$1/k$-Eulerian polynomials and k-inversion sequences

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Abstract

Let $\mathbf{s} = (s_1, s_2, \ldots, s_n, \ldots)$ be a sequence of positive integers. An $\mathbf{s}$-inversion sequence of length n is a sequence $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of nonnegative integers such that $0 \leqslant e_i < s_i$ for $1 \leqslant i \leqslant n$. When $s_i = (i-1)k+1$ for any $i \geqslant 1$, we call the $\mathbf{s}$-inversion sequences the k-inversion sequences. In this paper, we provide a bijective proof that the ascent number over k-inversion sequences of length n is equidistributed with a weighted variant of the ascent number of permutations of order n, which leads to an affirmative answer of a question of Savage (2016). A key ingredient of the proof is a bijection between k-inversion sequences of length n and $2 \times n$ arrays with particular restrictions. Moreover, we present a bijective proof of the fact that the ascent plateau number over k-Stirling permutations of order n is equidistributed with the ascent number over k-inversion sequences of length n.

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1 Introduction

Let $\mathfrak{S}_n$ be the symmetric group on the set $[n] = \{1, 2, \ldots, n\}$. Let

$$\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n.$$

A *descent* (resp. *excedance*, *ascent*) in π is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$ (resp. $\pi(i) > i$, $\pi(i) < \pi(i+1)$). A *left-to-right minimum* in π is an index i such that $\pi(i) < \pi(j)$ for any $j < i$ or $i = 1$. Let $\mathrm{Asc}\,(\pi)$ and $\mathrm{Lrm}\,(\pi)$ denote the set of ascents and left-to-right minima of π, respectively. For example, $\mathrm{Asc}\,(324165) = \{2, 4\}$ and $\mathrm{Lrm}\,(324165) = \{1, 2, 4\}$. Let $\mathrm{des}\,(\pi)$ (resp. $\mathrm{exc}\,(\pi)$, $\mathrm{asc}\,(\pi)$, $\mathrm{lrmin}(\pi)$, $\mathrm{cyc}\,(\pi)$) denote the number of descents (resp. excedances, ascents, left-to-right minima, cycles) of π. It is well known that descents and excedances are equidistributed over $\mathfrak{S}_n$. The classical Eulerian polynomial is defined by

$$A_n(x) = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)}.$$

The exponential generating function of $A_n(x)$ is given as follows:

$$A(x, z) = 1 + \sum_{n\geqslant 1} A_n(x)\frac{z^n}{n!} = \frac{1-x}{e^{z(x-1)} - x}.$$

In [11], Savage and Schuster introduced the concept of $\mathbf{s}$-inversion sequences in study of lecture hall polytopes. Let $\mathbf{s} = (s_1, s_2, \ldots, s_n, \ldots)$ be a sequence of positive integers. An $\mathbf{s}$-*inversion sequence* of length n is a sequence $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ of nonnegative integers such that $0 \leqslant e_i < s_i$ for $1 \leqslant i \leqslant n$. Let $\mathrm{I}_\mathrm{n}^{(\mathbf{s})}$ denote the set of $\mathbf{s}$-inversion sequences of length n. An ascent in $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ is an index $i \in \{0, 1, \ldots, n-1\}$ such that

$$\frac{e_i}{s_i} < \frac{e_{i+1}}{s_{i+1}},$$

with the convention that $e_0 = 0$ and $s_0 = 1$. Let $\mathrm{Asc}\,(\mathbf{e})$ be the set of ascents of $\mathbf{e}$ and let $\mathrm{asc}\,(\mathbf{e}) = |\mathrm{Asc}\,(\mathbf{e})|$.

The *s-inversion Eulerian polynomial* is defined by

$$E_n^{(\mathbf{s})}(x) = \sum_{\mathbf{e}\in\mathrm{I}_\mathrm{n}^{(\mathbf{s})}} x^{\mathrm{asc}\,(\mathbf{e})}.$$

Let

$$\mathcal{P}_n^{\mathbf{s}} = \{\lambda \in \mathbb{R}^n \mid 0 \leqslant \frac{\lambda_1}{s_1} \leqslant \frac{\lambda_2}{s_2} \leqslant \cdots \frac{\lambda_n}{s_n} \leqslant 1\}$$

be the $\mathbf{s}$-lecture hall polytope. Savage and Schuster [11, Theorem 5] showed that the Ehrhart series of $\mathcal{P}_n^{\mathbf{s}}$ is

$$\frac{E_n^{(\mathbf{s})}(x)}{(1-x)^{n+1}}.$$

Following [13, Section 3], the polynomial $E_n^{(\mathbf{s})}(x)$ is the h^*-polynomial of $\mathcal{P}_n^{\mathbf{s}}$. For some special cases of $\mathbf{s}$, the s-inversion Eulerian polynomial has been extensively studied. For example, for $\mathbf{s} = (1, 4, 3, 8, 5, 12, \ldots)$, i.e., $s_{2i} = 4i$ and $s_{2i-1} = 2i - 1$ for $i \geqslant 1$, Chen et al. [4] proved that the ascent number over $\mathrm{I}_{2\mathrm{n}}^{(\mathbf{s})}$ is equidistributed with the descent number over P_n, where P_n is the set of signed permutations on the multiset $\{1^2, 2^2, \ldots, n^2\}$.

In the following discussion, we always assume that $s_i = (i-1)k + 1$ for any $i \geqslant 1$, where k is a positive integer. For convenience, we write $\mathrm{I}_{\mathrm{n}}^{(\mathbf{s})}$ as $I_{n,k}$. In other words,

$$I_{n,k} = \{\mathbf{e} \in \mathbb{Z}^n : 0 \leqslant e_i \leqslant (i-1)k\}.$$

As usual, we call the $\mathbf{s}$-inversion sequences the *k-inversion sequences*. Following [12], the $1/k$-Eulerian polynomial $E_{n,k}(x)$ is defined by

$$E_{n,k}(x) = \sum_{\mathbf{e} \in \mathrm{I}_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})},$$

The exponential generating function of $E_{n,k}(x)$ is given as follows:

$$\sum_{n \geqslant 0} E_{n,k}(x) \frac{z^n}{n!} = \sqrt[k]{A(x, kz)}.$$

Using (1), Savage and Viswanathan [12, Section 1.5] found that

$$\sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{exc}\,(\pi)} k^{n - \mathrm{cyc}\,(\pi)}. \tag{1}$$

By using the fundamental transformation of Foata and Schützenberger [6], the pairs of statistics (exc , cyc) and (asc , lrmin) are equidistributed over $\mathfrak{S}_n$. Thus

$$\sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} k^{n - \mathrm{lrmin}(\pi)}. \tag{2}$$

It is well known that any permutation $\pi \in \mathfrak{S}_n$ can be encoded by its inversion sequence $\theta(\pi) = (e_1, e_2, \ldots, e_n) \in I_{n,1}$, where $e_i = |\{j \mid j < i \text{ and } \pi_j > \pi_i\}|$. Moreover, the map $\theta : \mathfrak{S}_n \mapsto I_{n,1}$ is a bijection.

Proposition 1. *For any $n \geqslant 1$, we have*

$$\sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{\mathbf{e} \in I_{n,1}} x^{n-1-\mathrm{asc}\,(\mathbf{e})} k^{n - \max(\mathbf{e})},$$

where $\max(\mathbf{e}) = |\{i \mid e_i = i - 1\}|$.

