of a bipartition, there is always only two possibilities for the nature of the nodes with content $j-1$ and $j+1$ in the same component, in respectively $\mathcal{E}_{\mathfrak{j}-1}(\boldsymbol{\lambda})$ and $\mathcal{E}_{\mathfrak{j}+1}(\boldsymbol{\lambda})$. They are given in the following table:

Possible nature of the node of content $j-1$	Nature of the node of content j	Possible nature of the node of content $j+1$
B_v or R	A	B_h or R
B_h or A	R	B_v or A
B_v or R	B_v	B_v or A
B_h or A	B_h	B_h or R

Example 2. We consider the bipartition $\boldsymbol{\lambda} = (3.3.1, 2.1)$ of $n = 10$ and $\mathbf{s} = (0, 1)$, $e = 3$. Here is the associated extended Young diagram with the residue of each node written in the associated box:

$\mathcal{E}_2(\boldsymbol{\lambda})$ consists in the following extended nodes:

- the nodes $(0, 5 + 3k, 1)$ with $k \in \mathbb{N}$ which are of nature B_h.

- the nodes $(0, 4 + 3k, 2)$ with $k \in \mathbb{N}$ which are of nature B_h.

- the nodes $(4 + 3k, 0, 1)$ with $k \in \mathbb{N}$ which are of nature B_v.

- the nodes $(5 + 3k, 0, 2)$ with $k \in \mathbb{N}$ which are of nature B_v.

- $(1, 3, 1)$ which is of nature B_v, $(3, 2, 1)$ which is of nature A, $(1, 2, 2)$ which is of nature R and $(3, 1, 2)$ which is of nature A.

2.2 Order on nodes of a bipartition

Let $\boldsymbol{\lambda}$ be a bipartition and γ_1 (resp. γ_2) be an extended node of $\boldsymbol{\lambda}$ or an addable node of $\boldsymbol{\lambda}$. We write $\gamma_1 <_{\mathbf{s}} \gamma_2$ if and only if

- $\mathrm{cont}(\gamma_1) < \mathrm{cont}(\gamma_2)$ or,

- $\mathrm{cont}(\gamma_1) = \mathrm{cont}(\gamma_2)$, $c_1 = 2$ and $c_2 = 1$.

Let now $\mathfrak{j} \in I$. Then the nodes in $\mathcal{E}_{\mathfrak{j}}(\boldsymbol{\lambda})$ are all comparable and we see that we have $\gamma_1 <_{\mathbf{s}} \gamma_2$ for such two nodes and that these two nodes are consecutive if we are in one of the following two cases:

- $\mathrm{cont}(\gamma_1) = \mathrm{cont}(\gamma_2)$, $c_1 = 2$, $c_2 = 1$,

- $\mathrm{cont}(\gamma_1) + e = \mathrm{cont}(\gamma_2)$, $c_1 = 1$, $c_2 = 2$, (if e is finite, if $e = \infty$ only the above case occurs because we only have two nodes with a given content).

2.3 Order on bipartitions

We now define a total order $\preceq_{\mathbf{s}}$ on the set of bipartitions depending on $\mathbf{s}$. To do this, let $\boldsymbol{\lambda} \in \mathcal{P}^2(n)$, consider the set of all the nodes on the vertical boundary of $\boldsymbol{\lambda}$ and write them in decreasing order with respect to $<_{\mathbf{s}}$ (such nodes are always comparable because there are no two nodes of the same content within the vertical boundary of a given component). We obtain an infinite sequence $(\gamma_i(\boldsymbol{\lambda}))_{i \in \mathbb{Z}_{>0}}$ which will be called the *boundary sequence*. We write $\boldsymbol{\lambda} \preceq_{\mathbf{s}} \boldsymbol{\mu}$ if and only $\boldsymbol{\lambda} = \boldsymbol{\mu}$ or if there exists $j > 0$ such that

$$\forall 0 < i < j, \ \gamma_i(\boldsymbol{\lambda}) = \gamma_i(\boldsymbol{\mu}), \ \text{and} \ \gamma_j(\boldsymbol{\lambda}) <_{\mathbf{s}} \gamma_j(\boldsymbol{\mu}).$$

This ordering works for arbitrary charges. In a special case, this specialises to the more familiar lexicographic ordering of Dipper–James–Mathas, as we will now show. Consider the lexicographic order on the set of bipartitions, that is, $\boldsymbol{\lambda} \leqslant \boldsymbol{\mu}$ if and only $\lambda = \mu$ or if there exists $j > 0$ such that

$$\forall 0 < i < j, \ \lambda_i^1 = \mu_i^1, \ \text{and} \ \lambda_j^1 < \mu_j^1$$

or $\lambda^1 = \mu^1$ and there exists $j > 0$ such that

$$\forall 0 < i < j, \ \lambda_i^2 = \mu_i^2, \ \text{and} \ \lambda_j^2 < \mu_j^2.$$

We write $\boldsymbol{\lambda} \prec_{\mathbf{s}} \boldsymbol{\mu}$ if $\boldsymbol{\lambda} \preceq_{\mathbf{s}} \boldsymbol{\mu}$ and $\boldsymbol{\lambda} \neq \boldsymbol{\mu}$. We have the following particular case.

Proposition 3. *Assume that* $\mathbf{s} = (s_1, s_2)$ *is such that* $s_1 - s_2 > n - 1$ *then we have*

$$\boldsymbol{\lambda} \preceq_{\mathbf{s}} \boldsymbol{\mu} \iff \boldsymbol{\lambda} \leqslant \boldsymbol{\mu}.$$

Proof. Assume that $s_1 - s_2 > n - 1$ then if $\gamma = (a, b, c)$ and $\eta = (a', b', c')$ are two nodes on the vertical boundary of $\boldsymbol{\lambda}$ we have that $\gamma \leqslant_{\mathbf{s}} \eta$ if and only if:

- $c' = 1$ and $c = 2$,

- $c = c'$ and $a < a'$.

This implies that the order $\preceq_{\mathbf{s}}$ is the same as the usual lexicographic order on bipartitions. $\square$

Remark 4. Of course, the order $\preceq_{\mathbf{s}}$ strongly depends on the choice of $\mathbf{s}$ and it does not correspond to the lexicographic order in general if $s_1 - s_2 \leqslant n - 1$.

2.4 Boundary sequence and nature of nodes

We study in details the relations between the boundary sequence and the nature of some nodes of a bipartition. Let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ be a bipartition and $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. For each $j \in \mathbb{Z}$ and each component, there is a unique extended node with content j which is in this component and which is either addable, or in the boundary of $\boldsymbol{\lambda}$.

Also note that this node is of nature B_v or R if and only if j appears in the boundary sequence. As a consequence, one may easily obtain the boundary sequence from a table, called the *table of natures*, listing all the nature of the extended nodes associated with each $j \in \mathbb{Z}$ (representing the content of the node) and $c \in \{1, 2\}$ (representing the component of the node) as in the following examples.

Example 5. Let $\mathbf{s} = (0, 1)$ and consider the bipartition $\boldsymbol{\lambda} = (6.1, 2.2)$.

$$\left(\begin{array}{c} \boxed{7} \ldots \ldots \\ \boxed{0}\boxed{1}\boxed{2}\boxed{3}\boxed{4}\boxed{5} \\ \boxed{-1} \\ \boxed{-3} \\ \vdots \end{array} , \begin{array}{c} \boxed{4}\boxed{5}\boxed{6}\boxed{7} \ldots \ldots \\ \boxed{1}\boxed{2} \\ \boxed{0}\boxed{1} \\ \boxed{-2} \\ \boxed{-3} \\ \vdots \end{array} \right)$$
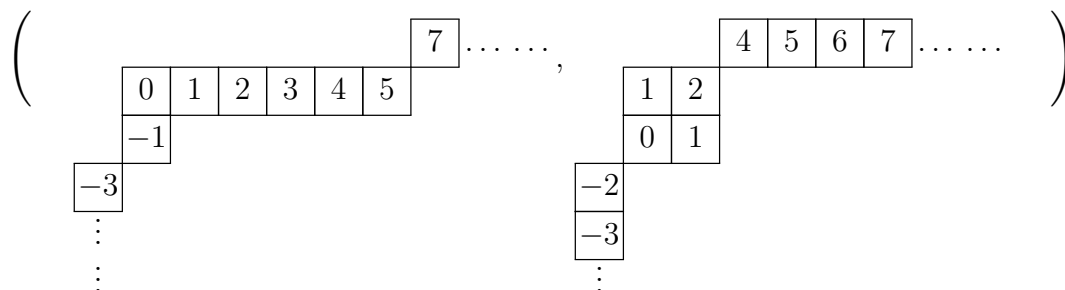
The table of natures gives the nature of all the (extended) nodes which are either addable, or in the boundary, of content between -3 and 6 written in increasing order with respect to $(0, 1)$ for $\boldsymbol{\lambda}$. The nodes of content greater than 6 are all virtual nodes of the horizontal boundary and the nodes of content less than -3 are all virtual nodes of the vertical boundary

Component	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
Content	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6
$(6.1, 2.2)$	$\mathbf{B_v}$	$\mathbf{B_v}$	$\mathbf{B_v}$	A	A	$\mathbf{R}$	B_h	A	$\mathbf{R}$	B_h	$\mathbf{B_v}$	B_h	A	B_h	B_h	B_h	B_h	$\mathbf{R}$	B_h	A

From this, we can write the infinite sequence of §2.3 which allows to compare bipartitions with respect to $\preceq_{\mathbf{s}}$. If a node is of nature B_v or R, then this means that this node is in this sequence otherwise, it is not. For example, take the bipartition $(6.3, 2.1)$ then with respect to $(0, 1)$ we have:

Component	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
Content	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6
$(6.3, 2)$	$\mathbf{B_v}$	$\mathbf{B_v}$	$\mathbf{B_v}$	A	$\mathbf{B_v}$	B_h	A	B_h	B_h	$\mathbf{R}$	$\mathbf{R}$	A	A	B_h	B_h	B_h	B_h	$\mathbf{R}$	B_h	A

and we immediately see that $(6.1, 2.2) \prec_{(0,1)} (6.3, 2)$.

