

2-Colorability of r -Uniform Hypergraphs

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Abstract

A hypergraph is properly 2-colorable if each vertex can be colored by one of two colors and no edge is completely colored by a single color. We present a complete algebraic characterization of the 2-colorability of r -uniform hypergraphs. This generalizes a well known algebraic characterization of k -colorability of graphs due to Alon, Tarsi, Lovász, de Loera, and Hillar. We also introduce a method for distinguishing proper 2-colorings called coloring schemes, and provide a decomposition of all proper 2-colorings into these schemes.

As an application, we present a new example of a 4-uniform non-2-colorable hypergraph on 11 vertices and 24 edges which is not isomorphic to a well-known construction by Seymour (1974) of a minimal non-2-colorable 4-uniform hypergraph. Additionally, we provide a heuristically constructed hypergraph which admits only specific coloring schemes. Further, we give an algebraic characterization of the coloring scheme known as a conflict-free coloring.

Mathematics Subject Classifications: 05C15

1 Introduction

Colorability of graphs has a rich and extensive history and includes many different techniques. Studying colorability through algebraic methods has been addressed by many, including Bayer, Alon, Tarsi, Lovász, de Loera, and Hillar (cf. [3] [1] [2] [14] [7] [8] [11] [12]).

In 1982 Bayer introduced a method of determining the 3-colorability of a graph by examining systems of polynomial equations using the division algorithm [3]. Alon and Tarsi used polynomials to prove several conjectures about the chromatic number of a graph [2]. In addition, they provided equivalent conditions for a graph to be not k -colorable using polynomial ideals.

Later, de Loera et al. and Hillar proved results concerning the algebraic characterization of a graph colorability [7], [12]. The main tools de Loera et al. use in their algebraic characterizations for the colorability of a graph are polynomial ideals and Gröbner bases. Gröbner bases were introduced by Buchberger in 1965 and have since become widely used in the study of polynomial ideals [4].

There are two goals for this paper: first, to extend the results mentioned above for graphs to the 2-colorability of uniform hypergraphs by generalizing the results of Hillar [12] to uniform hypergraphs; second, to introduce *coloring schemes*: a new classification of proper hypergraph colorings distinguishing different proper vertex colorings.

To illustrate we present two heuristically constructed 4-uniform hypergraphs and analyze their 2-colorability. The first is 2-colorable, however this hypergraph will only permit certain coloring schemes to be properly 2-colored. The second example is non-2-colorable having 24 edges and non-isomorphic to a well-known construction of Seymour [15].

The paper is structured in the following way. The rest of the introduction contains a description of results in the graph case and an informal description of coloring schemes which are properly defined later in Section 2. Sections 2 and 3 contain all relevant definitions and the statements of our main results. Section 5 is a brief overview of the algebraic tools required for the proofs of the main results. In Section 6 we provide proofs for the theorems stated in Sections 2 and 3, along with additional definitions, theorems, and technical lemmas required for the proofs. As an application of our results, Section 4 contains a new example of a minimal non-2-colorable 4-uniform hypergraph and Section 7 presents results on algebraic characterization of conflict-free colorings. Lastly we attach computational examples illustrating our theorems in the appendices.

1.1 2-Colorability

We generalize some of the results mentioned above to the 2-colorability of uniform hypergraphs and translate the notion of hypergraph vertex colorings to polynomial ideals. Let us introduce the known results for colorability of graphs which we wish to generalize. Let $G = (V, E)$ be a graph. We define its graph polynomial P_G in variables x_1, \dots, x_n , $n = |V(G)|$, as

$$P_G = P_G(x_1, \dots, x_n) = \prod_{\{i,j\} \in E(G)} (x_i - x_j).$$

To give an algebraic formulation of colorability of a graph on n vertices we will work in the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$. Let \mathcal{H} denote the set of graphs with vertices $\{1, \dots, n\}$ consisting of a clique of size $k + 1$ and isolated other vertices. Let the ideals $T_{n,k}$, $R_{n,k}$, and $R_{G,k}$ of the ring R be defined as in [12]:

$$\begin{aligned} T_{n,k} &= \langle P_G : G \in \mathcal{H} \rangle, \\ R_{n,k} &= \langle x_i^k - 1 : i \in [n] \rangle, \\ R_{G,k} &= R_{n,k} + \langle x_i^{k-1} + x_i^{k-2}x_j + \dots + x_i x_j^{k-2} + x_j^{k-1} : \{i, j\} \in E(G) \rangle. \end{aligned}$$

The main result of this paper, Theorem 2, extends the following result by Hillar and Windfeldt [12] to uniform hypergraphs.

Theorem 1 (Theorem 1.1, [12]). *Let $k < n$ be positive integers and G be a graph on n vertices. The following statements are equivalent:*

- (1) *The graph G is not k -colorable.*
- (2) *$\dim_{\mathbb{C}} R/R_{G,k} = 0$ as a vector space.*
- (3) *The constant polynomial 1 belongs to the ideal $R_{G,k}$.*
- (4) *The graph polynomial P_G belongs to the ideal $R_{n,k}$.*
- (5) *The graph polynomial P_G belongs to the ideal $T_{n,k}$.*

The algebraic techniques developed by de Loera and Hillar extended here give not only theoretical results, but also have algorithmic implications. Our proofs can be extended to determine the existence of 2-colorings of hypergraphs satisfying specified color patterns introduced in the next subsection.

1.2 Coloring Schemes

A vertex coloring of a hypergraph is exactly the same as that of a graph; vertices are assigned or labeled with a color. Such a coloring is considered *proper* if no edge in the graph is comprised completely of monochromatically colored vertices. For a graph, proper colorings may be more conveniently described by stating that adjacent vertices must be assigned different colors. However, in a hypergraph, (possibly) more than two vertices determine an edge, therefore a proper coloring may assign the same color to adjacent vertices. The generalization of a proper vertex coloring from graphs to hypergraphs leads to some interesting consequences concerning the *types* of proper coloring. We informally describe the idea of coloring schemes in this subsection with a formal definition in Section 3.

In a graph, adjacent vertices x and y may only be colored in one way: $c(x) \neq c(y)$, where $c(i)$ denotes the color assigned to vertex i ; thus, all vertices in the edge must be assigned *distinct* colors. When we consider a properly vertex colored hypergraph however, vertices in the same edge need not be assigned distinct colors. For example, suppose $e = (x, y, u, v)$ is an edge in a 4-uniform hypergraph, then each pair of vertices from the set $\{x, y, u, v\}$ are considered adjacent. Note that:

$$c(x) = c(y) = c(u) \neq c(v), \quad \text{and} \quad c(x) = c(y) \neq c(u) = c(v)$$

are two proper colorings of the vertices in the edge e . We refer to these different proper colorings as *color patterns*. Note, isomorphic 2-colorings do not effect the color patterns applied to an edge. For example, using colors A and B , the assignments:

$$A = c(x) = c(y) = c(u) \neq c(v) = B \quad \text{and} \quad B = c(x) = c(y) = c(u) \neq c(v) = A,$$

constitute two different isomorphic colorings of the vertices in the edge with the same color pattern. We will consider these two colorings contributions to the same color pattern.

Note further that in a proper vertex coloring of a hypergraph, the vertices of each edge may (or may not) be colored with a different color pattern, which gives rise to the idea of a *coloring scheme*. A coloring scheme may consist of a single color pattern applied to each edge, in which case we consider the coloring scheme to be *homogeneous*, or, several color patterns may be used to create a *heterogeneous* coloring scheme. Each individual proper coloring of a hypergraph corresponds to a single coloring scheme, however, that coloring is not unique to its coloring scheme since isomorphic colorings will not alter the color patterns nor the coloring scheme as stated above. A properly vertex colorable hypergraph may also admit *multiple* coloring schemes, and each scheme may be satisfied by several distinct (non-isomorphic OR isomorphic) colorings.

A properly vertex colored graph admits only one color pattern for the vertices of each edge, and thus has only one coloring scheme: the coloring scheme with that single color pattern. Likewise, a proper vertex 2-coloring of a 3-uniform hypergraph also has only one color pattern (two vertices of color A, and one of color B) and thus one coloring scheme. However, once the uniformity of the hypergraph grows larger than 3 (or more than 2 colors are used), the coloring schemes are no longer unique.

Consider a 4-uniform hypergraph and the two coloring patterns above: pattern 1 colors three vertices with color A, and one with color B; pattern 2 colors two vertices with each color. If the hypergraph may be properly colored where some edges are assigned pattern 1, and some are assigned pattern 2, then we say the hypergraph admits the (heterogeneous) coloring scheme consisting of pattern 1 and pattern 2. If a coloring properly colors the hypergraph and *all* edges are assigned a single pattern, then the hypergraph admits the (homogeneous) coloring scheme consisting of the single pattern.

One particular coloring scheme known as a conflict-free coloring has been studied recently. Conflict-free colorings of uniform hypergraphs were introduced in connection with applications to cellular networks [10] and later studied by Kostochka et al. [13]. We will examine conflict-free colorings in Section 7.

