

Greene–Kleitman invariants for Sulzgruber insertion

Alexander Garver* Rebecca Patrias†

Laboratoire de Combinatoire et d’Informatique Mathématique
Université du Québec à Montréal
Québec, Canada

{alexander.garver,patriasr}@lacim.ca

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Abstract

R. Sulzgruber’s rim hook insertion and the Hillman–Grassl correspondence are two distinct bijections between the reverse plane partitions of a fixed partition shape and multisets of rim-hooks of the same partition shape. It is known that Hillman–Grassl may be equivalently defined using the Robinson–Schensted–Knuth correspondence, and we show the analogous result for Sulzgruber’s insertion. We refer to our description of Sulzgruber’s insertion as diagonal RSK. As a consequence of this equivalence, we show that Sulzgruber’s map from multisets of rim hooks to reverse plane partitions can be expressed in terms of Greene–Kleitman invariants.

Mathematics Subject Classifications: 05E99, 05A19

1 Introduction

Reverse plane partitions are prominent combinatorial objects with connections to areas like symmetric functions and representation theory (see for example [9]). A generating function for reverse plane partitions was discovered by Stanley. Let $h(u)$ denote the hook length of the cell u in partition λ and $|\pi|$ denote the sum of the entries in reverse plane partition π .

Theorem 1. [9] *The generating function for reverse plane partitions of shape λ with respect to the sum of its entries is*

$$\sum_{\pi} q^{|\pi|} = \prod_{u \in \lambda} \frac{1}{1 - q^{h(u)}}.$$

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The first bijective proof of this generating function was found by Hillman and Grassl in [5]. The authors give a bijection between nonnegative integer arrays of shape λ —representing multisets of rim hooks of λ —and reverse plane partitions of λ , which is now known as the *Hillman–Grassl correspondence*. This correspondence has since been well studied, for example by Gansner in [2] and by Morales, Pak, and Panova in [7]. In particular, Gansner relates Hillman–Grassl to the Robinson–Schensted–Knuth (RSK) correspondence and shows that the correspondence has the same Greene–Kleitman invariants as RSK.

Recently, Sulzgruber defined a new bijection between multisets of rim hooks of λ and reverse plane partitions of λ that takes the form of a rim hook insertion algorithm, which we call *Sulzgruber insertion* [11]. It is easy to check that Sulzgruber insertion is distinct from Hillman–Grassl, and so his correspondence gives an alternative bijective proof of Theorem 1.

In this paper, we explicitly phrase Sulzgruber insertion in terms of RSK insertion with the goal of proving that Sulzgruber insertion can be expressed in terms of Greene–Kleitman invariants (see Theorem 23). Each diagonal of the reverse plane partition associated to a multiset of rim hooks encodes the Greene–Kleitman partition associated to a certain poset on a subset of the rim hooks—namely, the rim hooks whose support intersects the given diagonal. This idea is relevant to future work of the authors with Hugh Thomas that relates reverse plane partitions to the theory of quiver representations [3].

We remark that since announcing our paper on the arXiv, Sulzgruber has proven in [12, Theorem 7.4] that his insertion is equivalent to a bijection of Pak [8]. Thus a corollary of our work together with this result of Sulzgruber is the fact that Pak’s procedure has the same Greene–Kleitman invariant, which refines Theorem 6 of [8].

The paper proceeds as follows. In Section 2, we set up notation and review the necessary combinatorial background related to tableaux, the RSK correspondence, Knuth equivalence, and the Hillman–Grassl correspondence. Section 3 defines Sulzgruber insertion as well as its inverse. In Section 4, we define a rim hook insertion algorithm we call *diagonal RSK*, and we show that diagonal RSK is equivalent to Sulzgruber insertion. Section 5 reminds the reader how to phrase Hillman–Grassl in terms of RSK in the same spirit as our diagonal RSK and states the corresponding Greene–Kleitman invariants. We conclude by showing that Sulzgruber insertion has a Greene–Kleitman invariant in Section 6.

2 Preliminaries

2.1 Reverse plane partitions and rim hooks

A *partition* is a finite, weakly decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \dots \geq \lambda_k)$. Let $|\lambda| = \lambda_1 + \dots + \lambda_k$. To each partition, we associate a *Young diagram*: a left-justified array of boxes where row sizes weakly decrease from top to bottom. Throughout this paper, we refer to both the partition and its Young diagram as λ . We count the rows of a Young diagram from top to bottom and the columns from left to right, and we refer to a

box in a Young diagram by its row and column index. We let $\lambda' = (\lambda'_1, \dots, \lambda'_k)$ denote the *conjugate* or the *transpose* of partition λ —the partition obtained by swapping the roles of the rows and columns in λ . Thus λ'_i is the size of the i th column of λ .

We impose a partial ordering on the boxes of a Young diagram by defining $(i, j) \preceq (i', j')$ if $i \leq i'$ and $j \leq j'$; in other words, $(i, j) \preceq (i', j')$ if (i, j) is weakly northwest of (i', j') . We say that a box $(i, j) \in \lambda$ is in *diagonal* d of λ if $d = j - i$. With respect to the partial order \preceq , each diagonal d of λ has a unique maximal element $\mathbf{r}_d = (i, j)$, the southeasternmost entry in the diagonal. We refer to \mathbf{r}_d as a *border box* of λ .

A *rim hook* of a Young diagram λ is a connected strip of border boxes in λ such that deleting these boxes from λ yields a Young diagram. To each box (i, j) of λ , we associate the unique rim hook of λ that has its southwesternmost box in column j and northeasternmost box in row i . We denote the rim hook associated to box (i, j) of fixed shape λ by $h_{(i,j)}$.

A *reverse plane partition* of shape λ is an order-preserving map $\lambda \rightarrow \{0, 1, 2, \dots\}$, i.e., a filling of the boxes of a Young diagram with nonnegative integers such that entries weakly increase across rows and down columns. An \mathbb{N} -*tableau* is a filling of shape λ with nonnegative integers. A *semistandard Young tableau* of shape λ is a filling of the boxes of λ with positive integers such that entries weakly increase across rows and strictly increase down columns. A *standard Young tableau* is a semistandard Young tableau where the entries $1, 2, \dots, |\lambda|$ each appear exactly once. See Figure 1.

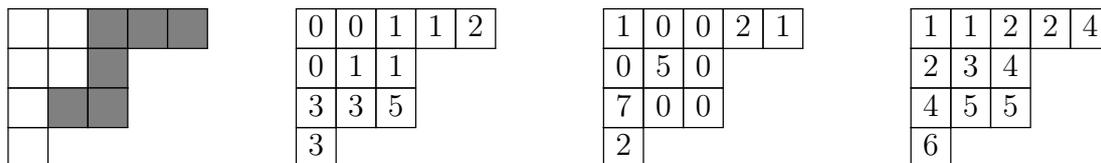


Figure 1: From left to right, we have the rim hook $h_{(1,2)}$ inside shape $\lambda = (5, 3, 3, 1)$, a reverse plane partition of shape λ , an \mathbb{N} -tableau of shape λ , and a semistandard Young tableau of shape λ .

