# Improved bounds for the extremal number of subdivisions

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**Abstract**

Let $H_t$ be the subdivision of $K_t$. Very recently, Conlon and Lee have proved that for any integer $t \geqslant 3$, there exists a constant $C$ such that $\mathrm{ex}(n, H_t) \leqslant Cn^{3/2-1/6^t}$. In this paper, we prove that there exists a constant $C'$ such that $\mathrm{ex}(n, H_t) \leqslant C'n^{3/2-\frac{1}{4t-6}}$.

**Mathematics Subject Classifications:** 05C35

## 1 Introduction

For a graph $H$, the extremal function $\mathrm{ex}(n, H)$ is defined to be the maximal number of edges in an $H$-free graph on $n$ vertices. This function is well understood for graphs $H$ with chromatic number at least three by the Erdős-Stone-Simonovits theorem [5, 3]. However, for bipartite graphs $H$, much less is known. For a survey on the subject, see [7]. One of the few general results, proved by Füredi [6], and reproved by Alon, Krivelevich and Sudakov [1] is the following.

**Theorem 1** (Füredi, Alon–Krivelevich–Sudakov)**.** *Let $H$ be a bipartite graph such that in one of the parts all the degrees are at most $r$. Then there exists a constant $C$ such that $\mathrm{ex}(n, H) \leqslant Cn^{2-1/r}$.*

Conlon and Lee [2] have conjectured that the only case when this is tight up to the implied constant is when $H$ contains a $K_{r,r}$ (it is conjectured [8] that $\mathrm{ex}(n, K_{r,r}) = \Omega(n^{2-1/r})$), and that for other graphs $H$ there exists some $\delta > 0$ such that $\mathrm{ex}(n, H) = O(n^{2-1/r-\delta})$.

The subdivision of a graph $L$ is the bipartite graph with parts $V(L)$ and $E(L)$ (the vertex set and the edge set of the graph $L$, respectively) where $v \in V(L)$ is joined to

$e \in E(L)$ if $v$ is an endpoint of $e$. In other words, it is the graph obtained by replacing every edge of $L$ by a path of length 2.

It is easy to see that any $C_4$-free bipartite graph in which every vertex in one part has degree at most two is a subgraph of $H_t$ for some positive integer $t$, where $H_t$ is the subdivision of $K_t$. Conlon and Lee have verified their conjecture in the $r = 2$ case by proving the following result.

**Theorem 2** (Conlon and Lee [2, Theorem 5.1])**.** *For any integer $t \geqslant 3$, there exists a constant $C_t$ such that* $\mathrm{ex}(n, H_t) \leqslant C_t n^{3/2-1/6^t}$.

They have observed the lower bound $\mathrm{ex}(n, H_t) \geqslant c_t n^{3/2-\frac{t-3/2}{t^2-t-1}}$ coming from the probabilistic deletion method, and have asked for an upper bound of the form $\mathrm{ex}(n, H_t) \leqslant C_t n^{3/2-\delta_t}$, where $1/\delta_t$ is bounded by a polynomial in $t$. We can prove such a bound even for a linear $\delta_t$.

**Theorem 3.** *For any integer $t \geqslant 3$, there exists a constant $C_t$ such that* $\mathrm{ex}(n, H_t) \leqslant C_t n^{1+\frac{t-2}{2t-3}} = C_t n^{3/2-\frac{1}{4t-6}}$.

It would be very interesting to know whether or not this bound is tight up to the implied constant. It certainly is tight for $t = 3$ as $\mathrm{ex}(n, C_6) = \Theta(n^{4/3})$.

We can in fact prove a slightly stronger result. For integers $s \geqslant 1$ and $t \geqslant 3$, let $L_{s,t}$ be the graph which is a $K_{s+t-1}$ with the edges of a $K_s$ removed. That is, the vertex set of $L_{s,t}$ is $S \cup T$ where $S \cap T = \emptyset$, $|S| = s$ and $|T| = t - 1$, and $xy$ is an edge if and only if $x \in T$ or $y \in T$. Let $L'_{s,t}$ be the subdivision of $L_{s,t}$.

**Theorem 4.** *For any two integers $s \geqslant 1$ and $t \geqslant 3$, there exists a constant $C_{s,t}$ such that* $\mathrm{ex}(n, L'_{s,t}) \leqslant C_{s,t} n^{3/2-\frac{1}{4t-6}}$.