In $\boldsymbol{\mu}$ is another bipartition, then we say that the two extended or addable nodes of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ with the same content and in the same component has the same *general nature* if the nature of these nodes are both in $\{R, B_v\}$ or both in $\{A, B_h\}$. Hence, one can compare two bipartitions with respect to $\prec_{\mathbf{s}}$ by looking at the table of natures of $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ and by checking when, starting from the left, the general nature of extended or addable nodes differs for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. In the above case, this happens for the content 1 and component 1.

3 Uglov bipartitions

In this section, we define the notion of Uglov bipartitions, recall the context in which they appear and recall several properties.

3.1 Definition

Let $\mathfrak{g}_e$ be the quantum affine algebra of type $A_{e-1}^{(1)}$. Let $\mathbf{s} \in \mathbb{Z}^2$. For each $n \in \mathbb{Z}_{\geqslant 0}$, we have a $\mathbb{C}$-vector space:

$$\mathcal{F}_n = \bigoplus_{\boldsymbol{\lambda} \in \mathcal{P}^2(n)} \mathbb{C}\boldsymbol{\lambda}.$$

The Fock space of level two is then the following $\mathbb{C}$-vector space:

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}_{\geqslant 0}} \mathcal{F}_n.$$

There is an action of $\mathfrak{g}$ on $\mathcal{F}$ which gives a structure of an integrable module. In particular, the action of the Chevalley operators e_{j} and f_{j} (with $\mathsf{j} \in I$) on the Fock space are given by:

$$\forall \mathsf{j} \in I, \forall \boldsymbol{\lambda} \in \mathcal{P}^2(n),\ f_{\mathsf{j}}\boldsymbol{\lambda} = \sum_{\mathcal{Y}(\boldsymbol{\mu})=\mathcal{Y}(\boldsymbol{\lambda})\sqcup\{\gamma\},\ \mathrm{res}(\gamma)=\mathsf{j}} \boldsymbol{\mu}, e_{\mathsf{j}}\boldsymbol{\lambda} = \sum_{\mathcal{Y}(\boldsymbol{\lambda})=\mathcal{Y}(\boldsymbol{\mu})\sqcup\{\gamma\},\ \mathrm{res}(\gamma)=\mathsf{j}} \boldsymbol{\mu}.$$

The submodule $V_{\mathbf{s}}$ generated by the empty bipartition is then an irreducible highest weight module with weight $\Lambda_{s_1} + \Lambda_{s_2}$ (where the Λ_i's, $i \in I$, denote the fundamental weights).

3.2 Canonical bases

Let $\boldsymbol{\lambda}$ be a bipartition, and $\mathsf{j} \in I$. We consider the set of addable and removable j-nodes of $\boldsymbol{\lambda}$. We define a word obtained by reading these nodes in the increasing order with respect to $\prec_{\mathbf{s}}$. If a removable j-node appears just before an addable j-node, we delete both and continue the same procedure as many times as possible. In the end, we reach a word of nodes such that the first p nodes are addable and the last q nodes are removable, for some $p, q \in \mathbb{N}$. If $p > 0$, let γ be the rightmost addable j-node. If it exists, the node γ is called the *good* j-node of $\boldsymbol{\lambda}$.

By definition, $\boldsymbol{\lambda}$ is said to be an *Uglov bipartition* of rank $n > 0$ if there exists a sequence of bipartitions $\boldsymbol{\lambda}^{[1]} := (\varnothing, \varnothing), \boldsymbol{\lambda}^{[2]}, \ldots, \boldsymbol{\lambda}^{[n]} := \boldsymbol{\lambda}$ such that for each $j \in \{1, \ldots, n\}$, the bipartition $\boldsymbol{\lambda}^{[j]}$ is in $\mathcal{P}^2(j)$ and such that for each $j \in \{2, \ldots, n\}$ $\boldsymbol{\lambda}^{[j]}$ is obtained by adding a good j-node to $\boldsymbol{\lambda}^{[j-1]}$.

The set of Uglov bipartitons of rank n is denoted by $\Phi_{\mathbf{s}}^e(n)$ and the set of all Uglov bipartitions by $\Phi_{\mathbf{s}}^e$. Note that $(\varnothing, \varnothing)$ is the unique Uglov bipartition with rank 0.

Let us come back to the submodule $V_{\mathbf{s}}$ of the Fock space. By [8, Th 6.6.14] (this is a result by Uglov using Kashiwara-Lusztig theory of canonical bases for quantum groups), there exists a basis, called the canonical basis for $V_{\mathbf{s}}$ (as a $\mathbb{C}$-vector space) which is labeled by the set of Uglov bipartitions:

$$\{G(\boldsymbol{\lambda}, \mathbf{s}) \mid \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e\}.$$

In fact, we even get that the set

$$\{G(\boldsymbol{\lambda}, \mathbf{s}) \mid \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e(n)\}$$

is a basis of $V_{\mathbf{s}} \cap \mathcal{F}_n$.

This basis enjoys nice properties which we not recall here (see [8, Th. 6;6;11]). All we need in the folllowing is:

Proposition 6 (Uglov [17]). *For each $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e(n)$, we have:*

$$G(\boldsymbol{\lambda}, \mathbf{s}) = \boldsymbol{\lambda} + \text{Linear combination of } \boldsymbol{\mu} \text{ of rank } n \text{ with } \boldsymbol{\mu} \prec_{\mathbf{s}} \boldsymbol{\lambda}.$$

Lemma 7. *For all sequences of residues* $(\mathrm{j}_1, \ldots, \mathrm{j}_n) \in I^n$, *the maximal element with respect to* $\prec_{\mathbf{s}}$ *in* $f_{\mathrm{j}_1} \ldots f_{\mathrm{j}_n}.\varnothing$ *is an Uglov bipartition of rank n.*

Proof. Each canonical basis element is a linear combination of bipartitions and the maximal one with respect to $\preceq_{\mathbf{s}}$ is an Uglov bipartition. As $f_{\mathrm{j}_1} \ldots f_{\mathrm{j}_n}.\varnothing$ is a linear combination of these canonical basis elements, the results follows $\qquad\square$

3.3 Bijections between Uglov bipartitions

Let $\widehat{\mathfrak{S}}_2$ be the (extended) affine symmetric group. We denote by $P_2 := \mathbb{Z}^2$ the $\mathbb{Z}$-module with standard basis $\{y_1, y_2\}$. We denote by σ_1 the generator of $\mathbb{Z}/2\mathbb{Z}$. Then $\widehat{\mathfrak{S}}_2$ can be seen as the semi-direct product $P_2 \rtimes \mathbb{Z}/2\mathbb{Z}$ where the relations are given by $\sigma_1 y_1 \sigma_1 = y_2$. This group acts faithfully on $\mathbb{Z}^2$ by setting for any $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$:

$$\begin{aligned}
\sigma_1.(s_1, s_2) &= (s_2, s_1), \\
y_1.(s_1, s_2) &= (s_1 + e, s_2), \\
y_2.(s_1, s_2) &= (s_1, s_2 + e).
\end{aligned}$$

Set $\tau := \sigma_1 y_1$ then $\widehat{\mathfrak{S}}_2$ is generated by σ_1 and τ. Moreover, we have a fundamental domain for this action given by:

$$\mathcal{S}_e := \left\{ (s_1, s_2) \in \mathbb{Z}^2 \mid 0 \leqslant s_1 \leqslant s_2 < e \right\}.$$

If $\mathbf{s}$ is in $\mathcal{S}_e$, the set $\Phi_{\mathbf{s}}^e$ has a nice non recursive definition (see [8, Def. 5.7.8]) but such definition is not available in the general case. Nevertheless, one can use the following method to compute the sets $\Phi_{\mathbf{s}}^e$ in the general case.