2.2 RSK insertion

There is a well-known bijective correspondence between words in the alphabet of positive integers and pairs consisting of a semistandard Young tableau and a standard Young tableau of the same shape called the Robinson–Schensted–Knuth (RSK) correspondence. We briefly review the correspondence here and refer the reader to [10] for more information.

Given a word $w = w_1 w_2 \cdots w_r$ in the positive integers, the RSK correspondence maps w to a pair of tableaux via a row insertion algorithm consisting of inserting a positive integer into a tableau. The algorithm for inserting positive integer k into a row of a semistandard tableau is as follows. If k is greater than or equal to all entries in the row, add a box labeled k to the end of the row. Otherwise, find the first y in the row with

$y > k$. Replace y with k in this box, and proceed to insert y into the next row. To insert k into semistandard tableau P , we start by inserting k into the first row of P . To create the *insertion tableau* of a word $w = w_1 w_2 \cdots w_r$, we first insert w_1 into the empty tableau, insert w_2 into the result of the previous insertion, insert w_3 into the result of the previous insertion, and so on until we have inserted all letters of w . We denote the resulting insertion tableau by $P(w)$. The insertion tableau will always be a semistandard tableau.

To obtain a standard Young tableau from w , we define the *recording tableau*, $Q(w)$, of w to be the tableau with the same shape as $P(w)$ and label s in the box added to $P(w_1 \cdots w_{s-1})$ during the insertion of w_s . For example, $w = 14252$ has insertion tableau and recording tableau

$$P(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 4 & 5 & \\ \hline \end{array} \qquad Q(w) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array}.$$

Given a pair (P, Q) of a semistandard and standard Young tableau of the same shape, one can easily reverse the procedure to recover w by using Q to determine which entry of P to reverse insert at each step.

We next discuss a few important properties of RSK insertion that we will use later. The first is known as the Greene–Kleitman invariant.

Theorem 2 ([4]). *Let $\delta_k(w)$ denote the largest cardinality of a disjoint union of k non-decreasing subwords of the word w with $\delta_0(w) = 0$, and let $\gamma_k(w)$ denote the largest cardinality of a disjoint union of k strictly decreasing subwords of w with $\gamma_0(w) = 0$. Then $P(w)$ has shape $(\lambda_1, \dots, \lambda_k)$ with*

$$\begin{aligned} \lambda_i &= \delta_i(w) - \delta_{i-1}(w) \text{ and} \\ \lambda'_i &= \gamma_i(w) - \gamma_{i-1}(w) \end{aligned}$$

for all $i = 1, \dots, k$.

We will also need the following easy lemma, which follows from the fact that larger numbers never bump smaller numbers in RSK insertion. Let $w|_{[1,k]}$ denote the restriction of the word w to the alphabet $[1, k]$ and $P|_{[1,k]}$ denote the restriction of semistandard Young tableau P to boxes containing $[1, k]$.

Lemma 3. *We have that $P(w|_{[1,k]}) = P(w)|_{[1,k]}$.*

Next, we define the *bumping path* of RSK insertion of a positive integer into a semistandard Young tableau T to be the set of cells of T that are added or whose entries change during the insertion process. We order the list from top to bottom row. For example, the bumping path for the insertion of 1 into $P(14252)$ shown above is $(1, 2), (2, 1), (3, 1)$. Let $(T \leftarrow i)$ denote the result of inserting i into semistandard Young tableau T . The following lemma can be found, for example, in [10].

Lemma 4. (a) *When we insert positive integer k into semistandard Young tableau T , the bumping path moves weakly left.*

(b) *Let T be a semistandard Young tableau and let $i \leq j$. Then the bumping path of i in $(T \leftarrow i)$ lies strictly to the left of the bumping path of j in $((T \leftarrow i) \leftarrow j)$.*

2.3 Knuth equivalence

We recall the Knuth equivalence relations on words in the alphabet of positive integers. The *elementary Knuth relations* are given below, where x, y, z are positive integers:

$$\begin{aligned} yzx &\equiv yxz && \text{whenever } x < y \leq z \\ xzy &\equiv zxy && \text{whenever } x \leq y < z. \end{aligned}$$

We then say that two words are *Knuth equivalent* if one can be obtained from the other by a finite sequence of the elementary Knuth relations. For example, $221343 \equiv 241233$ because of the sequence below, where the underlined triplets show to which three letters the equivalence was applied:

$$\underline{221}343 \equiv 21\underline{234}3 \equiv 21\underline{243}3 \equiv \underline{214}233 \equiv 241233.$$

One of the main motivations for Knuth equivalence is the connection to RSK insertion.

Theorem 5 ([6]). *Let u and v be two words. Then $P(u) = P(v)$ if and only if $u \equiv v$.*

We define the *reading word* of a tableau T , $r(T)$, to be the word obtained by reading the entries of T row by row from left to right starting with the bottom row. For example, $r(T) = 645523411224$ for the rightmost tableau T in Figure 1. The following fact is well known, and the interested reader may find a proof in [10].

Theorem 6. *For any word w , $r(P(w)) \equiv w$.*

We will use Theorem 6 in Section 4 in the following way. Suppose we have a word w on the set of positive integers $a_1 < \cdots < a_n$. In addition, suppose that $w|_{[a_k, a_n]}$ is weakly increasing for some $k \in [1, n]$. It follows from the definition of the Knuth relations that $u|_{[a_k, a_n]}$ will be weakly increasing for any $u \equiv w$. In particular, $r(P(w))|_{[a_k, a_n]}$ will be weakly increasing. This implies that the boxes of $P(w)$ with entries in $[a_k, a_n]$ form a horizontal strip (i.e., no two of the boxes lie in the same column). We state this as a lemma.

Lemma 7. *Suppose w is a word on positive integers $a_1 < \cdots < a_k < \cdots < a_n$ and $w|_{[a_k, a_n]}$ is weakly increasing. Then the boxes of $P(w)$ containing a_k, \dots, a_n form a horizontal strip.*

2.4 The Hillman–Grassl correspondence

We next review the Hillman–Grassl correspondence. We will start with a reverse plane partition π of shape λ and obtain a multiset of rim hooks of λ .

Let $\pi_0 = \pi$. Scan the columns of π_0 from bottom to top and left to right to find the cell with the first nonzero entry, and call this cell v_0 . We inductively form a lattice path L_0 in π starting at v_0 and taking only north and east steps as follows. Suppose we have constructed the path up to some box $v \in \lambda$.

- If there is a box directly above v and its entry is equal to the entry in v , the lattice path travels to the box directly above v .

- If there is not a box directly above v or its entry is strictly less than the entry in v and there is a box directly to the right of v , the path travels to the box directly right of v .
- If there is not a box directly above v or its entry is strictly less than the entry in v and there is not a box directly to the right of v , the path ends at v .