This result certainly implies Theorem 3 as $L_{1,t} = K_t$. Moreover, we can apply Theorem 4 to obtain good bounds on the extremal number of the subdivision of the complete bipartite graph $K_{a,b}$ as well. Let us write $H_{a,b}$ for the subdivision of $K_{a,b}$. Conlon and Lee [2, Theorem 4.2] have proved that for any $2 \leqslant a \leqslant b$ there exists a constant $C$ such that $\mathrm{ex}(n, H_{a,b}) \leqslant C n^{3/2-\frac{1}{12b}}$. They have also observed the lower bound $\mathrm{ex}(n, H_{a,b}) = \Omega_{a,b}(n^{3/2-\frac{a+b-3/2}{2ab-1}})$ (which follows from the probabilistic deletion method). Hence their upper bound is reasonably close to best possible when $a = b$, but is weak when $b$ is much larger then $a$.

Since $K_{a,b}$ is a subgraph of $L_{b,a+1}$, Theorem 4 implies the following result, by taking $s = b$ and $t = a + 1$.

**Corollary 5.** *For any two integers $2 \leqslant a \leqslant b$, there exists a constant $C_{a,b}$ such that* $\mathrm{ex}(n, H_{a,b}) \leqslant C_{a,b} n^{3/2-\frac{1}{4a-2}}$.

## 2 Proof of Theorem 4

We shall use the following lemma of Conlon and Lee [2, Lemma 2.3], which is a slight modification of a result of Erdős and Simonovits [4]. Let us say that a graph $G$ is *$K$-almost-regular* if $\max_{v\in V(G)} \deg(v) \leqslant K \min_{v\in V(G)} \deg(v)$. Moreover, following Conlon and Lee, we say that a bipartite graph $G$ with a bipartition $A \cup B$ is *balanced* if $\frac{1}{2}|B| \leqslant |A| \leqslant 2|B|$.

**Lemma 6.** *For any positive constant $\alpha < 1$, there exists $n_0$ such that if $n \geqslant n_0$, $C \geqslant 1$ and $G$ is an $n$-vertex graph with at least $Cn^{1+\alpha}$ edges, then $G$ has a $K$-almost-regular balanced bipartite subgraph $G'$ with $m$ vertices such that $m \geqslant n^{\frac{\alpha(1-\alpha)}{2(1+\alpha)}}$, $|E(G')| \geqslant \frac{C}{10}m^{1+\alpha}$ and $K = 60 \cdot 2^{1+1/\alpha^2}$.*

This reduces Theorem 4 to the following.

**Theorem 7.** *For every $K \geqslant 1$, and positive integers $s \geqslant 1, t \geqslant 3$, there exists a constant $c = c(s,t,K)$ with the following property. Let $n$ be sufficiently large and let $G$ be a balanced bipartite graph with bipartition $A \cup B$, $|B| = n$ such that the degree of every vertex of $G$ is between $\delta$ and $K\delta$, for some $\delta \geqslant cn^{\frac{t-2}{2t-3}}$. Then $G$ contains a copy of $L'_{s,t}$.*

Given a bipartite graph $G$ with bipartition $A \cup B$, the *neighbourhood graph* is the weighted graph $W_G$ on vertex set $A$ where the weight of the pair $uv$ is $d_G(u,v) = |N_G(u) \cap N_G(v)|$. Here and below $N_G(v)$ denotes the neighbourhood of the vertex $v$ in the graph $G$. For a subset $U \subset A$, we write $W(U)$ for the total weight in $U$, ie. $W(U) = \sum_{uv\in\binom{U}{2}} d_G(u,v)$.

We shall use the following simple lemma of Conlon and Lee [2, Lemma 2.4].

**Lemma 8.** *Let $G$ be a bipartite graph with bipartition $A \cup B$, $|B| = n$, and minimum degree at least $\delta$ on the vertices in $A$. Then for any subset $U \subset A$ with $\delta|U| \geqslant 2n$,*

$$\sum_{uv\in\binom{U}{2}} d_G(u,v) \geqslant \frac{\delta^2}{2n}\binom{|U|}{2}$$

In other words, the conclusion of Lemma 8 is that $W(U) \geqslant \frac{\delta^2}{2n}\binom{|U|}{2}$.

In the next definition, and in the rest of this paper, for a weighted graph $W$ on vertex set $A$, if $u, v \in A$, then $W(u,v)$ stands for the weight of $uv$. Moreover, we shall tacitly assume throughout the paper that $s \geqslant 1$ and $t \geqslant 3$ are fixed integers.

**Definition 9.** Let $W$ be a weighted graph on vertex set $A$ and let $u, v \in A$ be distinct. We say that $uv$ is a *light edge* if $1 \leqslant W(u,v) < \binom{s+t-1}{2}$ and that it is a *heavy edge* if $W(u,v) \geqslant \binom{s+t-1}{2}$.

