Flow polynomials of a signed graph

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Submitted: Oct 5, 2018; Accepted: Aug 9, 2019; Published: Aug 30, 2019

Abstract

For a signed graph G and non-negative integer d, it was shown by DeVos et al. that there exists a polynomial $F_d(G,x)$ such that the number of the nowhere-zero Γ-flows in G equals $F_d(G,x)$ evaluated at k for every Abelian group Γ of order k with $\epsilon(\Gamma)=d$, where $\epsilon(\Gamma)$ is the largest integer d for which Γ has a subgroup isomorphic to $\mathbb{Z}_2^d$. We define a class of particular directed circuits in G, namely the fundamental directed circuits, and show that all Γ-flows (not necessarily nowhere-zero) in G can be generated by these circuits. It turns out that all Γ-flows in G can be evenly partitioned into $2^{\epsilon(\Gamma)}$ classes specified by the elements of order 2 in Γ, each class of which consists of the same number of flows depending only on the order of Γ. Using an extension of Whitney's broken circuit theorem of Dohmen and Trinks, we give a combinatorial interpretation of the coefficients in $F_d(G,x)$ for $d=0$ in terms of broken bonds. Finally, we show that the sets of edges in a signed graph that contain no broken bond form a homogeneous simplicial complex.

Mathematics Subject Classifications: 05C21, 05C22, 05C31

1 Introduction

Nowhere-zero $\mathbb{Z}_k$-flows, or modular k-flows, in a graph were initially introduced by Tutte [17] as a dual problem to vertex-colouring of plane graphs. It has long been known that the number of nowhere-zero $\mathbb{Z}_k$-flows, or, more generally, nowhere-zero Γ-flows (flows with values in Γ) for an Abelian group Γ of order k is a polynomial function in k, which does not depend on the algebraic structure of the group [17]. An analog of a $\mathbb{Z}_k$-flow is an integer k-flow, or k-flow for short, in which values on edges are integers strictly less than k in absolute value. It is well known that a graph has a nowhere-zero k-flow if and

*Corresponding author, supported by NSFC grant No.11561058 and 11471273.

only if it has a nowhere-zero $\mathbb{Z}_k$-flow [16]. In [14], Kochol showed that the number of nowhere-zero k-flows is also a polynomial in k, although not the same polynomial as that for nowhere-zero $\mathbb{Z}_k$-flows.

The notion of a signed graph was introduced by Harary [11], initially as a model for social networks. In a similar way to flows in plane graphs, or more generally in graphs embedded in an orientable surface, the definition of $\mathbb{Z}_k$-flows in signed graphs is naturally considered for the study of graphs embedded in a non-orientable surface, where nowhere-zero $\mathbb{Z}_k$-flows emerge as the dual notion to local tensions [2, 13].

In contrast to ordinary graphs, the problem of counting nowhere-zero flows in a signed graph seems more complicated and there are relatively few results to be found in the literature. Applying the theory of counting lattice points in inside-out polytopes to signed graphs, Beck and Zaslavsky [1] showed that the number of nowhere-zero k-flows in a signed graph is a quasi-polynomial of period two, that is, given by a pair of polynomials, one for odd values of k and the other for even k. In the same paper, Beck and Zaslavsky also showed that there exists a polynomial $f(G, x)$ such that, for every odd integer k, the number of nowhere-zero Γ-flows in a signed graph G equals $f(G, x)$ evaluated at k for every Abelian group Γ with $|\Gamma| = k$. This result was recently extended by DeVos, Rollová and Šámal [6] to a general Abelian group: for any non-negative integer d, there exists a polynomial $f_d(G, x)$ such that the number of nowhere-zero Γ-flows in G is exactly $f_d(G, x)$ evaluated at n for every Abelian group Γ with $\epsilon(\Gamma) = d$ and $|\Gamma| = 2^d n$, where $\epsilon(\Gamma)$ is the largest integer d for which Γ has a subgroup isomorphic to $\mathbb{Z}_2^d$. More recently, Goodall et al. [9] (available from arXiv) gave an explicit expression for $f_d(G, x)$ in the form of an edge-subset expansion.

In this paper we focus on the combinatorial structure of Γ-flows in a signed graph G and the coefficients in the polynomial $f_d(G, x)$. For convenience, instead of working on $f_d(G, x)$, we will work on the polynomial $F_d(G, x)$ defined by $F_d(G, x) = f_d(G, 2^{-d}x)$ and call $F_d(G, x)$ the *d-type flow polynomial*, or simply the *flow polynomial* of G. It can be seen that $F_d(G, x)$ evaluated at k is exactly the number of nowhere-zero Γ-flows in G for every Abelian group Γ with $\epsilon(\Gamma) = d$ and $|\Gamma| = k$.

In the third section we introduce the fundamental directed circuits and the fundamental root circuit (a particular unbalanced circuit) in a signed graph G. We show that every Γ-flow (not necessarily nowhere-zero) in G can be generated by these circuits. More specifically, the values of the flows assigned to the fundamental directed circuits are elements of Γ, while the value assigned to the fundamental root circuit is an element of order 2 in Γ. As a consequence, all Γ-flows in G can be evenly partitioned into $2^{\epsilon(\Gamma)}$ classes specified by the elements of order 2 in Γ. Moreover, each class consists of the same number of flows, which depends only on the order of the group. This gives an explanation for why the number of the Γ-flows in a signed graph varies with different $\epsilon(\Gamma)$ and, also gives an answer to a problem posed by Beck and Zaslavsky in [1]. Further, this result also yields an explicit expression of the polynomial $F_d(G, x)$ obtained earlier by Goodall et al.