Next, associate a rim hook h_0 of λ to L_0 by taking the rim hook with southwesternmost and northeasternmost cells the same as the starting and ending cells of L_0 . Define a new reverse plane partition π_1 to be the result of subtracting 1 from each entry of π_0 that intersects L_0 .

Continue this process until π_k is the reverse plane partition filled with all zeros. The multiset of rim hooks of λ that corresponds to π under Hillman–Grassl is then $\mathcal{M} = \{h_0, \dots, h_{k-1}\}$.

Example 8. Starting with reverse plane partition $\pi = \pi_0$, we end with multiset \mathcal{M} .

$$\begin{array}{cccc}
 \pi_0 = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 1 & 2 & 2 \\ \hline 3 & 4 & \\ \hline \end{array} &
 \pi_1 = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \hline \end{array} &
 \pi_2 = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 1 & 2 & 2 \\ \hline 1 & 2 & \\ \hline \end{array} &
 \pi_3 = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 0 & 2 & \\ \hline \end{array} \\
 \\
 \pi_4 = \begin{array}{|c|c|c|} \hline 0 & 0 & 1 \\ \hline 0 & 1 & 1 \\ \hline 0 & 1 & \\ \hline \end{array} &
 \pi_5 = \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \\ \hline \end{array} &
 &
 \\
 \\
 \mathcal{M} = \left\{ \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}, \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \right\}
 \end{array}$$

One can also easily describe the inverse algorithm, which shows that the Hillman–Grassl correspondence is a bijection between reverse plane partitions of shape λ and multisets of rim hooks of λ .

3 Sulzgruber insertion

In [11], Sulzgruber defines a bijection between reverse plane partitions of shape λ and multisets of rim hooks of λ in the form of an insertion and reverse insertion procedure. We now follow his exposition to review how to insert a rim hook h into a reverse plane partition π of shape λ . We will think of h as a rigid object consisting of 3-dimensional cubes and of π as stacks of cubes lying on top of shape λ . For further details, see [11].

First try placing h on top of π such that the cubes of h are above their corresponding cells of λ . If the stacks of cubes below h all have the same height, we are done. This is because placing h as a rigid object on top of π yields a reverse plane partition; there are no empty spaces within any stack of cubes.

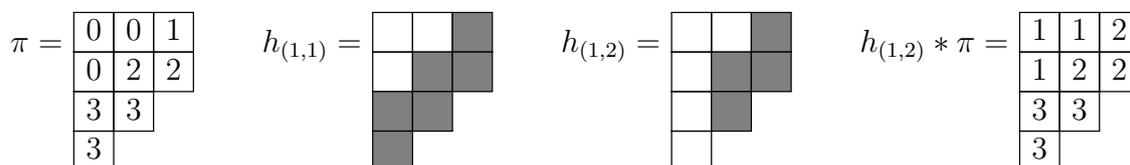
If the stacks of cubes below h do not all have the same height, let j denote the height of the stack of cubes lying below the northeasternmost block of h .

Split the rim hook h into two parts as follows. One part consists of the cubes of h that cover a stack of cubes of height j . This part of h will not move in the next step. Each cube in this part will result in adding 1 to the corresponding cell in π .

The second part of h consists of all other cubes in h —those that do not cover a stack of cubes of height j . Move this part of h diagonally northwest one cell and start this process over again. Note that many iterations of this process may be necessary to complete the insertion. If the end result is a reverse plane partition, then h may be inserted into π , and we denote the result by $h * \pi$. If this procedure fails to produce a reverse plane partition, then h may not be inserted into π .

Move the remaining portion of h diagonally northeast one cell and start the process over again. Note that multiple cuts and shifts may be necessary to complete this process. If the result is a reverse plane partition, then h may be inserted into π , and we denote the result by $h * \pi$. If this procedure fails to produce a reverse plane partition, then h may not be inserted into π .

Example 9. Suppose we have reverse plane partition π , and we wish to insert rim hook $h_{(1,1)}$. Placing $h_{(1,1)}$ on top of π , we see that the shortest stack of boxes below $h_{(1,1)}$ has height 1. We thus break off the segment of $h_{(1,1)}$ lying above stacks of size 1 (in this case, just the northeasternmost block of $h_{(1,1)}$), and insert this block in its place. This changes the 1 in position (1, 3) of π to a 2. We now shift the remaining blocks of $h_{(1,1)}$ northeast one box. We notice that the remaining blocks of $h_{(1,1)}$ are no longer contained in π . This means the insertion has failed, and $h_{(1,1)}$ may not be inserted into π .



Suppose instead we wish to insert rim hook $h_{(1,2)}$ into π . We again place $h_{(1,2)}$ on top of π , break off the northeasternmost cell, and shift the remaining three blocks of $h_{(1,2)}$ northeast one box. Each of the three blocks now lies over a stack of size 0. We can thus insert these blocks in their current position to obtain $h_{(1,2)} * \pi$.

Sulzgruber defines an insertion order for rim hooks of λ , which we denote by \leq . This order has the property that given rim hooks $h_1 \leq h_2 \leq \dots \leq h_k$ in the shape λ , the insertion of h_k into $h_{k-1} * \dots * h_2 * h_1 * 0$ always yields a reverse plane partition, where 0 denotes the reverse plane partition where each entry is 0. (Note that we have changed conventions slightly from those in [11].) After fixing λ , the *insertion order* for λ is as follows: First insert some non-negative number of copies of the rim hook $h_{(1,1)}$. Next insertion some non-negative number of copies of the rim hook $h_{(2,1)}$. Continue until you reach the bottom of the first column, and then start again from the top of the second column. Continue in this manner until you have inserted some non-negative number of

copies of the rim hook corresponding to the bottom cell of the rightmost column of λ . This insertion order is illustrated for shape $\lambda = (5, 4, 4, 3, 2)$ below.

1	6	11	15	18
2	7	12	16	
3	8	13	17	
4	9	14		
5	10			

Theorem 10. [11, Theorem 3] *Let λ be a partition and π be a reverse plane partition of shape λ . Then there exists a unique sequence h_1, \dots, h_s of rim hooks of λ such that $h_i \leq h_{i+1}$ in Sulzgruber's insertion order for $i \in [s - 1]$ and $\pi = h_s * \dots * h_2 * h_1 * 0$*

Remark 11. One can easily check that Sulzgruber insertion is not equivalent to the Hillman–Grassl correspondence by looking at the reverse plane partition of shape $(2, 2)$ with each box filled with 1. Hillman–Grassl associates this reverse plane partition with multiset $\{h_{(1,1)}, h_{(2,2)}\}$ while Sulzgruber associates it with $\{h_{(2,1)}, h_{(1,2)}\}$.

3.1 Sulzgruber reverse insertion

Given a reverse plane partition $\pi = h_s * \dots * h_2 * h_1 * 0$ of shape λ , we may identify the rim hook h_s using the reverse insertion procedure we now describe.