Proof. For any $\pi \in \mathfrak{S}_n$, let $\mathbf{e} = (e_1, e_2, \ldots, e_n) = \theta(\pi)$. Then $i \in \mathrm{Asc}\,(\pi)$ if and only if $e_i \geqslant e_{i+1}$, and $i \in \mathrm{Lrm}\,(\pi)$ if and only if either $i = 1$ or $e_i = i - 1$. Moreover, when $e_i \geqslant e_{i+1}$, we have

$$(i+1)e_i - ie_{i+1} \geqslant (i+1)e_i - ie_i = e_i \geqslant 0;$$

when $e_i < e_{i+1}$, we have $e_i + 1 \leqslant e_{i+1}$ and

$$(i+1)e_i - ie_{i+1} \leqslant (i+1)e_i - i(e_i+1) = e_i - i < 0.$$

This tells us that $e_i \geqslant e_{i+1}$ if and only if $i \notin \mathrm{Asc}\,(\mathbf{e})$. Hence,

$$\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} k^{n-\mathrm{lrmin}(\pi)} = \sum_{\mathbf{e} \in I_{n,1}} x^{n-1-\mathrm{asc}\,(\mathbf{e})} k^{n-\max\,(\mathbf{e})},$$

and it follows from (2)) that

$$\sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{\mathbf{e} \in I_{n,1}} x^{n-1-\mathrm{asc}\,(\mathbf{e})} k^{n-max(\mathbf{e})}. \qquad \square$$

Recently, Savage [10] gave a survey for the study of lecture hall partitions. In particular, she posed the following question.

Question 2 ([10, p. 466])**.** Is there a bijective proof of (1)?

A bijective proof of (1) may arouse interests in the study of the connections between **s**-lecture hall polytope and other structures. In this paper, we give a bijective proof of (1). It suffices to present a bijective proof of (2). The method is to present a series of three bijections: the first bijection maps k-inversion sequences to $2 \times n$ arrays with particular restrictions. The second bijection maps these $2 \times n$ arrays to k-colored permutations $B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix}$ in which $\pi \in \mathfrak{S}_n$ and $\mathbf{c}$ is a map from $[n]$ to $[k]$ satisfying $1 \leqslant c(\pi(i)) \leqslant k$ if $i \notin \mathrm{Lrm}\,(\pi)$, otherwise $c(\pi(i)) = 1$. The final bijection maps k-colored permutations to themselves, but in a way that will create the correct correspondence between ascents in the original k-inversion sequence and ascents in the final k-colored permutation.

2 A bijective proof of (2)

Given an inversion sequence $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in I_{n,k}$. Let

$$c_i = \begin{cases} 1 & \text{if } \; e_i = 0 \\ \lceil \frac{e_i}{i-1} \rceil & \text{if } \; e_i \geqslant 1 \end{cases} \quad \text{and } b_i = e_i - (c_i - 1)(i-1) + 1$$

for any $i = 1, 2, \ldots, n$. Denote by $A(\mathbf{e})$ the following array

$$\begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}.$$

Lemma 3. *For any* $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in I_{n,k}$*, we have* $A(\mathbf{e}) \in \mathcal{A}_{n,k}$*, where* $\mathcal{A}_{n,k}$ *is the set of* $2 \times n$ *arrays*

$$A = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$$

such that $c_1 = b_1 = 1$ *and for any* $2 \leqslant i \leqslant n$*,*

$$c_i \in \{1, 2, \ldots, k\} \; \text{and } b_i \in \begin{cases} \{1, 2, \ldots, i\} & \text{if} \;\; c_i = 1; \\ \{2, 3, \ldots, i\} & \text{if} \;\; c_i \geqslant 2. \end{cases}$$

Proof. Let $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in I_{n,k}$. For each $i = 1, 2, \ldots, n$, we have $0 \leqslant e_i \leqslant k(i-1)$. If $e_i = 0$, then $c_i = 1$; if $e_i \geqslant 1$, then $0 < \frac{e_i}{i-1} \leqslant k$ for any $i \geqslant 2$, and so $c_i = \lceil \frac{e_i}{i-1} \rceil \in \{1, 2, \ldots, k\}$.

When $c_i = 1$, we have $b_i = e_i + 1$. If $e_i = 0$, then $b_i = 1$. If $e_i > 0$, then $0 < \frac{e_i}{i-1} \leqslant 1$ since $c_i = \lceil \frac{e_i}{i-1} \rceil = 1$, and so $0 < e_i \leqslant i - 1$. This implies that $1 < b_i \leqslant i$. Thus, if $c_i = 1$, then $b_i \in [i]$.

When $c_i \geqslant 2$, we have $c_i - 1 < \frac{e_i}{i-1} \leqslant c_i$. This implies that $(c_i - 1)(i-1) < e_i \leqslant c_i(i-1)$. Hence, $1 < b_i \leqslant i$, and so $A(\mathbf{e}) \in \mathcal{A}_{n,k}$. $\square$

Define a map $\varphi_n : I_{n,k} \mapsto \mathcal{A}_{n,k}$ by letting $\varphi_n(\mathbf{e}) = A(\mathbf{e})$ for any $\mathbf{e} \in I_{n,k}$. For any $A \in \mathcal{A}_{n,k}$, we say that an index i is an *ascent* of A if either (i) $c_i < c_{i+1}$ or (ii) $c_i = c_{i+1}$ and $b_i < b_{i+1}$. Let $\mathrm{Asc}\,(A)$ be the set of ascents of A and let $\mathrm{asc}\,(A) = |\mathrm{Asc}\,(A)|$.

Example 4. Take $n = 17$ and $k = 2$. Let

$$\mathbf{e} = (0, 1, 3, 0, 5, 10, 3, 7, 16, 15, 0, 3, 13, 1, 2, 20, 12) \in I_{17,2}.$$

We have

$$A = A(\mathbf{e}) = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 & 6 & 4 & 8 & 9 & 7 & 1 & 4 & 2 & 2 & 3 & 6 & 13 \end{pmatrix} \in \mathcal{A}_{17,2}.$$

It is clear that $\mathrm{Asc}\,(A) = \{1, 2, 4, 5, 7, 8, 11, 12, 14, 15\}$ and $\mathrm{asc}\,(A) = 10$.

The following lemma is fundamental.

Lemma 5. *The map φ_n is a bijection from $I_{n,k}$ to $\mathcal{A}_{n,k}$. For any $\mathbf{e} \in I_{n,k}$, we have*

$$\mathrm{Asc}\,(\mathbf{e}) = \mathrm{Asc}\,(\varphi_n(\mathbf{e})).$$

Therefore,

$$\sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{A \in \mathcal{A}_{n,k}} x^{\mathrm{asc}\,(A)}.$$

Proof. For any

$$A = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \in \mathcal{A}_{n,k},$$

let $\theta_n(A) = (e'_1, e'_2, \ldots, e'_n)$, where $e'_i = b_i + (c_i - 1)(i-1) - 1$. It is clear that the map θ_n is the inverse of φ_n.