In the fifth section we give a combinatorial interpretation of the coefficients in $F_d(G, x)$ for $d = 0$. To this end, we apply Whitney's broken circuit theory [18]. In the study of graph coloring, one significance of Whitney's broken circuit theorem is that it gives a very nice 'cancellation' to reduce the terms in the chromatic polynomial (represented in the form of inclusion-exclusion) until the remaining terms cannot be cancelled out any further, and also gives a combinatorial interpretation for the coefficients of the polynomial [3, 4]. Using an extended form of Whitney's theorem given by Dohmen and Trinks [7], we show that $F_0(G, x)$ is a polynomial with leading term x^{m-n} and with its coefficients alternating in sign. More specifically, the coefficient of $(-1)^i x^{m-n-i}, i = 0, 1, \cdots, m-n$, is exactly the number of the sets consisting of i edges that contain no broken bond. As an example, we give an analytic expression of $F_0(G, x)$ for a class of signed graphs that contain no balanced circuit. Finally, we show that the broken bonds in a signed graph form a homogeneous simplicial complex of top dimension $m - n$. Thus, the coefficients of $F_0(G, x)$ are the simplex counts in each dimension of the complex.

2 Preliminaries

Graphs in this paper may contain parallel edges or loops. For a graph G, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. A *signed graph* is a pair (G, E_N), where $E_N \subseteq E(G)$ and the edges in E_N are negative while the other ones are positive. In the following, we will use G simply to denote a signed graph if no confusion can occur.

A *circuit* is a connected 2-regular graph. An *unbalanced circuit* in a signed graph is a circuit that has an odd number of negative edges. A *balanced circuit* is a circuit that is not unbalanced. A signed subgraph is *unbalanced* if it contains an unbalanced circuit; otherwise, it is *balanced*. In particular, a subgraph without negative edges is balanced. A *barbell* is the union of two unbalanced circuits C_1, C_2 and a (possibly trivial) path P with end vertices $v_1 \in V(C_1)$ and $v_2 \in V(C_2)$, such that $C_1 - v_1$ is disjoint from $P \cup C_2$ and $C_2 - v_2$ is disjoint from $P \cup C_1$. We call P the *barbell path* of the barbell. A *signed circuit* is either a balanced circuit or a barbell.

Given a signed graph G, *switching* at a vertex v is the inversion of the sign of each edge incident with v. Two signed graphs are said to be *switching-equivalent* if one can be obtained from the other by a series of switchings. It is known [13, 15, 23] and easy to see that equivalent signed graphs have the same sets of unbalanced circuits and the same sets of balanced circuits. This means, in particular, that a balanced signed graph G is switching-equivalent to the underlying unsigned graph of G.

Following [2, 21, 22], we introduce the notion of *half-edges* so as to orient a signed graph G: each edge $e = uv$ of G is viewed as composed of two half-edges, denoted by (u, e) and (v, e). An *orientation* $\tau(G)$ of a signed graph G is obtained by orienting all its edges. To *orient* a half edge (v, e) we mean to assign it with a sign $\tau(v, e)$, meaning that e points toward v if $\tau(u, e) = -1$ and points away v if $\tau(u, e) = 1$. An edge $e = uv$ is oriented if both (u, e) and (v, e) are oriented subject to $\tau(u, e)\tau(v, e) = -\sigma(e)$, where σ is the *sign function* defined by $\sigma(e) = 1$ when e is positive and $\sigma(e) = -1$ otherwise. Thus, the orientation of a positive edge is in the usual way and, in an orientation of a negative edge e, both the two half-edges point toward the end vertices of e, called *extroverted*, or both point toward the inside of e, called *introverted*. For an edge $e = uv$, we also write

$\tau(u, e)$ and $\tau(v, e)$ together as $\tau(e)$ for short. For a vertex v in an orientation of a signed graph, we denote by $E^+(v)$ (resp., $E^-(v)$) the set of the half edges (v, e) with $\tau(v, e) = 1$ (resp., $\tau(v, e) = -1$).

Let $D = \tau(G)$ be an orientation of G and Γ be an additive Abelian group. A map $\mathbf{f} : E(D) \to \Gamma$ is called a Γ-*flow* if the usual conservation law (Kirchhoff's law) is satisfied, that is, for each vertex v, the sum of $\mathbf{f}(e)$ over all the half edges (v, e) in $E^+(v)$ equals that in $E^-(v)$, i.e.,

$$\sum_{(v,e)\in E^+(v)} \mathbf{f}(e) = \sum_{(v,e)\in E^-(v)} \mathbf{f}(e).$$

A flow $\mathbf{f}$ is called nowhere-zero if $\mathbf{f}(e) \neq 0$ for each $e \in E(D)$. It is straightforward to see that the number of nowhere-zero Γ-flows is independent of the orientation of G. A signed graph is said to be Γ-*flow admissible* if it admits at least one nowhere-zero Γ-flow. It is clear that the property of 'Γ-flow admissible' is invariant under switching inversion.

3 Fundamental circuits in a signed graph

In this section we generalize the notion of fundamental circuits in graphs to signed graphs, which will play an important role in revealing the structural properties of Γ-flows in signed graphs.

For a signed graph G and a set F of edges, we denote by $G + F$ and $G - F$ the subgraphs obtained from G by adding and deleting the edges in F, respectively. Let $E_N = \{e_0, e_1, e_2, \cdots, e_{m_N-1}\}$ (the set of all negative edges of G), where $m_N = |E_N|$. In this section we always assume that G is unbalanced and, with no loss of generality, contains as few negative edges as possible in its switching equivalent class. Thus, $E_N \neq \emptyset$ and $G - E_N$ is connected [20].

Let T be a spanning tree of $G - E_N$. Choose an arbitrary edge e_0 from E_N and call $T_0 = T + e_0$ a *signed rooted tree* of G with root edge e_0 (note that a signed rooted tree we defined here is not a tree in the usual sense for graphs because it has a unique unbalanced circuit). Let $\overline{T}_0 = E(G) \setminus E(T_0)$. For any $e \in \overline{T}_0$, it is clear that $T_0 + e$ contains a unique signed circuit. We call this circuit a *fundamental circuit* and denote it by C_e. We can see that, if $e \in \overline{T}_0 \setminus E_N$ then C_e is a graph circuit (a circuit without negative edge) and if $e \in E_N \setminus \{e_0\}$ then C_e is a barbell or a balanced circuit with two negative edges e_0 and e.