We know that if $\mathbf{s}_1$ and $\mathbf{s}_2$ are in the same class modulo the action of $\widehat{\mathfrak{S}}_2$ then both modules $V_{\mathbf{s}_1}$ and $V_{\mathbf{s}_2}$ are isomorphic and there is a bijection

$$\Psi_{\mathbf{s}_1 \to \mathbf{s}_2}^e : \Phi_{\mathbf{s}_1}^e \to \Phi_{\mathbf{s}_2}^e,$$

which enjoys nice properties with respect to the module structure. In particular, this is a crystal isomorphism. This means that if $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}_1}^e$ and if $\boldsymbol{\lambda}' \in \Phi_{\mathbf{s}_1}^e$ is obtained from $\boldsymbol{\lambda}$ by removing a good j-node. Then $\Psi_{\mathbf{s}_1 \to \mathbf{s}_2}^e(\boldsymbol{\lambda})$ admits a removable good j-node and if we remove it, we obtain a bipartition $\boldsymbol{\mu}'$ satisfying $\Psi_{\mathbf{s}_1 \to \mathbf{s}_2}^e(\boldsymbol{\lambda}') = \boldsymbol{\mu}'$.

- if $\mathbf{s}_2 = \tau.\mathbf{s}_1$, by [12, Prop. 3.1(2)], we have

$$\Psi_{\mathbf{s}_1 \to \mathbf{s}_2}^e(\lambda^1, \lambda^2) = (\lambda^2, \lambda^1).$$

- if $\mathbf{s}_2 = \sigma_1.\mathbf{s}_1$, the bijection has been combinatorially described in [12]. We don't need the explicit form of this bijection but only some properties which will be recall once we need them.

Otherwise, the map $\Psi^e_{\mathbf{s}_1 \to \mathbf{s}_2}$ is a composition of maps of the above form. If $\boldsymbol{\lambda} \in \Phi^e_{\mathbf{s}_1}$ then the Uglov bipartitions $\boldsymbol{\mu} \in \Phi^e_{\sigma.\mathbf{s}_1}$ such that $\Psi^e_{\mathbf{s}_1 \to \sigma.\mathbf{s}_1}(\boldsymbol{\lambda}) = \boldsymbol{\mu}$ for all $\sigma \in \widehat{\mathfrak{S}}_2$ are said to be the bipartitions in the isomorphism class of $\boldsymbol{\lambda}$.

Example 8. Take $\mathbf{s} = (0, 1)$ and $e = 3$. We consider the bipartition $\boldsymbol{\lambda} = (6.1, 2.2)$ which is in $\Phi^e_{\mathbf{s}}$. Then one can compute $\Psi^e_{\mathbf{s} \to \sigma.\mathbf{s}}(\boldsymbol{\lambda})$ for $\sigma \in \widehat{\mathfrak{S}}_2$ using the algorithm in [12] or one can use the program given in [14]. We get for all $k \in \mathbb{Z}$:

$$\Psi^e_{(0,1) \to (1+3k,0)}(\boldsymbol{\lambda}) = \begin{cases} (5.2.1, 3) & \text{if } k \geqslant 0 \\ (2.2, 6.1) & \text{if } k = -1 \\ (2.1, 6.1.1) & \text{if } k < -1 \end{cases} \qquad \Psi^e_{(0,1) \to (0,1+3k)}(\boldsymbol{\lambda}) = \begin{cases} (3, 5.2.1) & \text{if } k > 1 \\ (6.1.1, 2.1) & \text{if } k < 0 \end{cases}$$

Thus one may recover the set of Uglov bipartitions $\Phi^e_{\mathbf{s}}$ for all $\mathbf{s} \in \mathbb{Z}^2$ from the set $\Phi^e_{\mathbf{s}'}$ where $\mathbf{s}' \in \mathcal{S}_e$ is in the orbit of $\mathbf{s}$ modulo the above action and from the use of the above known bijections.

Finally, note that, given a bipartition $\boldsymbol{\lambda} \in \Phi^e_{\mathbf{s}}$, the isomorphism $\Psi^e_{\mathbf{s} \to \sigma.\mathbf{s}}$ affects the table of natures for the j-nodes. In fact, this table may be easily obtained from the one of $\boldsymbol{\lambda}$ in the case where $\sigma := \tau$: we just have to translate the natures of the nodes by one box to the right for the nodes in component 2. For the nodes in component 1, we have to translate them by $2e - 1$ boxes. Here is an example for $e = 2$

Component	...	2	1	2	1	2	1	2	1	2	1	...
Content	...	-2	-2	-1	-1	0	0	1	1	2	2	...
$\boldsymbol{\lambda}$	...	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	.	...
$\Psi^e_{\mathbf{s} \to \tau.\mathbf{s}}(\boldsymbol{\lambda})$	...	?	X_1	?	X_3	X_2	X_5	X_4	X_7	X_6	X_9	...

Such table is less elementary in the case $\sigma = \sigma_1$ but a table in [13, §6.1.2] explains the different possibilities: for a given content $j \in \mathbb{Z}$ and two addable or extended nodes j-nodes γ_2 and γ_1 of content j in component 2 and 1, the natures of the two associated nodes in component 1 and 2 are transformed into nodes with specific natures:

For example if γ_2 is of nature R and γ_1 of nature B_h then the nodes γ_2' and γ_1' of content j in component 2 and 1 of $\Psi^e_{\mathbf{s} \to \sigma_1.\mathbf{s}}(\boldsymbol{\lambda})$ may be of nature R and B_h, or B_h and R.

Note however that there is a nice property which is verified by the bijection $\Psi^e_{\mathbf{s} \to \sigma_1.\mathbf{s}}$: by [15], this bijection does not depend on e and we thus have $\Psi^e_{\mathbf{s} \to \sigma.\mathbf{s}} = \Psi^\infty_{\mathbf{s} \to \sigma.\mathbf{s}}$.

Example 9. Consider the bipartition $\boldsymbol{\lambda} = (6.1, 2.2)$ which is in $\Phi^3_{(0,1)}$. We have already seen that $\boldsymbol{\mu} := \Psi^3_{(0,1) \to (1,0)}(5.2.1, 3)$. Write the Young tableau of these two bipartitions with the associated content:

Nodes in λ		Nodes in $\Psi^e_{s\to\sigma_1.s}(\lambda)$	
Component 2	Component 1	Component 2	Component 1
R	R	R	R
A	R	A	R
B_v	R	R	B_v
		B_v	R
B_h	R	B_h	R
R	A	R	A
		B_h	B_v
A	A	A	A
B_v	A	B_v	A
		A	B_v
B_h	A	B_h	A
R	B_h	R	B_h
		B_h	R
A	B_h	A	B_h
		B_h	A
B_v	B_h	R	A
		B_v	B_h
		B_h	B_v
B_h	B_h	B_h	B_h
R	B_v	R	B_v
A	B_v	A	B_v
B_v	B_v	B_v	B_v
B_h	B_v	B_h	B_v

$$\left(\ \boxed{\begin{array}{c}\end{array}}\ \boxed{7}\ldots\ldots\ ,\quad \boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}\ldots\ldots\ \right)$$

Component 2 (left): row $\boxed{0}\,\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}$ with $\boxed{7}$ above the rightmost; $\boxed{-1}$; $\boxed{-3}$; $\vdots$

Component 1 (right): $\boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}$; $\boxed{1}\,\boxed{2}$; $\boxed{0}\,\boxed{1}$; $\boxed{-2}$; $\boxed{-3}$; $\vdots$

$$\left(\ \boxed{7}\,\boxed{8}\ldots\ ,\quad \boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}\ldots\ \right)$$

Component 2 (left): $\boxed{7}\,\boxed{8}$ above; row $\boxed{1}\,\boxed{2}\,\boxed{3}\,\boxed{4}\,\boxed{5}$; $\boxed{0}\,\boxed{1}$; $\boxed{-1}$; $\boxed{-3}$; $\vdots$

Component 1 (right): $\boxed{4}\,\boxed{5}\,\boxed{6}\,\boxed{7}$; $\boxed{0}\,\boxed{1}\,\boxed{2}$; $\boxed{-2}$; $\boxed{-3}$; $\vdots$

The following table gives the nature of all the nodes of content between -3 and 6 written in increasing order with respect to $(0,1)$ for $\boldsymbol{\lambda}$ and $(1,0)$ for $\boldsymbol{\mu}$

Component	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1
Content	-3	-3	-2	-2	-1	-1	0	0	1	1	2	2	3	3	4	4	5	5	6	6
$(6.1, 2.2)$	$\mathbf{B_v}$	$\mathbf{B_v}$	$\mathbf{B_v}$	A	A	$\mathbf{R}$	B_h	A	$\mathbf{R}$	B_h	$\mathbf{B_v}$	B_h	A	B_h	B_h	B_h	B_h	$\mathbf{R}$	B_h	A
$(5.2.1, 3)$	$\mathbf{B_v}$	$\mathbf{B_v}$	$\mathbf{B_v}$	A	A	$\mathbf{R}$	B_h	A	B_h	$\mathbf{R}$	$\mathbf{R}$	A	A	B_h	B_h	B_h	B_h	$\mathbf{R}$	B_h	A

Remark 10. The isomorphism between $V_{\mathbf{s}_1}$ and $V_{\mathbf{s}_2}$ implies the following fact. Assume that $(\mathsf{j}_1, \ldots, \mathsf{j}_n)$ is a sequence of element in I then $f_{\mathsf{j}_1} \ldots f_{\mathsf{j}_n}.\varnothing$ writes as a linear combinaison of the canonical basis elements:

$$f_{\mathsf{j}_1} \ldots f_{\mathsf{j}_n}.\varnothing = \sum_{\boldsymbol{\lambda} \in \Phi^e_{\mathbf{s}_1}} a_{\boldsymbol{\lambda}} G(\boldsymbol{\lambda}, \mathbf{s}_1)$$