2 2-Colorability of Uniform Hypergraphs

Let $r \geq 2$ be a positive integer. An r -uniform hypergraph, $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$, consists of a set of vertices, $V(\mathcal{H})$, together with a collection of subsets of vertices each of cardinality r , called edges, $E(\mathcal{H})$.

Let \mathcal{H} be an r -uniform hypergraph on $n = |V(\mathcal{H})|$ vertices with $m = |E(\mathcal{H})|$ edges. For a positive integer k , a k -coloring of \mathcal{H} is defined to be a map $c: V(\mathcal{H}) \rightarrow \mathcal{C}$, $|\mathcal{C}| = k$. A *proper k -coloring* is a k -coloring of \mathcal{H} where no edge $e \in E(\mathcal{H})$ is monochromatic, i.e. no edge $e \in E(\mathcal{H})$ is contained in $c^{-1}(\iota)$ for some $\iota \in \mathcal{C}$. The smallest number of colors required in a proper coloring is called the chromatic number of \mathcal{H} , $\chi(\mathcal{H})$.

It is convenient for our purposes to restrict the colors to specific values and label vertices of a hypergraph by the first n positive integers, that is $V(\mathcal{H}) = [n]$. We define a 2-coloring as a map

$$c: V(\mathcal{H}) \rightarrow \{-1, 1\} .$$

Note that this formulation is equivalent to the definition above. Further, we associate the vertices of hypergraphs with variables in a polynomial ring. For each vertex $i \in V(\mathcal{H}) = [n]$ we assign a variable in the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ via the map

$$i \rightarrow x_i .$$

If c is a 2-coloring of \mathcal{H} , we naturally extend c to assign the values 1 and -1 to the variables x_i , and write $c(x_i) = c(i)$, $i \in [n]$, for the values assigned to the variables. With this convention, a 2-coloring of \mathcal{H} corresponds to an assignment of ± 1 to the variables x_1, \dots, x_n .

Though we are concerned with vertex colorings, we utilize edges to distinguish between proper and improper vertex colorings. We will use the following notation to keep track of vertices in a particular edge. For an edge $e \in E(\mathcal{H})$ let

$$e = \{e_1, e_2, \dots, e_r\},$$

where e_j represents a vertex in $V(\mathcal{H}) = [n]$.

Now let us introduce all necessary definitions to formulate our results. We call a partition of r a *proper 2-integer partition* if it consists of exactly two parts, $\{r_1, r_2\}$:

$$0 < r_2 \leq r_1 < r, \quad r_1 + r_2 = r.$$

Let $par(r, 2)$ be the set of all proper 2-integer partitions of r and note $|par(r, 2)| = \lfloor \frac{r}{2} \rfloor$.

Let $d(r, 2)$ be the set of all differences of proper 2-integer partitions of r (which by design will be non-negative), that is

$$\text{if } \{r_1, r_2\} \in par(r, 2), \quad r_1 \geq r_2, \quad \text{then, } r_1 - r_2 \in d(r, 2).$$

For an edge $e = \{e_1, \dots, e_r\} \in E(\mathcal{H})$ we define the *coloring polynomial* for edge e , f_e , by:

$$\begin{aligned} f_e = & (\omega_1 x_{e_1} + x_{e_2} + \dots + x_{e_r})(x_{e_1} + \omega_1 x_{e_2} + \dots + x_{e_r}) \cdots (x_{e_1} + x_{e_2} + \dots + \omega_1 x_{e_r}) \cdot \\ & (\omega_2 x_{e_1} + x_{e_2} + \dots + x_{e_r})(x_{e_1} + \omega_2 x_{e_2} + \dots + x_{e_r}) \cdots (x_{e_1} + x_{e_2} + \dots + \omega_2 x_{e_r}) \cdot \\ & \vdots \\ & (\omega_{p(r,2)} x_{e_1} + x_{e_2} + \dots + x_{e_r})(x_{e_1} + \omega_{p(r,2)} x_{e_2} + \dots + x_{e_r}) \cdot \\ & \qquad \qquad \qquad \cdots (x_{e_1} + x_{e_2} + \dots + \omega_{p(r,2)} x_{e_r}) \end{aligned}$$

where $\omega_j - 1 \in d(r, 2)$, $1 \leq j \leq \lfloor \frac{r}{2} \rfloor$, i.e. $d(r, 2) = \{\omega_1 - 1, \dots, \omega_{\lfloor \frac{r}{2} \rfloor} - 1\}$. The coloring polynomials are crucial in the definition of the *2-colorability ideal* $I_2(\mathcal{H})$ of \mathcal{H}

$$I_2(\mathcal{H}) = \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle f_e : e \in E(\mathcal{H}) \rangle.$$

These definitions are illustrated for the Fano plane in Example 1, Appendix A.

As an analogue to the commonly used graph polynomial, we define the *hypergraph polynomial for 2-colorability*, $P_2(\mathcal{H})$, by

$$P_2(\mathcal{H}) = \prod_{e \in E(\mathcal{H})} \left[\left(\sum_{e_j \in e} x_{e_j} - r \right) \cdot \left(\sum_{e_j \in e} x_{e_j} + r \right) \right] .$$

Here, \mathcal{H} is again an r -uniform hypergraph. A similar generalization of the graph polynomial was introduced by Vishwanathan [16].

We are now ready to state our first main result. The following theorem characterizes the 2-colorability of uniform hypergraphs and provides a generalization of Theorem 1.

Theorem 2. *Let \mathcal{H} be a uniform hypergraph on n vertices. Let $R = \mathbb{C}[x_1, \dots, x_n]$. Let $I_2(\mathcal{H})$ be the 2-colorability ideal of \mathcal{H} and let $P_2(\mathcal{H})$ be the hypergraph polynomial for \mathcal{H} . Then the following are equivalent:*

- (1) *The hypergraph \mathcal{H} is not 2-colorable.*
- (2) *The constant 1 is an element of the ideal $I_2(\mathcal{H})$.*
- (3) *$\dim_{\mathbb{C}} R/I_2(\mathcal{H}) = 0$ as a vector space.*
- (4) *The hypergraph polynomial $P_2(\mathcal{H})$ belongs to the ideal*

$$\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle .$$

3 2-Coloring Schemes

The algebraic encoding of vertex colorings into polynomial ideals allows us to decompose proper 2-colorings of \mathcal{H} into one or more coloring schemes. Furthermore, we are able to characterize whether a hypergraph admits a proper vertex coloring restricted to *specific* coloring schemes.

First we define a useful tool for distinguishing proper 2-colorings of a hypergraph. Let $e = \{e_1, \dots, e_r\} \in E(\mathcal{H})$ be an edge. We set

$$\alpha(e) = \sum_{j=1}^r x_{e_j}$$

to be the sum of the variables assigned to the vertices in e . The *edge sum* of an edge e in a hypergraph \mathcal{H} colored by a 2-coloring c , $\alpha_c(e)$, is defined to be the sum of the values c assigns to the vertices in e

$$\alpha_c(e) = \sum_{j=1}^r c(x_{e_j}) = \sum_{j=1}^r c(e_j) .$$

A *proper edge sum* is an edge sum of a properly 2-colored edge.

Since we are currently concerned with 2-colorings only, and thus restrict our colors to ± 1 , the set of all proper edge sums is determined completely by the color patterns associated with the uniformity of the hypergraph. Note, that there are $2 \cdot \lfloor \frac{r}{2} \rfloor$ proper edge sums that can appear in proper 2-colorings of r -uniform hypergraphs. These correspond to the $\lfloor \frac{r}{2} \rfloor$ proper 2-integer partitions of r , taken twice for whether the larger part is

assigned 1, or -1 in a coloring. In the case that r is even, the proper edge sum of 0 is counted twice.

Proper edge sums are the primary component in our second algebraic characterization of 2-colorings of \mathcal{H} . While similar in utility to the f_e polynomials used in the 2-colorability ideal, $I_2(\mathcal{H})$, proper edge sums allow us to distinguish between different coloring schemes.

For an r -uniform hypergraph, \mathcal{H} , let $A = \{a_1, a_2, \dots, a_{2\lfloor \frac{r}{2} \rfloor}\}$ be the possible proper edge sums for \mathcal{H} . As stated previously, A is generated by the possible color patterns for \mathcal{H} , with each color pattern contributing two proper edge sums (potentially repeating the sum 0 twice). Let $U \subseteq A$ be non-empty; we will refer to U as a *2-coloring scheme*. Note that we require the following condition on U : if $a \in U$, $a \neq 0$, then $-a \in U$ as well. This condition induces a correspondence between the pair $\{a, -a\}$ and the proper 2-integer partition of r where they originated.