Note that if there is a $K_{s+t-1}$ in $W_G$ formed by heavy edges, then clearly there is an $L_{s,t}$ in $W_G$ formed by heavy edges, therefore there is an $L'_{s,t}$ in $G$.

The next lemma is one of our key observations.

**Lemma 10.** *Let $G$ be an $L'_{s,t}$-free bipartite graph with bipartition $A \cup B$, $|B| = n$ and suppose that $W(A) \geqslant 8(s+t)^2 n$. Then the number of light edges in $W_G$ is at least $\frac{W(A)}{4(s+t)^3}$.*

*Proof.* Let $B = \{b_1, \ldots, b_n\}$. Let $k_i = |N_G(b_i)|$ and suppose that $k_i \geqslant 2(s+t-2)$ for some $i$. As $G$ is $L'_{s,t}$-free, there is no $K_{s+t-1}$ in $W[N_G(b_i)]$ formed by heavy edges. Thus, by Turán's theorem, the number of light edges in $N_G(b_i)$ is at least

$$(s+t-2)\binom{\frac{k_i}{s+t-2}}{2} = \frac{1}{2}k_i\left(\frac{k_i}{s+t-2} - 1\right) \geqslant \frac{k_i^2}{4(s+t-2)}.$$

But

$$\sum_{i: k_i < 2(s+t-2)} \binom{k_i}{2} < 4(s+t)^2 n \leqslant \frac{W(A)}{2},$$

so

$$\sum_{i: k_i \geqslant 2(s+t-2)} \binom{k_i}{2} \geqslant \frac{W(A)}{2}.$$

Since every light edge is present in at most $\binom{s+t-1}{2}$ of the sets $N_G(b_i)$, it follows that the total number of light edges is at least

$$\frac{1}{\binom{s+t-1}{2}} \sum_{i: k_i \geqslant 2(s+t-2)} \frac{k_i^2}{4(s+t-2)} \geqslant \frac{W(A)}{4(s+t)^3}.$$ □

**Corollary 11.** *Let $G$ be an $L'_{s,t}$-free bipartite graph with bipartition $A \cup B$, $|B| = n$, and minimum degree at least $\delta$ on the vertices in $A$. Then for any subset $U \subset A$ with $|U| \geqslant \frac{8(s+t)n}{\delta}$ and $|U| \geqslant 2$, the number of light edges in $W_G[U]$ is at least $\frac{\delta^2}{8(s+t)^3 n}\binom{|U|}{2}$.*

*Proof.* By Lemma 8, we have $W(U) \geqslant \frac{\delta^2}{2n}\binom{|U|}{2} \geqslant \frac{\delta^2}{8n}|U|^2 \geqslant 8(s+t)^2 n$. Now the result follows by applying Lemma 10 to the graph $G[U \cup B]$. □

We are now in a position to complete the proof of Theorem 7.

*Proof of Theorem 7.* Let $c$ be specified later and suppose that $n$ is sufficiently large. Assume, for contradiction, that $G$ is $L'_{s,t}$-free. We shall find distinct $u_1, \ldots, u_{t-1} \in A$ with the following properties.

(i) Each $u_i u_j$ is a light edge in $W_G$.

(ii) If $i, j, k$ are distinct, then $N_G(u_i) \cap N_G(u_j) \cap N_G(u_k) = \emptyset$.

(iii) For each $1 \leqslant i \leqslant t-1$, the number of $v \in A$ with the property that for every $j \leqslant i$, $u_j v$ is a light edge is at least $(\frac{\delta^2}{32(s+t)^3 n})^i \cdot |A|$.

As $n$ is sufficiently large, we have $|A| \geqslant n/2 \geqslant \frac{8(s+t)n}{\delta}$, therefore by Corollary 11 there are at least $\frac{\delta^2}{8(s+t)^3 n}\binom{|A|}{2}$ light edges in $A$, so we may choose $u_1 \in A$ such that the number of light edges $u_1 v$ is at least $\frac{\delta^2}{8(s+t)^3 n}(|A|-1) \geqslant \frac{\delta^2}{32(s+t)^3 n}|A|$.

