Monomial ideals of weighted oriented graphs

Yuriko Pitones* Enrique Reyes† Jonathan Toledo

Departamento de Matemáticas
Centro de Investigación y de Estudios Avanzados del IPN
Ciudad de México, México

{ypitones,ereyes,jtt}@math.cinvestav.mx

Submitted: Jan 26, 2019; Accepted: Aug 7, 2019; Published: Aug 30, 2019

Abstract

Let $I = I(D)$ be the edge ideal of a weighted oriented graph D whose underlying graph is G. We determine the irredundant irreducible decomposition of I. Also, we characterize the associated primes and the unmixed property of I. Furthermore, we give a combinatorial characterization for the unmixed property of I, when G is bipartite, G is a graph with whiskers or G is a cycle. Finally, we study the Cohen–Macaulay property of I.

Mathematics Subject Classifications: 05C22, 05E40, 13F20, 13H10

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph. A *weighted oriented graph* D whose *underlying graph* is G, is a triplet $(V(D), E(D), w)$ where $V(D) = V(G)$, $E(D) \subseteq V(D) \times V(D)$ such that $\{\{x, y\} \mid (x, y) \in E(D)\} = E(G)$ and w is a function $w : V(D) \to \mathbb{N}$. The vertex set of D and the edge set of D are $V(D)$ and $E(D)$, respectively. Some times for short we denote these sets by V and E respectively. The *weight* of $x \in V$ is $w(x)$. If $e = (x, y) \in E$, then x is the tail of e and y is the head of e. If $V(D) = \{x_1, \ldots, x_n\}$, then we consider the polynomial ring $R = K[x_1, \ldots, x_n]$ in n variables over a field K. In this paper, we introduce and study the edge ideal of D given by $I(D) = (x_i x_j^{w(x_j)} : (x_i, x_j) \in E(D))$ in R, (see Definition 16). These ideals generalize to the usual edge ideals of graphs, since if $w(x) = 1$ for each $x \in V(D)$, then $I(D) = I(G)$.

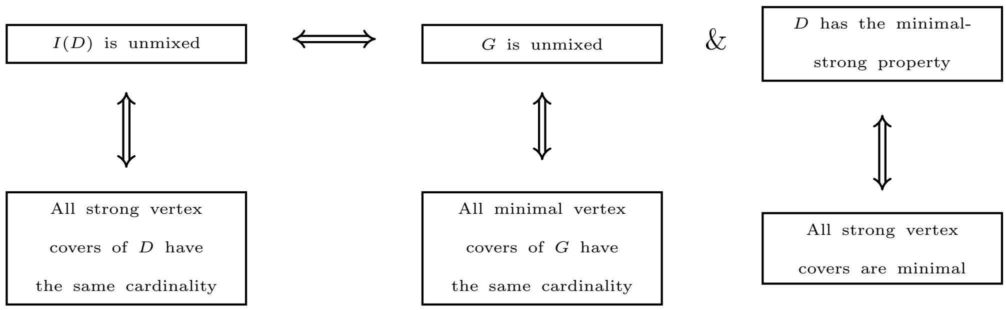
Furthermore, the study of these ideals have an important relation with coding theory, as we will now explain. If K is a finite field $\mathbb{F}_q$, $H_1 \subset \cdots \subset H_n$ is a nested sequence

*Supported by CONACYT.

†Partial supported by SNI- CONACYT.

of multiplicative subgroups of $\mathbb{F}_q^\star = \mathbb{F}_q \setminus \{0\}$ and $\mathcal{X}$ is the set in the projective space $\mathcal{X} = [(H_1 \cup \{0\}) \times \cdots \times (H_n \cup \{0\})]$, whose vanishing ideal is $I(\mathcal{X})$, then $I(D)$ is the initial ideal (with the lexicographic order) of $I(\mathcal{X})$, where $(x_i, x_j) \in E(D)$ if and only if $i < j$, $w(x_1) = 1$ and $w(x_\ell) = |H_\ell| + 1$ for $\ell > 1$. In this context, some basic parameters of the Reed–Muller code of $\mathcal{X}$ can be estimated examining $I(D)$, [2, 6, 11]. In particular, if $I(D)$ is Cohen–Macaulay, then $I(\mathcal{X})$ is Cohen–Macaulay.

In Section 2, we study the vertex covers of D. We introduce the notion of strong vertex cover (Definition 7) and we prove that a minimal vertex cover is strong. In Section 3, we characterize the irredundant irreducible decomposition of $I(D)$. We show that the minimal monomial irreducible ideals of $I(D)$ are associated with the strong vertex covers of D. In Section 4, we give the following characterization of the unmixed property of $I(D)$.