For a barbell consisting of two unbalanced circuits $u_1e_1u_2 \cdots u_pe_pu_1, v_1e'_1v_2 \cdots v_qe'_qv_1$ and a barbell path $u_1e''_1w_2 \cdots w_re''_rv_1$, we also write it as a closed walk [10]:

$$u_1e_1u_2 \cdots u_pe_pu_1e''_1w_2 \cdots w_re''_rv_1e'_1v_2 \cdots v_qe'_qv_1e''_rw_r \cdots w_2e''_1u_1.$$

Given a fixed orientation $D = \tau(G)$, a *fundamental directed circuit* $\overrightarrow{C}_e$ of G is the orientation τ' of a fundamental circuit $C_e = v_1e_1v_2e_2 \cdots v_ke_kv_1$ (where $e = e_1 = v_1v_2$) such that the direction of e is the same as that which it has in D and the directions of all other edges on $\overrightarrow{C}_e$ coincide consistently with e along with C_e, i.e., $\tau'(e_1) = \tau(e_1)$ and $\tau'(v_i, e_{i-1})\tau'(v_i, e_i) = -1, \tau'(v_i, e_i)\tau'(v_{i+1}, e_i) = -\sigma(e_i)$ for every $i \in \{1, 2, \cdots, k\}(\text{mod } k)$. Under this orientation, it can be seen that if C_e is an ordinary circuit then $\overrightarrow{C}_e$ is a usual directed circuit with direction coincident with D on e, and if C_e is a balanced circuit or a barbell (with two negative edges e_0 and e), then the direction of the two negative edges are always opposite, that is, e_0 is extroverted if and only if e is introverted, see Figure 1.

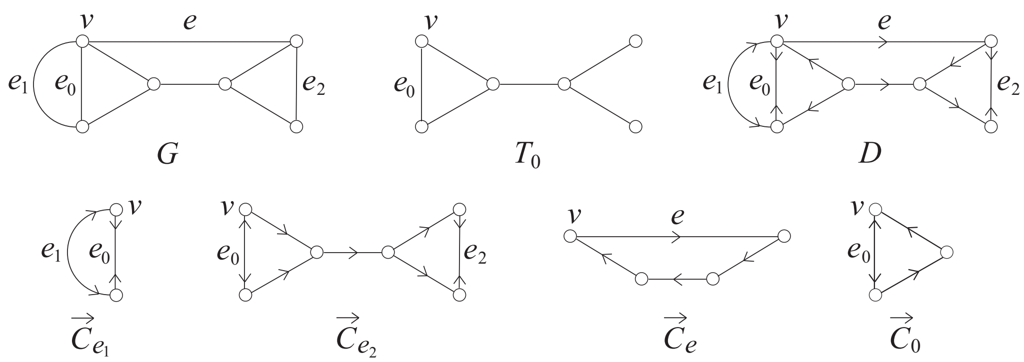

Figure 1. The edges e_0, e_1, e_2 are negative and e is positive.

For a fundamental circuit C_e, let C_e^D be the orientation D restricted to C_e. We associate with C_e a function $\mathbf{f}_e$ on $E(D)$ defined by

$$\mathbf{f}_e(a) = \begin{cases} 1, & \text{if } a \in \overrightarrow{C}_e; \\ -1, & \text{if } a \in C_e^D \setminus \overrightarrow{C}_e; \\ 2, & \text{if } a \in \overrightarrow{C}_e \text{ and } a \text{ is on the barbell path of } C_e; \\ -2, & \text{if } a \in C_e^D \setminus \overrightarrow{C}_e \text{ and } a \text{ is on the barbell path of } C_e; \\ 0, & \text{otherwise} \end{cases}$$

for any $a \in E(D)$, where 'a is on the barbell path of C_e' means that C_e is a barbell and a is an edge belonging to the barbell path of C_e.

From the above definition, it can be seen that $\mathbf{f}_e(e) = 1$ for any $e \in \overline{T}_0$.

Let C_0 be the unique (un-balanced) circuit in T_0 (i.e., formed by e_0 and T). Choose an arbitrary vertex v on C_0 and let $\overrightarrow{C}_0$ be the orientation of C_0 such that the direction of e_0 is extroverted and all other edges on C_0 are oriented so that $d^-(v) = 2, d^+(v) = 0$ and $d^-(u) = d^+(u) = 1$ for any vertex u on C_0 other than v, where $d^-(v)$ and $d^+(v)$ are the in-degree and out-degree of v on $\overrightarrow{C}_0$, respectively, see Figure 1. We call $\overrightarrow{C}_0$ the *fundamental root circuit* and associate it with a function $\mathbf{g}$ on $E(D)$ defined by

$$\mathbf{g}(e) = \begin{cases} 1, & \text{if } e \in \overrightarrow{C}_0; \\ -1, & \text{if } e \in C_0^D \setminus \overrightarrow{C}_0; \\ 0, & \text{otherwise} \end{cases}$$

for any $e \in E(D)$.

For convenience, in the following we regard each Γ-flow, each function $\mathbf{f}_e$ ($e \in \overline{T}_0$) and the function $\mathbf{g}$ as m-dimensional vectors indexed by $e \in E(G)$. Let $\mathcal{S}_G$ denote the class of all Γ-flows (not necessarily nowhere-zero) in G.

For a finite additive Abelian group Γ, let Γ_2 be the set of the elements of order 2 in Γ (including the zero element). Recalling that $\epsilon(\Gamma)$ is the largest integer d for which Γ has a subgroup isomorphic to $\mathbb{Z}_2^d$, we have $|\Gamma_2| = 2^{\epsilon(\Gamma)}$.

Theorem 1. *Let Γ be an additive Abelian group and let G be a connected unbalanced signed graph. Let T be a spanning tree of G consisting of positive edges and let $e_0 \in E_N$. Then*

$$\mathcal{S}_G = \{\gamma\mathbf{g} + \sum_{e\in\overline{T}_0} \gamma_e \mathbf{f}_e : \gamma \in \Gamma_2, \gamma_e \in \Gamma\}. \tag{1}$$

Proof. It is clear that

$$\gamma\mathbf{g} + \sum_{e\in\overline{T}_0} \gamma_e \mathbf{f}_e \tag{2}$$

is a Γ-flow for any $\gamma \in \Gamma_2$ and $\gamma_e \in \Gamma$. Let $\mathbf{f}$ be an arbitrary Γ-flow in G. We need only prove that $\mathbf{f}$ can be written as the combination (2).