We define the *2-coloring scheme ideal* of \mathcal{H} , $S_{2,U}(\mathcal{H})$, as

$$S_{2,U}(\mathcal{H}) = \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \left\langle \prod_{a \in U} \left(\sum_{j=1}^r x_{e_j} - a \right) : e \in E(\mathcal{H}) \right\rangle .$$

The 2-coloring scheme ideals distinguish different proper 2-colorings of \mathcal{H} and allow the 2-colorability ideal $I_2(\mathcal{H})$ to be decomposed algebraically.

Theorem 3. *Let \mathcal{H} be an r -uniform hypergraph. Let $A = \{a_1, a_2, \dots, a_{2\lfloor \frac{r}{2} \rfloor}\}$ consist of all possible proper edge sums in r -uniform hypergraphs as above. Then,*

$$I_2(\mathcal{H}) = \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} S_{2,U}(\mathcal{H}) .$$

In addition to the decomposition of $I_2(\mathcal{H})$, the 2-coloring scheme ideals allow us to determine when \mathcal{H} admits the coloring scheme U .

Corollary 4. *\mathcal{H} can be properly 2-colored by the coloring scheme U if and only if the ideal $S_{2,U}(\mathcal{H})$ does not contain the constant 1.*

We prove Theorems 2 and 3, and Corollary 4 in Section 6 along with all necessary supporting results. We illustrate our results on the following example of a 4-uniform hypergraph with 10 edges on 7 vertices in Appendix A:

$$\begin{aligned} & \{\{1 \ 2 \ 3 \ 4\}, \{1 \ 5 \ 6 \ 7\}, \\ & \{2 \ 5 \ 6 \ 7\}, \{3 \ 5 \ 6 \ 7\}, \\ & \{3 \ 4 \ 5 \ 6\}, \{3 \ 4 \ 6 \ 7\}, \\ & \{3 \ 4 \ 5 \ 7\}, \{1 \ 2 \ 3 \ 5\}, \\ & \{1 \ 2 \ 3 \ 6\}, \{1 \ 2 \ 3 \ 7\} \} \end{aligned}$$

For comparison, we also analyze the well-known Fano Plane and a 2-colorable sub-hypergraph which admits a single coloring scheme only.

4 A new minimum non-2-colorable 4-uniform hypergraph

In 1974, Seymour [15] proved that the minimum number of edges of a non-2-colorable 4-uniform hypergraph is 24. We present a new example of a non-2-colorable 4-uniform hypergraph on 11 vertices, $\{1, \dots, 11\}$, and 24 edges which is not isomorphic to the well-known example constructed by Seymour.

Let $n = 11$ be the number of vertices. Let us start with Seymour's construction. Let \mathcal{S} be a 4-uniform hypergraph on 11 vertices containing the following edges

$$E(\mathcal{S}) = \{ \{1, 2, 9, 10\}, \{1, 2, 10, 11\}, \{1, 2, 9, 11\}, \{3, 4, 9, 10\}, \{3, 4, 10, 11\}, \{3, 4, 9, 11\}, \\ \{5, 6, 9, 10\}, \{5, 6, 10, 11\}, \{5, 6, 9, 11\}, \{7, 8, 9, 10\}, \{7, 8, 10, 11\}, \{7, 8, 9, 11\}, \{2, 3, 5, 7\}, \\ \{1, 4, 5, 7\}, \{1, 3, 6, 7\}, \{2, 3, 6, 7\}, \{1, 4, 6, 7\}, \{2, 4, 6, 7\}, \{2, 4, 6, 8\}, \{1, 3, 6, 8\}, \{2, 4, 5, 8\}, \\ \{1, 4, 5, 8\}, \{2, 3, 5, 8\}, \{1, 3, 5, 8\} \}.$$

Seymour proved that \mathcal{S} is a non-2-colorable hypergraph. Note that \mathcal{S} has 24 edges and its degree sequence is

$$\{9, 9, 9, 9, 9, 9, 9, 9, 8, 8, 8\}.$$

Next, let us introduce our example: a 4-uniform hypergraph \mathcal{H} on 11 vertices with the edge set

$$E(\mathcal{H}) = \{ \{5, 6, 10, 11\}, \{1, 4, 5, 7\}, \{2, 3, 6, 7\}, \{3, 4, 10, 11\}, \{2, 4, 5, 8\}, \{1, 2, 9, 10\}, \\ \{1, 2, 10, 11\}, \{1, 3, 6, 8\}, \{5, 6, 9, 10\}, \{1, 4, 6, 7\}, \{7, 8, 9, 10\}, \{7, 8, 9, 11\}, \{2, 3, 5, 7\}, \\ \{3, 4, 9, 10\}, \{2, 4, 6, 8\}, \{7, 8, 10, 11\}, \{1, 2, 9, 11\}, \{5, 6, 9, 11\}, \{2, 4, 5, 6\}, \{1, 4, 5, 8\}, \\ \{3, 4, 9, 11\}, \{1, 3, 6, 7\}, \{1, 3, 5, 8\}, \{2, 3, 5, 8\} \}.$$

We claim this is a non-2-colorable hypergraph. The proof relies on Theorem 2 and Gröbner bases. We describe the 2-colorability ideal for \mathcal{H} in Appendix B. Note that \mathcal{H} has 24 edges and its degree sequence is

$$\{9, 9, 9, 9, 10, 9, 8, 9, 8, 8, 8\}.$$

Thus, we can conclude that the two hypergraphs \mathcal{S} and \mathcal{H} are non-isomorphic.

5 Algebraic Background

We recall some algebraic tools that will be required for the proofs in Section 6. We use standard algebraic notation, for reference see for example [5], and [9].

We will consider the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$. For an ideal I in R , the *radical* of I , denoted \sqrt{I} is the set

$$\sqrt{I} = \{f \in R : f^m \in I, \text{ for some } m \in \mathbb{Z}^+\}.$$

An ideal I is called *radical* if $I = \sqrt{I}$. The subset of \mathbb{C}^n consisting of the solutions common to all polynomials in I is the *variety* of I , denoted $\mathcal{V}(I)$. Conversely, given a

subset $V \subseteq \mathbb{C}^n$, the *vanishing ideal* is the set of all polynomials in $R = \mathbb{C}[x_1, \dots, x_n]$ that vanish at every point in V , and is denoted by $\mathcal{I}(V)$. The symbols \mathcal{V} and \mathcal{I} represent set maps which allow transition between ideals of the ring R and subsets of \mathbb{C}^n , and are related in the following way:

$$\mathcal{V}(\mathcal{I}(V)) = V, \quad \text{and} \quad \mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

Moreover, the maps are inclusion-reversing, a fact which we will utilize in the following way: if $J \subseteq I$ are ideals, then $\mathcal{V}(J) \supseteq \mathcal{V}(I)$ (see Chapter 4, Section 2, Theorem 7 of [5]). Finally, we say that an ideal I in the ring R is *zero-dimensional* (as an ideal), if its corresponding variety, $\mathcal{V}(I)$, contains only a finite number of points.

A *monomial ordering* on the monomials of R is a multiplicative well-ordering, $<$, on the set of monomials with the constant 1 being the least element. With a monomial ordering established we can define the *leading term* of any polynomial $f \in R$; the leading term of f , $LT(f)$, is the monomial in f that is largest with respect to $<$. Any monomial which is not a leading term of a polynomial in an ideal I is called a *standard monomial* and the set of all such monomials is denoted $\mathcal{B}_<(I)$.

In [12], Hillar and Windfeldt collect several results from commutative algebra and algebraic geometry that are useful in determining when an ideal is zero-dimensional and radical. We reproduce the lemma here for completeness.

Lemma 5 (Lemma 2.1, [12]). *Let I be a zero-dimensional ideal in the polynomial ring R and let $<$ be a fixed monomial ordering. Then,*

$$\dim_{\mathbb{C}} R/I = |\mathcal{B}_<(I)| \geq |\mathcal{V}(I)| \quad (\text{as a vector space}).$$

Moreover, the following are equivalent:

- (1) I is a radical ideal.
- (2) I contains a univariate square-free polynomial in each indeterminate.
- (3) $|\mathcal{B}_<(I)| = |\mathcal{V}(I)|$.

Proof. See [12], Lemma 2.1; [5] p. 229, Proposition 4; and [6] pp. 39-41, Proposition 2.7 and Theorem 2.10. \square

Given an ideal I in R , the ideal of leading terms is defined as $LT(I) = \langle LT(f) : f \in I \rangle$. A *Gröbner basis* for an ideal I is a finite set of generators $\{g_1, \dots, g_m\}$ for I whose leading terms generate the ideal of all leading terms in I , i.e.

$$I = \langle g_1, \dots, g_m \rangle \quad \text{and} \quad LT(I) = \langle LT(g_1), \dots, LT(g_m) \rangle.$$

A *reduced* Gröbner basis \mathcal{G} , is a Gröbner basis whose elements are all monic, and are such that no leading term in \mathcal{G} divides any other term in any polynomial in \mathcal{G} . Gröbner bases are a very useful tool for determining properties of polynomial ideals, including ideal membership, intersections, and equality between ideals.

