Subdivisions in digraphs of large out-degree or large dichromatic number*

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Abstract

In 1985, Mader conjectured the existence of a function f such that every digraph with minimum out-degree at least $f(k)$ contains a subdivision of the transitive tournament of order k. This conjecture is still completely open, as the existence of $f(5)$ remains unknown. In this paper, we show that if D is an oriented path, or an in-arborescence (i.e., a tree with all edges oriented towards the root) or the union of two directed paths from x to y and a directed path from y to x, then every digraph with minimum out-degree large enough contains a subdivision of D. Additionally, we study Mader's conjecture considering another graph parameter. The *dichromatic number* of a digraph D is the smallest integer k such that D can be partitioned into k acyclic subdigraphs. We show that any digraph with dichromatic number greater than $4^m(n-1)$ contains every digraph with n vertices and m arcs as a subdivision.

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1 Introduction

For each positive integer k, we denote by $[k]$ the subset of the natural numbers $\{1, \dots, k\}$.

Mader [19] established the following.

Theorem 1 (Mader [19])**.** *There exists an integer $g(k)$ such that every graph with minimum degree at least $g(k)$ contains a subdivision of K_k.*

For $k \leqslant 4$, we have $g(k) = k - 1$ as first proved by Dirac [11]; for $k = 5$, we have the estimate $6 \leqslant g(5) \leqslant 7$ by Thomassen [26, 29]. In general, the order of growth of $g(k)$ is k^2 as shown in [5] and [17].

Similarly, it would be interesting to find analogous results for digraphs. However, the obvious analogue that a digraph with sufficiently large minimum in- and out-degree contains a subdivision of the complete digraph of order n is false as shown by Mader [20].

Let γ be a digraph parameter. A digraph F is γ-maderian if there exists a least integer $\text{mader}_\gamma(F)$ such that every digraph D with $\gamma(D) \geqslant \text{mader}_\gamma(F)$ contains a subdivision of F.

For a digraph D, $\delta^+(D)$ (resp. $\delta^-(D)$) denote the minimum out-degree (resp. in-degree) and $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. A natural question is to ask which digraphs F are δ^+*-maderian* (resp. δ^0*-maderian*). Observe that every δ^+-maderian digraph is also δ^0-maderian and that $\text{mader}_{\delta^+} \geqslant \text{mader}_{\delta^0}$.

On the positive side, Mader conjectured that every acyclic digraphs is δ^+-maderian. Since every acyclic digraph is the subdigraph of the transitive tournament on the same order, it is enough to prove that transitive tournaments are δ^+-maderian.

Conjecture 2 (Mader [20])**.** There exists a least integer $\text{mader}_{\delta^+}(TT_k)$ such that every digraph D with $\delta^+(D) \geqslant \text{mader}_{\delta^+}(TT_k)$ contains a subdivision of TT_k.

Mader proved that $\text{mader}_{\delta^+}(TT_4) = 3$, but even the existence of $\text{mader}_{\delta^+}(TT_5)$ is still open.

This conjecture implies directly that transitive tournaments (and thus all acyclic digraphs) are δ^0-maderian.

Conjecture 3. There exists a least integer $\text{mader}_{\delta^0}(TT_k)$ such that every digraph D with $\delta^0(D) \geqslant \text{mader}_{\delta^0}(TT_k)$ contains a subdivision of TT_k.

In fact, Conjecture 3 is equivalent to Conjecture 2 because if transitive tournaments are δ^0-maderian, then $\text{mader}_{\delta^+}(TT_k) \leqslant \text{mader}_{\delta^0}(TT_{2k})$ for all k. Indeed, let D be a digraph with minimum out-degree $\text{mader}_{\delta^0}(TT_{2k})$, and let D' be the digraph obtained from disjoint copies of D and its converse (the digraph obtained by reversing all arcs) $\overline{D}$ by adding all arcs from $\overline{D}$ to D. Clearly, $\delta^0(D') \geqslant \text{mader}_{\delta^0}(TT_{2k})$. Therefore D' contains a subdivision of TT_{2k}. Hence, either D or $\overline{D}$ (and so D) contains a subdivision of TT_k.

Both conjectures are equivalent, but the above reasoning does not prove that a δ^0-maderian digraph is also δ^+-maderian. The case of *oriented trees* (i.e. orientations of undirected trees) is typical. Using a simple greedy procedure, one can easily find every oriented tree of order k in every digraph with minimum in- and out-degree k (so

$\text{mader}_{\delta^0}(T) = |T| - 1$ for any oriented tree T). On the other hand, it is still open whether oriented trees are δ^+-maderian and a natural important step towards Conjecture 2 would be to prove the following weaker one.

Conjecture 4. Every oriented tree is δ^+-maderian.

We give evidences to this conjecture. First, in Subsection 2.1, we prove that every *oriented path* (i.e. orientation of an undirected path) P is δ^+-maderian and that $\text{mader}_{\delta^+}(P) = |V(P)| - 1$. Next, in Subsection 2.2, we consider arborescences. An *out-arborescence* (resp. *in-arborescence*) is an oriented tree in which all arcs are directed away from (resp. towards) a vertex called the *root*. Trivially, the simple greedy procedure shows that $\text{mader}_{\delta^+}(T) = |T| - 1$ for every out-arborescence. In contrast, the fact that in-arborecences are δ^+-maderian is not obvious since we have no control on the in-degree of each vertex in a digraph of out-degree at least k. We show in Theorem 23 that the in-arborescences are δ^+-maderian.

In [21], Mader gave another partial result towards Conjecture 2. He proved the existence of a function $f_1(k)$ such that every digraph D with $\delta^+(D) \geqslant f_1(k)$ contains a pair of vertices u and v with k disjoint directed paths between u and v. It is however not known if we can insist on these paths to be arbitrarily long. In Theorem 24, we do the first step toward this question by proving it in the case $k = 2$.

Conjecture 2 states that the acyclic digraphs are δ^+-maderian. However, they are not the only ones. For example, it is folklore that every digraph with minimum out-degree at least 1 contains a directed cycle, which is a subdivision of $\vec{C}_2$, the directed 2-cycle. More generally, one can easily show, by considering a directed path of maximum length, that every digraph with minimum out-degree at least $k-1$ contains a directed cycle of length at least k. In other words, $\text{mader}_{\delta^+}(\vec{C}_k) = k - 1$, where $\vec{C}_k$ denotes the directed k-cycle. Furthermore Alon [3] showed that every digraph with minimum out-degree at least $64k$ contains k disjoint directed cycles, which forms a subdivision of the disjoint union of k copies of $\vec{C}_2$. A celebrated conjecture of Bermond and Thomassen [4] states that the bound $64k$ can be decreased to $2k - 1$.

Conjecture 5 (Bermond and Thomassen [4])**.** For every positive integer k, every digraph with minimum out-degree at least $2k - 1$ contains k disjoint directed cycles.

In [3], Alon also conjectured the following.

Conjecture 6 (Alon [3])**.** There exists a function h such that every digraph with minimum out-degree $h(k)$ has a partition (V_1, V_2) such that, for $i = 1, 2$, $D\langle V_i \rangle$ has minimum out-degree k.

The difficulty of this question is remarkable, as the existence of $h(2)$ still remains open. If true, this conjecture implies the following one.

Conjecture 7. If F_1 and F_2 are δ^+-maderian, then the disjoint union of F_1 and F_2 is also δ^+-maderian.

Partial positive answers to this conjecture can be obtained via the Erdős-Pósa Property. For a digraph F and an integer k, we denote by $k \times F$ the disjoint union of k copies of F. A digraph F is said to have the *Erdős-Pósa Property* if for every positive integer k, there exists $\phi(k)$, such that every digraph D either contains a subdivision of $k \times F$, or has a set S of size (at most) $\phi(k)$ such that $D - S$ contains no subdivision of F.