Since a Γ-flow is independent of the orientation D, to simplify our discussion we make the following assumption:

Assumption 1. In orientation D, the direction of the root edge e_0 is extroverted while the directions of all other negative edges are introverted.

For each negative edge $e_i = u_i v_i \in E_N$, insert a new vertex w_i into the middle of e_i so that the two half edges of e_i in D become two ordinary directed edges $w_i u_i$ (with direction from w_i to u_i) and $w_i v_i$ if $i = 0$, or $u_i w_i$ and $v_i w_i$ if $i \in \{1, 2, \cdots, m_N - 1\}$. Further, add a new vertex w to D and add the directed edges $e_0' = ww_0$ and $e_i' = w_i w$ for every $i \in \{1, 2, \cdots, m_N - 1\}$. The resulting graph, denoted by D^w, is a directed graph without negative edges, that is, D^w is an ordinary directed graph.

Let $\mathbf{f}^w$ be the extension of the function $\mathbf{f}$ from the edges of D to the edges of D^w defined by $\mathbf{f}^w(w_0u_0) = \mathbf{f}^w(w_0v_0) = \mathbf{f}(e_0), \mathbf{f}^w(e_0') = 2\mathbf{f}(e_0)$ and $\mathbf{f}^w(u_iw_i) = \mathbf{f}^w(v_iw_i) = \mathbf{f}(e_i)$, $\mathbf{f}^w(e_i') = 2\mathbf{f}(e_i)$ for $i \in \{1, 2, \cdots, m_N - 1\}$. It is clear that, except possibly for w, the conservation law is satisfied at all the vertices in D^w and therefore must be also satisfied at w. Thus, by the conservation law at w, we have

$$\mathbf{f}^w(e_0') = \sum_{i=1}^{m_N-1} \mathbf{f}^w(e_i')$$

or equivalently,

$$2\mathbf{f}(e_0) = \sum_{i=1}^{m_N-1} 2\mathbf{f}(e_i) = 2\sum_{e_i\in E_N^*} \mathbf{f}(e_i), \tag{3}$$

where $E_N^* = E_N \setminus \{e_0\} = \{e_1, e_2, \cdots, e_{m_N-1}\}$.

Further, we note that, for any $\gamma \in \Gamma$, the solution of the equation $2x = 2\gamma$ (in x) over Γ has the form $x = \gamma + \gamma_2$, where γ_2 is an element of order 2 (possibly the zero element), i.e., $\gamma_2 \in \Gamma_2$. Thus, (3) is equivalent to

$$\mathbf{f}(e_0) = \gamma_2 + \sum_{e_i \in E_N^*} \mathbf{f}(e_i), \tag{4}$$

where $\gamma_2 \in \Gamma_2$.

On the other hand, for any $e \in E_N^*$, by Assumption 1 and the definitions of $\overrightarrow{C}_e$ and $\mathbf{f}_e$, we have

$$\mathbf{f}_e(e_0) = \mathbf{f}_e(e_i) = 1. \tag{5}$$

In (2), we set $\gamma = \gamma_2$ and for $e \in \overline{T}_0$, set $\gamma_e = \mathbf{f}(e)$. Let

$$\mathbf{f}' = \mathbf{f} - (\gamma_2 \mathbf{g} + \sum_{e \in \overline{T}_0} \gamma_e \mathbf{f}_e). \tag{6}$$

Then for any $e \in \overline{T}_0$, by the definition of the vector $\mathbf{g}$ we have $\gamma_2 \mathbf{g}(e) = 0$ since e is not on C_0. This implies that $\mathbf{f}'(e) = 0$ for any $e \in \overline{T}_0$ because $\gamma_e = \mathbf{f}(e)$ and, as mentioned earlier, $\mathbf{f}_e(e) = 1$. Further, by (4), (5) and (6) we have

$$\begin{aligned}
\mathbf{f}'(e_0) &= \mathbf{f}(e_0) - (\gamma_2 \mathbf{g}(e_0) + \sum_{e \in \overline{T}_0} \gamma_e \mathbf{f}_e(e_0)) \\
&= \gamma_2 + \sum_{e \in E_N^*} \mathbf{f}(e) - (\gamma_2 \mathbf{g}(e_0) + \sum_{e \in \overline{T}_0 \setminus E_N^*} \gamma_e \mathbf{f}_e(e_0) + \sum_{e \in E_N^*} \gamma_e \mathbf{f}_e(e_0)) \\
&= \sum_{e \in E_N^*} \mathbf{f}(e) - \sum_{e \in E_N^*} \gamma_e \mathbf{f}_e(e_0) \\
&= \sum_{e \in E_N^*} \mathbf{f}(e)(1 - \mathbf{f}_e(e_0)) \\
&= 0,
\end{aligned}$$

where the third equality holds because $\mathbf{g}(e_0) = 1$ and $e_0 \notin C_e$ for any $e \in \overline{T}_0 \setminus E_N^*$ and, therefore $\mathbf{f}_e(e_0) = 0$; and the last two equalities hold because of (5) and $\gamma_e = \mathbf{f}(e)$ for any $e \in E_N^*$.

The above discussion means that $\mathbf{f}'$ evaluated at each edge outside of T is zero. Thus, we must have $\mathbf{f}' = \mathbf{0}$ (the vector of all zeros) because the values of $\mathbf{f}'$ at the edges of T are uniquely determined by those outside T. In conclusion, $\mathbf{f}$ is represented as the combination (2), which completes our proof. □

4 Classification of Γ-flows in a signed graph

From Theorem 1, we know that all Γ-flows in a connected unbalanced signed graph can be 'generated' by fundamental root circuit $\overrightarrow{C}_0$ and the fundamental directed circuits $\overrightarrow{C}_e, e \in \overline{T}_0$. This leads to the following classification of Γ-flows in a signed graph, which are specified by the elements of order 2 in Γ.