A final tool that we will utilize in a similar fashion as in [12] is the *ideal quotient* (or *colon ideal*). Given ideals I and J of R , the ideal quotient is the ideal

$$I : J = \{f \in R : fg \in I, \forall g \in J\}.$$

Moreover, given two varieties V and W ,

$$\mathcal{I}(V) : \mathcal{I}(W) = \mathcal{I}(V \setminus W).$$

See [5], pp. 191-193.

In addition we recall two results from polynomial algebra which allow us to apply our results to some basic examples. Both results can be found in any modern abstract algebra text, for example [9].

Theorem 6 (Weak Nullstellensatz). *The polynomials contained in the ideal I , specifically the generators of I , have a common solution, i.e. $\mathcal{V}(I) \neq \emptyset$, if and only if $1 \notin I$.*

A well known statement from the theory of Gröbner bases is the main tool in an algorithm for determining if a given hypergraph admits a 2-coloring with proper edge sums restricted to a given set.

Theorem 7. *The polynomials in an ideal $I = \langle f_1, \dots, f_m \rangle$ have a common solution if and only if any Gröbner basis for I is non-trivial.*

Using Theorem 7 in conjunction with Theorems 9 or 14 (Section 6), we can determine if a given hypergraph is 2-colorable or 2-colorable with the corresponding proper edge sums restricted to a certain set. Each of these tasks is completed by first determining the appropriate collection of proper edge sums, generating their associated 2-coloring scheme ideal, and finally computing a Gröbner basis for the ideal. If the associated ideal has a non-trivial Gröbner basis, then the hypergraph admits the coloring pattern in question. We give an example of this technique in Example 2 in Appendix A.

6 Colorability Proofs

Let $r \geq 2$ and n be positive integers. In this section, we let \mathcal{H} to be an r -uniform hypergraph on n vertices and $R = \mathbb{C}[x_1, \dots, x_n]$. Also let A be the set of all possible proper edge sums in an r -uniform hypergraph.

6.1 2-Colorability Proofs

We first establish the ideal characterization of a proper 2-coloring of \mathcal{H} by proving that the f_e polynomials encode proper vertex colorings.

Lemma 8. *Let e be an edge of a 2-colored r -uniform hypergraph. The polynomial f_e vanishes if and only if the edge e is properly 2-colored.*

Proof. Let $e \in E(\mathcal{H})$. Let c be a 2-coloring of \mathcal{H} .

(\Leftarrow) Assume c is a proper 2-coloring of \mathcal{H} . Then e has a proper edge sum equal to a where

$$a = \alpha_c(e) = \sum_{j=1}^r c(x_{e_j}) \quad \text{and} \quad -(r-2) \leq a \leq r-2.$$

Hence, $|a| \in d(r, 2)$ and we have that for some j , $1 \leq j \leq p(r, 2)$,

$$\omega_j - 1 = |a|.$$

Thus at least one of the factors

$$(\omega_j x_{e_1} + x_{e_2} + \cdots + x_{e_r})(x_{e_1} + \omega_j x_{e_2} + \cdots + x_{e_r}) \cdots (x_{e_1} + x_{e_2} + \cdots + \omega_j x_{e_r})$$

of f_e will be zero when evaluated at $(c(x_1), \dots, c(x_n))$.

Hence, f_e vanishes at $(c(x_1), \dots, c(x_n))$, i.e. $f_e(c(x_1), \dots, c(x_n)) = 0$.

(\Rightarrow) Assume $f_e(c) = 0$. For a contradiction, assume e is not properly colored by c . So all the vertices of e are assigned color 1 or all the vertices of e are assigned color -1 . Then each factor in

$$(\omega_j x_{e_1} + x_{e_2} + \cdots + x_{e_r})(x_{e_1} + \omega_j x_{e_2} + \cdots + x_{e_r}) \cdots (x_{e_1} + x_{e_2} + \cdots + \omega_j x_{e_r})$$

when evaluated at $(1, \dots, 1)$ or $(-1, \dots, -1)$ takes value either

$$\omega_j + r - 1 \quad \text{or} \quad -(\omega_j + r - 1).$$

Since $\omega_j - 1 \neq -r$ for all $\omega_j - 1 \in d(r, 2)$, no factor in f_e can be zero, thus $f_e(1, \dots, 1) \neq 0$ as well as $f_e(-1, \dots, -1) \neq 0$, a contradiction. So e is properly 2-colored by c . \square

With this lemma we can characterize the 2-colorability of \mathcal{H} using the ideal $I_2(\mathcal{H})$.

Theorem 9. *Let \mathcal{H} be an r -uniform hypergraph. The polynomials in the ideal $I_2(\mathcal{H})$ have a common solution if and only if \mathcal{H} is properly 2-colorable.*

Proof. (\Rightarrow) Let $c \in V(I_2(\mathcal{H}))$, $c = (c_1, \dots, c_n)$. Clearly the first n generators in $I_2(\mathcal{H})$ imply that $c_i = \pm 1$ for all $i \in [n]$. Hence, c is a 2-coloring of \mathcal{H} . Also, since $f_e(c) = 0$ for every $e \in E(\mathcal{H})$, every edges is properly colored by Lemma 8. Thus c is a proper 2-coloring of \mathcal{H} .

(\Leftarrow) Let $c = (c_1, \dots, c_n)$ be a proper 2-coloring of \mathcal{H} . Then each of the first n generators of $I_2(\mathcal{H})$, i.e. $x_i^2 - 1$, vanish at c . Further, by Lemma 8, each f_e , $e \in \mathcal{H}$, vanishes at c . Hence $c \in V(I_2(\mathcal{H}))$. \square

Remark 10. A note regarding Theorem 9 and the definition of the 2-colorability ideal $I_2(\mathcal{H})$ in the case r is even. Note that $0 \in d(r, 2)$ in this case. Let j be such that $\omega_j = r_1 - r_2 + 1 = 1$. We can modify the definition of the f_e polynomials in this case such that we need not cycle this particular ω_j coefficient through all r corresponding factors of f_e . Thus, in the case of r even, f_e can contain one factor of the form $(x_{e_1} + x_{e_2} + \cdots + x_{e_r})$ and the remaining $(r-1)$ factors where $\omega_j = 1$ can be omitted. This does not change the variety $\mathcal{V}(I_2(\mathcal{H}))$, however it does simplify some computations within $I_2(\mathcal{H})$.

The next theorem gives a characterization of 2-colorability using the hypergraph polynomial for 2-colorability $P_2(\mathcal{H})$.

Theorem 11. *Let \mathcal{H} be an r -uniform hypergraph and c be a 2-coloring of \mathcal{H} . Then c is not a proper 2-coloring if and only if $P_2(\mathcal{H})$ vanishes on c .*

Proof. (\Rightarrow) Let c be an improper coloring of \mathcal{H} . Then for some $\tilde{e} \in E(\mathcal{H})$, c colors \tilde{e} either

$$\{1, 1, \dots, 1\} \quad \text{or} \quad \{-1, -1, \dots, -1\}.$$

Thus the edge sum $\alpha_c(\tilde{e})$ equals either r or $-r$. Hence the factor

$$\left(\sum_{e_j \in \tilde{e}} x_{e_j} - r \right) \left(\sum_{e_j \in \tilde{e}} x_{e_j} + r \right),$$

vanishes on c . Thus, $P_2(\mathcal{H})$ vanishes on c . Thus $P_2(\mathcal{H}) = 0$.

(\Leftarrow) Let c be a 2-coloring of \mathcal{H} and assume that $P_2(\mathcal{H})$ vanishes on c . Then for some $e \in E(\mathcal{H})$, the factor

$$\left(\sum_{e_j \in e} x_{e_j} - r \right) \left(\sum_{e_j \in e} x_{e_j} + r \right)$$

must vanish. Thus the edge sum $\alpha_c(e)$ equals either r or $-r$. Since c is a 2-coloring, c colors e by either

$$\{1, 1, \dots, 1\} \quad \text{or} \quad \{-1, -1, \dots, -1\}.$$

Hence c is not a proper coloring. □

The following lemmas and their proofs are analogues of Lemmas 3.1 and 3.4 in [12].

Lemma 12. *For an r -uniform hypergraph \mathcal{H} , the varieties $\mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle)$, $\mathcal{V}(I_2(\mathcal{H}))$, and $\mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle)$ correspond to the sets of all 2-colorings of \mathcal{H} , the proper 2-colorings of \mathcal{H} , and the improper 2-colorings of \mathcal{H} , respectively.*

Proof. Let $n = |V(\mathcal{H})|$. Clearly, the set $\mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle)$ is the set of all possible n -tuples of ± 1 which represent all possible 2-colorings of a hypergraph on n vertices. By the construction of the f_e polynomials and the ideal $I_2(\mathcal{H})$ in Theorem 9, $\mathcal{V}(I_2(\mathcal{H}))$ is the set of all proper 2-colorings of \mathcal{H} .