Then we have:

$$f_{\mathsf{j}_1} \ldots f_{\mathsf{j}_n}.\varnothing = \sum_{\boldsymbol{\lambda} \in \Phi^e_{\mathbf{s}_1}} a_{\boldsymbol{\lambda}} G(\Psi^e_{\mathbf{s}_1 \to \sigma.\mathbf{s}_1}(\boldsymbol{\lambda}), \mathbf{s}_2)$$

In the case where $\mathbf{s} \in \mathcal{S}_e$, the Uglov bipartitions are then known as FLOTW bipartitions and they have a non recursive description given as follows:

Proposition 11 (Foda-Leclerc-Okado-Thibon-Welsh). *Assume that* $\mathbf{s} = (s_1, s_2) \in \mathcal{S}_e$. *The set* $\Phi^e_{\mathbf{s}}$ *of Uglov bipartitions is the set of bipartitions* $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ *such that:*

1. *for all* $i \in \mathbb{Z}_{>0}$, *we have:*

$$\lambda^1_i \geqslant \lambda^2_{i+s_2-s_1},$$
$$\lambda^2_i \geqslant \lambda^1_{i+e+s_1-s_2};$$

2. *for all* $k > 0$, *among the residues of the nodes of the vertical boundary of the form* (a, λ^c_a, c) *with* $a \in \mathbb{Z}_{>0}$, $c \in \{1, 2\}$ *and* $\lambda^c_a = k$, *at least one element of* $\{0, 1, \ldots, e-1\}$ *does not occur.*

3.4 Properties of Uglov bipartitions

In the following, we will need several technical properties of Uglov bipartitions. Let $\mathbf{s} \in \mathbb{Z}^2$ and $\boldsymbol{\lambda} \in \Phi^e_{\mathbf{s}}$.

12. Assume that for $\mathsf{j} \in I$, we have a list of exactly m normal j-nodes:

$$\eta_1 <_{\mathbf{s}} \eta_2 <_{\mathbf{s}} \ldots <_{\mathbf{s}} \eta_m,$$

then if follows from the definition of the crystal isomorphism (see §3.3) that for all $\mathbf{s}' \in \mathbb{Z}^2$ and $\Psi^e_{\mathbf{s} \to \sigma.\mathbf{s}'}(\boldsymbol{\lambda}) =: \boldsymbol{\mu}$, we have a list of exactly m normal j-nodes in $\boldsymbol{\mu}$:

$$\eta'_1 <_{\mathbf{s}} \eta'_2 <_{\mathbf{s}} \ldots <_{\mathbf{s}} \eta'_m$$

For all $1 \leqslant i \leqslant m$, one can thus canonically associate to the normal j-node η_i of $\boldsymbol{\lambda}$ the normal j-node η_i of $\boldsymbol{\mu}$.

13. Let $\gamma <_{\mathbf{s}} \gamma'$ be two removable j-nodes with $\mathsf{j} \in I$. We say that γ and γ' are (1)-*connected* if we have the following property: if we remove γ' from $\boldsymbol{\lambda}$ then there exists a set of nodes $(\gamma_1, \ldots, \gamma_e)$ in the vertical boundary of the extended Young diagram of the resulting bipartition $\boldsymbol{\lambda}'$, with for all $i \in \{1, \ldots, e-1\}$, we have $\mathrm{cont}(\gamma_{i+1}) = \mathrm{cont}(\gamma_i) + 1$ and $c_{i+1} \geqslant c_i$. In this case, we say that we have *period* in $\boldsymbol{\mu}$ and we have $\boldsymbol{\lambda}' \notin \Phi_{\mathbf{s}}^e$ by [15, Prop. 5.1]. In particular, it is easy to see that having a period for a bipartition satisfying (1) in Def. 11 is equivalent to violate the condition (2) in Def. 11.

Assume in addition that γ and γ' are normal j-nodes. Let $\mathbf{s}'$ be in the orbit of $\mathbf{s}$ and take $\boldsymbol{\mu} := \Psi_{\mathbf{s} \to \mathbf{s}'}^e(\boldsymbol{\lambda})$. Then γ and γ' correspond to two normal nodes in $\boldsymbol{\mu}$ which we denote by η and η'. It follows from the combinatorial description of the bijections that we cannot have a j-node of the vertical boundary η'', consecutive to η and such that $\eta \prec_{\mathbf{s}} \eta''$. Indeed, in this case, one may check that we obtain a period in $\boldsymbol{\mu}$ which contradicts the fact that $\boldsymbol{\mu} \in \Phi_{\mathbf{s}'}^e$

Example 14. The following example illustrates how the bijection $\Psi_{\mathbf{s} \to \sigma_1 \mathbf{s}}^e$ acts on the property of being (1)-connected. is

Let $\mathbf{s} = (0, 1)$, $e = 3$ and consider $\boldsymbol{\lambda} = (3.2.2.1.1, 3.3.1)$.
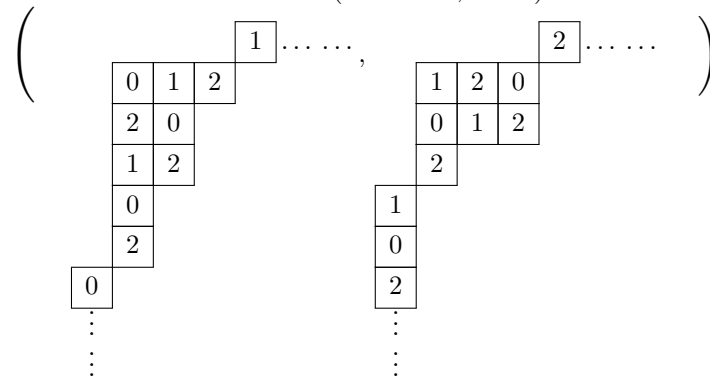
We see that the node $(1, 3, 1)$ is (1)-connected to $(3, 2, 1)$ which is itself (1)-connected to $(5, 1, 1)$; and $(2, 3, 2)$ is (1)-connected to $(3, 2, 1)$. Note that $\Psi_{(0,1) \to (1,0)}^3(3.2.2.1.1, 3.3.1) = (3.3.2.2.1.1, 3.1)$.
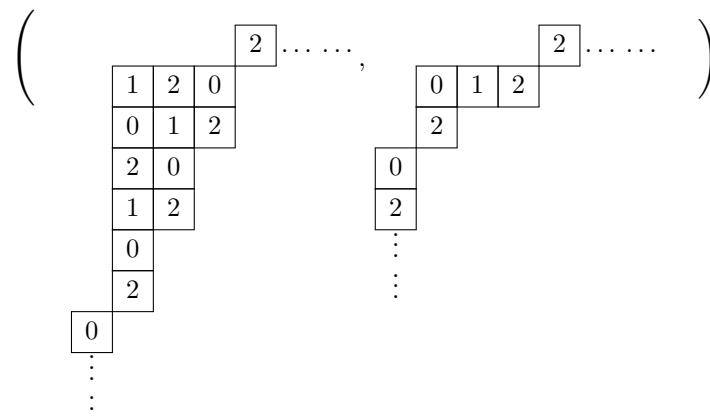
We see that the node $(2, 3, 1)$ is (1)-connected to $(4, 2, 1)$ which is itself (1)-connected to $(6, 1, 1)$; and $(1, 3, 2)$ is (1)-connected to $(4, 2, 1)$.

3.5 Main result

The main result of this paper is:

Theorem 15 (Generalized Dipper-James-Murphy's conjecture)**.** *Let $e \in \mathbb{Z}_{>1} \sqcup \{\infty\}$ and $\mathbf{s} \in \mathbb{Z}^2$ then $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e(n)$ if and only if there exists a sequence of residues $\mathfrak{i}_1, \dots, \mathfrak{i}_n$ in I and integers $c_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ for $\boldsymbol{\mu} \in \mathcal{P}^2(n)$ such that:*

$$f_{\mathfrak{i}_1} \dots f_{\mathfrak{i}_n}.\varnothing = c_{\boldsymbol{\lambda},\boldsymbol{\lambda}}\boldsymbol{\lambda} + \sum_{\boldsymbol{\mu} \prec_{\mathbf{s}} \boldsymbol{\lambda}} c_{\boldsymbol{\lambda},\boldsymbol{\mu}}\boldsymbol{\mu}.$$

with $c_{\boldsymbol{\lambda},\boldsymbol{\lambda}} \neq 0$.

See also section 4.4, for a reformulation of this result. We will also see several consequences of this result. The aim of this rest of the paper is to prove this Theorem thanks to the previous preparatory materials and to present two consequences.

4 Proof of the main result

We now prove Theorem 15. The main point is to associate to each Uglov bipartition, a certain sequence of residues which will play the role of the sequence $\mathfrak{i}_1, \dots, \mathfrak{i}_n$ in I in the Theorem. The important property about this sequence is that it will be an invariant on the isomorphism class of an Uglov bipartition.