Theorem 2. *Let Γ be an additive Abelian group of order k and let G be a connected unbalanced signed graph. Let T be a spanning tree of G consisting of positive edges and*

let $e_0 \in E_N$.
1). The flows in $\mathcal{S}_G$ are pairwise distinct and, therefore

$$|\mathcal{S}_G| = 2^{\epsilon(\Gamma)} k^{m-n}; \tag{7}$$

2). $\mathcal{S}_G$ can be evenly partitioned into $|\Gamma_2|$ classes specified by the elements in Γ_2, i.e., $\mathcal{S}_G = \bigcup_{\gamma\in\Gamma_2} \mathcal{S}_G(\gamma)$ and $|\mathcal{S}_G(\gamma)| = k^{m-n}$ for any $\gamma \in \Gamma_2$, *where*

$$\mathcal{S}_G(\gamma) = \{\gamma \mathbf{g} + \sum_{e\in\overline{T}_0} \gamma_e \mathbf{f}_e : \gamma_e \in \Gamma\}. \tag{8}$$

Proof. 1). We need only prove that

$$\gamma\mathbf{g} + \sum_{e\in\overline{T}_0} \gamma_e \mathbf{f}_e = \gamma'\mathbf{g} + \sum_{e\in\overline{T}_0} \gamma'_e \mathbf{f}_e \tag{9}$$

if and only if $\gamma = \gamma'$ and $\gamma_e = \gamma'_e$ for any $e \in \overline{T}_0$. For any $e \in \overline{T}_0$, by the definition of $\mathbf{g}$ and $\mathbf{f}_e$ we have $\mathbf{f}_e(e) = 1, \mathbf{g}(e) = 0$ and $\mathbf{f}_{e'}(e) = 0$ for any $e' \in \overline{T}_0$ with $e' \neq e$. Thus, (9) implies that $\gamma_e \mathbf{f}_e(e) = \gamma'_e \mathbf{f}_e(e)$ and, therefore $\gamma_e = \gamma'_e$ for any $e \in \overline{T}_0$. Consequently, again by (9), we have $\gamma\mathbf{g} = \gamma'\mathbf{g}$ and, therefore $\gamma = \gamma'$.
2). Since the flows in $\mathcal{S}_G$ are pairwise distinct, 2) follows directly. □

For a component ω of a signed graph G, let

$$\beta(\omega) = \begin{cases} m(\omega) - n(\omega) + 1, & \text{if } \omega \text{ is balanced;} \\ m(\omega) - n(\omega), & \text{if } \omega \text{ is unbalanced,} \end{cases} \tag{10}$$

where $m(\omega)$ and $n(\omega)$ are the number of edges and vertices in ω, respectively. In general, we let $\beta(G) = \sum \beta(\omega)$, where the sum is taken over all the components ω of G. Let $\kappa(G)$ be the number of unbalanced components and $F^*(G,\Gamma)$ be the number of Γ-flows (not necessarily nowhere-zero) in G.

Corollary 3. *Let G be a signed graph and let Γ be an additive Abelian group of order k. Then*

$$F^*(G,\Gamma) = 2^{\kappa(G)\epsilon(\Gamma)} k^{\beta(G)}. \tag{11}$$

Proof. If G is not connected then $F^*(G,\Gamma) = \prod F^*(\omega,\Gamma)$, where the product is taken over all the components ω of G. We need only consider the case when G is connected.

If G is unbalanced then (11) follows directly from (7). Now assume that G is balanced. Recall that a balanced signed graph is switching-equivalent to an ordinary graph. In this case it is known [12] that the number of Γ-flows (not necessarily nowhere-zero) in an ordinary graph is k^{m-n+1}, i.e., $F^*(G,\Gamma) = k^{m-n+1}$, where m and n are the numbers of edges and vertices in G, respectively. This agrees with (11) because $\kappa(G) = 0$ and $\beta(G) = m - n + 1$ when G is balanced. The proof is completed. □

Remark 1. When k (the order of Γ) is odd, Beck and Zaslavsky posed a problem (Problem 4.2, [1]): Is there any significance to $F^*(G,\Gamma)$ evaluated at even natural numbers? By Theorem 2 and Corollary 3 we can now give an answer to this problem. For simplicity, let's consider the case when G is connected and unbalanced. Since k is odd, we have $\epsilon(\Gamma)=0$ and therefore, $F^*(G,\Gamma)=k^{m-n}$. Thus, $F^*(G,\Gamma)$ evaluated at an even number h equals h^{m-n}, which is exactly the number of Γ'-flows in G divided by $2^{\epsilon(\Gamma')}$ for any group Γ' of order h. More specifically, by Theorem 2, $F^*(G,\Gamma)$ evaluated at h equals the number of those Γ'-flows in G which have the form

$$\mathbf{f}=\gamma\mathbf{g}+\sum_{e\in\overline{T}_0}\gamma_e\mathbf{f}_e,\quad \gamma_e\in\Gamma',$$

where γ is an arbitrary fixed element of order 2 in Γ' (in particular we may choose $\gamma=0$). $\square$

For any $e\in E(G)$, the number of Γ-flows in G with value 0 at e is clearly equal to $F^*(G-e,\Gamma)$. The polynomial $F_d(G,x)$ evaluated at $|\Gamma|$ counts the number of nowhere-zero Γ-flows. So by Corollary 3 and the principle of inclusion-exclusion, we get the following expression of $F_d(G,x)$ obtained earlier by Goodall et al.:

Corollary 4. *[9] For any signed graph G and non-negative integer d,*

$$F_d(G,x)=\sum_{F\subseteq E}(-1)^{|F|}2^{\kappa(G-F)d}x^{\beta(G-F)}.$$

We note that, if G is an ordinary graph then $\kappa(G-F)=0$ for any $F\subseteq E(G)$. Therefore, Corollary 4 generalizes the corresponding result for ordinary graphs [8, 12].