Similarly, Theorem 11 implies that each solution of the hypergraph polynomial $P_2(\mathcal{H})$ in $\{1, 1\}^n$ corresponds to an improper 2-coloring of \mathcal{H} . Thus $\mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle)$ is the set of all improper 2-colorings of \mathcal{H} . □

Lemma 12 and the fact that $I_2(\mathcal{H})$ is radical imply the following statement concerning the hypergraph chromatic polynomial, $\chi_{\mathcal{H}}(k)$, which is the univariate polynomial that counts the number of proper k -colorings for \mathcal{H} .

Lemma 13.

$$\chi_{\mathcal{H}}(2) = |\mathcal{V}(I_2(\mathcal{H}))| = \dim_{\mathbb{C}} R/I_2(\mathcal{H}),$$

and

$$2^n - \chi_{\mathcal{H}}(2) = \dim_{\mathbb{C}} R / (\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle).$$

Proof. These statements follow from the fact that the ideals $I_2(\mathcal{H})$, and $\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle$ are radical and from Lemma 12. \square

We now give a proof of Theorem 2.

Proof. (Theorem 2)

The equivalence of (1), (2), and (3) follow from Theorem 9 and the Weak Nullstellensatz, Theorem 6.

To finish the proof, we first show that (1) implies (4). Assume \mathcal{H} is not 2-colorable. Then for every coloring of \mathcal{H} ,

$$c \in \mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle),$$

c is not a proper coloring, so by Lemma 12,

$$c \in \mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle).$$

Thus,

$$\mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle) \supseteq \mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle),$$

and so,

$$\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle \subseteq \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle.$$

This implies

$$P_2(\mathcal{H}) \in \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle.$$

Next, we show that (4) implies (1). Assume

$$P_2(\mathcal{H}) \in \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle.$$

Then

$$\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle \subseteq \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle,$$

and

$$\mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle) \supseteq \mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle),$$

which, by Lemma 12, implies that every 2-coloring

$$c \in \mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle),$$

is also such that

$$c \in \mathcal{V}(\langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \langle P_2(\mathcal{H}) \rangle),$$

and hence is not proper. So \mathcal{H} is not 2-colorable. \square

6.2 Coloring Scheme Proofs

Moving now to our second main result, we establish the importance of the 2-coloring scheme ideals $S_{2,U}(\mathcal{H})$.

Theorem 14. *Let A be the set of all possible proper edge sums in r -uniform hypergraphs and U be a nonempty subset of A . Let \mathcal{H} be an r -uniform hypergraph on $n = |V(\mathcal{H})|$ vertices. Then the polynomials in the ideal $S_{2,U}(\mathcal{H})$ have a common solution if and only if the hypergraph \mathcal{H} can be properly 2-colored with the edge sums in U .*

Proof. Let $U \subseteq A$, U non-empty. Consider the ideal

$$S_{2,U}(\mathcal{H}) = \langle x_i^2 - 1 : i \in V(\mathcal{H}) \rangle + \left\langle \prod_{a \in U} \left(\sum_{j=1}^r x_{e_j} - a \right) : e \in E(\mathcal{H}) \right\rangle .$$

From the first set of generators we see that any common solution will be an n -tuple of 1's and -1's. Also, it is clear that for every edge, $e \in E(\mathcal{H})$

$$\prod_{a \in U} \left(\sum_{j=1}^r x_{e_j} - a \right) = 0 ,$$

if and only if one of the factors,

$$\sum_{j=1}^r x_{e_j} = a ,$$

for some $a \in U$. Since each of these factors is the sum of the variables representing the vertices in the edge e , this can happen if and only if the edge is properly 2-colored with the edges sum in U . \square

Theorem 14 also gives us the following corollaries.

Corollary 15. *Let A be the set of all possible proper edge sums in r -uniform hypergraphs and \mathcal{H} be an r -uniform hypergraph. Then*

$$\mathcal{V}(I_2(\mathcal{H})) = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(S_{2,U}(\mathcal{H})) .$$

Proof. This follows from Lemma 12 and Theorem 14. \square

We now prove Corollary 4 which determines if \mathcal{H} admits a coloring scheme U .

Proof. (Corollary 4.) Let $U \subseteq A$, U nonempty. By Theorem 14, \mathcal{H} can be properly 2-colored with the edge sums in U if and only if $\mathcal{V}(S_{2,U}(\mathcal{H})) \neq \emptyset$. Since $S_{2,U}(\mathcal{H}) \subseteq R$, $\mathcal{V}(S_{2,U}(\mathcal{H})) \neq \emptyset$ is equivalent to $S_{2,U}(\mathcal{H}) \neq R$ which is equivalent to $1 \notin S_{2,U}(\mathcal{H})$ by The Weak Nullstellensatz. \square

We can now prove our main decomposition theorem, Theorem 3.

Proof. (Theorem 3.)

Since the ideals $I_2(\mathcal{H})$ and $S_{2,U}(\mathcal{H})$ contain square-free univariate polynomials in each indeterminate, they are radical.

Also, since

$$\mathcal{V}(I_2(\mathcal{H})) = \bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(S_{2,U}(\mathcal{H})) ,$$

we have that

$$\begin{aligned} I_2(\mathcal{H}) &= \mathcal{I}(\mathcal{V}(I_2(\mathcal{H}))) \\ &= \mathcal{I} \left(\bigcup_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{V}(S_{2,U}(\mathcal{H})) \right) \\ &= \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} \mathcal{I}(\mathcal{V}(S_{2,U}(\mathcal{H}))) \\ &= \bigcap_{\substack{U \subseteq A \\ U \neq \emptyset}} S_{2,U}(\mathcal{H}) . \end{aligned} \quad \square$$

We illustrate Theorem 3 and Corollary 4 in Appendix A. Further, we use the scheme ideals and their properties to characterize conflict-free colorings in the next section.

7 Conflict-free coloring

A proper coloring of a uniform hypergraph \mathcal{H} is called a *conflict-free coloring* if each edge $e \in E(\mathcal{H})$ contains a vertex whose color does not get repeated in e . The smallest number of colors required in a conflict-free coloring is called the *conflict-free chromatic number* of \mathcal{H} , $\chi_{\text{CF}}(\mathcal{H})$. The conflict-free chromatic number was introduced by Even et al. in 2003 [10].

Our goal in this section is to show that we can recognize 2-colorable hypergraphs with $\chi_{\text{CF}}(\mathcal{H}) = 2$ or $\chi_{\text{CF}}(\mathcal{H}) \neq 2$. First we note that clearly, $\chi(\mathcal{H}) \leq \chi_{\text{CF}}(\mathcal{H})$, and we establish an equivalent condition for conflict-free colorings.

For an r -uniform 2-colorable hypergraph \mathcal{H} the only edge sums allowed in a conflict-free coloring are

$$a_1 = (r - 2) \text{ and } a_2 = -(r - 2) ,$$

which are given by the colorings of edges

$$\{1, 1, \dots, 1, -1\} \text{ and } \{-1, -1, \dots, -1, 1\} .$$

Thus, if $\chi_{\text{CF}}(\mathcal{H}) = 2$, then $\mathcal{V}(S_{2,\{a_1,a_2\}}(\mathcal{H})) \neq \emptyset$. Conversely, if $\mathcal{V}(S_{2,\{a_1,a_2\}}(\mathcal{H})) \neq \emptyset$, then \mathcal{H} is properly 2-colored with the edge sums a_1 and a_2 , and since this is a conflict-free coloring we have that $\chi_{\text{CF}}(\mathcal{H}) = 2$. Hence,

$$\chi_{\text{CF}}(\mathcal{H}) = 2 \text{ if and only if } \mathcal{V}(S_{2,\{a_1,a_2\}}(\mathcal{H})) \neq \emptyset .$$

Next we express this condition in terms of the ideal $S_{2,\{a_1,a_2\}}(\mathcal{H})$, cf. Corollary 4.

Theorem 16. *Let \mathcal{H} be an r -uniform hypergraph and $a_1 = r - 2$, $a_2 = -(r - 2)$ be the edge sums characterizing conflict-free colorings of \mathcal{H} . Assume that $\chi(\mathcal{H}) = 2$. Then $\chi_{\text{CF}}(\mathcal{H}) = 2$ if and only if $S_{2,\{a_1,a_2\}}(\mathcal{H})$ does not contain the constant 1. Here, $S_{2,\{a_1,a_2\}}(\mathcal{H})$ is the 2-coloring scheme ideal corresponding to the set $\{a_1, a_2\}$.*

Proof. Since a conflict-free coloring is a proper 2-coloring with edge sums in $\{a_1, a_2\}$, the theorem follows directly from Corollary 4. \square

See Example 2 in Appendix A for an illustration of Theorem 16 for a 4-uniform hypergraph on 7 vertices.

