# New Probabilistic Upper Bounds on the Domination Number of a Graph

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**Abstract**

A subset $S$ of vertices of a graph $G$ is a dominating set of $G$ if every vertex in $V(G) - S$ has a neighbor in $S$. The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. In this paper, we obtain new (probabilistic) upper bounds for the domination number of a graph, and improve previous bounds given by Arnautov (1974), Payan (1975), and Caro and Roditty (1985) for any graph, and Harant, Pruchnewski and Voigt (1999) for regular graphs.

**Mathematics Subject Classifications:** 05C69

## 1 Introduction

For graph theory notation and terminology not given here we refer to [11], and for the probabilistic methods notation and terminology we refer to [1]. We consider finite, undirected and simple graphs $G$ with vertex set $V = V(G)$ and edge set $E(G)$. The number of vertices of $G$ is called the *order* of $G$ and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of $v$ is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v$, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to $G$), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of $G$, respectively. For a subset $S$ of vertices of $G$, the subgraph of $G$ induced by $S$ is denoted by $G[S]$. A subset $S \subseteq V$ is a *dominating set* of $G$ if every vertex in $V - S$ has a neighbor in $S$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. For two subsets $A$ and $B$ of vertices of $G$, we say that $A$ dominates $B$ if any vertex of $B$ is adjacent to a vertex of $A$.

The following fundamental upper bound for the domination number was independently proved by Alon and Spencer [1], Arnautov [2], Lovász [12] and Payan [13].

**Theorem 1** ([1, 2, 12, 13]). *For any graph $G$ of order $n$ with $\delta(G) > 1$,*

$$\gamma(G) \leqslant \frac{1 + \ln(1 + \delta(G))}{1 + \delta(G)} n.$$

A deterministic algorithm to construct a dominating set satisfying the bound of Theorem 1 can be found in [1]. Using the same procedure of Alon and Spencer for the proof of Theorem 1, similar upper bounds for several domination variants have been obtained, (see, for example, [3, 4, 6, 7, 8, 9, 10, 11]. Caro and Roditty [5, 11] gave the following upper bound for the domination number of a graph which is one of the strongest known upper bounds for the domination number.

**Theorem 2** (Caro and Roditty [5, 11]). *For any graph $G$ of order $n$ with $\delta(G) \geqslant 1$,*

$$\gamma(G) \leqslant \left(1 - \frac{\delta(G)}{(\delta(G)+1)^{1+\frac{1}{\delta(G)}}}\right) n.$$

The (Alon-Spencer) procedure of proof for the above upper bounds (as is given in [1]) is as follows: A subset of vertices, say $A$, is chosen with probability $Pr(v \in A) = p$, where $0 < p < 1$, for any vertex $v \in V(G)$. Then in order to form a dominating set, a subset of $V(G) - A$ will be added to $A$. In this paper, we use the same Alon-Spencer procedure with the exception that we remove the non-necessary vertices from $A$. We thus reduce the cardinality of the dominating set. The organization of the paper is as follows. In Section 2, we prove two key lemmas. In Section 3, we present new probabilistic upper bounds on the domination number of a graph.

## 2 Key lemmas

**Lemma 3.** *Let $G$ be a graph on $n$ vertices with minimum degree $\delta > 1$ and maximum degree $\Delta$, and $0 < p < 1$. Let $A \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$, independently, with probability $p$, and let $A'' \subseteq A' \subseteq A$ be defined by $A' = \{v \in V(G) : N[v] \subseteq A\}$ and $A'' = \{v \in V(G) : N[v] \subseteq A'\}$ (see Figure 1 for an example). For any positive integer $s$, there is a subset $S \subseteq A'$ such that $S$ dominates $A''$ and $|S| \leqslant f(s-1)|A'|$, where $f(0) = p + (1-p)^{1+\delta}$ and*

$$\begin{aligned} f(j) &= p + (1-p)^{1+\delta} - \left(1 - p - (1-p)^{1+\delta}\right) \sum_{i=1}^{j} p^{i(1+\Delta)} \\ &= f(0) - (1 - f(0)) \sum_{i=1}^{j} p^{i(1+\Delta)} \end{aligned}$$

*for any positive integer $j > 0$.*

*Proof.* The proof is by an induction on $s$. Before starting the induction, we state the following simple but important claim.

Claim 1. $\deg_{G[A']}(v) = \deg_G(v) = \deg(v)$, for any vertex $v \in A''$.