Theorem 8. *If F is δ^+-maderian and has the Erdős-Pósa Property, then $k \times F$ is also δ^+-maderian for all positive integer k.*

Proof. Let F be a maderian digraph having the Erdős-Pósa Property for some function ϕ. Let D be a digraph with $\delta^+(D) \geqslant \phi(k) + \text{mader}_{\delta^+}(F)$. For every set S of size at most $\phi(k)$, $\delta^+(D - S) \geqslant \text{mader}_{\delta^+}(F)$, so $D - S$ contains a subdivision of F. Thus, since F has the Erdős-Pósa Property, D contains a subdivision of $k \times F$. □

Reed et al. [23] proved that $\vec{C}_2$ has the Erdős-Pósa Property, and using the Directed Grid Theorem [16], Akhoondian et al. [2] showed that many digraphs have the Erdős-Pósa Property, in particular all disjoint unions of directed cycles. Hence, Theorem 8 implies that disjoint unions of directed cycles are δ^+-maderian.

On the negative side, Thomassen [27] showed a construction of digraphs with arbitrarily large in- and out-degree and no directed cycles of even length (see also [10]). This gives a large class of digraphs that are not δ^0-maderian, namely: digraphs such that all its subdivisions have a directed cycle of even length. In particular, $\vec{K}_3$, the complete digraph on three vertices, belongs to this class. All digraphs in this class have been characterized by Seymour and Thomassen [25]. Morever, Devos et al. [10] showed the existence of digraphs with large minimum out-degree and in-degree such that there exists no pair of vertices u and v with four internally disjoint directed paths between them, two from u to v and two from v to u. In Theorem 28, we show that if the minimum out-degree is large enough then we can find two vertices u and v with three internally disjoint directed paths between them, one from u to v and two from v to u.

Note that every graph with chromatic number at least p has a subgraph with minimum degree at least $p-1$. This implies, by Theorem 1, that every graph with chromatic number at least $g(k)+1$ contains a subdivision of K_k. In the context of digraphs, there exist two natural analogues of the chromatic number. Given a digraph D, the *chromatic number* of D, denoted by $\chi(D)$, is simply the chromatic number of its underlying graph. The *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is the smallest integer k such that D admits a k-dicolouring. A *k-dicolouring* is a k-partition $\{V_1, \dots, V_k\}$ of $V(D)$ such that $D\langle V_i \rangle$ is acyclic for every $i \in [k]$. Hence, it is natural to ask which digraphs are χ-maderian and which ones are $\vec{\chi}$-maderian.

Burr [7] proved that every $(k-1)^2$-chromatic digraph contains every oriented forest of order k. Later on, Addario-Berry et al. [1] slightly improved this value to $k^2/2 - k/2 + 1$. This implies that every oriented forest is χ-maderian. Cohen et al. [9] showed that for any positive integer b, there are digraphs of arbitrarily high chromatic number that contain no oriented cycles with less than b blocks. (A *block* in an oriented path or cycle is a inclusion maximal directed subpath.) This directly implies that if a digraph is not an oriented forest, then it is not χ-maderian because it contains an oriented cycle, all subdivisions of which have the same number of blocks.

Theorem 9. *A digraph is χ-maderian if and only if it is an oriented forest.*

The χ-maderian digraphs are known but determining mader_χ for such digraphs is still open. Burr [7] made the following conjecture.

Conjecture 10 (Burr [7])**.** Every digraph with chromatic number $2k-2$ contains every oriented tree of order k as a subdigraph.

An interesting step towards Burr's conjecture is to prove the following consequence of it.

Conjecture 11. If T is an oriented tree of order k, then $\text{mader}_\chi(T) \leqslant 2k-2$.

In Section 3, we prove that every digraph is $\vec{\chi}$-maderian. Again determining $\text{mader}_{\vec{\chi}}$ for every digraph is still open. Since every digraph D of order n is a subdigraph of $\vec{K}_n$, the complete digraph of order n, and so $\text{mader}_{\vec{\chi}}(D) \leqslant \text{mader}_{\vec{\chi}}(\vec{K}_n)$, it is natural to focus on $\vec{K}_n$.

Problem 12. What is $\text{mader}_{\vec{\chi}}(\vec{K}_n)$?

In Subsection 3.1, we show $\text{mader}_{\vec{\chi}}(\vec{K}_n) \leqslant 4^{n^2-2n+1}(n-1)+1$ and more generally that if F is a digraph with n vertices, m arcs and c connected components, then $\text{mader}_{\vec{\chi}}(F) \leqslant 4^{m-n+c}(n-1)+1$ (Corollary 36). We also give better upper bounds on $\text{mader}_{\vec{\chi}}$ for some particular digraphs.

To prove Theorem 1, Mader showed a stronger result about the average degree. Recall that the *average degree* of a graph G is $\bar{d}(G) = 2|E(G)|/|V(G)|$. He proved that there exists a function $g'(k)$ such that every graph G with at least $\bar{d}(G) \geqslant g'(k)$ contains a subdivision of K_k. The *average out-degree* of a digraph D is $\bar{d}^+(D) = |A(D)|/|V(D)|$. (Note that this is equal to the average in-degree and half the average degree.) A digraph is $\overrightarrow{P}_3$*-free* if it does not contain $\overrightarrow{P}_3$ as a subdigraph, where $\overrightarrow{P}_3$ is the dipath on three vertices. There are bipartite graphs with arbitrarily large minimum degree and arbitrarily large girth (recall that the *girth* of a graph is the length of a smallest cycle or $+\infty$ if it is a forest). Orienting edges of such graphs from one part to the other result in $\overrightarrow{P}_3$-free digraphs with arbitrarily large average out-degree and arbitrarily large girth (the girth of a digraph is the girth of its underlying graph). Consequently, a digraph is $\bar{d}^+$-maderian only if it is an *antidirected forest*, that is, an oriented forest containing no $\overrightarrow{P}_3$ as a subdigraph. This simple necessary condition is also sufficient. Burr [8] showed that all antidirected forests are $\bar{d}^+$-maderian: for every antidirected forest F, $\text{mader}_{\bar{d}^+}(F) \leqslant 4|V(F)|-4$.

Theorem 13. *A digraph is $\bar{d}^+$-maderian if and only if it is an antidirected forest.*

Addario-Berry et al. [1] conjectured that the bound $4|V(F)|-4$ in Burr's result is not tight.

Conjecture 14 (Addario-Berry et al. [1])**.** Let D be a digraph. If $|A(D)| > (k-2)|V(D)|$, then D contains every antidirected tree of order k as a subdigraph.

The value $k-2$ in this conjecture would be best possible, since the oriented star S_k^+, consisting of a vertex dominating $k-1$ others, is not contained in any digraph in which every vertex has out-degree $k-2$. It is also tight because the complete digraph $\overleftrightarrow{K}_{k-1}$ has $(k-2)(k-1)$ arcs but trivially does not contain any oriented tree of order k.

As observed in [1], Conjecture 14 for oriented graphs implies Burr's conjecture (Conjecture 10) for antidirected trees and Conjecture 14 for symmetric digraphs is equivalent to the well-known Erdős-Sós conjecture.

Conjecture 15 (Erdős and Sós [12])**.** Let G be a graph. If $|E(G)| > \frac{1}{2}(k-2)|V(G)|$, then G contains every tree of order k.

Their conjecture has attracted a fair amount of attention over the last decades. Partial solutions are given in [6, 15, 24]. In the early 1990's, Ajtai, Komlós, Simonovits and Szemerédi announced a proof of this result for graphs of sufficiently large order.

Since k-connected and k-edge-connected graphs have minimum degree at least k, Theorem 1 implies that every graph G with connectivity (resp. edge-connectivity) at least $g(k)$ contains a subdivision of K_k. Let $\kappa(D)$, $\kappa'(D)$, be respectively the strong connectivity and the strong arc-connectivity of D.

Problem 16. Are all digraphs κ-maderian? κ'-maderian?

The following conjecture due to Thomassen [28] implies that all digraphs are κ-maderian.