Example. By Corollary 4, if G is the graph with two vertices joined by a negative edge and a positive edge then $F_d(G,x)=2^d-1$; if G is the graph consisting of two negative loops at a vertex then $F_d(G,x)=2^dx-2^{d+1}+1$; and if G is the graph consisting of a negative loop and a positive loop at a vertex then $F_d(G,x)=(2^d-1)(x-1)$.

5 Coefficients in $F_0(G,x)$

In this section we will give a combinatorial interpretation of the coefficients in $F_d(G,x)$ for $d=0$. We begin with the following extension of Whitney's broken circuit theorem given by Dohmen and Trinks.

Lemma 5. *[7] Let P be a finite linearly ordered set, $\mathscr{B}\subseteq 2^P\setminus\{\emptyset\}$ and Γ be an additive Abelian group. Let $f:2^P\to\Gamma$ be a mapping such that, for any $B\in\mathscr{B}$ and $A\supseteq B$,*

$$f(A)=f(A\setminus\{\max B\}).\tag{12}$$

Then

$$\sum_{A\in 2^P}(-1)^{|A|}f(A)=\sum_{A\in 2^P\setminus\mathscr{B}^*}(-1)^{|A|}f(A),\tag{13}$$

where $\max B$ *is the maximum element in* B *and*

$$\mathscr{B}^* = \{A : A \in 2^P, A \supseteq B \setminus \{\max B\} \text{ for some } B \in \mathscr{B}\}.$$

We call $\mathscr{B}$ in Lemma 5 a *broken system* of f and $B \setminus \{\max B\}$, or $B \setminus \max B$ for short, a *broken set* for any $B \in \mathscr{B}$.

To apply Lemma 5 we need to define a broken system and broken sets for signed graphs. We follow the idea of the notion of 'bonds' introduced in [5, 20]. For a signed graph G and $X \subseteq V(G)$, denote by $[X, X^C]$ the set of edges between X and its complement X^C, by $G[X]$ the subgraph of G induced by X, and by $E(X)$ the set of the edges in $G[X]$. A non-empty edge subset $B \subseteq E(G)$ is called a *cut* [5] or *improving set* [20] of G if it has the form $B = [X, X^C] \cup E_X$, where $X \subseteq V(G)$ is non-empty and $E_X \subseteq E(X)$ is minimal to have $G[X] - E_X$ balanced. A cut is called a *bond* of G if it is minimal. We note that, in the case when G is balanced, we have $E_X = \emptyset$ by the minimality of E_X and, therefore a bond is exactly a usual bond as in an ordinary graph. In this sense, the notion 'bond' for signed graph is a very nice extension of that for ordinary graphs [12].

By the definition of a broken set, it is not difficult to see that if B is a bond then, for any $e \in B$,

$$\beta(G - B) = \beta(G - (B \setminus \{e\})). \tag{14}$$

On the other hand, by Corollary 4, we have

$$F_0(G, x) = \sum_{F \subseteq E} (-1)^{|F|} x^{\beta(G-F)}.$$

Thus, an edge subset of G is a broken set for $F_0(G, x)$ if it has the form $B \setminus \max B$ for some $B \subseteq E(G)$ such that, for any $A \supseteq B$,

$$\beta(G - A) = \beta(G - (A \setminus \max B)). \tag{15}$$

On the other hand, by (14), for any bond B we have

$$\beta(G - B) = \beta(G - (B \setminus \max B)).$$

Moreover, it is not difficult to see that, for any $A \supseteq B$, (15) is satisfied by A and B. Thus, $B \setminus \max B$ is a broken set for $F_0(G, x)$ for any bond B and is called a *broken bond* of G. Then by Lemma 5 we immediately have the following result.

Theorem 6. *For any signed graph* G *with a linear order* $\prec$ *on* $E(G)$,

$$F_0(G, x) = \sum_{F \in 2^{E(G)} \setminus \mathscr{B}^*} (-1)^{|F|} x^{\beta(G-F)}, \tag{16}$$

where $\mathscr{B}$ *is the class of bonds of* G *and*

$$\mathscr{B}^* = \{F : F \in 2^{E(G)}, F \supseteq B \setminus \max B \text{ for some } B \in \mathscr{B}\}.$$

Remark 2. If G is balanced, then each broken bond is exactly a usual broken bond of an ordinary graph. In this case, (16) is still valid. Thus, Theorem 6 is a generalization of that for ordinary graph [12]. Further, in a very special case when an unbalanced signed graph G contains an edge whose removal leaves a balanced graph, the empty set is a broken bond and therefore any set of edges (including the empty set) contains a broken bond. This case means that $\mathscr{B}^* = 2^{E(G)}$ and thus, $F_0(G,x) = 0$, which coincides with the obvious fact that such a G is not Γ-flow admissible when $|\Gamma|$ is odd. □

Proposition 7. *For any signed graph G and $F \subseteq E(G)$, if F contains no broken bond then each component of $G - F$ is unbalanced, unless G is balanced.*

Proof. To the contrary suppose that one component ω of $G - F$ is balanced. Let $B = [V(\omega), \overline{V(\omega)}] \cup E_F$, where E_F is the set of edges in F whose two end vertices are both in ω. Then B is a bond since ω is balanced and thus $B \setminus \max B$ is a broken bond. Notice that $B \setminus \max B \subset B \subseteq F$, which contradicts that F contains no broken bond. □

Let $\eta(G)$ be the number of those edges e such that there is an edge e' with $e \prec e'$ satisfying one of the following three conditions:
1). one of e and e' is a cut edge and $G - \{e, e'\}$ has a balanced component;
2). $\{e, e'\}$ is an edge cut and $G - \{e, e'\}$ has a balanced component;
3). $\{e, e'\}$ is contained in a component ω of G and $\omega - \{e, e'\}$ is balanced.