For any $\mathbf{e} = (e_1, e_2, \ldots, e_n) \in I_{n,k}$, if $i \in \mathrm{Asc}\,(\mathbf{e})$, then $\frac{e_i}{k(i-1)+1} < \frac{e_{i+1}}{ki+1}$. For any $i \in [n]$, let

$$c_i = \begin{cases} 1 & \text{if } \; e_i = 0 \\ \lceil \frac{e_i}{i-1} \rceil & \text{if } \; e_i \geqslant 1 \end{cases} \quad \text{and } b_i = e_i - (c_i - 1)(i-1) + 1.$$

Since $e_i = b_i + (c_i - 1)(i-1) - 1$, we get

$$\begin{aligned} & e_i(ki+1) - e_{i+1}(k(i-1)+1) \\ = \;& (ki+1)((c_i - c_{i+1})(i-1) + b_i - b_{i+1}) + k(b_{i+1} - 1) - (c_{i+1} - 1). \end{aligned}$$

When $c_i < c_{i+1}$, we have $c_{i+1} \geqslant 2$, $b_{i+1} \geqslant 2$ and $b_i \leqslant i$. So,

$$e_i = (c_i - 1)(i-1) + b_i - 1 \leqslant c_i(i-1)$$

and

$$e_{i+1} = (c_{i+1} - 1)i + b_{i+1} - 1 \geqslant (c_{i+1} - 1)i + 1.$$

Hence,

$$\begin{aligned} & e_{i+1}(k(i-1)+1) - e_i(ki+1) \\ \geqslant\ & ((c_{i+1}-1)i+1)(k(i-1)+1) - c_i(i-1)(ki+1) \\ =\ & ki(i-1)(c_{i+1} - c_i - 1) + (k-1)(i-1) + (c_{i+1} - c_i)i + c_i \\ >\ & 0. \end{aligned}$$

When $c_i > c_{i+1}$, we have $c_i \geqslant 2$, $c_{i+1} < k$, $b_i \geqslant 2$. Moreover, we have $b_{i+1} \leqslant i$, since $b_{i+1} = i+1$ if and only if $c_{i+1} = k$. So,

$$e_i = (c_i - 1)(i-1) + b_i - 1 \geqslant (c_i - 1)(i-1) + 1$$

and

$$e_{i+1} = (c_{i+1} - 1)i + b_{i+1} - 1 \leqslant c_{i+1}i - 1.$$

Hence,

$$\begin{aligned} & e_{i+1}(k(i-1)+1) - e_i(ki+1) \\ \leqslant\ & (c_{i+1}i - 1)(k(i-1)+1) - ((c_i-1)(i-1)+1)(ki+1) \\ =\ & (ki+1)(i-1)(c_{i+1} - c_i + 1) + (c_{i+1} + k - 2ki - 2) \\ <\ & 0. \end{aligned}$$

When $c_i = c_{i+1}$ and $b_i < b_{i+1}$, we have

$$\begin{aligned} e_i &= (c_i - 1)(i-1) + b_i - 1 \\ &= (c_{i+1} - 1)(i-1) + b_i - 1 \\ &\leqslant (c_{i+1} - 1)(i-1) + b_{i+1} - 2 \\ &= e_{i+1} - c_{i+1}. \end{aligned}$$

Hence,

$$\begin{aligned} & e_{i+1}(k(i-1)+1) - e_i(ki+1) \\ \geqslant\ & e_{i+1}(k(i-1)+1) - (e_{i+1} - c_{i+1})(ki+1) \\ =\ & (ki+1)c_{i+1} - ke_{i+1} \\ \geqslant\ & (ki+1)c_{i+1} - kic_{i+1} = c_{i+1} > 0. \end{aligned}$$

When $c_i = c_{i+1}$ and $b_i \geqslant b_{i+1}$, we have

$$\begin{aligned} e_i &= (c_i - 1)(i-1) + b_i - 1 \\ &= (c_{i+1} - 1)(i-1) + b_i - 1 \\ &\geqslant (c_{i+1} - 1)(i-1) + b_{i+1} - 1 \\ &= e_{i+1} - c_{i+1} + 1. \end{aligned}$$

Hence,

$$\begin{aligned} & e_{i+1}(k(i-1)+1) - e_i(ki+1) \\ \leqslant\ & e_{i+1}(k(i-1)+1) - (e_{i+1} - c_{i+1} + 1)(ki+1) \\ =\ & (ki+1)(c_{i+1} - 1) - ke_{i+1} \\ =\ & k(1 - b_{i+1}) + c_{i+1} - 1 \leqslant 0, \end{aligned}$$

in which the last inequality is easily checked by using Lemma 3. Thus, we have $i \in \operatorname{Asc}(\mathbf{e})$ if and only if $i \in \operatorname{Asc}(\varphi_n(\mathbf{e}))$. $\square$

A *block k-colored permutation* on the set $[n]$ is a pair $B = \binom{\mathbf{c}}{\pi}$ such that $\pi \in \mathfrak{S}_n$ and $\mathbf{c}$ is a map from $[n]$ to $[k]$ which satisfies $1 \leqslant c(\pi(i)) \leqslant k$ if $i \notin \operatorname{Lrm}(\pi)$, otherwise $c(\pi(i)) = 1$. Let $\mathcal{B}_{n,k}$ be the set of block k-colored permutations on the set $[n]$. We write an element $B = \binom{\mathbf{c}}{\pi}$ in $\mathcal{B}_{n,k}$ as the following $2 \times n$ array

$$B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} c(\pi(1)) & c(\pi(2)) & \dots & c(\pi(n)) \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}.$$

For example, consider the permutation $\pi = 324165$ and let $k = 3$. Then the following 2×6 array

$$B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 1 & 1 & 2 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix}$$

is a block 3-colored permutation on the set $[6]$.

Given an array

$$A = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathcal{A}_{n,k},$$

we construct a permutation in $\mathfrak{S}_n$ by the following **Algorithm B**. Let

$$\pi^{(i)} = \pi^{(i)}(1)\pi^{(i)}(2)\cdots\pi^{(i)}(i)$$

denote the permutation in $\mathfrak{S}_i$ obtained by the algorithm at time i.

Algorithm B.

- Step 1. Let $\pi^{(1)}(1) = b_1$.
- Step 2. At time $i \geqslant 2$, suppose that $\pi^{(i-1)}$ is determined. For each $j = 1, 2, \ldots, i-1$, if $\pi^{(i-1)}(j) \geqslant b_i$, then let $\pi^{(i)}(j) = \pi^{(i)}(j) + 1$; otherwise, let $\pi^{(i)}(j) = \pi^{(i-1)}(j)$. Finally, set $\pi^{(i)}(i) = b_i$. Thus we get

$$\pi^{(i)} = \pi^{(i)}(1)\pi^{(i)}(2)\cdots\pi^{(i)}(i-1)\pi^{(i)}(i) \in \mathfrak{S}_i.$$

Iterating Step 2 until $i = n$, we obtain a permutation $\pi^{(n)} \in \mathfrak{S}_n$. Let $\mathbf{c}$ be a map from $[n]$ to $\mathbb{N}$ such that $c(\pi(i)) = c_i$ and let $B_A = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix}$. Then

$$B_A = \begin{pmatrix} \mathbf{c} \\ \pi^{(n)} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ \pi^{(n)}(1) & \pi^{(n)}(2) & \ldots & \pi^{(n)}(n) \end{pmatrix}.$$

Example 6. Let A be the 2×17 array given in Example 4. By **Algorithm B**, we have