4.1 Admissible residue sequence: FLOTW case

In this subsection, we let $\mathbf{s} = (s_1, s_2) \in \mathcal{S}_e$ and $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e$. First let us give another definition which will be necessary to define our sequence of nodes. Let $\gamma_1 := (a, b, c)$ be a removable $\mathfrak{j}$-node for some $\mathfrak{j} \in I$.

- If $c = 1$ and $\lambda_a^1 = \lambda_{a+s_2-s_1}^2$ then we set $\gamma_2 = (a + s_2 - s_1, \lambda_{a+s_2-s_1}^2, 2)$.

- If $c = 2$ and $\lambda_a^2 = \lambda_{a+e+s_1-s_2}^1$ then we set $\gamma_2 = (a + e + s_1 - s_2, \lambda_{a+e+s_1-s_2}^1, 1)$.

then we say that γ_1 and γ_2 are (2)-*connected*. If γ and η are two removable $\mathfrak{j}$-nodes then we set $\gamma \equiv \eta$ if γ and η are (1) or (2) *connected* and we consider the transitive closure of this relation.

Now let $\boldsymbol{\lambda} = (\lambda^1, \lambda^2)$ be in $\Phi_{\mathbf{s}}^e$. Consider the maximal removable node $\gamma_1 = (a, b, c)$ with respect to $<_{\mathbf{s}}$ and denote by $\mathfrak{j}$ its residue. Note that by the definition of FLOTW l-partition, there cannot exist a node on the vertical boundary with the same residue greater than γ_1. We now consider the sequence of removable $\mathfrak{j}$-nodes given by all the removable $\mathfrak{j}$-nodes in the equivalence class. We write it $(\gamma_1, \dots, \gamma_r)$ (written in increasing order).

Proposition 16. *Under the above hypotheses, we have the following properties:*

1. *for all $k = 1, \ldots, r$ the bipartition obtained by removing the $\mathfrak{j}$-nodes $(\gamma_k, \ldots, \gamma_r)$ from $\boldsymbol{\lambda}$ is in $\Phi^e_{\mathbf{s}}$.*

2. *γ_1 is greater than any addable $\mathfrak{j}$-node of $\boldsymbol{\lambda}$ and any $\mathfrak{j}$-node of the vertical boundary of $\boldsymbol{\lambda}$*

3. *if there exists a non virtual $\mathfrak{j}$-node of the horizontal boundary which is greater than γ_1 then γ_1 is (1) connected with a node of the sequence $(\gamma_2, \ldots, \gamma_{r-1})$*

Proof. Point 1 just follows from the definition of FLOTW bipartitions. Point 2 has already been stated (it also follows from [13, Lemma 4.2.5] and [13, Lemma 4.2.6]). Let us consider the last point. Assume that there exists a non virtual $\mathfrak{j}$-node of the horizontal boundary which is greater than γ_1 and take the smallest such node (note that such node cannot be greater than γ_r). This situation occurs only in the case where this node is between two nodes γ_i and γ_j, with $j < i$ which are (1)-connected. From the properties of FLOTW bipartitions, we see that γ_j must be (1)-connected with a node γ_k, with $k < j$ etc. (if it is (1)-connected to its successive node then it must be (1)-connected with another by the definition of FLOTW bipartitions) and then the result follows. $\qquad\square$

Example 17. We illustrate the above proof with an example : let $e = 3$, $\mathbf{s} = (0, 2)$ and $\boldsymbol{\lambda} = (3.1.1, 3.2.2.1.1)$ which is in $\Phi^e_{\mathbf{s}}$. Write its extended Young diagram:
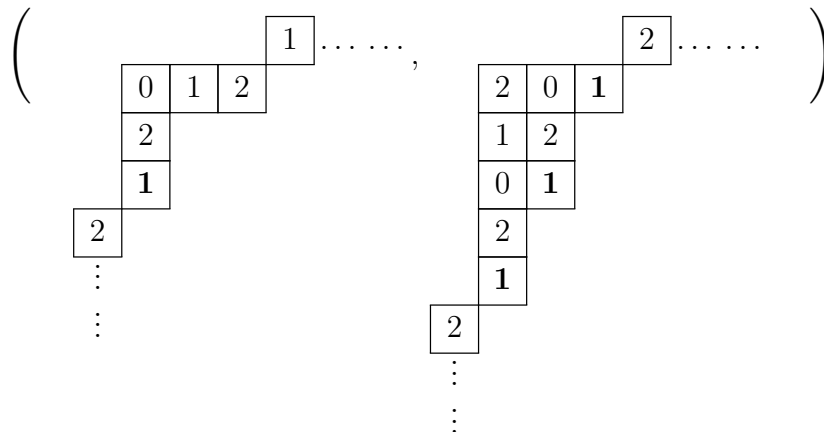
We here have $\gamma_1 = (5, 1, 2)$, $\gamma_2 = (3, 1, 1)$, $\gamma_3 = (3, 2, 2)$, $\gamma_4 = (1, 3, 2)$. We have a node $(1, 2, 1)$ of nature B_h between γ_3 and γ_4 and we see that γ_1 is (1)-connected with γ_3 which is itself (1)-connected to γ_4.

The admissible residue sequence $\mathrm{Adm}(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ is then defined recursively as follows. Let $\boldsymbol{\lambda}'$ be the FLOTW bipartitions obtained after removing the $\mathfrak{j}$-nodes $(\gamma_1, \ldots, \gamma_r)$ from $\boldsymbol{\lambda}$

$$\mathrm{Adm}(\boldsymbol{\lambda}) = \mathrm{Adm}(\boldsymbol{\lambda}'), \underbrace{\mathfrak{j}, \ldots, \mathfrak{j}}_{r \text{ times}}$$

4.2 Admissible residue sequence: general case

We now explain how one can extend the notion of an admissible residue sequence to the case of an arbitrary Uglov bipartition.

Lemma 18. *Let* $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$ *and and let* $\boldsymbol{\lambda}$ *be in* $\Phi_{\mathbf{s}}^e$. *For* $\mathrm{j} \in I$, *denote by*

$$\gamma_1 <_{\mathbf{s}} \gamma_2 <_{\mathbf{s}} \ldots <_{\mathbf{s}} \gamma_N$$

the normal j-*nodes of* $\boldsymbol{\lambda}$. *Let* $\sigma \in \widehat{\mathfrak{S}}_2$. *Then* $\boldsymbol{\mu} := \Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^e(\boldsymbol{\lambda})$ *admits exactly* N *normal* j-*nodes:*

$$\eta_1 <_{\sigma.\mathbf{s}} \eta_2 <_{\sigma.\mathbf{s}} \ldots <_{\sigma.\mathbf{s}} \eta_N.$$

Assume that there exists $1 \leqslant m \leqslant N$ *such that, for all* $N \geqslant k \geqslant m$, *the bipartition* $\boldsymbol{\lambda}'$ *obtained by removing* $\gamma_m, \ldots \gamma_N$ *from* $\boldsymbol{\lambda}$ *is in* $\Phi_{\mathbf{s}}^e$. *Then the bipartition* $\boldsymbol{\mu}'$ *obtained by removing* $\eta_m, \ldots \eta_N$ *from* $\boldsymbol{\mu}$ *is in* $\Phi_{\sigma.\mathbf{s}}^e$ *and we have* $\boldsymbol{\mu}' = \Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^e(\boldsymbol{\lambda}')$

Proof. First note that it is sufficient to prove the lemma in the case where $\sigma = \sigma_1$ (in the case where $s_1 \leqslant s_2$) and $\sigma = \tau$. The lemma is trivial in the case where $\sigma = \tau$ because then $\boldsymbol{\mu} = (\lambda^2, \lambda^1)$ and the bijection leave the order on j-nodes invariant (see [13, §6.1.1]). Let us thus consider the case $\sigma = \sigma_1$ and $s_1 \leqslant s_2$. Taking the notations of the theorem, we need to show that $\Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^e(\boldsymbol{\lambda}')$ is the bipartition obtained by removing the $N - m + 1$ greatest normal j-nodes of $\boldsymbol{\mu}$. To do this, we will essentially use the main result of [15], which has been already mentioned and which asserts that the isomorphism $\Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^e$ does not depend on e. We can thus take $e = \infty$ to compute it. In this case, the residue of the node becomes its content and we thus have only two possible nodes with a given residue.

Assume that the node γ_m is on component 1 and that its content is $j \in \mathbb{Z}$. By [15, Prop. 4.1.1], we have that $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^\infty$ and γ_m is a good j-node for $e = \infty$. Moreover, we have $\boldsymbol{\mu} = \Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^\infty(\boldsymbol{\lambda})$ and the bipartition $\boldsymbol{\mu}'$ obtained by removing the unique j-node from $\boldsymbol{\mu}$ satisfies $\boldsymbol{\mu}' = \Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^\infty(\boldsymbol{\lambda}')$ where $\boldsymbol{\lambda}'$ is the bipartition obtained by removing γ_m from $\boldsymbol{\lambda}$. This follows from the fact that $\Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^\infty = \Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^e$ is a crystal isomorphism. In addition, we have $\boldsymbol{\mu}' = \Psi_{\mathbf{s} \to \sigma.\mathbf{s}}^e(\boldsymbol{\lambda}')$ as this map does not depend on e.