Conjecture 17 (Thomassen [28])**.** There exists $f_1(k)$ such that if $\kappa(D) \geqslant f_1(k)$, and $x_1, \dots, x_k$ and $y_1, \dots, y_k$ are distinct vertices of D, then D contains k disjoint dipaths $P_1, \dots, P_k$ such that P_i goes from x_i to y_i for all $i \in [k]$.

This conjecture would also imply the following one due to Lovász [18].

Conjecture 18 (Lovász [18])**.** There exists an integer p such that every p-strongly connected digraph has an even directed cycle.

2 Subdivision in digraphs with large minimum out-degree

2.1 Subdivisions of oriented paths

Let $P = (x_1, x_2, \cdots, x_n)$ be an oriented path. We say that P is an (x_1, x_n)-*path*. The vertex x_1 is the *initial vertex* of P and x_n its *terminal vertex*. P is a *directed path* or simply a *dipath*, if $x_i \to x_{i+1}$ for all $i \in [n-1]$.

Let k_1 be nonnegative integer and $k_2, \dots, k_\ell$ be positive integers. We denote by $P(k_1, k_2, \dots, k_\ell)$ the path obtained from an undirected path $(v_1 v_2 \dots v_{\ell+1})$ by replacing, for every $i \in [\ell]$, the edge (v_i, v_{i+1}) by a directed path of length k_i from v_i to v_{i+1} if i is odd, and from v_{i+1} to v_i if i is even. (If $k_1 = 0$, then $v_1 = v_2$.)

Theorem 19. *Let $P(k_1, k_2, \dots, k_\ell)$ be a path, and let D be a digraph with $\delta^+(D) \geqslant \sum_{i=1}^{\ell} k_i$. For every $v \in V(D)$, D contains a path $P(k'_1, k'_2, \dots, k'_\ell)$ with initial vertex v such that $k'_i \geqslant k_i$ if i is odd, and $k'_i = k_i$ otherwise.*

Proof. By induction on ℓ. If $\ell = 1$, then the result holds trivially. Assume now $\ell \geqslant 2$, and suppose that, for every path $P(x_1, x_2, \dots, x_t)$ with $t < \ell$ and every digraph G with $\delta^+(G) \geqslant \sum_{i=1}^{t} x_i$, G contains a path $P(x'_1, x'_2, \dots, x'_t)$ starting at any vertex of G such that $x'_i \geqslant x_i$ if i is odd, and $x'_i = x_i$ otherwise.

Let v be a vertex of D. Since $\delta^+(D) \geqslant \sum_{i=1}^{\ell} k_i$, there exists a (v, u)-dipath $P_{v,u}$ in D of length exactly k_1, for some vertex $u \in V(D)$. Let $D' = D - (P_{v,u} - u)$, let C be the connected component of D' containing u, and let H be a *sink* strong component of C (i.e. a strong component without arcs leaving it) that is reachable by a directed path in C starting at u. We denote by $P_{u,x}$ a (u, x)-dipath in C such that $V(P_{u,x}) \cap V(H) = \{x\}$.

Note that no vertex of H dominates a vertex in $V(P_{u,x}) \setminus \{x\}$ since H is a sink strong component. Thus, $\delta^+(H) \geqslant \delta^+(D) - k_1 \geqslant k_2$. As a consequence, H contains a directed cycle of length at least k_2. Using this and the fact that H is strongly connected, we conclude that there exists a dipath $P_{y,x}$ in H from a vertex $y \in V(H) \setminus \{x\}$ to x of length exactly k_2. Let $G = H - (P_{y,x} - y)$. One may easily verify that $\delta^+(G)$ is at least $\delta^+(D) - k_1 - k_2 \geqslant \sum_{i=3}^{\ell} k_i$.

Let $Q_{v,y} = P_{v,u} P_{u,x} P_{y,x}$. Note that $Q_{v,y}$ is a path $P(k'_1, k_2)$ starting at v with $k'_1 \geqslant k_1$. Therefore, the result follows immediately if $\ell = 2$. Suppose now that $\ell \geqslant 3$. By the induction hypothesis, G contains a path $W_y := P(k'_3, \dots, k'_\ell)$ with initial vertex y such that $k'_i \geqslant k_i$ if i is odd, and $k'_i = k_i$ otherwise. Therefore, $Q_{v,y} W_y$ is the desired path $P(k'_1, k'_2, \dots, k'_\ell)$ with initial vertex v. □

Since $\sum_{i=1}^{\ell} k_i = |V(P(k_1, k_2, \dots, k_\ell))| - 1$, and that the complete digraph $\vec{K}_k$ on k vertices has minimum out-degree $k - 1$ and contains no path on more than k vertices, we obtain the following corollary.

Corollary 20. $\text{mader}_{\delta^+}(P) = |V(P)| - 1$ *for every oriented path P.*

2.2 Subdivision of in-arborescences

The aim of this subsection is to prove that the in-arborescences are δ^+-maderian. We need some preliminary results. The first one is the vertex and directed version of the celebrated Menger's theorem [22]. (See also [14] for a short proof.)

Theorem 21 (Menger's theorem)**.** *Let D be a digraph, and let $S, T \subseteq V(D)$. The maximum number of vertex-disjoint (S, T)-dipaths is equal to the minimum size of an (S, T)-vertex-cut.*

Lemma 22. *Let D be a digraph with at least one arc, let $S \subseteq V(D)$ be a nonempty subset of vertices of in-degree 0 in D, and let $T \subseteq V(D)$ such that $T \cap S = \varnothing$. If $d^+(v) \geqslant \Delta^-(D)$ for all $v \in V(D) \setminus T$, then there exist $|S|$ vertex-disjoint (S, T)-dipaths in D.*

Proof. Suppose to the contrary that there do not exist $|S|$ vertex-disjoint (S,T)-dipaths in D. By Menger's theorem, there exists a set of vertices $X \subseteq V(D)$ with cardinality $|X| < |S|$ which is an (S,T)-vertex-cut. Let C be the set of vertices in $D - X$ that are reachable in D by a dipath with initial vertex in $S \setminus X$. Set $k = |X \cap S|$.

Let us count the number $a(C,X)$ of arcs with tail in C and head in X. Since the vertices in S have in-degree 0 and every vertex in C has out-degree at least $\Delta^-(D)$,

$$a(C,X) \geqslant |C| \cdot \Delta^-(D) - [|C| - (|S| - k)] \cdot \Delta^-(D) = (|S| - k) \cdot \Delta^-(D).$$

Moreover, $a(C,X)$ is at most the number of arcs with head in X which is at most $(|X| - k) \cdot \Delta^-(D)$, because the vertices in $S \cap X$ have in-degree 0. Hence $(|S| - k) \cdot \Delta^-(D) \leqslant a(C,X) \leqslant (|X| - k) \cdot \Delta^-(D)$. This is a contradiction to $|X| < |S|$, since $\Delta^-(D) \geqslant 1$ because D has at least one arc. □

Let k and ℓ be positive integers. The *ℓ-branching in-arborescence of depth k*, denoted by $B(k,\ell)$, is defined by induction as follows.

- $B(0,\ell)$ is the in-arborescence with a single vertex, which is the root and the leaf of $B(0,\ell)$.
- $B(k,\ell)$ is obtained from $B(k-1,\ell)$ by taking each leaf of $B(k-1,\ell)$ in turn and adding ℓ new vertices dominating this leaf. The root of $B(k,\ell)$ is the one of $B(k-1,\ell)$, and the leaves of $B(k,\ell)$ are the newly added vertices, that is, those not in $V(B(k-1,\ell))$.

The number of vertices of $B(k,\ell)$ is denoted by $b(k,\ell)$; so $b(k,\ell) = \sum_{i=0}^{k} \ell^i = \frac{\ell^{k+1}-1}{\ell-1}$.