$\pi^{(1)}$	1																
$\pi^{(2)}$	1	2															
$\pi^{(3)}$	1	3	2														
$\pi^{(4)}$	2	4	3	1													
$\pi^{(5)}$	3	5	4	1	2												
$\pi^{(6)}$	3	5	4	1	2	6											
$\pi^{(7)}$	3	6	5	1	2	7	4										
$\pi^{(8)}$	3	6	5	1	2	7	4	8									
$\pi^{(9)}$	3	6	5	1	2	7	4	8	9								
$\pi^{(10)}$	3	6	5	1	2	8	4	9	10	7							
$\pi^{(11)}$	4	7	6	2	3	9	5	10	11	8	1						
$\pi^{(12)}$	5	8	7	2	3	10	6	11	12	9	1	4					
$\pi^{(13)}$	6	9	8	3	4	11	7	12	13	10	1	5	2				
$\pi^{(14)}$	7	10	9	4	5	12	8	13	14	11	1	6	3	2			
$\pi^{(15)}$	8	11	10	5	6	13	9	14	15	12	1	7	4	2	3		
$\pi^{(16)}$	9	12	11	5	7	14	10	15	16	13	1	8	4	2	3	6	
$\pi^{(17)}$	9	12	11	5	7	15	10	16	17	14	1	8	4	2	3	6	13

,

and so

$$B_A = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 9 & 12 & 11 & 5 & 7 & 15 & 10 & 16 & 17 & 14 & 1 & 8 & 4 & 2 & 3 & 6 & 13 \end{pmatrix}.$$

Lemma 7. *For any $A \in \mathcal{A}_{n,k}$, we have $B_A \in \mathcal{B}_{n,k}$.*

Proof. Fix an array $A = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathcal{A}_{n,k}$. For each $i \in [n]$, let

$$A^{(i)} = \begin{pmatrix} c_1 & c_2 & \dots & c_i \\ b_1 & b_2 & \dots & b_i \end{pmatrix}.$$

Then $A^{(i)} \in \mathcal{A}_{i,k}$. We prove the lemma by induction. For each i, suppose that

$$B_{A^{(i)}} = \begin{pmatrix} \mathbf{c}^{(i)} \\ \pi^{(i)} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_i \\ \pi^{(i)}(1) & \pi^{(i)}(2) & \dots & \pi^{(i)}(i) \end{pmatrix}.$$

Clearly, $B_{A^{(1)}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in \mathcal{B}_{1,k}$. Suppose that $B_{A^{(i)}} \in \mathcal{B}_{i,k}$. Let us consider $A^{(i+1)}$. By **Algorithm B**, we have

$$\mathrm{Lrm}\,(\pi^{(i+1)}) = \begin{cases} \mathrm{Lrm}\,(\pi^{(i)}) \cup \{i+1\} & \text{if} \quad b_{i+1} = 1 \\ \mathrm{Lrm}\,(\pi^{(i)}) & \text{if} \quad b_{i+1} \geqslant 2 \end{cases}$$

Note that $c_{i+1} = 1$ if $b_{i+1} = 1$. By induction hypothesis, we obtain $B_{A^{(i+1)}} \in \mathcal{B}_{i+1,k}$. □

Define a map $\alpha_n : \mathcal{A}_{n,k} \mapsto \mathcal{B}_{n,k}$ by letting $\alpha_n(A) = B_A$ for any $A \in \mathcal{A}_{n,k}$. For any $B \in \mathcal{B}_{n,k}$, we say that an index i is an ascent of B if either (i) $c(\pi(i)) < c(\pi(i+1))$ or (ii) $c(\pi(i)) = c(\pi(i+1))$ and $\pi(i) < \pi(i+1)$. Let $\mathrm{Asc}\,(B)$ be the set of ascents of B and $\mathrm{asc}\,(B) = |\mathrm{Asc}\,(B)|$. For example, for the array $B = B_A$ that is given in Example 6, we have $\mathrm{Asc}\,(B) = \{1, 2, 4, 5, 7, 8, 11, 12, 14, 15\}$ and $\mathrm{asc}\,(B) = 10$.

Lemma 8. *The map $\alpha_n : \mathcal{A}_{n,k} \mapsto \mathcal{B}_{n,k}$ is a bijection. For any $A \in \mathcal{A}_{n,k}$, we have $\mathrm{Asc}\,(A) = \mathrm{Asc}\,(\alpha_n(A))$. Therefore,*

$$\sum_{A \in \mathcal{A}_{n,k}} x^{\mathrm{asc}\,(A)} = \sum_{B \in \mathcal{B}_{n,k}} x^{\mathrm{asc}\,(B)}.$$

Proof. Given an array

$$B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} c(\pi(1)) & c(\pi(2)) & \dots & c(\pi(n)) \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix} \in \mathcal{B}_{n,k},$$

we first construct a sequence $b_1, b_2, \dots, b_n$ by the following **Algorithm B***. Let

$$\pi^{(n-i+1)} = \pi^{(n-i+1)}(1)\pi^{(n-i+1)}(2) \cdots \pi^{(n-i+1)}(n-i+1)$$

denote the permutation in $\mathfrak{S}_{n-i+1}$ obtained by the algorithm at time i.

Algorithm B*.

- Step 1. Let $\tilde{\pi}^{(n)}(j) = \pi(j)$ for any $j = 1, 2, \ldots, n$.
- Step 2. At time $i \geqslant 1$, suppose that
$$\tilde{\pi}^{(n-i+1)} = \tilde{\pi}^{(n-i+1)}(1)\tilde{\pi}^{(n-i+1)}(2) \cdots \tilde{\pi}^{(n-i+1)}(n-i+1)$$
is determined. Set $b_{n-i+1} = \tilde{\pi}^{(n-i+1)}(n-i+1)$. For each $j = 1, 2, \ldots, n-i$, if $\tilde{\pi}^{(n-i+1)}(j) > b_{n-i+1}$, then let $\tilde{\pi}^{(n-i)}(j) = \tilde{\pi}^{(n-i+1)}(j) - 1$; otherwise, let $\tilde{\pi}^{(n-i)}(j) = \tilde{\pi}^{(n-i+1)}(j)$. We have
$$\tilde{\pi}^{(n-i)} = \tilde{\pi}^{(n-i)}(1)\tilde{\pi}^{(n-i)}(2) \cdots \tilde{\pi}^{(n-i)}(n-i) \in \mathfrak{S}_{n-i}.$$

Iterating Step 2 until $i = n$, we obtain a sequence $b_1, b_2, \ldots, b_n$. Let $c(\pi(i)) = c_i$ for each i. We obtain a $2 \times n$ array $A = \begin{pmatrix} c_1 & c_2 & \ldots & c_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$. It is clear that $b_i = 1$ if and only if $i \in \mathrm{Lrm}\,(\pi)$. Hence, $A \in \mathcal{A}_{n,k}$. For example, let us consider the array
$$B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 1 & 1 & 2 \\ 3 & 2 & 4 & 1 & 6 & 5 \end{pmatrix} \in \mathcal{B}_{6,3}.$$

By **Algorithm B***, we have

$\tilde{\pi}^{(6)}$	3	2	4	1	6	5	$b_6 = 5$
$\tilde{\pi}^{(5)}$	3	2	4	1	5		$b_5 = 5$
$\tilde{\pi}^{(4)}$	3	2	4	1			$b_4 = 1$
$\tilde{\pi}^{(3)}$	2	1	3				$b_3 = 3$
$\tilde{\pi}^{(2)}$	2	1					$b_2 = 1$
$\tilde{\pi}^{(1)}$	1						$b_1 = 1$

,

and so
$$A = \begin{pmatrix} 1 & 1 & 3 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 & 5 & 5 \end{pmatrix} \in \mathcal{A}_{6,3}.$$