If γ_m is on component 2 and if its content is $j \in \mathbb{Z}$ and if there is no other removable node of content j, we conclude in the same manner. If we have two removable nodes γ_m and γ_{m+1} with content $j \in \mathbb{Z}$, we argue again in the same manner by removing these two nodes (which are are successively good nodes in the case $e = \infty$). We conclude by induction. $\qquad \square$

Let $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. Let $\boldsymbol{\mu}$ be in $\Phi_{\mathbf{s}}^e$. Let $\mathbf{s}'$ be the element in $\mathcal{S}_e$ which is in the orbit of $\mathbf{s}$ modulo the action of $\widehat{\mathfrak{S}}_2$. Let $\boldsymbol{\lambda} := \Psi_{\mathbf{s} \to \mathbf{s}'}^e(\boldsymbol{\mu})$. Consider the admissible residue sequence of $\boldsymbol{\lambda}$, then we define the admissible residue sequence of $\boldsymbol{\mu}$ to be this residue sequence. We know that the normal j-nodes:

$$\gamma_1 <_{\mathbf{s}'} \gamma_2 <_{\mathbf{s}'} \ldots <_{\mathbf{s}'} \gamma_N$$

associated to the admissible residue sequence of $\boldsymbol{\lambda}$ are canonically associated to normal j-nodes:

$$\eta_1 <_{\mathbf{s}} \eta_2 <_{\mathbf{s}} \ldots <_{\mathbf{s}} \eta_N$$

of $\boldsymbol{\mu}$. By the result above, if we remove these nodes from $\boldsymbol{\mu}$ then we still have an Uglov bipartition which is in the isomorphism class of the bipartition obtained by removing $(\gamma_1, \gamma_2, \ldots, \gamma_N)$ from $\boldsymbol{\lambda}$. This sequence of nodes have others interesting properties.

Proposition 19. *Under the above hypotheses,*

1. *there is no addable* j*-node in* $\boldsymbol{\mu}$ *greater than* η_1.

2. *If there is a non virtual* j*-node of nature* B_h *greater than* η_1 *in* $\boldsymbol{\mu}$ *then there is no* j*-node of nature* B_v *greater than* η_1.

Proof. We argue by contradiction and assume first that there is a non virtual j-node of nature B_h and a j-node of nature B_v greater than η_1. We have a sequence of bipartitions:

$$\boldsymbol{\lambda}[1] := \boldsymbol{\lambda}, \boldsymbol{\lambda}[2], \ldots, \boldsymbol{\lambda}[m] := \boldsymbol{\mu}$$

and a sequence of elements in $\mathbb{Z}^2$:

$$\mathbf{s}[1] := \mathbf{s}, \mathbf{s}[2], \ldots, \mathbf{s}[m] := \mathbf{s}'$$

where $s[j] := \sigma.s[j-1]$ with $\sigma = \sigma_1$ or τ and such that $\boldsymbol{\lambda}[j] = \Psi^e_{\mathbf{s}[j-1] \to \mathbf{s}[j]}(\boldsymbol{\lambda}[j-1])$ for $j = 2, \ldots, m$. Keeping the notation of this section, by the table in §3.3, there is a non virtual j-node of nature B_h greater than γ_1. Thus by Proposition 16, γ_1 must be (1)-connected with one of the other nodes γ_j Now, by the definition of the vertical boundary, with the node of nature B_v comes two nodes on the vertical boundary with residue j and $\mathsf{j} - 1$. By the discussion in §13, there exists $k \in \{1, \ldots, m\}$, such that in $\boldsymbol{\lambda}[k] \in \Phi^e_{\mathbf{s}[k]}$, we have a j-node γ'_1 associated to γ_1 which is (1)-connected with another node and such that there exist a j-node of the vertical boundary η which is consecutive to γ'_1 and such that $\gamma'_1 <_{\mathbf{s}} \eta$. By the discussion in §13, this is impossible for an Uglov bipartition.

The first point follows in fact from the second: if we have such an addable node, this means that there exists $j \in \{1, \ldots, m\}$ such that in $\boldsymbol{\lambda}[j]$, the associated j-nodes

$$\eta'_1 <_{\mathbf{s}} \eta_2 <_{\mathbf{s}} \ldots <_{\mathbf{s}} \eta'_N$$

are such that we have a sequence of two consecutive nodes of nature, respectively B_v and B_h greater than η'_1 (see the table in §3.3). This is thus impossible by the above discussion. $\qquad\square$

Example 20. Take $\mathbf{s} = (0, 1)$ and $e = 3$. We consider the bipartition $\boldsymbol{\lambda} = (6.1, 2.2)$ which is in $\Phi^e_{\mathbf{s}}$. Then one can compute $\Psi^e_{\mathbf{s} \to \sigma.\mathbf{s}}(\boldsymbol{\lambda})$ for $\sigma \in \widehat{\mathfrak{S}}_2$ using the algorithm in [12] or one can use the program given in [14]. We get for all $k \in \mathbb{Z}$:

$$\Psi^e_{(0,1)\to(1+3k,0)}(\boldsymbol{\lambda}) = \begin{cases} (5.2.1, 3) & \text{if } k \geqslant 0 \\ (2.2, 6.1) & \text{if } k = -1 \\ (2.1, 6.1.1) & \text{if } k < -1 \end{cases} \qquad \Psi^e_{(0,1)\to(0,1+3k)}(\boldsymbol{\lambda}) = \begin{cases} (3, 5.2.1) & \text{if } k > 1 \\ (6.1.1, 2.1) & \text{if } k < 0 \end{cases}$$

We can compute the admissible sequence of residues associated to $\boldsymbol{\lambda}$ and thus to all the bipartitions in its isomorphism class. It is given by:

$$1, 0, 2, 2, 1, 1, 2, 0, 1, 1, 2.$$

$$\left(\quad \boxed{\begin{array}{cccccc} & & & & & \boxed{1}\ \ldots\ldots \\ \boxed{0}\ \boxed{1}\ \boxed{2}\ \boxed{0}\ \boxed{1}\ \boxed{2} \\ \boxed{2} \\ \boxed{0} \\ \vdots \end{array}} \ , \quad \boxed{\begin{array}{cc} & \boxed{1}\ \ldots\ldots \\ \boxed{1}\ \boxed{2} \\ \boxed{0}\ \boxed{1} \\ \boxed{1} \\ \vdots \end{array}} \quad \right)$$

4.3 The proof

To prove our theorem, we will show the following result. Let $e \in \mathbb{Z}_{>1} \sqcup \{\infty\}$ and $\mathbf{s} \in \mathbb{Z}^2$. Let $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e(n)$ and denote $\mathrm{Adm}(\boldsymbol{\lambda}) = \mathfrak{i}_1, \ldots, \mathfrak{i}_n$. Then there exist integers $c_{\boldsymbol{\lambda},\boldsymbol{\mu}}$ for $\boldsymbol{\mu} \in \mathcal{P}^2(n)$ such that:

$$f_{\mathfrak{i}_n} \ldots f_{\mathfrak{i}_1}.\varnothing = c_{\boldsymbol{\lambda},\boldsymbol{\lambda}}\boldsymbol{\lambda} + \sum_{\boldsymbol{\mu} \prec_{\mathbf{s}} \boldsymbol{\lambda}} c_{\boldsymbol{\lambda},\boldsymbol{\mu}}\boldsymbol{\mu}.$$

with $c_{\boldsymbol{\lambda},\boldsymbol{\lambda}} \neq 0$.

Note that the admissible sequence of residue is by definition an invariant on the isomorphism class of an Uglov bipartition. Thus, it suffices to prove the following three properties:

1. The theorem is true in the case where $\mathbf{s} = (s_1, s_2) \in \mathcal{S}_e$,

2. If the theorem is true for all Uglov bipartitions $\boldsymbol{\lambda} \in \Phi_e^{\mathbf{s}}(n)$ with $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$ such that $s_1 \leqslant s_2$ then it is true for all Uglov bipartitions $\boldsymbol{\lambda} \in \Phi_e^{\sigma_1.\mathbf{s}}(n)$.

3. If the theorem is true for all Uglov bipartitions $\boldsymbol{\lambda} \in \Phi_e^{\mathbf{s}}(n)$ with $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$ such that $s_1 \geqslant s_2$ then it is true for all Uglov bipartitions $\boldsymbol{\lambda} \in \Phi_e^{\tau.\mathbf{s}}(n)$.

Point 1 is in fact a weak version of [8, Lemma 5.7.20] (the partial order $\ll_{\mathbf{m}}$ used in the book satisfies $\boldsymbol{\lambda} \ll_{\mathbf{m}} \boldsymbol{\mu} \Rightarrow \boldsymbol{\lambda} \prec_{\mathbf{s}} \boldsymbol{\mu}$). So we need to prove 2 and 3. To do this, assume that $\mathbf{s} = (s_1, s_2) \in \mathbb{Z}^2$. Let $n \in \mathbb{Z}_{>0}$. One may assume that our result is true for all Uglov bipartitions $\boldsymbol{\lambda} \in \Phi_e^{\mathbf{s}}(n)$. We show that the result is still true for the Uglov bipartitions $\Psi_{\mathbf{s}\to\sigma_1.\mathbf{s}}^e(\boldsymbol{\lambda}) =: \boldsymbol{\lambda}^\sigma$ (in which case we assume $s_1 \leqslant s_2$) and $\Psi_{\mathbf{s}\to\tau.\mathbf{s}}^e(\boldsymbol{\lambda}) =: \boldsymbol{\lambda}^\tau$ (in which case we assume $s_1 \geqslant s_2$).