We now start the induction. For the base step of the induction, assume that $s = 1$. We show that there is a subset $S \subseteq A'$ such that $S$ dominates $A''$ and $|S| \leqslant f(0)|A'| = \Big(p + (1-p)^{1+\delta}\Big)|A'|$. Create a subset $A_1 \subseteq A'$ by choosing each vertex $v \in A'$, independently, with probability $p$. Let $B_1 \subseteq A''$ be the set of all vertices of $A''$ that are not dominated by $A_1$. Then $S_1 = A_1 \cup B_1$ dominates any vertex of $A''$. We now compute the expectation of $|S_1|$. Clearly, $\mathbb{E}(|A_1|) = |A'|p$. For a vertex $v \in A'$, using Claim 1,

$$Pr(v \in B_1) = (1-p)^{1+\deg_{G[A']}(v)} = (1-p)^{1+\deg_G(v)} \leqslant (1-p)^{1+\delta}. \tag{1}$$

Thus $\mathbb{E}(|B_1|) \leqslant |A'|(1-p)^{1+\delta}$. Now by the linearity property of the expectation, we have

$$\begin{aligned}
\mathbb{E}(|S_1|) = \mathbb{E}(|A_1 \cup B_1|) &= \mathbb{E}(|A_1|) + \mathbb{E}(|B_1|) \\
&\leqslant |A'|p + |A'|(1-p)^{1+\delta} = |A'|\Big(p + (1-p)^{1+\delta}\Big) = f(0)|A'|.
\end{aligned}$$

Since for a random variable $X$, always $Pr(X < \mathbb{E}(X)) > 0$, there is a subset $S \subseteq A'$ such that $S$ dominates $A''$ and $|S| \leqslant f(0)|A'|$, and thus the base step of the induction holds.

Assume that the result holds for any positive integer $s'$ with $s' \leqslant s$. We prove the assertion for $s+1$. Create a subset $A_1 \subseteq A'$ by choosing each vertex $v \in A'$, independently, with probability $p$. Let $B_1 \subseteq A''$ be the set of all vertices of $A''$ that are not dominated by $A_1$. Let $A_1' = \{v \in A_1 : N_{G[A']}[v] \subseteq A_1\}$, and $A_1'' = \{v \in A_1 : N_{G[A']}[v] \subseteq A_1'\}$. Clearly, $A_1' \subseteq A''$, since $A_1 \subseteq A'$. Thus by Claim 1, for any vertex $v \in A_1'$, we have $\deg_{G[A']}(v) = \deg(v)$. In particular, $G[A_1']$ is a graph with minimum degree at least $\delta(G) > 1$. By applying the inductive hypothesis on the graph $G[A_1']$, there is a subset $S_s \subseteq A_1'$ such that $S_s$ dominates $A_1''$ and $|S_s| \leqslant f(s-1)|A_1'|$. Then $S_{s+1} = (A_1 - A_1') \cup S \cup B_1$ dominates $A''$. Clearly, $\mathbb{E}(|A_1|) = |A'|p$, $\mathbb{E}(|B_1|) \leqslant |A'|(1-p)^{1+\delta}$, and $\mathbb{E}(|A_1'|) \geqslant |A'|p^{1+\Delta}$. Thus

$$\begin{aligned}
\mathbb{E}(|S_{s+1}|) &= \mathbb{E}(|(A_1 - A_1') \cup S_s \cup B_1|) \\
&= \mathbb{E}(|A_1|) - \mathbb{E}(|A_1'|) + \mathbb{E}(|S_s|) + \mathbb{E}(|B_1|) \\
&\leqslant |A'|p + |A'|(1-p)^{1+\delta} - \mathbb{E}(|A_1'|) + f(s-1)\mathbb{E}(|A_1'|) \\
&= |A'|p + |A'|(1-p)^{1+\delta} - (1 - f(s-1))\mathbb{E}(|A_1'|) \\
&\leqslant |A'|p + |A'|(1-p)^{1+\delta} - (1 - f(s-1))|A'|p^{1+\Delta}
\end{aligned}$$