Corollary 8. *Let G be an unbalanced, Γ-flow admissible ($|\Gamma|$ is odd) signed graph with n vertices and m edges. Then for any linear order $\prec$ on $E(G)$,*

$$F_0(G,x) = a_0 x^{m-n} - a_1 x^{m-n-1} + a_2 x^{m-n-2} - \cdots + (-1)^{m-n} a_{m-n}, \tag{17}$$

where, for each $i \in \{0, 1, \cdots, m-n\}$, a_i is the number of edge subsets of G having i edges and containing no broken bond as a subset. In particular,
1). $a_i > 0$ for every $i = 0, 1, 2, \cdots, m-n$;
2). $a_0 = 1$;
3). $a_1 = m - \eta(G)$;

Proof. Let $F \subseteq E(G)$ be an edge subset that contains no broken bond. Since G is unbalanced, by Proposition 7 every component ω of $G - F$ is unbalanced. Thus, $\beta(\omega) = m(\omega) - n(\omega)$ due to (10). Therefore,

$$\beta(G - F) = \sum_{\omega} \beta(\omega) = m(G - F) - n(G - F) = m - n - |F|,$$

where the sum is taken over all the components of $G - F$. This equation means that the value of $\beta(G - F)$ is determined uniquely by the number of edges in F, as long as F contains no broken bond. So by Theorem 6, the coefficient of $(-1)^i x^{m-n-i}$ in $F_0(G,x)$ counts exactly those edge subsets F which have i edges and contain no broken bond. Thus, (17) follows directly.
1). We first show that there is an edge set F with n edges that contains no broken bond. By the definition of a broken bond, an edge set F contains no broken bond if and only if

$E(G)\setminus F$ contains at least one edge from each broken bond of G. Let F^* be maximum such that $E(G)\setminus F^*$ contains at least one edge from each broken bond of G (such F^* clearly exists because $E(G)\setminus\emptyset$ does). Let ω be a component of $G-F^*$. Then by Proposition 7, ω contains at least one unbalanced circuit, say C_u. We claim that ω does not contain any other circuit.

Suppose to the contrary that C is a circuit in ω with $C\neq C_u$. Since C is a circuit, the property that $G-F^*$ contains at least one edge from each broken bond is still satisfied by $G-F^*-\max C$ because any bond containing $\max C$ must contain another edge e on C with, of course, $e\prec\max C$. This contradicts our assumption that F^* is maximum. Our claim follows.

In a word, each component ω of $G-F^*$ contains exactly one unbalanced circuit and no any other circuit. This means that $m(\omega)=n(\omega)$ and, therefore $m(G-F^*)=n$, i.e., $|F^*|=m-n$. Thus, $a_{m-n}>0$. Further, if an edge subset F contains no broken bond then any subset of F contains neither broken bond, which implies $a_i>0$ for any i with $0\leqslant i\leqslant m-n$.

2). Since G is flow-admissible, as pointed out in Remark 2, G contains no edge whose removal leaves a balanced graph. This means that the empty set is not a broken bond. Thus, a_0 equals the number of the edge subsets of G having 0 edges, that is, the unique empty set.

3). Now we consider the coefficient a_1. From the above discussion we see that a_1 equals the number of the edges that are not broken bond. On the other hand, an edge e is a broken bond if there is e' such that $B=\{e,e'\}$ is a bond and $e'=\max B$. By the definition of a bond, $B=\{e,e'\}$ must satisfy one of the above three conditions and, vice versa. □

Remark 3. Corollary 8 remains a natural question: How about the case when $|\Gamma|$ is even? Indeed, by Lemma 5 and Corollary 4, if an edge subset is a broken set for $F_d(G,x)$ then it must have the form of $B\setminus\max B$ satisfying both (15) and

$$\kappa(G-A)d=\kappa(G-(A\setminus\max B))d \tag{18}$$

for any $A\supseteq B$. When $d=0$, (18) always holds and, hence the family of broken sets can be chosen to be the one consisting of all broken bonds since (15) is satisfied by every broken bond. For $d>0$, it seems not easy to find such a B that satisfy both (15) and (18), unless the signed graph G admits some particular properties as shown in the following, for an example:

For a bond $B=[X,X^C]\cup E_X$, it is not difficult to see that if $\max B\in[X,X^C]$ then $\kappa(G-A)d=\kappa(G-(A\setminus\max B))d$ for any $A\supseteq B$. In this case, $B\setminus\max B$ is also a broken set for $F_d(G,x)$ with arbitrary nonnegative integer d. This implies that if G has an edge-ordering such that $\max B\in[X,X^C]$ for every bond B then the broken bond expansion (16) for $F_0(G,x)$ can be generalized to

$$F_d(G,x)=\sum_{F\in 2^{E(G)}\setminus\mathscr{B}^*}(-1)^{|F|}2^{\kappa(G-F)d}x^{\beta(G-F)} \tag{19}$$

for any $d \geqslant 0$. Even so, it does not mean that (19) can yield an interpretation for the coefficients of $F_d(G,x)$, unless $\kappa(G-F)$ is determined uniquely by the number of edges in F, for an example (in the following section we will give such an example). □

6 Applications

The broken bond expansions in Corollary 8 and (19) allow us to calculate $F_0(G,x)$ and $F_d(G,x)$ for some particular signed graphs. For our first application, we consider a class of signed graphs which, oppositely to ordinary graphs, do not contain any balanced circuit, and which are Γ-flow admissible.

For a tree T, let G_T be the signed graph obtained from T by replacing each of its end vertices (the vertices of degree 1) with an unbalanced circuit. It is clear that G_T contains no balanced circuit.

Let $v_1, v_2, \cdots, v_p$ be the vertices in T that have degree at least 3 and let $d_1, d_2, \cdots, d_p$ be their degrees, respectively. Choosing an arbitrary leaf vertex r of T as the root, we get a rooted tree (here the 'rooted tree' is not the same thing as the 'signed rooted tree' defined earlier). For a vertex v_i (with degree at least 3) and an edge e incident with v_i, we call e *the father* of *the family* v_i if e is nearer to the root than other edges incident with v_i and call every edge other than the father a *child* of the family v_i. In particular, we call the set of all the children of v_i *the children class* of v_i and denote it by $C(v_i)$.