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8 Appendix A

In this appendix we present computation illustrating Theorems 2, 3, 14, 16, and Corollary 4 for some small hypergraphs. Example 1 utilizes the Fano Plane and illustrates how edge sums are used to create coloring schemes. Example 2 provides a hypergraph which is 2-colorable only with certain coloring schemes, one homogeneous, one heterogeneous.

Example 1.

The Fano Plane, FP , is a 3-uniform hypergraph on 7 vertices with 7 edges:

$$\{\{1, 2, 5\}, \{1, 3, 7\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{2, 4, 7\}, \{5, 6, 7\}\}.$$

Since $r = 3$, we have $par(r, 2) = \{\{1, 2\}\}$, $p(r, 2) = |par(r, 2)| = 1$, and $d(r, 2) = \{1\}$, the corresponding 2-colorability ideal $I_2(FP)$ is given by

$$\begin{aligned} I_2(FP) = & \langle x_i^2 - 1 : i \in [7] \rangle + \\ & \langle (2x_1 + x_2 + x_5)(x_1 + 2x_2 + x_5)(x_1 + x_2 + 2x_5), \\ & (2x_1 + x_3 + x_7)(x_1 + 2x_3 + x_7)(x_1 + x_3 + 2x_7), \\ & (2x_1 + x_4 + x_6)(x_1 + 2x_4 + x_6)(x_1 + x_4 + 2x_6), \\ & (2x_2 + x_3 + x_6)(x_2 + 2x_3 + x_6)(x_2 + x_3 + 2x_6), \\ & (2x_3 + x_4 + x_5)(x_3 + 2x_4 + x_5)(x_3 + x_4 + 2x_5), \\ & (2x_2 + x_4 + x_7)(x_2 + 2x_4 + x_7)(x_2 + x_4 + 2x_7), \\ & (2x_5 + x_6 + x_7)(x_5 + 2x_6 + x_7)(x_5 + x_6 + 2x_7) \rangle. \end{aligned}$$

Using a Gröbner basis package in a computer algebra system, like Mathematica or Singular, we can show that the ideal $I_2(\mathcal{H})$ has Gröbner basis $\{1\}$ and thus is non-2-colorable.

Let the Modified Fano Plane be the Fano Plane with an edge removed, FP' , with edges:

$$\{\{1, 3, 7\}, \{1, 4, 6\}, \{2, 3, 6\}, \{3, 4, 5\}, \{2, 4, 7\}, \{5, 6, 7\}\}.$$

The corresponding 2-colorability ideal is given by

$$\begin{aligned}
 I_2(FP') = & \langle x_i^2 - 1 : i \in [7] \rangle + \\
 & \langle (2x_1 + x_3 + x_7)(x_1 + 2x_3 + x_7)(x_1 + x_3 + 2x_7), \\
 & (2x_1 + x_4 + x_6)(x_1 + 2x_4 + x_6)(x_1 + x_4 + 2x_6), \\
 & (2x_2 + x_3 + x_6)(x_2 + 2x_3 + x_6)(x_2 + x_3 + 2x_6), \\
 & (2x_3 + x_4 + x_5)(x_3 + 2x_4 + x_5)(x_3 + x_4 + 2x_5), \\
 & (2x_2 + x_4 + x_7)(x_2 + 2x_4 + x_7)(x_2 + x_4 + 2x_7), \\
 & (2x_5 + x_6 + x_7)(x_5 + 2x_6 + x_7)(x_5 + x_6 + 2x_7) \rangle.
 \end{aligned}$$

The reduced Gröbner basis for $I_2(FP')$ with respect to the monomial ordering

$$x_1 > x_2 > \cdots > x_7$$

is:

$$\begin{aligned}
 & \{ x_7^2 - 1, x_6^2 - 1, x_5^2 - 1, x_4^2 - 1, x_3^2 - 1, \\
 & x_5x_6 + x_5x_7 + x_6x_7 + 1, \\
 & x_4x_6 - x_4x_7 - 2x_5x_7 - x_6x_7 - 1, \\
 & x_4x_5 + x_4x_7 + x_5x_7 + 1, \\
 & x_3x_6 - x_3x_7 - 2x_5x_7 - x_6x_7 - 1, \\
 & x_3x_5 + x_3x_7 + x_5x_7 + 1, \\
 & x_3x_4 - x_3x_7 - x_4x_7 - 2x_5x_7 - 1, \\
 & x_2 - x_5, x_1 - x_5 \}.
 \end{aligned}$$

Thus we see that FP' is 2-colorable.

Since FP' is 3-uniform, there are only 2 possible proper edge sums for a 2-coloring:

$$\text{the edge colored with two } 1\text{'s and one } -1 \Rightarrow a_1 = \sum_{i=1}^3 c(x_{e_i}) = 1,$$

and

$$\text{the edge colored with one } 1 \text{ and two } -1\text{'s} \Rightarrow a_2 = \sum_{i=1}^3 c(x_{e_i}) = -1.$$

Hence, $A = \{1, -1\}$ and the possible non-empty subsets U of A are

$$\{1\}, \{-1\} \text{ and } \{1, -1\}.$$

Note that there is only one possible coloring scheme corresponding to the single proper 2-integer partition of 3, making the coloring scheme homogeneous. Hence the only 2-

coloring scheme ideal for FP' is:

$$S_{2,\{-1,1\}}(FP') = \langle x_i^2 - 1 : i \in [7] \rangle + \\
\langle (x_1 + x_3 + x_7 - 1)(x_1 + x_3 + x_7 + 1), \\
(x_1 + x_4 + x_6 - 1)(x_1 + x_4 + x_6 + 1), \\
(x_2 + x_3 + x_6 - 1)(x_2 + x_3 + x_6 + 1), \\
(x_3 + x_4 + x_5 - 1)(x_3 + x_4 + x_5 + 1), \\
(x_2 + x_4 + x_7 - 1)(x_2 + x_4 + x_7 + 1), \\
(x_5 + x_6 + x_7 - 1)(x_5 + x_6 + x_7 + 1) \rangle.$$

The reduced Gröbner bases for $S_{2,\{-1,1\}}(FP')$ with respect to the monomial ordering $x_1 > x_2 > \dots > x_7$, is:

$$\{x_7^2 - 1, x_6^2 - 1, x_5^2 - 1, x_4^2 - 1, x_3^2 - 1, \\
x_5x_6 + x_5x_7 + x_6x_7 + 1, \\
x_4x_6 - x_4x_7 - 2x_5x_7 - x_6x_7 - 1, \\
x_4x_5 + x_4x_7 + x_5x_7 + 1, \\
x_3x_6 - x_3x_7 - 2x_5x_7 - x_6x_7 - 1, \\
x_3x_5 + x_3x_7 + x_5x_7 + 1, \\
x_3x_4 - x_3x_7 - x_4x_7 - 2x_5x_7 - 1, \\
x_2 - x_5, x_1 - x_5\}.$$

We see that the Gröbner bases for $I_2(FP')$ and $S_{2,\{1,-1\}}(FP')$ with respect to the monomial ordering $x_1 > x_2 > \dots > x_7$, are identical.

Example 2.

Let \mathcal{H} stand for a 4-uniform hypergraph on 7 vertices, $\{1, \dots, 7\}$, and the following edge set

$$\{\{1 \ 2 \ 3 \ 4\}, \{1 \ 5 \ 6 \ 7\}, \\
\{2 \ 5 \ 6 \ 7\}, \{3 \ 5 \ 6 \ 7\}, \\
\{3 \ 4 \ 5 \ 6\}, \{3 \ 4 \ 6 \ 7\}, \\
\{3 \ 4 \ 5 \ 7\}, \{1 \ 2 \ 3 \ 5\}, \\
\{1 \ 2 \ 3 \ 6\}, \{1 \ 2 \ 3 \ 7\}\}.$$

The 2-colorability ideal of \mathcal{H} equals the following.