Observe that every in-arborescence T is a subdigraph of $B(k,\ell)$ with $\ell = \Delta^-(T)$ and k the maximum length of a dipath in T. Therefore to prove that in-arboresences are δ^+-maderian, it suffices to show that $B(k,\ell)$ is δ^+-maderian for all k and ℓ.

We define a recursive function $f : \mathbb{N} \to \mathbb{N}$ as follows. For all positive integers k and ℓ such that $\ell \geqslant 2$, $f(1,\ell) = \ell$ and, for $k \geqslant 2$, we define

$$f(k,\ell) = t(k,\ell) \cdot (\ell - 1) \cdot k + t(k,\ell),$$

where $t(k,\ell) := f(k-1,\ b(k-1,\ell) \cdot (\ell-1) + 1) \cdot b(k-1,\ell)$.

If $\mathcal{D}$ is a family of digraphs, a *packing* of elements of $\mathcal{D}$ is the disjoint union of copies of elements of $\mathcal{D}$.

Theorem 23. *Let k and $\ell \geqslant 2$ be positive integers, and let D be a digraph with $\delta^+(D) \geqslant f(k,\ell)$. Then D contains a subdivision of $B(k,\ell)$.*

Proof. We prove the result by induction on k and ℓ. If $k = 1$, then $\delta^+(D) \geqslant \ell$. Thus, $\Delta^-(D) \cdot |V(D)| \geqslant \sum_{v \in V(D)} d^-(v) = \sum_{v \in V(D)} d^+(v) \geqslant \ell \cdot |V(D)|$. Hence there is a vertex with in-degree at least ℓ in D and, consequently, the result follows when $k = 1$. Assume now $k \geqslant 2$, and suppose that, for every positive integers $k' < k$ and ℓ', and every digraph H with $\delta^+(H) \geqslant f(k',\ell')$, H contains a subdivision of the ℓ'-branching in-arborescence of depth k'.

Let $\mathcal{F}$ be a packing of ℓ-branching in-arborescence subdigraphs of any non-zero depth in D that covers the maximum number of vertices. We denote by $U \subseteq V(D)$ the set of vertices not covered by $\mathcal{F}$, that is, $U = V(D) \setminus \bigcup_{A \in \mathcal{F}} V(A)$. Let r_A denote the root of the in-arborescence A, for each $A \in \mathcal{F}$, and let $R = \{r_A \in V(D)\colon A \in \mathcal{F}\}$ be the set of the roots of the arborescences in $\mathcal{F}$.

We now construct the digraph H with vertex set R such that there exists an arc (r_A, r_B) in H if and only if r_A dominates some vertex of $V(B)$ in D.

Observe that, for every $v \in R$ such that $d^+_H(v) < t(k,\ell)/b(k-1,\ell)$, we have, in the digraph D, that $d^+_U(v) \geqslant t(k,\ell)\cdot(\ell-1)\cdot k$ since $\delta^+(D) \geqslant t(k,\ell)\cdot(\ell-1)\cdot k + t(k,\ell)$. We define

$$X = \{v \in R\colon d^+_U(v) \geqslant t(k,\ell)\cdot(\ell-1)\cdot k\}.$$

Let D' be the digraph obtained from $D[U \cup X]$ by removing all arcs with head in X. From D', we construct a digraph G by replacing every vertex $v \in X$ by $t(k,\ell)$ new vertices $v_1, \ldots, v_{t(k,\ell)}$, and adding, for each $i \in [t(k,\ell)]$, at least $(\ell-1)\cdot k$ arcs from v_i to $N^+_{D'}(v)$ in such a way that $d^-_{D'}(u) = d^-_G(u)$, for all $u \in N^+_{D'}(v)$. In other words, we "redistribute" the out-neighbours of v in D' among its $t(k,\ell)$ copies in G so that every copy has out-degree at least $(\ell-1)\cdot k$, and the in-degrees of vertices belonging to U are not changed. Let $S \subseteq V(G)$ be the set of vertices that replaced those of X, that is, $S = \bigcup_{v \in X} \{v_1, \ldots, v_{t(k,\ell)}\}$. Let T be the set of vertices in U that have large out-degree outside U in the digraph D, more formally, $T = \{v \in U\colon d^+_{V(D)\setminus U}(v) \geqslant t(k,\ell)+1\}$.

For every $i \in [k-1]$, let $\mathcal{F}_i = \{A \in \mathcal{F}\colon A \text{ has depth exactly } i\}$. Note that $\{\mathcal{F}_i\}_{i\in[k-1]}$ forms a partition of the packing $\mathcal{F}$. Additionally, observe that, due to the maximality of $\mathcal{F}$, every vertex in U is dominated by at most $\ell-1$ vertices belonging to U, and by at most $\ell-1$ roots of in-arborescences in $\mathcal{F}_i$, for each $i \in [k-1]$. Thus, the in-degree in G of every vertex belonging to U is at most $(\ell-1)+(\ell-1)\cdot(k-1) = (\ell-1)\cdot k$. Therefore, we have $\Delta^-(G) \leqslant (\ell-1)\cdot k$. Moreover, since $\delta^+(D) \geqslant t(k,\ell)\cdot(\ell-1)\cdot k + t(k,\ell)$, we have $d^+_G(v) \geqslant t(k,\ell)\cdot(\ell-1)\cdot k$ for every $v \in U \setminus T$. Hence, $d^+_G(v) \geqslant (\ell-1)\cdot k$, for every $v \in V(G)\setminus T$. By Lemma 22, there exists a set $\mathcal{P}$ of $|S|$ vertex-disjoint paths from S to T in G.

Note that, in D, every vertex belonging to T has at least $t(k,\ell)+1$ out-neighbours in $V(D)\setminus U$. Therefore one can greedily extend each path of $\mathcal{P}$ with an out-neighbour of its terminal vertex in $V(D)\setminus U$ in order to obtain a set $\mathcal{P}'$ of $|S|$ vertex-disjoint paths from S to $V(D)\setminus U$ such that for any $v \in X$ all the paths in $\mathcal{P}'$ with initial vertex v have distinct terminal vertices (and different from v).

We now construct the digraph M on the vertex set R where there exists an arc from $v = r_A$ to r_B in M whenever

either r_A dominates some vertex of $V(B)$ in D,

or there is a dipath from some v_i to $V(B)$ in $\mathcal{P}'$.

Since, for each $v \in X$, all vertices in $\{v_i\}_{i\in[t(k,\ell)]}$ are the initial vertices of vertex-disjoint dipaths in $\mathcal{P}'$, we obtain $\delta^+(M) \geqslant t(k,\ell)/b(k-1,\ell)$. Using the induction hypothesis on M, we are ready to prove the result.

Let $p = b(k-1,\ell)\cdot(\ell-1)+1$. By the induction hypothesis, M contains a subdivision T of $B(k-1,p)$. Let R' be the set of branching vertices of T, that is, $R' = \{r \in T : d^-_T(r) = p\}$. We assume that each in-arborescence of $\mathcal{F}$ has at most $b(k-1,\ell)$ vertices, as any larger arborescence would yield the theorem. Thus, for each $r \in R'$, there exists a vertex h_r in the in-arborescence rooted at r, and ℓ vertices of $V(T)$ $x_{r,1},\dots,x_{r,\ell}$ such that for every $i \in [\ell]$, either h_r is dominated in D by $x_{r,i}$ or one of the paths of $\mathcal{P}'$ is from $x_{r,i}$ to h_r. Similarly, for each $r \in V(T)$ with in-degree 1, there exists a vertex h_r in the in-arborescence rooted at r and a vertex $x_{r,1}$ of $V(T)$ such that either h_r is dominated in D by $x_{r,1}$ or one of the paths of $\mathcal{P}'$ is from $x_{r,1}$ to h_r. Consider the subdigraph of T obtained by removing, for each $r \in T$, all arcs with head r but the ℓ arcs from $x_{r,1},\dots,x_{r,\ell}$ to r if $r \in R'$ or the arc from $x_{r,1}$ to h_r, if r is a vertex of T of degree 1. We denote by T' the component of this subdigraph containing the root of T. One may easily verify that T' is a subdivision of $B(k-1,\ell)$. Let P_r be the path from h_r to r in the in-arborescence corresponding to r, for every $r \in V(T')$ such that either $r \in R'$ or r has in-degree 1.