First, notice that the southwesternmost box of any rim hook can be one of two types of cell of λ . It is either an *outer corner* of λ (i.e., a cell c such that $\lambda - c$ is still a partition), or it is a cell of λ lying in the same row as an outer corner and at the bottom of its column. We will categorize any cell that is an outer corner or in the same diagonal as an outer corner as \mathcal{O} and any cell not in \mathcal{O} lying in the same row as an outer corner at the bottom of its column or any cell in the same diagonal as such a cell as \mathcal{A} . For example, we have the following categories.

		\mathcal{A}	\mathcal{A}	\mathcal{O}	
\mathcal{O}			\mathcal{A}	\mathcal{A}	\mathcal{O}
	\mathcal{O}				
\mathcal{O}		\mathcal{O}			
\mathcal{A}	\mathcal{O}				

The reverse insertion proceeds as follows:

1. We scan π diagonal by diagonal starting with the northeasternmost diagonal and reading through each diagonal southeast to northwest until we find the first box b such that either
 - (a) $b \in \mathcal{O}$ and the entry in b is strictly greater than the entry to the left of b or
 - (b) $b \in \mathcal{A}$ and the entry in b is strictly greater than the entry above b and strictly greater than the entry to the left of b .

If there is no box above or to the left of b , we consider the entry to be 0. Sulzgruber calls any box satisfying (a) or (b) a *candidate*. Box b is the first candidate in the described diagonal reading order.

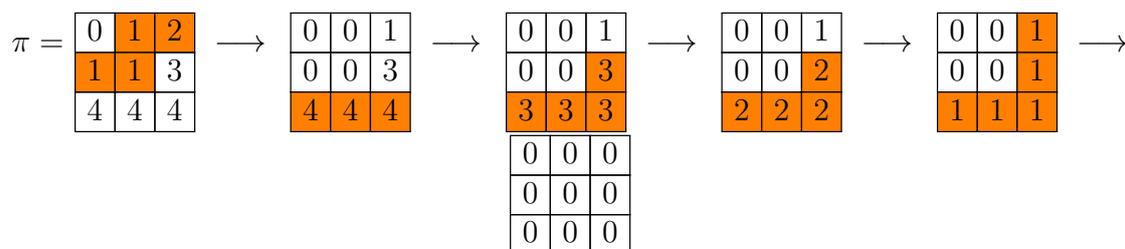
2. We inductively form a path P in λ starting at b and ending at a border box of λ that travels either one box right or one box up at each step using the following rules. Suppose we have constructed the path up to some box $b' \in \lambda$.
 - (a) If there is a box of λ above b' and its entry is equal to the entry in b' , the path travels next to the box above b' .
 - (b) If there is a box of λ to the right of b' and the box above b' either does not exist or has entry strictly smaller than the entry in b' , the path travels to the box to the right of b' .

If there is neither a box above b' nor to the right of b' , the path ends at b' .

There is then a unique rim hook of λ with the same number of boxes as P and the same northeasternmost cell as P ; this is h_s , the last rim hook inserted to obtain $\pi = h_s * \dots * h_1 * 0$. We obtain $h_{s-1} * \dots * h_1 * 0$ by subtracting one from the entry in each box in $\pi \cap P$.

Remark 12. This description of reverse insertion differs slightly from that of [11] but is easily seen to be equivalent. The key observation is that if the entry above b' is equal to the entry in b' , the path must move upward in order for the result of the subtraction to be a reverse plane partition.

Example 13. Performing this reverse insertion on π below gives the path P highlighted in orange. This path tells us that the last rim hook inserted to obtain π was $h_{(1,2)}$. Continuing this process, we see that $\pi = h_{(1,2)} * h_{(3,1)} * h_{(2,1)} * h_{(2,1)} * h_{(1,1)} * 0$.



4 Diagonal RSK

We next introduce an additional bijection between multisets of rim hooks of λ and reverse plane partitions of shape λ that uses the RSK algorithm, which we will show is equivalent to Sulzgruber's bijection. We call this bijection *diagonal RSK*. The analogue of this approach for the Hillman–Grassl bijection is described by Gansner in [1] and also by Morales–Pak–Panova in [7]; this explicit description of Hillman–Grassl using RSK is reviewed in Section 5. We phrase our insertion very explicitly to aid in later proofs.

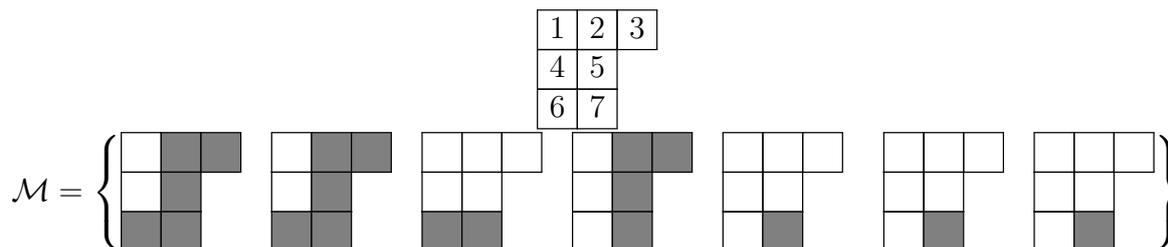
We will use the same insertion order as Sulzgruber, as described in Section 3. In addition, we give each rim hook a distinct positive integer label. To obtain the labels, simply fill the boxes of $\lambda = (\lambda_1, \dots, \lambda_k)$ in increasing order such that row 1 contains exactly $1, 2, \dots, \lambda_1$, row 2 contains exactly $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$, etc. (This filling is sometimes known as the *superstandard filling* of λ .) Now define the label for rim hook $h_{(i,j)}$ to be the label of cell (i, j) of λ . For example, the rim hook shown in Figure 1 has label 2.

To obtain a reverse plane partition from a multiset \mathcal{M} of rim hooks of λ , first order the multiset of rim hooks to be inserted in increasing insertion order. Reading off their labels gives a word, which we call $w(\mathcal{M})$.

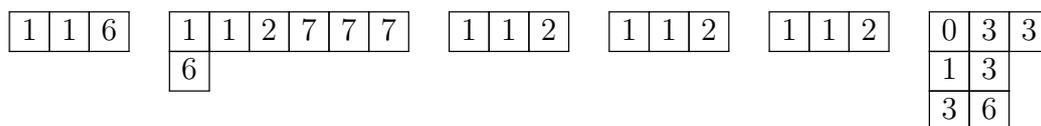
We assign a subword of this word to each diagonal as follows. Fix a diagonal i . The j th letter in the $w(\mathcal{M})$ is included in the subword for diagonal i exactly when the corresponding rim hook has a cell in diagonal i . (See Example 14.) We denote the resulting word by $w(\mathcal{M})_i$.