$$I_2(\mathcal{H}) = \langle x_i^2 - 1 : i \in [7] \rangle +$$

$$\begin{aligned}
& \langle (x_1 + x_2 + x_3 + x_4)(3x_1 + x_2 + x_3 + x_4)(x_1 + 3x_2 + x_3 + x_4)(x_1 + x_2 + 3x_3 + x_4)(x_1 + x_2 + x_3 + 3x_4), \\
& (x_1 + x_5 + x_6 + x_7)(3x_1 + x_5 + x_6 + x_7)(x_1 + 3x_5 + x_6 + x_7)(x_1 + x_5 + 3x_6 + x_7)(x_1 + x_5 + x_6 + 3x_7), \\
& (x_2 + x_5 + x_6 + x_7)(3x_2 + x_5 + x_6 + x_7)(x_2 + 3x_5 + x_6 + x_7)(x_2 + x_5 + 3x_6 + x_7)(x_2 + x_5 + x_6 + 3x_7), \\
& (x_3 + x_5 + x_6 + x_7)(3x_3 + x_5 + x_6 + x_7)(x_3 + 3x_5 + x_6 + x_7)(x_3 + x_5 + 3x_6 + x_7)(x_3 + x_5 + x_6 + 3x_7), \\
& (x_3 + x_4 + x_5 + x_6)(3x_3 + x_4 + x_5 + x_6)(x_3 + 3x_4 + x_5 + x_6)(x_3 + x_4 + 3x_5 + x_6)(x_3 + x_4 + x_5 + 3x_6), \\
& (x_3 + x_4 + x_6 + x_7)(3x_3 + x_4 + x_6 + x_7)(x_3 + 3x_4 + x_6 + x_7)(x_3 + x_4 + 3x_6 + x_7)(x_3 + x_4 + x_6 + 3x_7), \\
& (x_3 + x_4 + x_5 + x_7)(3x_3 + x_4 + x_5 + x_7)(x_3 + 3x_4 + x_5 + x_7)(x_3 + x_4 + 3x_5 + x_7)(x_3 + x_4 + x_5 + 3x_7), \\
& (x_1 + x_2 + x_3 + x_5)(3x_1 + x_2 + x_3 + x_5)(x_1 + 3x_2 + x_3 + x_5)(x_1 + x_2 + 3x_3 + x_5)(x_1 + x_2 + x_3 + 3x_5), \\
& (x_1 + x_2 + x_3 + x_6)(3x_1 + x_2 + x_3 + x_6)(x_1 + 3x_2 + x_3 + x_6)(x_1 + x_2 + 3x_3 + x_6)(x_1 + x_2 + x_3 + 3x_6), \\
& (x_1 + x_2 + x_3 + x_7)(3x_1 + x_2 + x_3 + x_7)(x_1 + 3x_2 + x_3 + x_7)(x_1 + x_2 + 3x_3 + x_7)(x_1 + x_2 + x_3 + 3x_7) \rangle
\end{aligned}$$

Using a computer algebra system, we find that the reduced Gröbner basis for $I_2(\mathcal{H})$ with respect to the monomial ordering $x_1 > x_2 > \dots > x_7$ is

$$\begin{aligned}
& \{x_7^2 - 1, x_6^2 - 1, x_5^2 - 1, x_4^2 - 1, x_3^2 - 1, x_2^2 - 1, x_1^2 - 1, \\
& x_5x_6x_4 + x_5x_7x_4 + x_6x_7x_4 + x_4 - x_5 - x_6 - x_5x_6x_7 - x_7, \\
& x_5x_6x_3 + x_5x_7x_3 + x_6x_7x_3 + x_3 + x_5 + x_6 + x_5x_6x_7 + x_7, \\
& x_3x_4 + x_5x_4 + x_6x_4 + x_7x_4 + x_3x_5 + x_3x_6 + x_3x_7 + 1, \\
& x_5x_6x_2 + x_5x_7x_2 + x_6x_7x_2 + x_2 + x_5 + x_6 + x_5x_6x_7 + x_7, \\
& x_5x_6x_1 + x_5x_7x_1 + x_6x_7x_1 + x_1 + x_5 + x_6 + x_5x_6x_7 + x_7, \\
& x_1x_2 + x_3x_2 + x_1x_3 - x_5x_6 - x_5x_7 - x_6x_7\},
\end{aligned}$$

and conclude that \mathcal{H} is 2-colorable.

Here $r = 4$, so the possible 2-integer partitions which yield coloring schemes are $\{1, 3\}$, and $\{2, 2\}$ which produce the proper edge sums ± 2 and 0 respectively. The associated homogeneous 2-coloring scheme ideals are:

$$\begin{aligned}
S_{2, \{-2, 2\}}(\mathcal{H}) = & \langle x_i^2 - 1 : i \in [7] \rangle + \\
& \langle (x_1 + x_2 + x_3 + x_4 - 2)(x_1 + x_2 + x_3 + x_4 + 2), \\
& (x_1 + x_5 + x_6 + x_7 - 2)(x_1 + x_5 + x_6 + x_7 + 2), \\
& (x_2 + x_5 + x_6 + x_7 - 2)(x_2 + x_5 + x_6 + x_7 + 2), \\
& (x_3 + x_5 + x_6 + x_7 - 2)(x_3 + x_5 + x_6 + x_7 + 2), \\
& (x_3 + x_4 + x_5 + x_6 - 2)(x_3 + x_4 + x_5 + x_6 + 2), \\
& (x_3 + x_4 + x_6 + x_7 - 2)(x_3 + x_4 + x_6 + x_7 + 2), \\
& (x_3 + x_4 + x_5 + x_7 - 2)(x_3 + x_4 + x_5 + x_7 + 2), \\
& (x_1 + x_2 + x_3 + x_5 - 2)(x_1 + x_2 + x_3 + x_5 + 2), \\
& (x_1 + x_2 + x_3 + x_6 - 2)(x_1 + x_2 + x_3 + x_6 + 2), \\
& (x_1 + x_2 + x_3 + x_7 - 2)(x_1 + x_2 + x_3 + x_7 + 2) \rangle;
\end{aligned}$$

and,

$$S_{2,\{0\}}(\mathcal{H}) = \langle x_i^2 - 1 : i \in [7] \rangle + \langle x_1 + x_2 + x_3 + x_4, \\ x_1 + x_5 + x_6 + x_7, x_2 + x_5 + x_6 + x_7, x_3 + x_5 + x_6 + x_7, \\ x_3 + x_4 + x_5 + x_6, x_3 + x_4 + x_6 + x_7, x_3 + x_4 + x_5 + x_7, \\ x_1 + x_2 + x_3 + x_5, x_1 + x_2 + x_3 + x_6, x_1 + x_2 + x_3 + x_7 \rangle.$$

The only heterogeneous coloring scheme for this uniformity is:

$$S_{2,\{-2,0,2\}}(\mathcal{H}) = \langle x_i^2 - 1 : i \in [7] \rangle + \\ \langle (x_1 + x_2 + x_3 + x_4 - 2)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2 + x_3 + x_4 + 2), \\ (x_1 + x_5 + x_6 + x_7 - 2)(x_1 + x_5 + x_6 + x_7)(x_1 + x_5 + x_6 + x_7 + 2), \\ (x_2 + x_5 + x_6 + x_7 - 2)(x_2 + x_5 + x_6 + x_7)(x_2 + x_5 + x_6 + x_7 + 2), \\ (x_3 + x_5 + x_6 + x_7 - 2)(x_3 + x_5 + x_6 + x_7)(x_3 + x_5 + x_6 + x_7 + 2), \\ (x_3 + x_4 + x_5 + x_6 - 2)(x_3 + x_4 + x_5 + x_6)(x_3 + x_4 + x_5 + x_6 + 2), \\ (x_3 + x_4 + x_6 + x_7 - 2)(x_3 + x_4 + x_6 + x_7)(x_3 + x_4 + x_6 + x_7 + 2), \\ (x_3 + x_4 + x_5 + x_7 - 2)(x_3 + x_4 + x_5 + x_7)(x_3 + x_4 + x_5 + x_7 + 2), \\ (x_1 + x_2 + x_3 + x_5 - 2)(x_1 + x_2 + x_3 + x_5)(x_1 + x_2 + x_3 + x_5 + 2), \\ (x_1 + x_2 + x_3 + x_6 - 2)(x_1 + x_2 + x_3 + x_6)(x_1 + x_2 + x_3 + x_6 + 2), \\ (x_1 + x_2 + x_3 + x_7 - 2)(x_1 + x_2 + x_3 + x_7)(x_1 + x_2 + x_3 + x_7 + 2) \rangle;$$

which coincides with the 2-colorability ideal.

Using a computer algebra system we find that the coloring scheme ideals have the following Gröbner bases with respect to the monomial ordering $x_1 > x_2 > \dots > x_7$.

$$S_{2,\{-2,2\}}(\mathcal{H}) : \{x_7^2 - 1, x_6 - x_7, x_5 - x_7, x_4 - x_7, x_3 + x_7, x_2 + x_7, x_1 + x_7\} \\ S_{2,\{0\}}(\mathcal{H}) : \{1\} \\ S_{2,\{-2,0,2\}}(\mathcal{H}) : \{x_7^2 - 1, x_6^2 - 1, x_5^2 - 1, x_4^2 - 1, x_3^2 - 1, x_2^2 - 1, x_1^2 - 1, \\ x_5x_6x_4 + x_5x_7x_4 + x_6x_7x_4 + x_4 - x_5 - x_6 - x_5x_6x_7 - x_7, \\ x_5x_6x_3 + x_5x_7x_3 + x_6x_7x_3 + x_3 + x_5 + x_6 + x_5x_6x_7 + x_7, \\ x_3x_4 + x_5x_4 + x_6x_4 + x_7x_4 + x_3x_5 + x_3x_6 + x_3x_7 + 1, \\ x_5x_6x_2 + x_5x_7x_2 + x_6x_7x_2 + x_2 + x_5 + x_6 + x_5x_6x_7 + x_7, \\ x_5x_6x_1 + x_5x_7x_1 + x_6x_7x_1 + x_1 + x_5 + x_6 + x_5x_6x_7 + x_7, \\ x_1x_2 + x_3x_2 + x_1x_3 - x_5x_6 - x_5x_7 - x_6x_7\}$$

Thus, we conclude that \mathcal{H} is 2-colorable with the homogeneous coloring scheme associated with the 2-integer partition of 4: $\{1, 3\}$ and the heterogeneous coloring scheme associated with both proper 2-integer partitions of 4 (which corresponds to the general 2-colorability); however, the 2-integer partition of 4: $\{2, 2\}$ produces a homogeneous coloring scheme which will not properly 2-color \mathcal{H} .