Let Q be the in-arborescence of D obtained from T' in the following way. For each $r \in V(T')$ we add h_r to T', and for every in-neighbour $x_{r,i}$ of r in T' we either add the arc from $x_{r,i}$ to h_r if it exists in D or the path of $\mathcal{P}'$ from $x_{r,i}$ to h_r. Additionally, we remove all arcs with head r in T', and link h_r to r by using the dipath P_r. Finally, for each $r \in V(T')$ that is a leaf, we replace r by its corresponding in-arborescence belonging to $\mathcal{F}$.

By this construction, we have that Q is a subdigraph of D such that every internal vertex has either in-degree ℓ or 1. Furthermore, it has depth at least k. Therefore, by possibly pruning some levels of Q, we obtain a subdivision of $B(k,\ell)$. □

2.3 Cycles with two blocks

We denote by $C(k_1,k_2)$ the digraph which is the union of two internally disjoint dipaths, one of length k_1 and one of length k_2 with the same initial vertex and same terminal vertex. $C(k_1,k_2)$ may also be seen as an oriented cycles with two blocks, one of length k_1 and one of length k_2. Recall that the *blocks* of an oriented cycle are its maximal directed subpaths.

Theorem 24. *Let D be a digraph with $\delta^+(D) \geqslant 2(k_1+k_2)-1$. Then D contains a subdivision of $C(k_1,k_2)$.*

Proof. Let us assume, without loss of generality, that $k_1 \geqslant k_2$. Let ℓ be a positive integer. An (ℓ,k_1,k_2)-*fork* is a digraph obtained from the union of three disjoint dipaths $A = (a_0,a_1,\cdots,a_\ell)$, $B^1 = (b^1_1,\cdots,b^1_{k_1-1})$ and $B^2 = (b^2_1,\cdots,b^2_{k_2-1})$ by adding the arcs (a_ℓ,b^1_1) and (a_ℓ,b^2_1).

Since a $(1,k_1,k_2)$-fork has k_1+k_2+1 vertices and $\delta^+(D) \geqslant k_1+k_2+1$, the digraph D contains a $(1,k_1,k_2)$-fork as a subdigraph. Let $\ell \geqslant 1$ be the largest integer such that D contains an (ℓ,k_1,k_2)-fork as a subdigraph. Let F be such a fork. For convenience, we denote its subpaths and vertices by their labels in the above definition.

If there exist $i,j \in \{0,\dots,\ell\}$, where $i \leqslant j$ (resp. $j \leqslant i$), such that $a_i \in N^+(b^1_{k_1-1})$ and $a_j \in N^+(b^2_{k_2-1})$, then the union of the dipaths $(a_\ell,B^1,a_i,\cdots,a_j)$ and (a_ℓ,B^2,a_j) (resp. (a_ℓ,B^1,a_i) and $(a_\ell,B^2,a_j,\cdots,a_i)$) is a subdivision of $C(k_1,k_2)$.

Suppose now that $b^1_{k_1-1}$ has no out-neighbour in $\{a_0, \cdots, a_{\ell-1}\}$, that is, $N^+(b^1_{k_1-1}) \cap (A \setminus \{a_\ell\}) = \varnothing$ (the case $N^+(b^2_{k_2-1}) \cap (A \setminus \{a_\ell\}) = \varnothing$ is similar). Since $|B^1 \cup B^2 \cup \{a_\ell\}| = k_1 + k_2 - 1$ and $\delta^+(D) \geqslant 2(k_1 + k_2 - 1) + 1$, $b^1_{k_1-1}$ has two distinct out-neighbours, say c^1_1 and c^2_1, not in F.

Let $i_1 \geqslant 1$ be the largest integer such that there exist two disjoint dipaths C^1 and C^2 in $D - F$ with initial vertex c^1_1 and c^2_1, respectively, and on i_1 and $i_2 = \min\{k_2, i_1\}$ vertices. Set $C^1 = (c^1_1, \cdots, c^1_{i_1})$ and $C^2 = (c^2_1, \cdots, c^2_{i_2})$. By maximality of ℓ, if $i_1 \geqslant k_2$, then $i_1 < k_1 - 1$. Otherwise, the union of $A \cup B^1$, C^1, C^2, $(b^1_{k_1-1}, c^1_1)$ and $(b^1_{k_1-1}, c^2_1)$ would contain an $(\ell + k_1 - 1, k_1, k_2)$-fork, contradicting the maximality of ℓ.

Suppose to the contrary that both $c^1_{i_1}$ and $c^2_{i_2}$ have no out-neighbour in $A \setminus \{a_\ell\}$. Since $|V(B^1) \cup V(B^2) \cup \{a_\ell\} \cup V(C^1) \cup V(C^2)| = k_1 + k_2 + i_1 + i_2 - 1 < 2(k_1 + k_2) - 2$ (because $i_1 + i_2 < k_1 + k_2 - 1$) and $\delta^+(D) \geqslant 2(k_1 + k_2 - 1) + 1$, there exist $c^1_{i_1+1}, c^2_{i_2+1} \in V(D - (F \cup C^1 \cup C^2))$ such that $(c^1_{i_1}, c^1_{i_1+1}), (c^2_{i_2}, c^2_{i_2+1}) \in A(D)$ and $c^1_{i_1+1} \neq c^2_{i_2+1}$. This contradicts the maximality of i_1. Henceforth, we assume that $c^1_{i_1}$ has an out-neighbour $a_j \in A \setminus \{a_\ell\}$ for some $0 \leqslant j < \ell$. The case in which $c^2_{i_2}$ has an out-neighbour in $A \setminus \{a_\ell\}$ is similar.

If $b^2_{k_2-1}$ has also an out-neighbour $a_m \in A \setminus \{a_\ell\}$, then the union of the dipaths $(a_\ell, B^1, C^1, a_j, \ldots, a_m)$ and (a_ℓ, B^2, a_m) (if $m \geqslant j$) or of the dipaths (a_ℓ, B^1, C^1, a_j) and $(a_\ell, B^2, a_m, \ldots, a_j)$ (if $m < j$) is a subdivision of $C(k_1, k_2)$.

If $b^2_{k_2-1}$ has an out-neighbour $z \in V(C^1 \cup C^2)$, say $z = c^1_h$ for some $h \leqslant i_1$ (the case in which $z \in V(C^2)$ is similar), then the union of the dipaths (a_ℓ, B^2, c^1_h) and $(a_\ell, B^1, c^1_1, \ldots, c^1_h)$ is a subdivision of $C(k_1, k_2)$.

So, we may assume that $b^2_{k_2-1}$ has no out-neighbour in $A \setminus \{a_\ell\} \cup C^1 \cup C^2$. Hence, $b^2_{k_2-1}$ has two distinct out-neighbours, say c^3_1 and c^4_1, not in $F \cup C^1 \cup C^2$. Let $i_3 \geqslant 1$ be the largest integer such that there exist two disjoint dipaths C^3 and C^4 in $D - (F \cup C^1 \cup C^2)$ with initial vertex c^3_1 and c^4_1, respectively, and on i_3 and $i_4 = \min\{k_2, i_3\}$ vertices. By the maximality of ℓ, if $i_4 \geqslant k_2$ then $i_3 < k_1 - 1$ since otherwise the union of $A \cup B^2$, C^3, C^3, $(b^2_{k_2-1}, c^3_1)$ and $(b^2_{k_2-1}, c^4_1)$ would contain an $(\ell + k_1 - 1, k_1, k_2)$-fork, contradicting the maximality of ℓ.