By the argument above, it suffices to consider $\sigma \in \{\sigma_1, \tau\}$. We remove the a_k greatest removable $\mathfrak{j}_k$-nodes from $\boldsymbol{\lambda}^\sigma$. Let $\widetilde{\boldsymbol{\lambda}}^\sigma$ be the resulting Uglov bipartition. Assume that $\boldsymbol{\mu}^\sigma$ is the maximal element with respect to $\prec_{\sigma.\mathbf{s}}$ appearing in $f_{\mathfrak{i}_1}^{a_1} \ldots f_{\mathfrak{i}_k}^{a_k}.\varnothing$ and that $\boldsymbol{\mu}^\sigma \neq \boldsymbol{\lambda}^\sigma$. The aim is to show a contradiction.

By induction, we have that $\widetilde{\boldsymbol{\mu}}^\sigma \prec_{\mathbf{s}} \widetilde{\boldsymbol{\lambda}}^\sigma$. This implies that there exists $j \in \mathbb{Z}$ and a component $c \in \{1, 2\}$ such that:

- The node of $\widetilde{\boldsymbol{\lambda}}^\sigma$ with content j and component c is of nature B_v or B_h,

- The node of $\widetilde{\boldsymbol{\mu}}^\sigma$ with content j and component c is of nature A and the node of $\boldsymbol{\mu}^\sigma$ with content j and component c is of nature R.

- All the nodes in $\boldsymbol{\lambda}^\sigma$ and $\boldsymbol{\mu}^\sigma$ greater than the one of content j and component c have the same general nature.

21 (Case 1). The node of $\widetilde{\boldsymbol{\lambda}}^\sigma$ with content j and component c is of nature B_h.

- Assume that $c = 2$, the table of natures reads as follows:

Component	...	2	1	2	1	...
Content	...	$j-1$	$j-1$	j	j	...
$\boldsymbol{\mu}^\sigma$	...	X_1	X_2	R	X_3	...
$\boldsymbol{\lambda}^\sigma$	...	Y_1	Y_2	B_h	Y_3	...

As we have $\widetilde{\boldsymbol{\mu}}^\sigma \prec_{\sigma_1.\mathbf{s}} \widetilde{\boldsymbol{\lambda}}^\sigma$ and because of our assumptions, we must have the following table of natures:

Component	...	2	1	2	1	...
Content	...	$j-1$	$j-1$	j	j	...
$\widetilde{\boldsymbol{\mu}}^\sigma$	...	$\begin{cases} B_v \\ R \end{cases}$	$\begin{cases} B_h \\ A \end{cases}$	A	$\begin{cases} B_h \\ R \end{cases}$	...
$\widetilde{\boldsymbol{\lambda}}^\sigma$	...	$\begin{cases} B_h \\ A \end{cases}$	$\begin{cases} B_v \\ R \end{cases}$	B_h	$\begin{cases} B_v \\ A \end{cases}$	...

Assume that the node of $\widetilde{\boldsymbol{\lambda}}^\sigma$ with content j in component 1 is of nature A then the one in $\widetilde{\boldsymbol{\mu}}^\sigma$ is of nature B_h and we have $Y_3 = R$ and $X_3 = B_h$, we thus have $\boldsymbol{\mu}^\sigma \prec_{\sigma.\mathbf{s}} \boldsymbol{\lambda}^\sigma$.

Assume that the node of $\widetilde{\boldsymbol{\lambda}}^\sigma$ with content j in component 1 is of nature B_v. Then $Y_3 = B_v$ and $X_3 = R$. We immediately see that $\boldsymbol{\mu}^{\sigma_1} \prec_{\sigma_1.\mathbf{s}} \boldsymbol{\lambda}^{\sigma_1}$. Let us consider the case $\sigma = \tau$. By Proposition 19, we have that the node of nature B_h in component 2 of $\widetilde{\boldsymbol{\lambda}}^\tau$ is virtual. Thus the node in component 1 and content $j - e$ of $\widetilde{\boldsymbol{\lambda}}$ is of nature B_h and virtual. This implies that the node in component 1 and content $j - e$ of $\boldsymbol{\lambda}$ is of nature B_h and virtual.

Note that we have $s_2 \geqslant s_1$, so one may consider the map $\Psi^e_{\sigma_1.\mathbf{s} \to \mathbf{s}}$ which has been explicitly described combinatorially (because then $\sigma_1.\mathbf{s} =: (s_2, s_1)$). The algorithm for the computation of this bijection shows the following. If a pair of consecutive nodes in components 2 and 1 are transformed into a pair of nodes in components 2 and 1 with nature $X \in \{A, R, B_h, B_v\}$ and B_h (with B_h virtual), this implies that the two nodes are of nature B_h (with B_h virtual). We deduce that the node in component 1 and content $j - e$ of $\boldsymbol{\lambda}$ is also of nature B_h and virtual. But then $Y_3 \neq B_v$ and we get a contradiction.

- Assume that $c = 1$, the table of natures reads as follows:

Component	...	2	1	2	1	...
Content	...	$j-1$	$j-1$	j	j	...
$\boldsymbol{\mu}^\sigma$	...	X_1	X_2	X_3	R	...
$\boldsymbol{\lambda}^\sigma$	...	Y_1	Y_2	Y_3	B_h	...

If $X_3 \in \{B_v, R\}$ then we immediately see that that $\boldsymbol{\lambda}^{\sigma_1} \prec_{\sigma_1.s} \boldsymbol{\mu}^\sigma$. If $X_3 \in \{A, B_h\}$ then if $Y_3 \in \{A, B_h\}$, we again conclude in the same manner. We have to consider the case where $Y_3 \in \{B_v, R\}$, we again conclude that $\boldsymbol{\lambda}^{\sigma_1} \prec_{\sigma_1.s} \boldsymbol{\mu}^{\sigma_1}$ (because of the table giving the modification of the nature of nodes by isomorphism, the nature of the nodes of content j in component 1 must be R for $\boldsymbol{\mu}^{\sigma_1}$ and B_h for $\boldsymbol{\lambda}^{\sigma_1}$). Thus we have $\boldsymbol{\lambda}^{\sigma_1} <_{\sigma_1.s} \boldsymbol{\mu}^{\sigma_1}$. The result is clear for $\boldsymbol{\mu}^\tau$ and $\boldsymbol{\lambda}^\tau$.

22 (Case 2). The node of $\widetilde{\boldsymbol{\lambda}}^\sigma$ with content j and component c is of nature B_v.

- Assume that $c = 2$, the node of content j in component 2 in $\boldsymbol{\mu}$ is R thus it implies that the node of content $j - 1$ in component 2 in $\boldsymbol{\mu}$ is B_h or A. As the node of content j in component 2 in $\boldsymbol{\lambda}$ is B_v thus it implies that the node of content $j - 1$ in component 2 in $\boldsymbol{\lambda}$ is B_v or R.

Component	...	2	1	2	1	...
Content	...	$j - 1$	$j - 1$	j	j	...
$\boldsymbol{\mu}^\sigma$	...	$\begin{cases} B_h \\ A \end{cases}$	X_1	R	X_2	...
$\boldsymbol{\lambda}^\sigma$	...	$\begin{cases} B_v \\ R \end{cases}$	Y_1	B_v	Y_2	...

But as X_2 and Y_2 have the same general nature, we must have $Y_1 \in \{B_h, A\}$ and $X_1 \in \{B_v, R\}$. This implies that $Y_2 \in \{R, B_h\}$ and $X_2 \in \{A, B_v\}$. But $Y_2 = R$ is impossible and $Y_2 = B_h$ implies $X_2 = A$ which contradicts the maximality of $\boldsymbol{\mu}^\sigma$.

- Assume that $c = 1$, again, we see that we have the following configuration:

Component	...	2	1	2	1	...
Content	...	$j - 1$	$j - 1$	j	j	...
$\boldsymbol{\mu}^\sigma$	...	X_1	$\begin{cases} B_h \\ A \end{cases}$	X_2	R	...
$\boldsymbol{\lambda}^\sigma$	...	Y_1	$\begin{cases} B_v \\ R \end{cases}$	Y_2	B_v	...

We have that $X_2 \in \{R, B_v\}$ and $Y_2 \in \{A, B_h\}$. One can obtain the desired contradiction using the same reasoning as in case 1 with $c = 2$.

Remark 23. Of course, one can ask if a similar result and proof can be obtained in the case of multipartitions. Most of the results presented in the last sections are still true but one cannot argue as in this section to conclude.