Now suppose that $2 \leqslant i \leqslant t-1$, and that $u_1, \ldots, u_{i-1}$ have been constructed satisfying (i),(ii) and (iii). Let $U_0$ be the set of vertices $v \in A$ with the property that $u_j v$ is a light edge for every $j \leqslant i-1$. By (iii), we have $|U_0| \geqslant (\frac{\delta^2}{32(s+t)^3 n})^{i-1}|A|$. Now let $U$ consist of those $v \in U_0$ for which $N_G(u_j) \cap N_G(u_k) \cap N_G(v) = \emptyset$ holds for all $1 \leqslant j < k \leqslant i-1$. Since $u_j u_k$ is a light edge for any $1 \leqslant j < k \leqslant i-1$, we have that $d_G(u_j, u_k) < \binom{s+t-1}{2}$. But the degree of every $b \in B$ is at most $K\delta$, therefore the number of $v \in A$ for which $N_G(u_j) \cap N_G(u_k) \cap N_G(v) \neq \emptyset$ is at most $\binom{s+t-1}{2}K\delta$, so $|U_0 \setminus U| \leqslant \binom{i-1}{2}\binom{s+t-1}{2}K\delta$. But note that for sufficiently large $n$, we have $(\frac{\delta^2}{32(s+t)^3 n})^{i-1}|A| \geqslant 2\binom{i-1}{2}\binom{s+t-1}{2}K\delta$ because $\delta = o((\delta^2/n)^{t-2}n)$ and $\delta = o((\delta^2/n)n)$. Thus,

$$|U| \geqslant \frac{1}{2}|U_0| \geqslant \frac{1}{2}\Big(\frac{\delta^2}{32(s+t)^3 n}\Big)^{i-1}|A|.$$

But for sufficiently large $c = c(s,t,K)$, we have $\frac{1}{2}(\frac{\delta^2}{32(s+t)^3 n})^{i-1}|A| \geqslant \frac{8(s+t)n}{\delta}$. Indeed, this is obvious when $\delta^2 \geqslant 32(s+t)^3 n$, and otherwise, using $\delta \geqslant cn^{\frac{t-2}{2t-3}}$, we have

$$\frac{1}{2}\Big(\frac{\delta^2}{32(s+t)^3 n}\Big)^{i-1}|A| \geqslant \frac{1}{2}\Big(\frac{\delta^2}{32(s+t)^3 n}\Big)^{t-2}|A| \geqslant \frac{1}{4(32(s+t)^3)^{t-2}} \cdot \frac{\delta^{2t-4}}{n^{t-3}} \geqslant \frac{8(s+t)n}{\delta}$$

Thus, by Corollary 11, there exists some $u_i \in U$ with at least $\frac{\delta^2}{8(s+t)^3 n}(|U|-1) \geqslant (\frac{\delta^2}{32(s+t)^3 n})^i|A|$ light edges adjacent to it in $U$. This completes the recursive construction of the vertices $\{u_j\}_{1 \leqslant j \leqslant t-1}$.

By (iii) for $i = t-1$, there is a set $V \subset A$ consisting of at least $(\frac{\delta^2}{32(s+t)^3 n})^{t-1}|A|$ vertices $v$ such that for every $j \leqslant t-1$, $u_j v$ is a light edge. We shall now prove that there exist distinct $v_1, \ldots, v_s \in V$ such that $N_G(u_i) \cap N_G(u_j) \cap N_G(v_k) = \emptyset$ for all $i \neq j$, and $N_G(u_i) \cap N_G(v_j) \cap N_G(v_k) = \emptyset$ for all $j \neq k$. It is easy to see that this suffices since then there is a copy of $L'_{s,t}$ in $G$, which is a subdivision of the copy of $L_{s,t}$ in $W_G$ whose vertices are $v_1, \ldots, v_s, u_1, \ldots, u_{t-1}$.

We shall now choose $v_1, \ldots, v_s$ one by one. Since every $u_i u_j$ is a light edge, the number of those $v \in A$ with $N_G(u_i) \cap N_G(u_j) \cap N_G(v) \neq \emptyset$ for some $i \neq j$ is at most $\binom{t-1}{2}\binom{s+t-1}{2}K\delta$. Moreover, given any choices for $v_1, \ldots, v_{k-1} \in V$, as each $u_i v_j$ is a light edge, the number of those $v \in A$ with $N_G(u_i) \cap N_G(v_j) \cap N_G(v) \neq \emptyset$ for some $i, j$ is at most $(t-1)(k-1)\binom{s+t-1}{2}K\delta$. Therefore as long as $|V| > \binom{t-1}{2}\binom{s+t-1}{2}K\delta + (t-1)(s-1)\binom{s+t-1}{2}K\delta$, suitable choices for $v_1, \ldots, v_s$ can be made. Since $|V| \geqslant (\frac{\delta^2}{32(s+t)^3 n})^{t-1}|A|$, this last inequality holds for large enough $c = c(s,t,K)$. $\square$

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