For sake of contradiction, assume that both $c^3_{i_3}$ and $c^4_{i_4}$ have no out-neighbour in $(A \setminus \{a_\ell\}) \cup C^1 \cup C^2$. Because $|V(B^1) \cup V(B^2) \cup V(C^3) \cup V(C^4) \cup \{a_\ell\}| = k_1 + k_2 + i_3 + i_4 - 1 < 2(k_1 + k_2) - 2$ (since $i_3 + i_4 < k_1 + k_2 - 1$) and $\delta^+(D) \geqslant 2(k_1 + k_2 - 1) + 1$, then there exist distinct vertices $c^3_{i_3+1}, c^4_{i_4+1} \in V(D - (F \cup C^1 \cup C^2 \cup C^3 \cup C^4))$ (if $i_3 \geqslant k_2$, we only define c_{i_3+1}) such that $(c^3_{i_3}, c^3_{i_3+1}), (c^4_{i_4}, c^4_{i_4+1}) \in A(D)$. This contradicts the maximality of i_3.

So one of $c^3_{i_3}, c^4_{i_4}$ has an out-neighbour in $(A \setminus \{a_\ell\}) \cup C^1 \cup C^2$. We assume that it is $c^3_{i_3}$; the case when it is $c^4_{i_4}$ is similar.

If $c^3_{i_3}$ has an out-neighbour $a_q \in A \setminus \{a_\ell\}$ (for some $q < \ell$), then the union of either the dipaths $(a_\ell, B^1, C^1, a_j, \ldots, a_q)$ and (a_ℓ, B^2, C^3, a_q) (if $q \geqslant j$), or the dipaths (a_ℓ, B^1, C^1, a_j) and $(a_\ell, B^2, C^3, a_q, \cdots, a_j)$ (if $q < j$), is a subdivision of $C(k_1, k_2)$.

If $c^3_{i_3}$ has an out-neighbour $c^1_h \in V(C^1)$ for some $1 \leqslant h \leqslant i_1$, then the union of the dipaths $(a_\ell, B^1, c^1_1, \cdots, c^1_h)$ and $(a_\ell, B^2, C^3, c^1_h)$ is a subdivision of $C(k_1, k_2)$. Similarly, we find a subdivision of $C(k_1, k_2)$ if $c^3_{i_3}$ has an out-neighbour in C^2. □

Theorem 24 shows that an oriented cycle with two blocks C is maderian and

$$\text{mader}_{\delta^+}(C) \leqslant 2|V(C)| - 1.$$

A natural question is to ask whether this upper bound is tight or not.

Problem 25. What it the value of $\text{mader}_{\delta^+}(C(k_1, k_2))$?

Proposition 26. *For any positive integer k,* $\text{mader}_{\delta^+}(C(k,1)) = \text{mader}_{\delta^0}(C(k,1)) = k$*.*

Proof. The complete digraph on k vertices has minimum in- and out-degree $k-1$, and it trivially contains no subdivision of $C(k,1)$ because it has less vertices than $C(k,1)$. Hence $\text{mader}_{\delta^+}(C(k,1)) \geqslant \text{mader}_{\delta^0}(C(k,1)) \geqslant k$.

Consider now a digraph D with $\delta^+(D) \geqslant k$. Let P be a longest dipath in D and let u be its terminal vertex. Necessarily $N^+(u) \subseteq V(P)$. Let v (resp. w) be the first (resp. last) vertex of $N^+(u)$ along P. The path $P[v,w]$ contains all vertices of $N^+(u)$, so it has length at least $k-1$. Hence the union of $(u,v) \cup P[v,w]$ and (u,w) is a subdivision of $C(k,1)$. □

A step further towards Conjecture 2 would be to prove that every oriented cycle is δ^+-maderian. We even conjecture that for every oriented cycle C $\text{mader}_{\delta^+}(C) \leqslant 2|V(C)|-1$.

Conjecture 27. Let D be a digraph with $\delta^+(D) \geqslant 2k-1$. Then D contains a subdivision of any oriented cycle of order k

2.4 Three dipaths between two vertices

A slight adaptation of the proof of Theorem 24 leads to a stronger result. Let k_1, k_2, k_3 be positive integers. Let $P(k_1, k_2; k_3)$ be the digraph formed by three internally disjoint paths between two vertices x, y, two (x,y)-dipaths, one of size at least k_1, the other of size at least k_2, and one (y,x)-dipath of size at least k_3. When we want to insist on the vertices x and y, we denote it by $P_{xy}(k_1, k_2; k_3)$.

Theorem 28. *Let k_1, k_2, k_3 be positive integers with $k_1 \geqslant k_2$. Let D be a digraph with $\delta^+(D) \geqslant 3k_1 + 2k_2 + k_3 - 5$. Then D contains $P(k_1, k_2; k_3)$.*

Proof. Let ℓ be an integer. An $(\ell, k_3; k_1, k_2)$*-fork* is a digraph obtained from the union of four disjoint directed paths $P = (p_1, \ldots, p_\ell)$, $A = (a_1, \cdots, a_{k_3-1})$, $B^1 = (b^1_1, \cdots, b^1_{k_1-1})$ and $B^2 = (b^2_1, \cdots, b^2_{k_2-1})$ by adding the arcs (p_ℓ, a_1), (a_{k_3-1}, b^1_1) and (a_{k_3-1}, b^2_1).

Since a $(1, k_3; k_1, k_2)$-fork has $k_1 + k_2 + k_3 - 2$ vertices and $\delta^+(D) \geqslant k_1 + k_2 + k_3 - 2$, D contains a $(1, k_3; k_1, k_2)$-fork as a subdigraph. So, let $\ell \geqslant 1$ be the largest integer such that D contains an $(\ell, k_3; k_1, k_2)$-fork as a subdigraph. Let F be such a fork. For convenience, we denote its subpaths and vertices by their labels in the above definition.

If there exist $i, j \in [\ell]$, with $i \leqslant j$, such that $p_i \in N^+(b^1_{k_1-1})$ and $p_j \in N^+(b^2_{k_2-1})$ or $p_i \in N^+(b^2_{k_2-1})$ and $p_j \in N^+(b^1_{k_1-1})$, then F contains a $P_{a_{k_3-1}p_j}(k_1, k_2; k_3)$.

So, let us assume that $b^1_{k_1-1}$ has no out-neighbour in P (the case where $b^2_{k_2-1}$ has no out-neighbour in P is similar). Since $|A \cup B^1 \cup B^2| = k_1+k_2+k_3-3$ and $\delta^+(D) \geqslant k_1+k_2+k_3-1$, $b^1_{k_1-1}$ has two distinct out-neighbours, say c^1_1 and c^2_1, not in F.

Let $i_1 \geqslant 1$ be the largest integer such that there exist two disjoint directed paths C^1 and C^2 in $D - F$ with initial vertex c^1_1 and c^2_1 respectively on i_1 and $i_2 = \min\{k_2-1, i_1\}$ vertices. If $i_1 \geqslant k_1 - 1$, then $i_2 \geqslant k_2 - 1$, and thus $P \cup A \cup B^1 \cup C_1 \cup C_2$ would contain a fork that contradicts the maximality of ℓ. Hence we may assume that $i_1 \leqslant k_1 - 2$ (and in particular $|V(C_1) \cup V(C_2)| \leqslant k_1 + k_2 - 3$).

For sake of contradiction, assume that both $c^1_{i_1}$ and $c^2_{i_2}$ have no out-neighbour in P. Since $|V(A) \cup V(B^1) \cup V(B^2) \cup V(C^1) \cup V(C^2)| \leqslant 2k_1 + 2k_2 + k_3 - 6 < \delta^+(D) - 2$, then there exist $c^1_{i_1+1}, c^2_{i_2+1} \in V(D - (F \cup C^1 \cup C^2))$ such that $(c^1_{i_1}, c^1_{i_1+1}), (c^2_{i_2}, c^2_{i_2+1}) \in A(D)$ and $c^1_{i_1+1} \neq c^2_{i_2+1}$. This contradicts the maximality of i_1. Henceforth, we assume that $c^1_{i_1}$ has an out-neighbour $p_i \in P$ (the case in which $c^2_{i_2}$ has an out-neighbour in P is similar).