But

$$\begin{aligned}
(1 - f(s-1))p^{1+\Delta} &= \left[1 - p - (1-p)^{1+\delta} + \Big(1 - p - (1-p)^{1+\delta}\Big)\sum_{i=1}^{s-1} p^{i(1+\Delta)}\right]p^{1+\Delta} \\
&= \Big(1 - p - (1-p)^{1+\delta}\Big)\sum_{i=1}^{s} p^{i(1+\Delta)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}(|S_{s+1}|) &\leqslant |A'|p + |A'|(1-p)^{1+\delta} - (1 - f(s-1))|A'|p^{1+\Delta} \\
&= |A'|p + |A'|(1-p)^{1+\delta} + |A'|\left(1 - p - (1-p)^{1+\delta}\right)\sum_{i=1}^{s} p^{i(1+\Delta)} \\
&= |A'|\left[p + (1-p)^{1+\delta} + \left(1 - p - (1-p)^{1+\delta}\right)\sum_{i=1}^{s} p^{i(1+\Delta)}\right] \\
&= f(s)|A'|.
\end{aligned}$$

Thus there is a subset $S \subseteq A'$ such that $S$ dominates $A''$ and $|S| \leqslant f(s)|A'|$. This completes the proof. $\square$

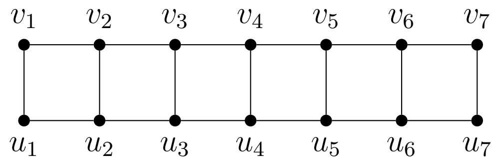


Figure 1. If the chosen set $A$ is the set of all vertices of degree three in the above graph, then $A' = \{v_3, v_4, v_5, u_3, u_4, u_5\}$ and $A'' = \{v_4, u_4\}$.

With the same argument in the proof of Lemma 3, and the fact that $1 - x \leqslant e^{-x}$ for $0 \leqslant x \leqslant 1$, we obtain the following.

**Lemma 4.** *Let $G$ be a graph on $n$ vertices with minimum degree $\delta > 1$ and maximum degree $\Delta$, and $0 < p < 1$. Let $A \subseteq V(G)$ be a set obtained by choosing each vertex $v \in V(G)$, independently, with probability $p$, $A' = \{v \in V(G) : N[v] \subseteq A\}$, and $A'' = \{v \in V(G) : N[v] \subseteq A'\}$. For any positive integer $s$, there is a subset $S \subseteq A'$ such that $S$ dominates $A''$ and $|S| \leqslant f(s-1)|A'|$, where $f(0) = p + e^{-p(1+\delta)}$ and*

$$f(j) = p + e^{-p(1+\delta)} - \left(1 - p - e^{-p(1+\delta)}\right)\sum_{i=1}^{j} p^{i(1+\Delta)}$$

*for any positive integer $j > 0$.*

## 3 New bounds on the domination number

In this section we apply Lemmas 3 and 4 to improve the bounds for the domination number given in Theorems 1 and 2.

**Theorem 5.** *If $G$ is a graph on $n$ vertices with minimum degree $\delta > 1$ and maximum degree $\Delta$, then for any integer $k \geqslant 1$,*

$$\gamma(G) \leqslant \frac{n}{\delta+1}\left[\ln(\delta+1) + 1 - (\delta - \ln(1+\delta))\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)}{1+\delta}\right)^{i(1+\Delta)}\right].$$

*Proof.* Let $G$ be a graph on $n$ vertices with minimum degree $\delta > 1$ and maximum degree $\Delta$, and let $k \geqslant 1$ be a positive integer. Create a subset $A \subseteq V(G)$ by choosing each vertex $v \in V(G)$ independently with probability $p = \frac{\ln(1+\delta)}{1+\delta}$. Let $B = V(G) - N[A]$, $A' = \{v \in V(G) : N[v] \subseteq A\}$, and $A'' = \{v \in V(G) : N[v] \subseteq A'\}$. Clearly, $\deg_{G[A']}(v) = \deg(v)$ for any vertex $v \in A''$. Note that any vertex of $A' - A''$ is dominated by some vertex of $A - A'$. By applying Lemma 3 with $p = \frac{\ln(1+\delta)}{1+\delta}$, there is set $S \subseteq A'$ such that $S$ dominates any vertex of $A''$, and $|S| \leqslant f(k-1)|A'|$, where $f(0) = \frac{1}{\delta+1}\Big(\ln(\delta+1)+1\Big)$ and

$$f(j) = \frac{1}{\delta+1}\left[\ln(\delta+1) + 1 - (\delta - \ln(1+\delta))\sum_{i=1}^{j}\left(\frac{\ln(1+\delta)}{1+\delta}\right)^{i(1+\Delta)}\right]$$

for any positive integer $j > 0$. Evidently, $D = (A - A') \cup B \cup S$ is a dominating set in $G$. We compute the expectation of $|D|$ as follows. Note that

$$\begin{aligned}
|D| = |(A - A') \cup B \cup S| &= |A - A'| + |B| + |S| \\
&= |A| - |A'| + |B| + |S| \\
&\leqslant |A| + |B| - |A'| + f(k-1)|A'| \\
&= |A| + |B| - (1 - f(k-1))|A'|.
\end{aligned}$$