Define a map $\theta_n : \mathcal{B}_{n,k} \mapsto \mathcal{A}_{n,k}$ by letting $\theta_n(B) = A$ for any $B \in \mathcal{B}_{n,k}$. We claim that
$$\theta_n(\alpha_n(A)) = A$$
for any $A \in \mathcal{A}_{n,k}$. We prove this claim by induction on n. Clearly,
$$\theta_1\left(\alpha_1\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)\right) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Fix an array $A = \begin{pmatrix} c_1 & c_2 & \ldots & c_n & c_{n+1} \\ b_1 & b_2 & \ldots & b_n & b_{n+1} \end{pmatrix} \in \mathcal{A}_{n+1,k}$. Suppose that
$$B = \alpha_{n+1}(A) = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} c(\pi(1)) & c(\pi(2)) & \ldots & c(\pi(n)) & c(\pi(n+1)) \\ \pi(1) & \pi(2) & \ldots & \pi(n) & \pi(n+1) \end{pmatrix}$$

and

$$A' = \theta_{n+1}(B) = \begin{pmatrix} c_1 & c_2 & \dots & c_n & c_{n+1} \\ b'_1 & b'_2 & \dots & b'_n & b'_{n+1} \end{pmatrix}.$$

For the array A, suppose that

$$\pi^{(n)} = \pi^{(n)}(1)\pi^{(n)}(2)\cdots\pi^{(n)}(n)$$

is the permutation obtained by **Algorithm B** at time n, and for the array B suppose that

$$\tilde{\pi}^{(n)} = \tilde{\pi}^{(n)}(1)\tilde{\pi}^{(n)}(2)\dots,\tilde{\pi}^{(n)}(n)$$

is the permutation obtained by **Algorithm B*** at time 2. Note that $\pi(n+1) = b_{n+1}$ at the time $n+1$ of **Algorithm B** and $b'_{n+1} = \pi(n+1)$ at the time 1 of **Algorithm B***. So, we have $b_{n+1} = b'_{n+1}$ and $\tilde{\pi}^{(n)} = \pi^{(n)}$. Furthermore, let

$$A^{(n)} = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \mathcal{A}_{n,k}$$

and

$$\begin{pmatrix} \mathbf{c}^{(n)} \\ \pi^{(n)} \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ \pi^{(n)}(1) & \pi^{(n)}(2) & \dots & \pi^{(n)}(n) \end{pmatrix}.$$

Algorithm B tells us that $\alpha_n(A^{(n)}) = \begin{pmatrix} \mathbf{c}^{(n)} \\ \pi^{(n)} \end{pmatrix} = \begin{pmatrix} \mathbf{c}^{(n)} \\ \tilde{\pi}^{(n)} \end{pmatrix}$. **Algorithm B*** tells us that

$$\theta_n\left(\begin{pmatrix} \mathbf{c}^{(n)} \\ \tilde{\pi}^{(n)} \end{pmatrix}\right) = \begin{pmatrix} c_1 & c_2 & \dots & c_n \\ b'_1 & b'_2 & \dots & b'_n \end{pmatrix}.$$

By the induction hypothesis, we have $\theta_n(\alpha_n(A^{(n)})) = A^{(n)}$. Hence,

$$\theta_{n+1}(\alpha_{n+1}(A)) = A,$$

which implies that $\alpha_n : \mathcal{A}_{n,k} \mapsto \mathcal{B}_{n,k}$ is a bijection. For any $A \in \mathcal{A}_{n,k}$, by using **Algorithm B**, we immediately get $\mathrm{Asc}\,(A) = \mathrm{Asc}\,(\alpha_n(A))$. $\square$

Let $B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} \in \mathcal{B}_{n,k}$. Note that $1 \in \mathrm{Lrm}\,(\pi)$. Suppose that $\mathrm{Lrm}\,(\pi) = \{k_1, k_2, \dots, k_r\}$ with $1 = k_1 < k_2 < \dots < k_r$ and let $k_{r+1} = n+1$. For each $i = 1, 2, \dots, r$, let τ_i be the subsequence $\pi(k_i)\pi(k_i+1)\dots\pi(k_{i+1}-1)$ of π. We call the sequence

$$\tau_1, \tau_2, \dots, \tau_r$$

the left-to-right minimum decomposition of π and the subsequence τ_i the i-th block of π. For convenience, let $l_i = k_{i+1} - k_i$, which is called the length of τ_i, and let $\tau_{i,j} = \pi(k_i + j - 1)$ for $i = 1, 2, \dots, r$ and $j = 1, 2, \dots l_i$. Thus, we have $\tau_i = (\tau_{i,1}, \tau_{i,2}, \dots, \tau_{i,l_i})$. We now define a total order in the set $\left\{\begin{pmatrix} c(\tau_{i,j}) \\ \tau_{i,j} \end{pmatrix} \mid j = 1, 2, \dots, l_i\right\}$: for any j_1, j_2 with $j_1 \neq j_2$, we say

$j_1 \prec j_2$ if the indices j_1 and j_2 satisfy either (i) $c(\tau_{i,j_1}) < c(\tau_{i,j_2})$ or (ii) $c(\tau_{i,j_1}) = c(\tau_{i,j_2})$ and $\tau_{i,j_1} < \tau_{i,j_2}$. Let ϕ_i be an increasing bijection from the set $\left\{ \begin{pmatrix} c(\tau_{i,j}) \\ \tau_{i,j} \end{pmatrix} \mid j = 1, 2, \ldots, l_i \right\}$ to the set $\{\tau_{i,j} \mid j = 1, 2, \ldots, l_i\}$, i.e.,

$$\phi_i \left(\begin{pmatrix} c(\tau_{i,j_1}) \\ \tau_{i,j_1} \end{pmatrix} \right) < \phi_i \left(\begin{pmatrix} c(\tau_{i,j_2}) \\ \tau_{i,j_2} \end{pmatrix} \right)$$

if $\begin{pmatrix} c(\tau_{i,j_1}) \\ \tau_{i,j_1} \end{pmatrix} \prec \begin{pmatrix} c(\tau_{i,j_2}) \\ \tau_{i,j_2} \end{pmatrix}$ for any $j_1 \neq j_2$. For any $j = 1, 2, \ldots, l_i$, let

$$\tilde{\tau}_{i,j} \quad = \quad \phi_i \left(\begin{pmatrix} c(\tau_{i,j}) \\ \tau_{i,j} \end{pmatrix} \right).$$

Then we get a sequence $\tilde{\tau}_i$ as follows:

$$\tilde{\tau}_i = (\tilde{\tau}_{i,1}, \tilde{\tau}_{i,2}, \ldots, \tilde{\tau}_{i,l_i}).$$

Since $\tau_{i,1}$ is a left-to-right minimum of π, we have $c(\tau_{i,1}) = 1$, and so

$$\tilde{\tau}_{i,1} = \phi_i \left(\begin{pmatrix} c(\tau_{i,1}) \\ \tau_{i,1} \end{pmatrix} \right) = \tau_{i,1}.$$

Moreover, $\tilde{\tau}_{i,j} < \tilde{\tau}_{i,j+1}$ if and only if the index j satisfies either (I) $c(\tau_{i,j}) < c(\tau_{i,j+1})$ or (II) $c(\tau_{i,j}) = c(\tau_{i,j+1})$ and $\tau_{i,j} < \tau_{i,j}$. Finally, let

$$\tau = \tilde{\tau}_{1,1} \cdots \tilde{\tau}_{1,l_1} \tilde{\tau}_{2,1} \cdots \tilde{\tau}_{2,l_2} \cdots \tilde{\tau}_{r,1} \cdots \tilde{\tau}_{r,l_r} \in \mathfrak{S}_n$$

and

$$\tilde{B} = \begin{pmatrix} \mathbf{c}' \\ \tau \end{pmatrix} = \begin{pmatrix} c(\tilde{\tau}_{1,1}) & \ldots & c(\tilde{\tau}_{1,l_1}) & c(\tilde{\tau}_{2,1}) & \ldots & c(\tilde{\tau}_{2,l_2}) & \ldots & c(\tilde{\tau}_{r,1}) & \ldots & c(\tilde{\tau}_{r,l_r}) \\ \tilde{\tau}_{1,1} & \ldots & \tilde{\tau}_{1,l_1} & \tilde{\tau}_{2,1} & \ldots & \tilde{\tau}_{2,l_2} & \ldots & \tilde{\tau}_{r,1} & \ldots & \tilde{\tau}_{r,l_r} \end{pmatrix}.$$

We immediately get the following lemma.