4.4 Reformulation

One can rephrase the main result as follows in terms of Young diagrams (see [8, §3.5.10]). To do this, let us introduce some more notations. Let $\boldsymbol{\lambda}$ be a bipartition of n. A bijection

$\mathfrak{s} : \mathcal{Y}(\boldsymbol{\lambda}) \to \{1, \ldots, n\}$ is called a $\boldsymbol{\lambda}$-*tableau* and we also say that $\boldsymbol{\lambda}$ is the shape of $\mathfrak{s}$. A $\boldsymbol{\lambda}$-tableau $\mathfrak{s}$ is called *row-standard* if the sequence $\mathfrak{s}(a, 1, c)$, $\mathfrak{s}(a, 2, c)$, $\ldots$ is strictly increasing for each a and $c \in \{1, 2\}$. The *residue sequence* of a $\boldsymbol{\lambda}$-tableau $\mathfrak{s}$ is defined by $\eta_e(\mathfrak{s}) = (\mathfrak{i}_1, \ldots, \mathfrak{i}_n)$ where $\mathfrak{i}_j$ is the residue of the node which is filled by the number j in $\mathfrak{s}$. The theorem now becomes:

Corollary 24. *Let $e \in \mathbb{Z}_{>1} \sqcup \{\infty\}$ and $\mathbf{s} \in \mathbb{Z}^2$ then $\boldsymbol{\lambda}$ is in $\Phi_{\mathbf{s}}^e$ if and only if for all bipartition $\boldsymbol{\mu}$ admitting a row standard $\boldsymbol{\mu}$-tableau $\mathfrak{s}$ such that $\eta_e(\mathfrak{s}) = \mathrm{Adm}(\boldsymbol{\lambda})$, we have $\boldsymbol{\mu} \prec_{\mathbf{s}} \boldsymbol{\lambda}$.*

5   Consequences

We quickly show how we obtain an application on the computation of canonical bases (this application is exactly the same as the one presented in [13]). Let $\mathbf{s} := (s_1, s_2) \in \mathbb{Z}^2$ and $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e$, consider the associated admissible residue sequence:

$$\mathrm{Adm}(\boldsymbol{\lambda}) = \underbrace{\mathfrak{j}_1, \ldots, \mathfrak{j}_1}_{a_1}, \ldots, \underbrace{\mathfrak{j}_k, \ldots, \mathfrak{j}_k}_{a_k},$$

where $\mathfrak{j}_m \in I$ and $a_m \in \mathbb{Z}_{>0}$ for all $m \in \{1, \ldots, k\}$ and where we assume that $\mathfrak{j}_m \neq \mathfrak{j}_{m+1}$ for all $s \in \{1, \ldots, k-1\}$. We have now to work on the quantum group $\mathcal{U}_v$ of affine type A (which can be seen as a deformation of $\mathfrak{g}_e$). It acts on the Fock space (where v is an indeterminate). With the same proof as [8, Thm 6.4.2], we obtain that

$$f_{\mathfrak{j}_1}^{(a_1)} \ldots f_{\mathfrak{j}_k}^{(a_k)} \varnothing = \boldsymbol{\lambda} + \sum_{\boldsymbol{\mu} \prec_{\mathbf{s}} \boldsymbol{\lambda}} c_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(v) \boldsymbol{\mu},$$

for Laurent polynomials $c_{\boldsymbol{\lambda}, \boldsymbol{\mu}}(v)$ and where the $f_{\mathfrak{i}}^{(a)}$'s for $a \in \mathbb{Z}_{>0}$ stand for the divided powers of the Chevalley operators (see [8]). The specialization at $v = 1$ of the above expression corresponds to the elements of the Theorem. As a consequence, as in [13], we obtain a LLT algorithm-like for the computation of the canonical basis elements.

One can also deduce from that the existence of basic sets for Hecke algebras of type B_n. We refer to [8] for motivations and results around this theory. Let u be an indeterminate, let V_1 and V_2 be two indeterminates, let R be a commutative ring with unit such that $\mathbb{Z} \subset R \subset \mathbb{C}$ and let $A := R[u^{\pm 1}, V_1, V_2]$. Let K be the field of fractions of A let $\mathbf{H}_n$ be the Hecke algebra of type B_n over A with generators $\{T_i \mid i = 0, \ldots, n-1\}$ and relations:

- $(T_i - u)(T_i + 1) = 0$ for $i = 1, \ldots, n-1$,

- $(T_0 - V_1)(T_0 - V_2) = 0$,

- the type B braid relations: $T_i T_j = T_j T_i$ with $|i - j| > 1$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leqslant i \leqslant n-2$, and $T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0$.

Let us denote $\mathbf{H}_{K,n} = K \otimes_A \mathbf{H}_n$ then the simple $\mathbf{H}_{K,n}$-modules are parametrized by the set of bipartitions. By Tits deformation theorem (see [8, §1.2]), we have:

$$\mathrm{Irr}(\mathbf{H}_{K,n}) = \{E^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{P}^2(n)\}$$

Let $\theta : A \to L$ is a specialisation for a field L and assume that:

$$\theta(u) \neq 1, \ \theta(V_1) = \theta(u)^{s_1}, \theta(V_2) = \theta(u)^{s_2}$$

Let $e > 1$ be the order of $\theta(u) \in L^{\times}$. The associated specalized $\mathbf{H}_{L,n} = L \otimes_A \mathbf{H}_n$ is non semisimple in general and the representation theory is controled by the decomposition matrix.

For each $\boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e$, one can define the representation of $\mathbf{H}_{K,n}$:

$$P_{\boldsymbol{\lambda}} = E^{\boldsymbol{\lambda}} \oplus \bigoplus_{\boldsymbol{\mu} \prec_{\mathbf{s}} \boldsymbol{\lambda}} c_{\boldsymbol{\lambda},\boldsymbol{\mu}} E^{\boldsymbol{\mu}}.$$

The set of representations

$$\{P_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e\}$$

satisfies the hypotheses of [8, Prop. 3.4.5]. This implies that $\mathbf{H}_{k,n}$ admits a basic set which is given by:

$$\{E_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Phi_{\mathbf{s}}^e\}$$

with respect to the partial order $\prec_{\mathbf{s}}$. This means that the associated decomposition matrix is lower unitriangular and thus that $\Phi_{\mathbf{s}}^e(n)$ is a natural parametrization set for the simple $\mathbf{H}_{L,n}$-modules. Note that this result is independent of the characteristic. We refer to [4] for an analogous result obtained by a completely different process.

Acknowledgements

References

[1] S. Ariki, Representations of quantum algebras and combinatorics of Young tableaux. Translated from the 2000 Japanese edition and revised by the author. University Lecture Series, 26. American Mathematical Society, Providence, RI, (2002).

[2] S. Ariki and N. Jacon, Dipper-James-Murphy's conjecture for Hecke algebras of type B. Representation Theory of Algebraic Groups and Quantum Groups, *Progress in Mathematics*, 2011, Volume 284, 17-32.

[3] A. Ariki, V. Kreiman, S. Tsuchioka, On the tensor product of two basic representations of $U_v(\widehat{sl}_e)$. *Advances in Mathematics*, Volume 218, Issue 1, 1 May 2008, Pages 28-86.

[4] C. Bowman, The many graded cellular bases of Hecke algebras. arXiv:1702.06579., 2017.

[5] M. Chlouveraki, I. Gordon and S. Griffeth, Cell modules and canonical basic sets for Hecke algebras from Cherednik algebras. New trends in noncommutative algebra, 77-89, *Contemporary Mathematics*, 562, Amer. Math. Soc., Providence, RI, 2012.

[6] R. Dipper, G. D. James and E. Murphy, Hecke Algebras of Type B_n at Roots of Unity. *Proceedings of the London Mathematical Society* s3-70(3):505-528.

[7] M. Geck and N. Jacon, Canonical basic sets in type B. *Journal of Algebra* 306 (2006), 104-127.

[8] M. Geck and N. Jacon, Representations of Hecke algebras at roots of unity. Algebra and Applications, Springer-Verlag London, Ltd., London, 2011.

[9] T.Gerber, G. Hiss and N. Jacon, Harish-Chandra series in finite unitary groups and crystal graphs. *International Mathematics Research Notices* 22 (2015), 12206-12250.

[10] J. Hu, On a generalisation of the Dipper-James-Murphy conjecture, *Journal of Combinatorial Theory Series A* 118 (2011), no. 1, 78-93.

[11] J. Hu, K. Zhou and D. Wang, On the generalised Dipper-James-Murphy conjecture in quantum characteristic 2, *Monatshefte für Mathematik* 181(3), 2016.

[12] N. Jacon, Crystal graphs of irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of level two and Uglov bipartitions. *Journal of Algebraic Combinatorics* 27 (2008), no. 2, 143-162

[13] N. Jacon, Kleshchev multipartitions and extended Young diagram. *Advances in Mathematics*, 339, 1 December 2018, pages 367-403.

[14] N. Jacon, GAP program "crystal.g" for the computation of crystal isomorphisms in affine type A. Available at http://njacon.perso.math.cnrs.fr/.

[15] N. Jacon and C. Lecouvey, Crystal isomorphisms for irreducible highest weight $\mathcal{U}_q(\widehat{\mathfrak{sl}}_e)$-modules of higher level. *Algebras and Representation Theory* 13, 2010, no. 4, 467-489.

[16] B.Leclerc and H. Miyachi, Constructible characters and canonical bases. *Journal of Algebra* 277(2004), no. 1, p. 298-317.

[17] D. Uglov, Canonical Bases of Higher-Level q-Deformed Fock Spaces and Kazhdan-Lusztig Polynomials. Physical Combinatorics pp 249-299, *Progress in Mathematics* book series (PM, volume 191).