If $b^2_{k_2-1}$ has also an out-neighbour $p_j \in P$, then $F \cup C_1$ contains a $P_{a_{k_3}p_j}(k_1, k_2; k_3)$ if $i \leqslant j$, and a $P_{a_{k_3}p_i}(k_1, k_2; k_3)$ if $j \leqslant i$.

So, we may assume that $b^2_{k_2-1}$ has no out-neighbour in P. Hence, $b^2_{k_2-1}$ has two distinct out-neighbours, say c^3_1 and c^4_1, not in $F \cup C^1$. Let $i_3 \geqslant 1$ be the largest integer such that there exist two disjoint dipaths C^3 and C^4 in $D - (F \cup C^1)$ with initial vertex c^3_1 and c^4_1 respectively on i_3 and $i_4 = \min\{k_2 - 1, i_3\}$ vertices. If $i_3 \geqslant k_1$, then $i_4 \geqslant k_2 - 1$ and thus $P \cup A \cup B^2 \cup C_3 \cup C_4$ contains a fork that contradicts the maximality of F. Thus, we may assume that $i_3 \leqslant k_1 - 2$. In particular $|V(C_3) \cup V(C_4)| \leqslant k_1 + k_2 - 3$.

Suppose to the contrary that both $c^3_{i_3}$ and $c^4_{i_4}$ have no out-neighbour in P, where $c^3_{i_3}$ and $c^4_{i_4}$ are the last vertices of C^3 and C^4. Note that $|V(A) \cup V(B^1) \cup V(B^2) \cup V(C^1) \cup V(C^3) \cup V(C^4)| \leqslant 3k_1 + 2k_2 + k_3 - 7 \leqslant \delta^+(D) - 2$. Hence, there exist distinct vertices $c^3_{i_3+1}, c^4_{i_4+1} \in V(D - (F \cup C^1 \cup C^3 \cup C^4))$ such that $(c^3_{i_3}, c^3_{i_3+1}), (c^4_{i_4}, c^4_{i_4+1}) \in A(D)$. This contradicts the maximality of i_3.

Therefore, one of $c^3_{i_3}, c^4_{i_4}$ has an out-neighbour in p_j in P. We assume that it is $c^3_{i_3}$; the case when it is $c^4_{i_4}$ is similar. We conclude that $F \cup C_1 \cup C_3$ contains a $P_{a_{k_3}p_j}(k_1, k_2; k_3)$ if $i < j$, and a $P_{a_{k_3}p_i}(k_1, k_2; k_3)$ if $j < i$. □

3 Subdivisions in digraphs with large dichromatic number

Recall that a k-*dicolouring* is a k-partition $\{V_1, \ldots, V_k\}$ of $V(D)$ such that $D\langle V_i \rangle$ is acyclic for every $i \in [k]$, and that the *dichromatic number* of D is the minimum k such that D admits a k-dicolouring. In this section, we first prove that every digraph is $\vec{\chi}$-maderian. We need some preliminaries. The first one is an easy lemma, whose proof is left to the reader.

Lemma 29. *The dichromatic number of a digraph is the maximum of the dichromatic numbers of its strong components.*

Our proof is based on levelling. Forthwith, we introduce the necessary definitions. Given a digraph D, the *distance* from a vertex x to another y, denoted by $\mathrm{dist}_D(x, y)$ or simply $\mathrm{dist}(x,y)$ when D is clear from the context, is the minimum length of an (x,y)-dipath or $+\infty$ if no such dipath exists. An *out-generator* in D is a vertex u such that, for every $x \in V(D)$, there exists an (u,x)-dipath in D. Analogously, an *in-generator* in D is a vertex u such that, for every $x \in V(D)$, there exists an (x,u)-dipath in D. For simplicity, we call a vertex *generator* if it is an in- or out-generator. Observe that every vertex in a strong digraph is an in- and out-generator.

Let D be a digraph. Let w, u be in- and out-generators of D, respectively. We remark that w and u are not necessarily different. For every nonnegative integer i, the *ith out-level from u* in D is the set $L_i^{u,+} = \{v \in V(D) \mid \mathrm{dist}_D(u,v) = i\}$, and the *$i$th in-level from w* in D is the set $L_i^{w,-} = \{v \in V(D) \mid \mathrm{dist}_D(v,w) = i\}$. Note that $\bigcup_i L_i^{u,+} = \bigcup_i L_i^{w,-} = V(D)$.

An *out-Breadth-First-Search Tree* or *out-BFS-tree* T^+ with *root* u, is a subdigraph of D spanning $V(D)$ such that T^+ is an oriented tree and, for every $v \in V(D)$, $dist_{T^+}(u,v) = \mathrm{dist}_D(u,v)$. Similarly, an *in-Breadth-First-Search Tree* or *in-BFS-tree* T^- with *root* w, is a subdigraph of D spanning $V(D)$ such that T^- is an oriented tree and, for every $v \in V(D)$, $dist_{T^-}(v,w) = \mathrm{dist}_D(v,w)$.

It is well-known that if D has an out-generator, then there exists an out-BFS-tree rooted at this vertex. Likewise, if D has an in-generator, then there exists an in-BFS-tree rooted at this generator.

Let T denote an in- or out-BFS-tree rooted at u. For any vertex x of D, there is a single (u,x)-dipath in T if T is an out-BFS-tree, and a single (x,u)-dipath in T if T is an in-BFS-tree. The *ancestors* or *successors* of x in T are naturally defined. If y is an ancestor of x, we denote by $T[y,x]$ the (y,x)-dipath in T. If y is a successor of x, we denote by $T[x,y]$ the (x,y)-dipath in T.

Lemma 30. *Let D be a strong digraph and let T be an in- or out-BFS-tree in D. There is a level L such that $\vec{\chi}(D\langle L\rangle) \geqslant \vec{\chi}(D)/2$.*

Proof. First, let us suppose, without loss of generality, that T is an out-BFS-tree in D. The proof when T is an in-BFS-tree is analogous.

Let D_1 and D_2 be the subdigraphs of D induced by the vertices of odd and even levels, respectively. Since there is no arc from L_i to L_j for every $j \geqslant i+2$, the strong components of D_1 and D_2 are contained in the levels. Hence, by Lemma 29, $\vec{\chi}(D_1) = \max\{\vec{\chi}(D\langle L_i\rangle) \mid i \text{ is odd}\}$ and $\vec{\chi}(D_2) = \max\{\vec{\chi}(D\langle L_i\rangle) \mid i \text{ is even}\}$. Moreover, note that $V(D_1) \cup V(D_2) = V(D)$ because D is strong. Therefore, $\vec{\chi}(D) \leqslant \vec{\chi}(D_1) + \vec{\chi}(D_2) \leqslant 2 \cdot \max\{\vec{\chi}(D\langle L_i\rangle) \mid i \in \mathbb{N}\}$. ☐

Lemma 31. *Let F be a digraph and let $a = xy$ be an arc in $A(F)$. If $F - a$ is $\vec{\chi}$-maderian, then F is $\vec{\chi}$-maderian, and* $\mathrm{mader}_{\vec{\chi}}(F) \leqslant 4 \cdot \mathrm{mader}_{\vec{\chi}}(F-a) - 3$.

Proof. Let $c = \mathrm{mader}_{\vec{\chi}}(F-a)$ and let D be a digraph with $\vec{\chi}(D) \geqslant 4c-3$. We shall prove that D contains a subdivision of F. Note that the case $c = 1$ is trivial, so we can assume $c \geqslant 2$.