Next, use RSK insertion to obtain a semistandard Young tableau corresponding to each diagonal: $P(w(\mathcal{M})_i)$. To build the reverse plane partition of shape λ corresponding to \mathcal{M} , record the shape of $P(w(\mathcal{M})_i)$ in diagonal i by entering the size of the largest part in the southeasternmost cell and proceeding up the diagonal. Given multiset $\mathcal{M} = \{h_1 \leq h_2 \leq \dots \leq h_k\}$, we denote the result of this insertion by $h_k \star \dots \star h_2 \star h_1 \star 0$.

Example 14. Suppose $\lambda = (3, 2, 2)$ and the multiset of rim hooks is as shown below, where they are written in increasing insertion order. The corresponding multiword is $w(\mathcal{M}) = 1162777$.



Then $w(\mathcal{M})_{-2} = 116$, $w(\mathcal{M})_{-1} = 1162777$, and $w(\mathcal{M})_0 = w(\mathcal{M})_1 = w(\mathcal{M})_2 = 112$. This gives the five semistandard Young tableaux below, which lead to the reverse plane partition shown on the right.



Proposition 15. Suppose $h_1 \leq h_2 \leq \dots \leq h_k$ are rim hooks in partition λ . Then $h_k \star \dots \star h_2 \star h_1 \star 0$ is a reverse plane partition of shape λ .

In order to prove the previous proposition as well as the equivalence of Sulzgruber insertion and diagonal RSK, it will be helpful to examine how $w(\mathcal{M})_d$ is related to $w(\mathcal{M})_{d+1}$. We accomplish this with the following easy lemmas. Recall that \mathbf{r}_d denotes the border box on diagonal d .

Lemma 16. *Suppose \mathbf{r}_d is directly left of \mathbf{r}_{d+1} . Then $w(\mathcal{M})_{d+1} = w(\mathcal{M})_d u$, where either u is the empty word or u is weakly increasing. The word u is constructed from the alphabet of labels of rim hooks corresponding to boxes in the same column as \mathbf{r}_{d+1} .*

Proof. Every rim hook containing \mathbf{r}_d also contains \mathbf{r}_{d+1} , and the rim hooks containing \mathbf{r}_{d+1} but not \mathbf{r}_d are inserted after all of the rim hooks containing \mathbf{r}_d and with labels in increasing order. \square

Lemma 17. *Suppose \mathbf{r}_d is directly below \mathbf{r}_{d+1} . Then $w(\mathcal{M})_{d+1}$ is a subword of $w(\mathcal{M})_d$: the restriction of $w(\mathcal{M})_d$ to the interval of labels of rim hooks containing \mathbf{r}_{d+1} gives $w(\mathcal{M})_{d+1}$. In addition, the letters in $w(\mathcal{M})_d$ that are not in $w(\mathcal{M})_{d+1}$ appear in $w(\mathcal{M})_d$ in weakly increasing order.*

Proof. In this situation, any rim hook containing \mathbf{r}_{d+1} contains \mathbf{r}_d . In addition, any rim hook that contains \mathbf{r}_d but not \mathbf{r}_{d+1} has label strictly greater than all of the rim hooks containing \mathbf{r}_{d+1} , and the labels of these rim hooks increase with insertion order. The result follows. \square

Example 18. Consider $\lambda = (3, 2, 2)$ as in Example 14 and suppose \mathcal{M} contains one copy of each rim hook of λ . Then we have $w(\mathcal{M})_{-2} = 146$ and $w(\mathcal{M})_{-1} = 146257$, so we see $w(\mathcal{M})_{-2}$ as a prefix of $w(\mathcal{M})_{-1}$. We also have $w(\mathcal{M})_0 = 1425$, so we see that $w(\mathcal{M})_{-1}$ restricted to $[1, 5]$ gives $w(\mathcal{M})_0$.

Proof of Proposition 15. We first show entries are weakly increasing along rows and down columns. We do this by considering the entries of adjacent diagonals d and $d + 1$ of $h_k \star \cdots \star h_2 \star h_1 \star 0$. First suppose \mathbf{r}_d is directly left of \mathbf{r}_{d+1} . It follows from Lemma 16 and Theorem 2 that the shape of $P(w(\mathcal{M})_d)$ is contained in the shape of $P(w(\mathcal{M})_{d+1})$, and so the entries in diagonal d are weakly less than the entries to their right in diagonal $d + 1$.

We next argue that the entries in diagonal d are weakly greater than the entries above them. By Lemma 16, $w(\mathcal{M})_{d+1} = w(\mathcal{M})_d u$ for weakly increasing word u . The desired result then follows from Lemma 7.

If \mathbf{r}_d is directly below \mathbf{r}_{d+1} , it follows from Lemma 17 that the restriction of $P(w(\mathcal{M})_d)$ to the letters inserted on diagonal $d + 1$ gives $P(w(\mathcal{M})_{d+1})$. This means the entries in diagonal d will be weakly greater than the entries directly above them.

By Lemma 7 and Lemma 17, the letters in $w(\mathcal{M})_d$ not inserted on diagonal $d + 1$ form a horizontal strip in the insertion tableau $P(w(\mathcal{M})_d)$. It follows that the i th row of $P(w(\mathcal{M})_d)$ must be weakly shorter than the $(i - 1)$ th row of $P(w(\mathcal{M})_{d+1})$, and so entries in diagonal d will be weakly less than the entries to their right in diagonal $d + 1$.

It remains to show that the number of nonzero parts in the shape of $P(w(\mathcal{M})_d)$ is no larger than the number of boxes of λ in diagonal d . It follows from Theorem 2 that the number of parts of $P(w)$ is equal to the minimal number of disjoint non-decreasing sequences in w necessary to partition the word w . Consider some diagonal $j - i$ with border box in position (i, j) and consider the word \tilde{w} obtained by reading the rim hook labels in boxes

$$(1, 1), (2, 1), \dots, (i, 1), (1, 2), (2, 2), \dots, (i, 2), \dots, (1, j), \dots, (i, j).$$

For example, the corresponding word for diagonal -1 in Example 14 is $\tilde{w} = 146257$. This word can clearly be written as the disjoint union of j non-decreasing sequences:

$$\{(1, 1), \dots, (i, 1)\}, \{(1, 2), \dots, (i, 2)\}, \dots, \{(1, j), \dots, (i, j)\}.$$

It can also be written as the disjoint union of i non-decreasing sequences:

$$\{(1, 1), (1, 2), \dots, (1, j)\}, \{(2, 1), (2, 2), \dots, (2, j)\}, \dots, \{(i, 1), (i, 2), \dots, (i, j)\}.$$

Hence the minimal number of disjoint non-decreasing sequences it takes to partition the word is at most $\min(i, j)$. Now $w(\mathcal{M})_{j-i}$ can be obtained from \tilde{w} by deleting entries and repeating existing entries, and can thus still be covered by $\min(i, j)$ non-decreasing sequences. \square

We develop the inverse procedure in the next section.