9 Appendix B

The 2-colorability ideal for a non-2-colorable 4-uniform hypergraph \mathcal{H} on 11 vertices and 24 edges introduced in Section 4

$$\{x_1^2 - 1, x_2^2 - 1, x_3^2 - 1, x_4^2 - 1, x_5^2 - 1, x_6^2 - 1, x_7^2 - 1, x_8^2 - 1, x_9^2 - 1, x_{10}^2 - 1, x_{11}^2 - 1, \\
(x_5 + x_6 + x_{10} + x_{11})(3x_5 + x_6 + x_{10} + x_{11})(x_5 + 3x_6 + x_{10} + x_{11})(x_5 + x_6 + 3x_{10} + x_{11})(x_5 + x_6 + x_{10} + 3x_{11}), \\
(x_1 + x_4 + x_5 + x_7)(3x_1 + x_4 + x_5 + x_7)(x_1 + 3x_4 + x_5 + x_7)(x_1 + x_4 + 3x_5 + x_7)(x_1 + x_4 + x_5 + 3x_7), \\
(x_2 + x_3 + x_6 + x_7)(3x_2 + x_3 + x_6 + x_7)(x_2 + 3x_3 + x_6 + x_7)(x_2 + x_3 + 3x_6 + x_7)(x_2 + x_3 + x_6 + 3x_7), \\
(x_3 + x_4 + x_{10} + x_{11})(3x_3 + x_4 + x_{10} + x_{11})(x_3 + 3x_4 + x_{10} + x_{11})(x_3 + x_4 + 3x_{10} + x_{11})(x_3 + x_4 + x_{10} + 3x_{11}), \\
(x_2 + x_4 + x_5 + x_8)(3x_2 + x_4 + x_5 + x_8)(x_2 + 3x_4 + x_5 + x_8)(x_2 + x_4 + 3x_5 + x_8)(x_2 + x_4 + x_5 + 3x_8), \\
(x_1 + x_2 + x_9 + x_{10})(3x_1 + x_2 + x_9 + x_{10})(x_1 + 3x_2 + x_9 + x_{10})(x_1 + x_2 + 3x_9 + x_{10})(x_1 + x_2 + x_9 + 3x_{10}), \\
(x_1 + x_2 + x_{10} + x_{11})(3x_1 + x_2 + x_{10} + x_{11})(x_1 + 3x_2 + x_{10} + x_{11})(x_1 + x_2 + 3x_{10} + x_{11})(x_1 + x_2 + x_{10} + 3x_{11}), \\
(x_1 + x_3 + x_6 + x_8)(3x_1 + x_3 + x_6 + x_8)(x_1 + 3x_3 + x_6 + x_8)(x_1 + x_3 + 3x_6 + x_8)(x_1 + x_3 + x_6 + 3x_8), \\
(x_5 + x_6 + x_9 + x_{10})(3x_5 + x_6 + x_9 + x_{10})(x_5 + 3x_6 + x_9 + x_{10})(x_5 + x_6 + 3x_9 + x_{10})(x_5 + x_6 + x_9 + 3x_{10}), \\
(x_1 + x_4 + x_6 + x_7)(3x_1 + x_4 + x_6 + x_7)(x_1 + 3x_4 + x_6 + x_7)(x_1 + x_4 + 3x_6 + x_7)(x_1 + x_4 + x_6 + 3x_7), \\
(x_7 + x_8 + x_9 + x_{10})(3x_7 + x_8 + x_9 + x_{10})(x_7 + 3x_8 + x_9 + x_{10})(x_7 + x_8 + 3x_9 + x_{10})(x_7 + x_8 + x_9 + 3x_{10}), \\
(x_7 + x_8 + x_9 + x_{11})(3x_7 + x_8 + x_9 + x_{11})(x_7 + 3x_8 + x_9 + x_{11})(x_7 + x_8 + 3x_9 + x_{11})(x_7 + x_8 + x_9 + 3x_{11}), \\
(x_2 + x_3 + x_5 + x_7)(3x_2 + x_3 + x_5 + x_7)(x_2 + 3x_3 + x_5 + x_7)(x_2 + x_3 + 3x_5 + x_7)(x_2 + x_3 + x_5 + 3x_7), \\
(x_3 + x_4 + x_9 + x_{10})(3x_3 + x_4 + x_9 + x_{10})(x_3 + 3x_4 + x_9 + x_{10})(x_3 + x_4 + 3x_9 + x_{10})(x_3 + x_4 + x_9 + 3x_{10}), \\
(x_2 + x_4 + x_6 + x_8)(3x_2 + x_4 + x_6 + x_8)(x_2 + 3x_4 + x_6 + x_8)(x_2 + x_4 + 3x_6 + x_8)(x_2 + x_4 + x_6 + 3x_8), \\
(x_7 + x_8 + x_{10} + x_{11})(3x_7 + x_8 + x_{10} + x_{11})(x_7 + 3x_8 + x_{10} + x_{11})(x_7 + x_8 + 3x_{10} + x_{11})(x_7 + x_8 + x_{10} + 3x_{11}), \\
(x_1 + x_2 + x_9 + x_{11})(3x_1 + x_2 + x_9 + x_{11})(x_1 + 3x_2 + x_9 + x_{11})(x_1 + x_2 + 3x_9 + x_{11})(x_1 + x_2 + x_9 + 3x_{11}), \\
(x_5 + x_6 + x_9 + x_{11})(3x_5 + x_6 + x_9 + x_{11})(x_5 + 3x_6 + x_9 + x_{11})(x_5 + x_6 + 3x_9 + x_{11})(x_5 + x_6 + x_9 + 3x_{11}), \\
(x_2 + x_4 + x_5 + x_6)(3x_2 + x_4 + x_5 + x_6)(x_2 + 3x_4 + x_5 + x_6)(x_2 + x_4 + 3x_5 + x_6)(x_2 + x_4 + x_5 + 3x_6), \\
(x_1 + x_4 + x_5 + x_8)(3x_1 + x_4 + x_5 + x_8)(x_1 + 3x_4 + x_5 + x_8)(x_1 + x_4 + 3x_5 + x_8)(x_1 + x_4 + x_5 + 3x_8), \\
(x_3 + x_4 + x_9 + x_{11})(3x_3 + x_4 + x_9 + x_{11})(x_3 + 3x_4 + x_9 + x_{11})(x_3 + x_4 + 3x_9 + x_{11})(x_3 + x_4 + x_9 + 3x_{11}), \\
(x_1 + x_3 + x_6 + x_7)(3x_1 + x_3 + x_6 + x_7)(x_1 + 3x_3 + x_6 + x_7)(x_1 + x_3 + 3x_6 + x_7)(x_1 + x_3 + x_6 + 3x_7), \\
(x_1 + x_3 + x_5 + x_8)(3x_1 + x_3 + x_5 + x_8)(x_1 + 3x_3 + x_5 + x_8)(x_1 + x_3 + 3x_5 + x_8)(x_1 + x_3 + x_5 + 3x_8), \\
(x_2 + x_3 + x_5 + x_8)(3x_2 + x_3 + x_5 + x_8)(x_2 + 3x_3 + x_5 + x_8)(x_2 + x_3 + 3x_5 + x_8)(x_2 + x_3 + x_5 + 3x_8)\}$$

Using a computer algebra system, we find that the reduced Gröbner basis for this ideal is trivial for any monomial ordering. Hence, hypergraph \mathcal{H} is not 2-colorable as claimed in Section 4.

Corrigendum – submitted June 3, 2024

After the initial publishing of this article, an error was brought to our attention regarding the minimality of the number of hyperedges in a non-2-colorable 4-uniform hypergraph. In [2], Seymour established that the minimum number of hyperedges in a non-2-colorable 4-uniform hypergraph is 23.

We note that the example provided in this paper is indeed not isomorphic to the construction of Abbott and Hanson, [1].

Moreover, the technique used to establish non-2-colorability in this article is unaffected by the error.

References

- [1] H. L. Abbott and D. Hanson, On a combinatorial problem of Erdős. *Canadian Mathematical Bulletin*, 12 (1969), 823–829.
- [2] P.D. Seymour, A note on a combinatorial problem of Erdős and Hajnal. *J. London Mathematical Society* (2), 8 (1974), 681–682.