Lemma 2.1. For any $B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} \in \mathcal{B}_{n,k}$, let $\tilde{B} = \begin{pmatrix} \mathbf{c}' \\ \tau \end{pmatrix}$. Then $\tilde{B} \in \mathcal{B}_{n,k}$, $\mathrm{Lrm}\,(\pi) = \mathrm{Lrm}\,(\tau)$, $\mathrm{Asc}\,(B) = \mathrm{Asc}\,(\tau)$ and $c'(i) = c(i)$ for any $i = 1, 2, \ldots, n$.

Example 9. Consider

$$B = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 9 & 12 & 11 & 5 & 7 & 15 & 10 & 16 & 17 & 14 & 1 & 8 & 4 & 2 & 3 & 6 & 13 \end{pmatrix}.$$

The left-to-right minimum decomposition of π is

$$\tau_1 = 9, 12, 11; \tau_2 = 5, 7, 15, 10, 16, 17, 14; \tau_3 = 1, 8, 4, 2, 3, 6, 13.$$

For the block τ_1, we have $\binom{1}{9} \prec \binom{1}{12} \prec \binom{2}{11}$ and list the increasing bijection ϕ_1 in the following table:

j	1	2	3
$\binom{c(\tau_{1,j})}{\tau_{1,j}}$	$\binom{1}{9}$	$\binom{1}{12}$	$\binom{2}{11}$
$\phi_1(\binom{c(\tau_{1,j})}{\tau_{1,j}})$	9	11	12

Table 1. The increasing bijection ϕ_1.

Hence,

$$\tilde{\tau}_1 = (9, 11, 12).$$

For the block τ_2, we have $\binom{1}{5} \prec \binom{1}{10} \prec \binom{1}{16} \prec \binom{2}{7} \prec \binom{2}{14} \prec \binom{2}{15} \prec \binom{2}{17}$ and list the increasing bijection ϕ_2 in the following table:

j	1	4	5	2	7	3	6
$\binom{c(\tau_{2,j})}{\tau_{2,j}}$	$\binom{1}{5}$	$\binom{1}{10}$	$\binom{1}{16}$	$\binom{2}{7}$	$\binom{2}{14}$	$\binom{2}{15}$	$\binom{2}{17}$
$\phi_2\left(\binom{c(\tau_{2,j})}{\tau_{2,j}}\right)$	5	7	10	14	15	16	17

Table 2. The increasing bijection ϕ_2.

Hence,

$$\tilde{\tau}_2 = (5, 14, 16, 7, 10, 17, 15).$$

For the block τ_3, we have $\binom{1}{1} \prec \binom{1}{2} \prec \binom{1}{3} \prec \binom{1}{8} \prec \binom{1}{13} \prec \binom{2}{4} \prec \binom{2}{6}$ and list the increasing bijection ϕ_3 in the following table:

j	1	4	5	2	7	3	6
$\binom{c(\tau_{3,j})}{\tau_{3,j}}$	$\binom{1}{1}$	$\binom{1}{2}$	$\binom{1}{3}$	$\binom{1}{8}$	$\binom{1}{13}$	$\binom{2}{4}$	$\binom{2}{6}$
$\phi_3\left(\binom{c(\tau_{1,j})}{\tau_{3,j}}\right)$	1	2	3	4	6	8	13

Table 3. The increasing bijection ϕ_3.

Hence,

$$\tilde{\tau}_3 = (1, 4, 8, 2, 3, 13, 6).$$

So $\tau = 9, 11, 12, 5, 14, 16, 7, 10, 17, 15, 1, 4, 8, 2, 3, 13, 6$ and

$$\tilde{B} = \binom{\mathbf{c}}{\tau} = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 2 \\ 9 & 11 & 12 & 5 & 14 & 16 & 7 & 10 & 17 & 15 & 1 & 4 & 8 & 2 & 3 & 13 & 6 \end{pmatrix}.$$

Define a map $\beta_n : \mathcal{B}_{n,k} \mapsto \mathcal{B}_{n,k}$ by letting $\beta_n(B) = \tilde{B}$ for any $B \in \mathcal{B}_{n,k}$.

Lemma 10. *The map β_n is a bijection from $\mathcal{B}_{n,k}$ to itself.*

Proof. We establish the inverse ψ_n of β_n as follows. For any

$$B = \binom{\mathbf{c}}{\tau} = \begin{pmatrix} c(\tau(1)) & c(\tau(2)) & \dots & c(\tau(n)) \\ \tau(1) & \tau(2) & \dots & \tau(n) \end{pmatrix} \in \mathcal{B}_{n,k},$$

let $\tau_1, \tau_2, \dots, \tau_r$ be the left-to-right minimum decomposition of τ. Suppose that

$$\tau_i = (\tau_{i,1}, \tau_{i,2}, \dots, \tau_{i,l_i}).$$

where l_i is the length of the subsequence τ_i. Recall that ϕ_i is the increasing bijection from from the set $\left\{ \binom{c(\tau_{i,j})}{\tau_{i,j}} \mid j = 1, 2, \dots, l_i \right\}$ to the set $\{\tau_{i,j} \mid j = 1, 2, \dots, l_i\}$. Let $\binom{c(\tilde{\pi}_{i,j})}{\tilde{\pi}_{i,j}} = \phi_i^{-1}(\tau_{i,j})$ for each $j = 1, 2, \dots, l_i$ and

$$\psi_n(B) = \binom{\mathbf{c}'}{\pi} = \begin{pmatrix} c(\tilde{\pi}_{1,1}) & \dots & c(\tilde{\pi}_{1,l_1}) & c(\tilde{\pi}_{2,1}) & \dots & c(\tilde{\pi}_{2,l_2}) & \dots & c(\tilde{\pi}_{r,1}) & \dots & c(\tilde{\tau}_{r,l_r}) \\ \tilde{\pi}_{1,1} & \dots & \tilde{\pi}_{1,l_1} & \tilde{\pi}_{2,1} & \dots & \tilde{\pi}_{2,l_2} & \dots & \tilde{\pi}_{r,1} & \dots & \tilde{\pi}_{r,l_r} \end{pmatrix}.$$

For example, let us consider the array

$$B = \binom{\mathbf{c}}{\tau} = \begin{pmatrix} 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 2 \\ 9 & 11 & 12 & 5 & 14 & 16 & 7 & 10 & 17 & 15 & 1 & 4 & 8 & 2 & 3 & 13 & 6 \end{pmatrix} \in \mathcal{B}_{17,2}$$

The left-to-right minimum decomposition of τ is

$$\tau_1 = 9, 11, 12; \tau_2 = 5, 14, 16, 7, 10, 17, 15; \tau_3 = 1, 4, 8, 2, 3, 13, 6.$$

For the block τ_1, the increasing bijection ϕ_1 is listed in Table 1. Hence, we have

$\tau_{1,j}$	9	11	12
$\binom{c(\tilde{\pi}_{1,j})}{\tilde{\pi}_{1,j}} = \phi_1^{-1}(\tau_{1,j})$	$\binom{1}{9}$	$\binom{1}{12}$	$\binom{2}{11}$

.