4.1 Diagonal RSK is equivalent to Sulzgruber insertion

In this section, we will show that diagonal RSK is equivalent to the Sulzgruber insertion by showing that the reverse insertion procedure is the same in both cases. The fact that the two insertion procedures are equivalent may be initially surprising because the entries on diagonal d of a reverse plane partition seemingly cannot be computed using Sulzgruber's insertion without knowing what happens in diagonal $d + 1$, while any one diagonal of a reverse plane partition can clearly be computed on its own in diagonal RSK. Recall that we use $*$ for Sulzgruber insertion and \star for diagonal RSK.

Theorem 19. *Suppose $h_1 \leq h_2 \leq \dots \leq h_k$ are rim hooks in partition λ . Then*

$$h_k \star \dots \star h_2 \star h_1 \star 0 = h_k * \dots * h_2 * h_1 * 0.$$

In other words, diagonal RSK coincides with Sulzgruber's insertion.

Proof. To prove the result, we show that the reverse insertion procedure for \star is the same as that of $*$. We first assume for notational simplicity that λ is an $s \times s$ square. In this case, it is easy to label the rim hooks of λ , as shown in Figure 2. However, the following argument may be used for any partition λ .

We briefly review the ideas of Lemmas 16 and 17 in the case of the $s \times s$ square. Consider a diagonal $d < 0$ (whose southeasternmost box is in column $d + s$). Then $w(\mathcal{M})_{d+1} = w(\mathcal{M})_d u$, where u is a possibly empty, nondecreasing word in the alphabet $\alpha s + (d + s + 1)$, where $0 \leq \alpha \leq s - 1$. Next consider a diagonal $d > 0$ (whose southeasternmost box is in row $s - d$). Then $w(\mathcal{M})_{d-1}|_{[1, (s-d)s]} = w(\mathcal{M})_d$. In particular, $P(w(\mathcal{M})_{d-1})_{[1, (s-d)s]} = P(w(\mathcal{M})_d)$ by Lemma 3.

Suppose we are given a reverse plane partition of λ , $\pi = h_k \star \dots \star h_2 \star h_1 \star 0$, where $h_1 \leq h_2 \leq \dots \leq h_k$ are rim hooks in λ . Let \mathcal{M} be the multiset $\{h_1, \dots, h_k\}$. We wish to recover h_k and $\pi' = h_{k-1} \star \dots \star h_2 \star h_1 \star 0$.

1	2	s
s + 1	s + 2	2s
⋮	⋱	⋱	⋱	⋮
⋮	⋱	⋱	⋱	⋮
s ² - s + 1	s ² - s + 2	s ²

Figure 2: Labelling of the rim hooks of λ .

Let ℓ denote the label of h_k , and suppose rim hook h_k has southwesternmost box in diagonal d_b . Note that it is possible that $w(\mathcal{M})$ includes more than one copy of ℓ ; in this argument, ℓ always refers to the copy corresponding to h_k .

Note that d_b is the southwesternmost diagonal of λ where ℓ is inserted. Let b denote the box in diagonal d_b of π corresponding to the row of $P(w(\mathcal{M})_{d_b})$ where the insertion of ℓ terminated, and suppose b is in position (p, q) in λ . In other words, if

$$P(w(\{h_1, \dots, h_{k-1}, h_k\})_{d_b}) / P(w(\{h_1, \dots, h_{k-1}\})_{d_b})$$

is a single box in row r , then b is the r th box from the bottom of diagonal d_b in π . In our setup, $r = s + 1 - p$; we will use this later in the proof. Let $\pi(i, j)$ denote the entry in position (i, j) of π and similarly for π' .

Notice that $\pi(p, q) > \pi(p, q - 1)$. Indeed, since π' is a reverse plane partition, we have that

$$\pi(p, q - 1) = \pi'(p, q - 1) \leq \pi'(p, q) = \pi(p, q) - 1.$$

(Recall here that if $q = 1$, we consider $\pi(p, q - 1) = 0$ and if $p = 1$, we consider $\pi(p - 1, q) = 0$.)

In addition, as explained in Section 3.1, box b must be one of two types. Either $d_b = 0$ and contains the unique outer corner of λ or $d_b < 0$. In the first case, we again say b is of type \mathcal{O} , while in the latter case, we say b is of type \mathcal{A} .

If b is of type \mathcal{A} , then $\pi(p, q) > \pi(p - 1, q)$. To see why this is true, first recall that the border box \mathbf{r}_{d_b} is in position $(s, d_b + s)$ in λ . Since h_k was the largest rim hook inserted (with respect to insertion order), none of the rim hooks corresponding to boxes of λ in column $d_b + s + 1$ have been inserted. Thus by Lemma 16, $w(\mathcal{M})_{d_b} = w(\mathcal{M})_{d_b+1}$. This

implies the insertion of ℓ into $P(w(\{h_1, \dots, h_{k-1}\})_{d_b+1})$ terminated in row $s+1-p$. Thus $\pi(p, q) = \pi'(p, q) + 1$, $\pi(p, q+1) = \pi'(p, q+1) + 1$, and $\pi(p-1, q) = \pi'(p-1, q)$, and so

$$\pi(p-1, q) = \pi'(p-1, q) \leq \pi'(p, q) < \pi(p, q).$$

We have shown that box b must fit into one of the following two categories:

- (i) box b is of type \mathcal{O} and $\pi(p, q) > \pi(p, q-1)$, or
- (ii) box b is of type \mathcal{A} and $\pi(p, q) > \pi(p, q-1)$ and $\pi(p, q) > \pi(p-1, q)$.

Following Sulzgruber's terminology, we call such a box b of π a *candidate*.

To begin the reverse insertion of h_k , scan the reverse plane partition π southeast to northwest along each diagonal starting with the northeasternmost diagonal and locate the first candidate b in some position (p, q) . We next argue that this first candidate b is in the southwesternmost diagonal where h_k has support and that the insertion of ℓ into $P(w(\{h_1, \dots, h_{k-1}\})_{d_b})$ terminated in row $s+1-p$. The latter statement tells us that b is correctly positioned in its diagonal d_b .

The fact that $\pi(p, q) > \pi(p, q-1)$ means that $w(\mathcal{M})_{d_b} \neq w(\mathcal{M})_{d_b-1}$ and the fact that b is the first box in our scan with this property means that $w(\mathcal{M})_{d_b} = w(\mathcal{M})_{d_b+i}$ for all $i > 0$. By Lemma 16, this implies we have inserted a nonempty submultiset of the rim hooks $d_b+s, d_b+2s, \dots, d_b+s^2$ and have not inserted any rim hooks with labels $d_b+i+s, d_b+i+2s, \dots, d_b+i+s^2$ for $0 < i \leq -d_b$. The fact that b is the southeasternmost candidate in its diagonal implies that it is the place where the insertion of the largest (in insertion order) of this submultiset terminated by part (b) of Lemma 4. Each of the rim hooks $d_b+s, d_b+2s, \dots, d_b+s^2$ has the property that its southwesternmost box is on diagonal d_b , i.e., that d_b is the southwesternmost diagonal of λ where the corresponding label is inserted. This proves the desired property.