By the linearity property of the expectation, $\mathbb{E}(|D_1|) \leqslant \mathbb{E}(|A|) + \mathbb{E}(|B|) - (1 - f(k-1))\mathbb{E}(|A'|)$. It is routine to see that $\mathbb{E}(|A|) = np$, $\mathbb{E}(|B|) \leqslant n(1-p)^{1+\delta}$, and $\mathbb{E}(|A'|) \geqslant np^{1+\Delta}$. Now by the linearity property of the expectation, the fact that $1 - x \leqslant e^{-x}$, for $0 \leqslant x \leqslant 1$, and a simple calculation, we obtain that

$$\begin{aligned}
\gamma(G) \leqslant \mathbb{E}(|D|) &\leqslant \mathbb{E}(|A|) + \mathbb{E}(|B|) - (1 - f(k-1))\mathbb{E}(|A'|) \\
&\leqslant np + n(1-p)^{1+\delta} - n(1 - f(k-1))p^{1+\Delta} \\
&\leqslant np + ne^{-p(1+\delta)} - n(1 - f(k-1))p^{1+\Delta} \\
&\leqslant \frac{n}{\delta+1}(\ln(\delta+1) + 1) - n(1 - f(k-1))p^{1+\Delta} \\
&= \frac{n}{\delta+1}\left[\ln(\delta+1) + 1 - (\delta - \ln(1+\delta))\sum_{i=1}^{k}\left(\frac{\ln(1+\delta)}{1+\delta}\right)^{i(1+\Delta)}\right].
\end{aligned}$$

Thus there is a dominating set of the desired cardinality. □

It is clear that Theorem 5 is an improvement of Theorem 1, since $\delta - \ln(1+\delta) > 0$.

**Theorem 6.** *If $G$ is a graph with minimum degree $\delta > 1$ and maximum degree $\Delta$, then for any integer $k \geqslant 1$,*

$$\gamma(G) \leqslant \left[1 - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}} - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}}\sum_{i=1}^{k}\left(1 - \frac{1}{(\delta+1)^{\frac{1}{\delta}}}\right)^{i(1+\Delta)}\right]n.$$

*Proof.* The proof is similar to the proof of Theorem 5 with the two exceptions that we apply Lemma 4 for $p = 1 - \frac{1}{(\delta+1)^{\frac{1}{\delta}}}$, and define $f(0) = 1 - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}}$ and

$$f(j) = \left\lceil 1 - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}} - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}} \sum_{i=1}^{j} \left(1 - \frac{1}{(\delta+1)^{\frac{1}{\delta}}}\right)^{i(1+\Delta)} \right\rceil.$$

for any positive integer $j$. Therefore, we omit the proof. □

It is clear that Theorem 6 is an improvement of Theorem 2.

## 4 Concluding remarks

An approach to obtain an upper bound on the domination number of a graph is given by Harant et al. [10]. They presented an algorithm that constructs a dominating set of cardinality at most

$$n\left(1 - \left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\right) + \sum_{v \in V(G)} \left(\frac{1}{\delta+1}\right)^{\frac{\deg(v)+1}{\delta}} \tag{2}$$

and then showed that

$$\gamma(G) \leqslant \sum_{v \in V(G)} \left(1 - \left(\frac{1}{\deg(v)+1}\right)^{\frac{1}{\delta}} + \left(\frac{1}{\deg(v)+1}\right)^{\frac{1}{\delta}} \prod_{u \in N(v)} \left(\frac{1}{\deg(u)+1}\right)^{\frac{1}{\delta}}\right). \tag{3}$$

It is straightforward to see that the bound of Theorem 6 improves both bounds given in (2) and (3) for regular graphs. Note that if $G$ is regular then both (2) and the right side of (3) are equal to $n\left(1 - \frac{\delta}{(\delta+1)^{1+\frac{1}{\delta}}}\right)$, and thus the improvements are obvious.

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