Let $\prec$ be an ordering on $E(G_T)$ such that no child is greater than its father and no edge on an unbalanced circuit is greater than one on T. Under this ordering, we can see that $\max B$ must be an edge on T for any bond $B = [X, X^C] \cup E_X$ in G_T, meaning that $\max B \in [X, X^C]$. Thus, the expansion (19) holds for G_T. Further, in the following we give an analytic expression of $F_0(G_T, x)$ for any tree T and expression of $F_d(G_T, x)$ when T is a star.

Let F be an edge set of G_T that contains no broken bond. By Corollary 8, F contributes $(-1)^{|F|}x^{m-n-|F|}$ to $F_0(G_T, x)$, where $m = |E(G_T)|, n = |V(G_T)|$. On the other hand, by our definition of $\prec$, F contains no broken bond if and only if F contains neither an edge from an unbalanced circuit nor a children class of a family. For any vertex v_i, let $F_i = F \cap C(v_i)$. In particular, let $F_r = F \cap \{e_r\}$, where e_r is the unique edge incident with the root r. Thus, the contribution of F to $F_0(G_T, x)$ can be specified as

$$x^{m-n}(-1)^{|F_r|}x^{-|F_r|}\prod_{i=1}^{p}(-1)^{|F_i|}x^{-|F_i|}. \tag{20}$$

On the other hand, we notice that $m-n = (d_1-2)+(d_2-2)+\cdots+(d_p-2)+1$. Rewrite (20) as

$$(-1)^{|F_r|}x^{1-|F_r|}\prod_{i=1}^{p}(-1)^{|F_i|}x^{d_i-2-|F_i|}.$$

In this product, the factors $(-1)^{|F_r|}x^{1-|F_r|}$ and $(-1)^{|F_i|}x^{d_i-2-|F_i|}$ can be regarded as the contributions of F restricted to $\{e_r\}$ and $C(v_i)$, respectively. Since $F \cap \{e_r\} = \emptyset$ or

$F \cap \{e_r\} = \{e_r\}$, all the possible contributions of F restricted to $\{e_r\}$ can be represented as $(-1)^{|\emptyset|}x^{1-|\emptyset|} + (-1)^{|\{e_r\}|}x^{1-|\{e_r\}|} = x - 1$.

In general, for each i, since v_i has exactly $d_i - 1$ children, all the possible contributions of F restricted to $C(v_i)$ equals

$$x^{d_i-2} - \binom{d_i-1}{1}x^{d_i-3} + \cdots + (-1)^{d_i-2}\binom{d_i-1}{d_i-2}.$$

Thus, the total contribution of all F that contains no broken bond equals

$$F_0(G_T, x) = (x-1)\prod_{i=1}^{p}(x^{d_i-2} - \binom{d_i-1}{1}x^{d_i-3} + \cdots + (-1)^{d_i-2}\binom{d_i-1}{d_i-2}). \tag{21}$$

When T is a star, every edge on an unbalanced circuit of G_T is a broken bond. Therefore, the number of unbalanced components in $G_T - F$ is determined uniquely by $|F|$, i.e., $\kappa(G_T - F) = |F| + 1$. So by (19), the coefficient of $(-1)^i x^{m-n-i}$ in $F_d(G_T, x)$ equals $2^{(i+1)d}a_i$, where $i \in \{0, 1, 2, \cdots, m-n\}$ and a_i is defined as in Corollary 8. Further, since T is a star, we have $p = 1$ in (21) and, hence $a_0 = 1$, $a_{d_1-1} = \binom{d_1-1}{d_1-2}$ and $a_i = \binom{d_1-1}{i-1} + \binom{d_1-1}{i}$ for $i \in \{1, 2, \cdots, d_1 - 2\}$.

Our second application is to show that the broken bonds in a signed graph have the topological structure of a homogeneous simplicial complex. A finite collection $\mathscr{S}$ of finite sets is called a *simplicial complex* if $S \in \mathscr{S}$ implies $T \in \mathscr{S}$ for any $T \subseteq S$. A simplicial complex is *homogeneous* [19] or *pure* [3] if all the maximal simplices have the same dimension (cardinality). A classic example of a homogeneous simplicial complex related to a graph is the broken-circuit complex [3, 4]. It has been shown [19] that the class $\mathfrak{B}(G)$ consisting of all the edge subsets of an ordinary graph G that contain no broken circuit is a homogeneous simplicial complex of top dimension $|V(G)| - 1$ and, moreover, the coefficients of the chromatic polynomial of G are the simplex counts in each dimension of $\mathfrak{B}(G)$.

Let $\mathfrak{F}(G)$ be the class consisting of all the edge subsets of a signed graph G that contain no broken bond.

Corollary 9. *Let G be an unbalanced signed graph with n vertices, m edges and with a linear order $\prec$ on $E(G)$. Then*
1). $\mathfrak{F}(G)$ is a homogeneous simplicial complex, i.e., every simplex is a subset of some simplex of top dimension $m - n$;
2). An edge set F is a simplex of top dimension $m - n$ of $\mathfrak{F}(G)$ if and only if $E(G) \setminus F$ contains at least one edge from each broken bond of G and each component $G - F$ contains exactly one unbalanced circuit;
3). For each $i \in \{0, 1, 2, \cdots, m-n\}$, the coefficient a_i in $F_0(G, x)$ is the number of the i-dimensional simplexes in $\mathfrak{F}(G)$.

Proof. 1). It is obvious that $\mathfrak{F}(G)$ is a simplicial complex. We prove that $\mathfrak{F}(G)$ is homogeneous.

Let F be a set of edges that contains no broken bond. If $|F| = m - n$ then we are done. We now assume that $|F| < m - n$, i.e., $|E(G - F)| > n$. In this case, it can be seen that there is a component ω in $G - F$ which contains at least two circuits C and C'. By Proposition 7, one of these two circuits, say C, is unbalanced. So by the same argument as that in Corollary 8, we can find an edge e in C' such that $G - F - e$ still contains an edge from each broken bond. Replacing F by $F \cup \{e\}$, the assertion follows by repeating this procedure, until $|F| = m - n$.

2) and 3) follows directly by Corollary 8. □

Acknowledgements

The authors would like to thank the anonymous reviewer for her/his careful reading of our original manuscript and constructive suggestions which is very helpful for revising and improving the manuscript.

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