We next define a lattice path in λ starting at b , ending at a box in column s of λ , and taking upward and rightward steps in λ as follows. Suppose we have constructed the path up to some box b' in position (p', q') in λ .

- If $\pi(p', q') = \pi(p'-1, q')$, the next box in the path is in position $(p'-1, q')$.
- If $\pi(p', q') \neq \pi(p'-1, q')$ and $(p', q'+1) \in \lambda$, the next box in the path is in position $(p', q'+1)$.
- If $\pi(p', q') \neq \pi(p'-1, q')$ and $(p', q'+1) \notin \lambda$, the path terminates at b' .

We claim that this path exactly identifies the boxes where the insertion of ℓ terminated in each diagonal where it was inserted. In other words, subtracting one from each of the entries of the boxes of π in the path yields π' and the boxes in the path uniquely identify h_k .

If $\pi(p', q') = \pi(p'-1, q')$, we must include the box above b' in the path in order for π' to be a reverse plane partition, which we know is true. Therefore the next box in the path is the box above b' .

If $\pi(p', q') \neq \pi(p' - 1, q')$ and $(p', q' + 1) \in \lambda$, we show why

$$P(w(\mathcal{M})_{d_{b'+1}})/P(w(\{h_1, \dots, h_{k-1}\})_{d_{b'+1}})$$

is a box in the row corresponding to box $(p', q' + 1)$ of λ , i.e., $\pi(p', q' + 1) = \pi'(p', q' + 1) + 1$. Note that since $(p', q' + 1) \in \lambda$, $s + 1 - p' \neq 1$, ℓ is not the largest element of $w((M))_{d_{b'}}$. It follows that $\ell \in [1, (s - d_{b'})s]$ and was thus inserted in diagonal $d_{b'} + 1$.

Suppose first that $\mathbf{r}_{d_{b'}}$ is in the rightmost column s . Let c denote the entry in the box $P(w(\mathcal{M})_{d_{b'}})/P(w(\{h_1, \dots, h_{k-1}\})_{d_{b'}})$. In other words, c is the entry of the box where the insertion of ℓ into $P(w(\{h_1, \dots, h_{k-1}\})_{d_{b'}})$ terminated, which is in row $s + 1 - p'$. Note that $\pi(p' - 1, q') < \pi(p', q')$ means that $c \in [(s - d_{b'})s + 1, s^2]$, since $P(w(\mathcal{M})_{d_{b'}})_{[1, (s - d_{b'})s]} = P(w(\mathcal{M})_{d_{b'+1}})$.

We show that the entry that bumped c was also inserted into diagonal $d_{b'} + 1$, i.e., the entry that bumped c is in $[1, s(s - d_{b'})]$. Indeed, $P(w(\mathcal{M})_{d_{b'}})_{[(s - d_{b'})s + 1, s^2]}$ is a horizontal strip with weakly increasing reading word by Lemma 7, Lemma 17, and Theorem 6. Thus c could not have been bumped by anything in $[(s - d_{b'})s + 1, s^2]$ by part (a) of Lemma 4. Now

$$P(w(\mathcal{M})_{d_{b'}})_{[1, (s - d_{b'})s]} = P(w(\mathcal{M})_{d_{b'+1}})$$

and similarly

$$P(w(\{h_1, \dots, h_{k-1}\})_{d_{b'}})_{[1, (s - d_{b'})s]} = P(w(\{h_1, \dots, h_{k-1}\})_{d_{b'+1}})$$

imply that the insertion of ℓ into $P(w(\{h_1, \dots, h_{k-1}\})_{d_{b'+1}})$ terminated in row $s - p'$.

Now suppose border box $\mathbf{r}_{d_{b'}}$ is not in the rightmost column. Since $w(\mathcal{M})_{d_{b'}} = w(\mathcal{M})_{d_{b'+1}}$, it is clear that the insertion of ℓ terminated in row $s + 1 - p'$ of $P(w(\mathcal{M})_{d_{b'+1}})$, and thus we go right in the path.

If $\pi(p', q') \neq \pi(p' - 1, q')$ and $(p', q' + 1) \notin \lambda$, then clearly b' is in the rightmost column of λ . This means ℓ ends up in the first row of $P(w(\mathcal{M})_{d_{b'}})$. Since $\pi(p', q') \neq \pi(p' - 1, q')$ and $P(w(\mathcal{M})_{d_{b'}})_{[1, (s - d_{b'})s]} = P(w(\mathcal{M})_{d_{b'+1}})$, this implies $\ell \notin [1, (s - d_{b'})s]$ and so is not inserted into diagonal $d_{b'} + 1$ by Lemma 17. This justifies why our path must stop.

Once the path is determined, we can identify h_k as the rim hook whose northeastmost box is the box at the end of the path and that contains the same number of boxes as the path. It is clear that this reverse insertion procedure is the same as that for Sulzgruber's $*$ insertion, and thus the two insertions agree for the square.

If our partition λ is not a square, we may apply the same arguments to show the result with the following small modifications: the rim hook indexing will be much less convenient, and so notation will become more complicated; and we must define \mathcal{O} and \mathcal{A} as we did in Section 3.1 instead of using the simplified definition for the square. \square

In the next section, we review the analogous result for Hillman–Grassl, which can be found for example in [7]. In this case, the Greene–Kleitman invariant for Hillman–Grassl was known, and the explicit RSK interpretation is obtained as a corollary.

5 Hillman–Grassl as RSK

One can describe the Hillman–Grassl correspondence similarly using a variant of the diagonal RSK defined above. We next remind the reader of such a description, and we give it in the same spirit as diagonal RSK. We will refer to it as *HG diagonal RSK*. We present this description of Hillman–Grassl to emphasize the fact that both Hillman–Grassl and Sulzgruber insertion are determined by a linear ordering of the cells of the Young diagram (the insertion order) along with a compatible labeling of rim hooks.

To obtain the Hillman–Grassl insertion order for shape λ , label the cells of λ top to bottom along columns starting with the rightmost column. The labeling for the rim hooks is obtained by labeling the cells of λ right to left along rows starting with the top row. See Figure 3, where we see, for example, that inside $\lambda = (3, 3, 2)$, rim hook $h_{(2,1)}$ is seventh in the insertion order and has label 6.

6	3	1
7	4	2
8	5	

3	2	1
6	5	4
8	7	

Figure 3: HG diagonal RSK insertion order (left) and rim hook labeling (right).

Given a multiset of rim hooks \mathcal{M} of λ , we can again form a word corresponding to each diagonal of λ in an analogous way:

1. First, order the multiset \mathcal{M} by insertion order.
2. Create a word $w_{HG}(\mathcal{M})$ by recording the corresponding label for each rim hook in the ordered multiset \mathcal{M} .
3. Obtain word $w_{HG}(\mathcal{M})_i$ corresponding to diagonal i by restricting $w_{HG}(\mathcal{M})$ to the labels of rim hooks with support on diagonal i .