By Lemma 29, we may assume that D is strong. Let u be a vertex in D and T_u an out-BFS-tree with root u. By Lemma 30, there is a level L^u such that $\vec{\chi}(D\langle L^u\rangle) \geqslant 2c-1$. Since $c \geqslant 2$, $L^u \neq u$ and thus, by Lemma 29, there is a strong component C of $D\langle L^u\rangle$ such that $\vec{\chi}(C) = \vec{\chi}(D\langle L^u \rangle) \geqslant 2c-1$. Since D is strong, there is a shortest (v,u)-dipath P in D such that $V(P) \cap V(C) = \{v\}$. Let T_v be an in-BFS-tree in C rooted at v. By Lemma 30, there is a level L^v of T_v such that $\vec{\chi}(D\langle L^v \rangle) \geqslant c$. Now since $\text{mader}_{\vec{\chi}}(F-a) = c$, $D\langle L^v \rangle$ contains a subdivision S of $F - a$. With a slight abuse of notation, let us call x and y the vertices in S corresponding to the vertices x and y of F. Now $T_v[x,v] \cup P \cup T_u[u,y]$ is a directed (x,y)-walk with no internal vertex in L^v. Hence it contains an (x,y)-dipath Q whose internal vertices are not in S. Therefore, $S \cup Q$ is a subdivision of F in D. □

Theorem 32. *Every digraph F is $\vec{\chi}$-maderian. More precisely,* $\text{mader}_{\vec{\chi}}(F) \leqslant 4^m(n-1)+1$*, where $m = |A(F)|$ and $n = |V(F)|$.*

Proof. We prove the result by induction on m. If $m = 0$, then F is an empty digraph which is trivially $\vec{\chi}$-maderian and $\text{mader}_{\vec{\chi}}(F) = n$. If $m > 0$, then consider an arc $a \in A(F)$. By Lemma 31, we obtain $\text{mader}_{\vec{\chi}}(F) \leqslant 4 \cdot \text{mader}_{\vec{\chi}}(F-a) - 3$. By the induction hypothesis, $\text{mader}_{\vec{\chi}}(F-a) \leqslant 4^{m-1}(n-1)+1$. Therefore, $\text{mader}_{\vec{\chi}}(F) \leqslant 4^m(n-1)+1$. □

Observe that Theorem 32 generalizes the consequence of Theorem 1, stating that every graph with sufficiently large chromatic number contains a subdivision of K_k. In fact this statement corresponds to the case of symmetric digraphs of Theorem 32.

3.1 Better bounds

The bound on $\text{mader}_{\vec{\chi}}$ given in Theorem 32 is not optimal. The aim of this subsection is to find better upper bounds.

A digraph is *k-$\vec{\chi}$-critical* if $\vec{\chi}(D) = k$ and $\vec{\chi}(D') < k$ for every proper subdigraph D' of D.

Proposition 33. *If D is k-$\vec{\chi}$-critical, then $\delta^0(D) \geqslant k-1$.*

Proof. Let v be a vertex of D. Since D is k-$\vec{\chi}$-critical, $\vec{\chi}(D-v) \leqslant k-1$, so $D - v$ admits a $(k-1)$-dicolouring $\{V_1, \dots, V_{k-1}\}$. Thus, for each $i \in [k-1]$, $D\langle V_i \cup \{v\} \rangle$ has a directed cycle that contains v. Therefore, v has an in-neighbour and an out-neighbour in each V_i. □

Corollary 34. $\text{mader}_{\vec{\chi}}(F) \leqslant \text{mader}_{\delta^0}(F) + 1$ *for every digraph F.*

Corollary 35. $\text{mader}_{\vec{\chi}}(F) = |V(F)|$ *for every oriented forest F.*

Let us denote by $\text{cc}(F)$ the number of connected components of F, that are the connected components of the underlying graph.

Corollary 36. *For every digraph F, we have* $\text{mader}_{\vec{\chi}}(F) \leqslant 4^{m-n+\text{cc}(F)}(n-1)+1$*, where $m = |A(F)|$ and $n = |V(F)|$.*

Proof. The proof is identical to the one of Theorem 32, but instead of starting the induction with empty digraphs, we start it with a forest that is the union of spanning trees of the connected components. □

Corollary 36 implies that $\text{mader}_{\vec{\chi}}(\vec{K}_n) \leqslant 4^{n(n-2)+1}(n-1)+1$. On the other hand, we have $\text{mader}_{\vec{\chi}}(\vec{K}_n) \geqslant \Omega(\frac{n^2}{\log n})$. Indeed, consider a tournament T on p vertices with a subdivision S of $\vec{K}_n$. For every two distinct vertices u, v of $\vec{K}_n$, at least one of the arcs (u,v), (v,u) is subdivided in S. Hence, S has at least $n+\binom{n}{2} = \binom{n+1}{2}$ vertices, so $p \geqslant \binom{n+1}{2}$. Erdős and Moser [13] proved that for every integer p, there exists a tournament T_p on p vertices with no transitive tournament of order $2\log p + 1$. Thus $\vec{\chi}(T_p) \geqslant \frac{p}{2\log p}$. Now set $p = \binom{n+1}{2} - 1$. The tournament T_p contains no subdivision of K_n and $\vec{\chi}(T_p) \geqslant \frac{p}{2\log p}$. Hence $\text{mader}_{\vec{\chi}}(\vec{K}_n) \geqslant \frac{p}{2\log p} + 1 \geqslant \Omega(\frac{n^2}{\log n})$.

A *k-source* in a digraph is a vertex x with in-degree 0 and out-degree at most k; a *k-sink* in a digraph is a vertex x with out-degree 0 and in-degree at most k. A digraph is *k-reducible* if it can be reduced to the empty digraph by repeated deletion of k-sources or k-sinks. For instance, the 1-reducible digraphs are the oriented forests.

Lemma 37. *Let F be a digraph having a 2-source x. Then*

$$\text{mader}_{\vec{\chi}}(F) \leqslant 2\,\text{mader}_{\vec{\chi}}(F-x) - 1.$$

Proof. Suppose that $F-x$ is $\vec{\chi}$-maderian and $\text{mader}_{\vec{\chi}}(F-x) = c$. We shall prove that F is $\vec{\chi}$-maderian and $\text{mader}_{\vec{\chi}}(F) \leqslant 2c-1$. Note that the case $c=1$ is trivial so we can assume $c \geqslant 2$.

Let D be a digraph with $\vec{\chi}(D) \geqslant 2c-1$. By Lemma 29, and since $c \geqslant 2$, we may assume that D is strong. Let u be a vertex in D, and let T be a BFS-tree with root u. By Lemma 30, there is a level such that $\vec{\chi}(D\langle L\rangle) \geqslant c$. Consequently, $D\langle L\rangle$ contains a subdivision S of $F-x$. Let y_1 and y_2 be the vertices in S corresponding to the two out-neighours of x in F. Let v be the least common ancestor of y_1 and y_2 and, for $i \in \{1,2\}$, let P_i be the (v, y_i)-dipath in T. Therefore, we conclude that the digraph $S \cup P_1 \cup P_2$ is a subdivision of F in D. □

Corollary 38. *The following statements hold.*

(a) $\text{mader}_{\vec{\chi}}(F) \leqslant 2^{|V(F)|-2}+1$ *for every 2-reducible digraph F or order at least 2.*

(b) $\text{mader}_{\vec{\chi}}(C) \leqslant 2\cdot|V(C)|-3$ *for every oriented cycle C of order at least 3.*

Proof. Statement (a) follows by induction on $|V(F)|$. Observe that the result trivially holds when $|V(F)| = 2$.

To prove statement (b), consider the following two complementary cases. If C is directed, say $C = \vec{C}_k$, then $\text{mader}_{\vec{\chi}}(C) \leqslant \text{mader}_{\delta^0}(C) + 1 \leqslant \text{mader}_{\delta^+}(C) \leqslant k \leqslant 2k-3$. If C is not directed, then it contains a 2-source x. Hence $C-x$ is an oriented path, and, by Corollary 35, it follows that $\text{mader}_{\vec{\chi}}(C-x) = |C-x| = |C|-1$. □

Conjecture 39. $\text{mader}_{\vec{\chi}}(C) \leqslant |C|$ for every oriented cycle C.

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