For the block τ_2, the increasing bijection ϕ_2 is listed in Table 2. Hence, we have

$\tau_{2,j}$	5	14	16	7	10	17	15
$\binom{c(\tilde{\pi}_{2,j})}{\tilde{\pi}_{2,j}} = \phi_2^{-1}(\tau_{2,j})$	$\binom{1}{5}$	$\binom{2}{7}$	$\binom{2}{15}$	$\binom{1}{10}$	$\binom{1}{16}$	$\binom{2}{17}$	$\binom{2}{14}$

.

For the block τ_3, the increasing bijection ϕ_3 is listed in Table 3. Hence, we have

$\tau_{3,j}$	1	4	8	2	3	13	6
$\binom{c(\tilde{\pi}_{3,j})}{\tilde{\pi}_{3,j}} = \phi_3^{-1}(\tau_{3,j})$	$\binom{1}{1}$	$\binom{1}{8}$	$\binom{2}{4}$	$\binom{1}{2}$	$\binom{1}{3}$	$\binom{2}{6}$	$\binom{1}{13}$

.

Thus,

$$\psi_{17}(B) = \begin{pmatrix} 1 & 1 & 2 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 9 & 12 & 11 & 5 & 7 & 15 & 10 & 16 & 17 & 14 & 1 & 8 & 4 & 2 & 3 & 6 & 13 \end{pmatrix}. \quad \square$$

The proof of (2): By Lemma 5 and Lemma 8, we have

$$\sum_{\mathbf{e}\in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})} = \sum_{A\in\mathcal{A}_{n,k}} x^{\mathrm{asc}\,(A)} = \sum_{B\in\mathcal{B}_{n,k}} x^{\mathrm{asc}\,(B)}.$$

By Lemma 2.1 and Lemma 10, we obtain

$$\sum_{B\in\mathcal{B}_{n,k}} x^{\mathrm{asc}\,(B)} = \sum_{\binom{\mathbf{c}}{\pi}\in\mathcal{B}_{n,k}} x^{\mathrm{asc}\,(\pi)} = \sum_{\pi\in\mathfrak{S}_n} x^{\mathrm{asc}\,(\pi)} k^{n-\mathrm{lrmin}(\pi)}.$$

3 k-inversion sequences and k-Stirling permutations

A *k-Stirling permutation* of order n is a permutation of the multiset

$$\{\underbrace{1,\ldots,1}_{k},\underbrace{2,\ldots,2}_{k},\ldots,\underbrace{n,\ldots,n}_{k}\}$$

such that for each i, $1 \leqslant i \leqslant kn$, all entries between the two occurrences of i are larger than i. Let $\mathcal{Q}_n^{(k)}$ be the set of k-Stirling permutations of order n and write an element σ in $\mathcal{Q}_n^{(k)}$ as $\sigma = \sigma_1\sigma_2\cdots\sigma_{kn}$. For example, $\mathcal{Q}_2^{(2)} = \{1122, 1221, 2211\}$. It is easy to obtain that the cardinality $|\mathcal{Q}_n^{(k)}|$ of $\mathcal{Q}_n^{(k)}$ is $\prod_{i=1}^{n}((i-1)k+1)$, and so $|\mathcal{Q}_n^{(k)}| = |I_{n,k}|$.

Let $\sigma = \sigma_1\sigma_2\cdots\sigma_{kn} \in \mathcal{Q}_n^{(k)}$. For any integer $a \in [n]$, we say that a is an *ascent plateau* of σ if there is an index $i \in \{1, 2, \ldots, (n-1)k\}$ such that

$$\sigma_{i-1} < \sigma_i = \sigma_{i+1} = \ldots = \sigma_{i+k-1} = a.$$

Let $AP(\sigma)$ be the set of all ascent plateaus of σ and $ap(\sigma) = |AP(\sigma)|$.

In [8], Ma and Yeh provided a constructive proof that the number of ascent plateaus of 2-Stirling permutations of order n is equidistributed with a weighted variant of the number of excedances in permutations of length n, where the weight is $2^{n-\mathrm{cyc}\,(\pi)}$. Very recently, Duh et al. [5, Lemma 8] established a bijection between 2-colored permutations and Stirling permutations. Expanding [5, Lemma 8] and combining Lemma 5 and Lemma 8, in this section, we will present a bijective proof that the ascent plateau number over k-Stirling permutations of order n is equidistributed with the ascent number over k-inversion sequences of length n.

Given a $\sigma = \sigma_1\sigma_2\cdots\sigma_{kn} \in \mathcal{Q}_n^{(k)}$, a *left-to-right minimum* in σ is an index $i \in [kn-1]$ such that $\sigma(i) < \sigma(j)$ for any $j < i$ or $i = 1$. Denote by $\mathrm{Lrm}\,(\sigma)$ the set of left to right minimums of σ and $\mathrm{Lrm}^*(\sigma) = \{\sigma_i \mid i \in \mathrm{Lrm}\,(\sigma)\}$. A block of σ is a substring which begins with a left-to-right minimum, and contains exactly this one left-to-right minimum; moreover, the substring is maximal, i.e., not contained in any larger such substring. It is easily derived by induction that any k-Stirling permutation has a unique decomposition as a sequence of blocks.

Example 11. Consider the 3-stirling permutation $\sigma = 334666443225552777111 \in \mathcal{Q}_7^{(3)}$. We have $Lrm(\sigma) = \{1, 10, 19\}$, $Lrm^*(\sigma) = \{3, 2, 1\}$, $AP(\sigma) = \{6, 5, 7\}$. The block decomposition of σ is [334666443][225552777][111].

Lemma 12. *There is a bijection $\Phi = \Phi_n^{(k)}$ from $\mathcal{Q}_n^{(k)}$ to $\mathcal{B}_{n,k}$ such that*

$$\mathrm{Lrm}^*(\sigma) = \mathrm{Lrm}^*(\pi) \text{ and } AP(\sigma) = Asc^*(\pi),$$

where the permutation π satisfies $\Phi(\sigma) = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix}$, $\mathrm{Lrm}^*(\pi) = \{\pi(i) \mid i \in \mathrm{Lrm}(\pi)\}$ and $Asc^*(\pi) = \{\pi(i+1) \mid i \in \mathrm{Asc}(\pi)\}$.

Proof. We construct the bijection $\Phi = \Phi_n^{(k)}$ from $\mathcal{Q}_n^{(k)}$ to $\mathcal{B}_{n,k}$ as follows.