As before, we then perform RSK insertion with each word $w_{HG}(\mathcal{M})_i$, and we form the reverse plane partition corresponding to \mathcal{M} by recording the row sizes of the insertion tableau $P(w_{HG}(\mathcal{M})_i)$ on diagonal i .

We now state the Greene–Kleitman invariants for HG diagonal RSK, which will lead to the proof that HG diagonal RSK is equivalent to Hillman–Grassl. Let $B = (b_{ij})$ be a nonnegative integer matrix. An A_{HG} -chain in the matrix B is a sequence $V = ((i_1, j_1), \dots, (i_k, j_k))$ of positions in B where $i_1 \leq i_2 \leq \dots \leq i_k$ and $j_1 \geq j_2 \geq \dots \geq j_k$, where (i, j) can be used at most b_{ij} times. Let $|V|$ denote the cardinality of a A_{HG} -chain. Define $a_k^{HG}(B)$ to be the maximum of $|V_1| + \dots + |V_k|$, where the maximum is over all collections of k A_{HG} -chains such that (i, j) is used at most b_{ij} times in the collection and $|V_i|$ denotes the cardinality of V_i . Let $a_0^{HG}(B) = 0$.

Similarly, define a C_{HG} -chain in the matrix B to be a sequence

$$U = ((i_1, j_1), \dots, (i_k, j_k))$$

of *distinct* positions in B where $i_1 < i_2 < \dots < i_k$ and $j_1 < j_2 < \dots < j_k$, where (i, j) can be used only if $b_{ij} \neq 0$. Define $c_k^{HG}(B)$ to be the maximum of $|U_1| + \dots + |U_k|$, where the maximum is over all collections of k C_{HG} -chains such that (i, j) is used at most once in the collection and $|U_i|$ denotes the cardinality of U_i . Let $c_0^{HG}(B) = 0$.

Remark 20. A C_{HG} -chain is classically referred to as a D -chain, where the “ D ” stands for “descending”. Since we use “ d ” to denote a diagonal of a Young diagram, we use the C_{HG} -chain (and C_S -chain in the next section) terminology to avoid confusion.

Given a partition λ and a multiset \mathcal{M} of rim hooks of λ , form an \mathbb{N} -tableau $T_{\mathcal{M}}$ by entering in each box of λ the number of times the corresponding rim hook appears in \mathcal{M} . For each diagonal $d = j - i$ of λ , define the matrix B_d to be the $i \times j$ -matrix obtained by restricting $T_{\mathcal{M}}^{HG}$ to the boxes weakly northwest of \mathbf{r}_d .

Theorem 21. *Let λ be a partition, \mathcal{M} be a multiset of rim hooks of λ . Let π denote the reverse plane partition obtained from \mathcal{M} using HG diagonal RSK, and let μ^d be the partition obtained by reading the entries of π in diagonal d . Then*

$$\begin{aligned} a_k^{HG}(B_d) &= \mu_1^d + \mu_2^d + \dots + \mu_k^d \text{ and} \\ c_k^{HG}(B_d) &= (\mu^d)'_1 + (\mu^d)'_2 + \dots + (\mu^d)'_k. \end{aligned}$$

In other words,

$$\begin{aligned} \mu_i^d &= a_i^{HG}(B_d) - a_{i-1}^{HG}(B_d) \text{ and} \\ (\mu^d)'_i &= c_i^{HG}(B_d) - c_{i-1}^{HG}(B_d) \end{aligned}$$

Proof. The result follows immediately from Theorem 2 and the fact that A_{HG} -chains (resp., C_{HG} -chains) in B_d correspond exactly to nondecreasing (resp., decreasing) sequences in $w_{HG}(\mathcal{M})_d$. \square

The following corollary is well known and can be found, for example, in [7].

Corollary 22. *The Hillman–Grassl correspondence is equivalent to HG diagonal RSK.*

Proof. As stated in [1, 2], the Hillman–Grassl correspondence has the same Greene–Kleitman invariants as given in Theorem 21. \square

6 Greene–Kleitman invariants for Sulzgruber insertion

We now state the Greene–Kleitman invariants for Sulzgruber insertion in the same spirit as those stated for Hillman–Grassl by Gansner in [1, 2] and for HG diagonal RSK in Theorem 21.

Let $B = (b_{ij})$ be a nonnegative integer matrix. An A_S -chain in the matrix B is a sequence $W = ((i_1, j_1), \dots, (i_k, j_k))$ of positions in B where $i_1 \leq i_2 \leq \dots \leq i_k$ and $j_1 \leq j_2 \leq \dots \leq j_k$, where (i, j) can be used at most b_{ij} times. Let $|W|$ denote the cardinality of a A_S -chain. Define $a_k^S(B)$ to be the maximum of $|W_1| + \dots + |W_k|$, where

the maximum is over all collections of k A_S -chains such that (i, j) is used at most b_{ij} times in the collection and $|W_i|$ denotes the cardinality of W_i . Let $a_0^S(B) = 0$.

Similarly, define a C_S -chain in the matrix B to be a sequence $R = ((i_1, j_1), \dots, (i_k, j_k))$ of *distinct* positions in B where $i_1 > i_2 > \dots > i_k$ and $j_1 < j_2 < \dots < j_k$, where (i, j) can be used only if $b_{ij} \neq 0$. Define $c_k^S(B)$ to be the maximum of $|R_1| + \dots + |R_k|$, where the maximum is over all collections of k C_S -chains such that (i, j) is used at most once in the collection and $|R_i|$ denotes the cardinality of R_i . Let $c_0^S(B) = 0$.

Given a partition λ and a multiset \mathcal{M} of rim hooks of λ , we again form an \mathbb{N} -tableau $T_{\mathcal{M}}$ by entering in each box of λ the number of times the corresponding rim hook appears in \mathcal{M} . For each diagonal $d = j - i$ of λ , define the matrix B_d to be the $i \times j$ matrix obtained by restricting $T_{\mathcal{M}}$ to the boxes weakly northwest of \mathbf{r}_d .

Theorem 23. *Let λ be a partition, \mathcal{M} be a multiset of rim hooks of λ . Let π denote the reverse plane partition obtained from \mathcal{M} using Sulzgruber insertion, and let μ^d be the partition obtained by reading the entries of π in diagonal d . Then*

$$\begin{aligned} a_k^S(B_d) &= \mu_1^d + \mu_2^d + \dots + \mu_k^d \text{ and} \\ c_k^S(B_d) &= (\mu^d)'_1 + (\mu^d)'_2 + \dots + (\mu^d)'_k. \end{aligned}$$

In other words,

$$\begin{aligned} \mu_i^d &= a_i^S(B_d) - a_{i-1}^S(B_d) \text{ and} \\ (\mu^d)'_i &= c_i^S(B_d) - c_{i-1}^S(B_d) \end{aligned}$$

Proof. The result follows immediately from Theorem 2, Theorem 19, and the fact that A_S -chains (resp., C_S -chains) in B_d correspond exactly to nondecreasing (resp., decreasing) sequences in $w(\mathcal{M})_d$. \square

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