When $n = 1$, let $\Phi_1^{(k)}(\underbrace{1 \cdots 1}_{k}) = 1$. Fix $n \geqslant 2$, and assume that $\Phi_{n-1}^{(k)}$ is the bijection between $\mathcal{Q}_{n-1}^{(k)}$ to $\mathcal{B}_{n-1,k}$. Let $\sigma' \in \mathcal{Q}_n^{(k)}$ be obtained from some $\sigma \in \mathcal{Q}_{n-1}^{(k)}$ by inserting the substring $\underbrace{nn \cdots n}_{k}$ into σ. By the assumption, we have

$$\Phi_{n-1}^{(k)}(\sigma) = \begin{pmatrix} \mathbf{c} \\ \pi \end{pmatrix} \in \mathcal{B}_{n-1,k},\ Lrm^*(\sigma) = Lrm^*(\pi) \text{ and } AP(\sigma) = Asc^*(\pi).$$

If $\underbrace{nn \cdots n}_{k}$ is placed at the front of σ, that is, $\sigma' = \underbrace{nn \cdots n}_{k}\sigma$ then we let $\Phi_n^{(k)}(\sigma') = \begin{pmatrix} 1 & \mathbf{c} \\ n & \pi \end{pmatrix}$. Note that $\mathrm{Lrm}^*(\sigma') = \{n\} \cup \mathrm{Lrm}^*(\sigma), AP(\sigma') = AP(\sigma)$ and

$$\mathrm{Lrm}^*(n\pi) = \{n\} \cup \mathrm{Lrm}^*(\pi), Asc^*(n\pi) = Asc^*(\pi).$$

In this case, we have $Lrm^*(\sigma') = Lrm^*(n\pi)$ and $AP(\sigma') = Asc^*(n\pi)$.

Otherwise, suppose σ' is obtained from by inserting $\underbrace{nn \cdots n}_{k}$ into the i-th block of σ. Let $m \in \mathrm{Lrm}(\sigma)$ be the left-to-right minimum contained in the i-th block of σ. There are three possible cases. We construct $\Phi_n^{(k)}(\sigma') = \begin{pmatrix} \mathbf{c}' \\ \pi' \end{pmatrix}$ as follows:

1. If $\underbrace{nn \cdots n}_{k}$ is inserted at the end of the i-th block, then let π' be obtained by inserting the integer n at the end of the i-th block of π and

$$c'(j) = \begin{cases} c(j) & \text{if } j \neq n \\ k & \text{if } j = n \end{cases}.$$

Note that $\mathrm{Lrm}^*(\sigma') = \mathrm{Lrm}^*(\sigma), AP(\sigma') = \{n\} \cup AP(\sigma)$ and

$$\mathrm{Lrm}^*(\pi') = \mathrm{Lrm}(\pi), Asc^*(\pi') = \{n\} \cup Asc^*(\pi).$$

In this case, we have $\mathrm{Lrm}^*(\sigma') = \mathrm{Lrm}^*(\pi')$ and $AP(\sigma') = Asc^*(\pi')$.

2. If $\underbrace{nn\cdots n}_{k}$ is inserted immediately before the p-th σ_m for some $p = 2, 3, \ldots, k$, then let π' be obtained by inserting the integer n at the end of the i-th block of π and

$$c'(j) = \begin{cases} c(j) & \text{if} \quad j \neq n \\ k+1-p & \text{if} \quad j = n \end{cases}.$$

Note that $\mathrm{Lrm}^*(\sigma') = \mathrm{Lrm}^*(\sigma)$, $AP(\sigma') = \{n\} \cup AP(\sigma)$ and

$$\mathrm{Lrm}^*(\pi') = \mathrm{Lrm}(\pi), Asc^*(\pi') = \{n\} \cup Asc^*(\pi).$$

In this case, we have $\mathrm{Lrm}^*(\sigma') = \mathrm{Lrm}^*(\pi')$ and $AP(\sigma') = Asc^*(\pi')$.

3. If $\underbrace{nn\cdots n}_{k}$ is inserted immediately before the p-th integer b, $b \neq \sigma_m$, for some $p = 1, 2, \ldots, k$, then let π' be obtained by inserting n into the i-th block of π such that n is immediately before b and

$$c'(j) = \begin{cases} c(j) & \text{if} \quad j \neq n \\ p & \text{if} \quad j = n \end{cases}.$$

Note that $\mathrm{Lrm}^*(\sigma') = \mathrm{Lrm}^*(\sigma)$ and $\mathrm{Lrm}^*(\pi') = \mathrm{Lrm}^*(\pi)$. When $b \in AP(\sigma)$, we have $AP(\sigma') = (AP(\sigma) \setminus \{b\}) \cup \{n\}$ and $Asc^*(\pi') = (Asc^*(\pi) \setminus b) \cup \{n\}$; When $b \notin AP(\sigma)$, we have $AP(\sigma') = AP(\sigma) \cup \{n\}$ and $Asc^*(\pi') = Asc^*(\pi) \cup \{n\}$. In this case, we have $\mathrm{Lrm}^*(\sigma') = \mathrm{Lrm}^*(\pi')$ and $AP(\sigma') = Asc^*(\pi')$.

The above argument shows that $\Phi_n^{(k)} \in \mathcal{B}_{n,k}$, and that $\Phi_n^{(k)}$ is injective from $\mathcal{Q}_n^{(k)}$ to $\mathcal{B}_{n,k}$. Lemmas 5 and 8 tells us that $|I_{n,k}| = |\mathcal{B}_{n,k}|$, and so the cardinality of $\mathcal{Q}_n^{(k)}$ is the same as that of $\mathcal{B}_{n,k}$. Thus, $\Phi_n^{(k)}$ must be a bijection between $\mathcal{Q}_n^{(k)}$ and $\mathcal{B}_{n,k}$. By induction, we see that $\Phi_n^{(k)}$ is the desired bijection between k-Stirling permutations and block k-colored permutations. □

Example 13. Consider $\sigma = 226662555133444311 \in \mathcal{Q}_6^{(3)}$. The correspondence between σ and $\Phi_6^{(3)}(\sigma) = \begin{pmatrix} 1 & 3 & 1 & 1 & 3 & 2 \\ 2 & 5 & 6 & 1 & 4 & 3 \end{pmatrix}$ is built up as follows:

$$\begin{aligned}
111 &\Leftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
222111 &\Leftrightarrow \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \\
222133311 &\Leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \\
222133444311 &\Leftrightarrow \begin{pmatrix} 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix} \\
222555133444311 &\Leftrightarrow \begin{pmatrix} 1 & 3 & 1 & 3 & 2 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix} \\
226662555133444311 &\Leftrightarrow \begin{pmatrix} 1 & 3 & 1 & 1 & 3 & 2 \\ 2 & 5 & 6 & 1 & 4 & 3 \end{pmatrix}
\end{aligned}$$

Theorem 14. *$\Psi_n = \varphi_n^{-1} \circ \alpha_n^{-1} \circ \beta_n^{-1} \circ \Phi_n^{(k)}$ is a bijection from $\mathcal{Q}_n^{(k)}$ to $I_{n,k}$ such that $ap(\sigma) = \mathrm{asc}\,(\Psi_n(\sigma))$.*

Proof. Combining Lemmas 5, 8, 2.1, 10 and 12, we have Ψ_n is a bijection from $\mathcal{Q}_n^{(k)}$ to $I_{n,k}$ such that $ap(\sigma) = \mathrm{asc}\,(\Psi_n(\sigma))$. □

Corollary 15. *For any $n \geqslant 1$ and $k \geqslant 1$, we have $\sum_{\sigma \in \mathcal{Q}_n^{(k)}} x^{ap(\sigma)} = \sum_{\mathbf{e} \in I_{n,k}} x^{\mathrm{asc}\,(\mathbf{e})}$.*

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