Also, if G is bipartite, G is a graph with whiskers or G is a cycle, we give an effective (combinatorial) characterization of the unmixed property. In Section 5, we study the Cohen–Macaulayness of $I(D)$. In particular, we prove that if $I(D)$ is Cohen–Macaulay, then $I(D)$ is unmixed and $I(G)$ is Cohen–Macaulay. Furthermore we characterize the Cohen–Macaulayness when G is a path or G is a complete graph. For these families of graphs we prove that unmixedness and Cohen–Macaulayness of $I(D)$ are equivalent. This equivalence is known for some monomial ideals, for example the monomial ideal associated to: König clutters without 3, 4-cycles [12]; special admissible clutters [7]; graphs with girt greater than 7 [3, Corollary 36]; and graphs with whiskers [13, Theorem 2.1]. Finally, we give an example of D where Cohen–Macaulay property of $I(D)$ depends on the characteristic of the field K.

2 Weighted oriented graphs and their vertex covers

In this section we study the vertex covers of a weighted oriented graph D. Furthermore, we define the strong vertex covers and we characterize when $V(D)$ is a strong vertex cover of D. In this paper we denote the set $\{x \in V(D) \mid w(x) \neq 1\}$ by $V^+(D)$ or V^+.

Definition 1. A *vertex cover* C of D is a subset of V, such that if $(x, y) \in E$, then $x \in C$ or $y \in C$. A vertex cover C of D is *minimal* if each proper subset of C is not a vertex cover of D.

Remark 2. C is a minimal vertex cover of D if and only if C is a minimal vertex cover of G.

Definition 3. Let x be a vertex of a weighted oriented graph D. The sets $N_D^+(x) = \{y : (x,y) \in E(D)\}$ and $N_D^-(x) = \{y : (y,x) \in E(D)\}$ are called the *out-neighbourhood* and the *in-neighbourhood* of x, respectively. Furthermore, the *neighbourhood* of x is the set $N_D(x) = N_D^+(x) \cup N_D^-(x)$ and $\deg_D(x) = |N_D(x)|$.

Definition 4. Let C be a vertex cover of a weighted oriented graph D. We define

- $L_1(C) := \{x \in C \mid N_D^+(x) \cap C^c \neq \emptyset\}$,
- $L_2(C) := \{x \in C \mid x \notin L_1(C) \text{ and } N_D^-(x) \cap C^c \neq \emptyset\}$,
- $L_3(C) := C \setminus (L_1(C) \cup L_2(C))$,

where C^c is the complement of C, i.e. $C^c = V \setminus C$.

Proposition 5. *If C is a vertex cover of D, then*

$$L_3(C) = \{x \in C \mid N_D(x) \subset C\}.$$

Proof. If $x \in L_3(C)$, then $N_D^+(x) \subseteq C$, since $x \notin L_1(C)$. Furthermore $N_D^-(x) \subseteq C$, since $x \notin L_2(C)$. Hence $N_D(x) \subset C$, since $x \in C \setminus N_D(x)$. Now, if $x \in C$ and $N_D(x) \subset C$, then $x \notin L_1(C) \cup L_2(C)$. Therefore $x \in L_3(C)$. □

Proposition 6. *Let C be a vertex cover of D. Hence, $L_3(C) = \emptyset$ if and only if C is a minimal vertex cover of D.*

Proof. $\Rightarrow$) If $x \in C$, then by Proposition 5 we have $N_D(x) \not\subset C$, since $L_3(C) = \emptyset$. Thus, there is $y \in N_D(x) \setminus C$ implying $C \setminus \{x\}$ is not a vertex cover. Therefore, C is a minimal vertex cover.

$\Leftarrow$) If $x \in L_3(C)$, then by Proposition 5, $N_D(x) \subseteq C \setminus \{x\}$, since $x \notin N_D(x)$. Hence, $C \setminus \{x\}$ is a vertex cover. A contradiction, since C is minimal. Therefore $L_3(C) = \emptyset$.

Definition 7. A vertex cover C of D is *strong* if for each $x \in L_3(C)$ there is $(y,x) \in E(D)$ such that $y \in L_2(C) \cup L_3(C)$ with $y \in V^+$ (i.e. $w(y) \neq 1$).

Remark 8. Let C be a vertex cover of D. Hence, by Proposition 5 and since $C = L_1(C) \cup L_2(C) \cup L_3(C)$, we have that C is strong if and only if for each $x \in C$ such that $N(x) \subset C$, there exist $y \in N^-(x) \cap (C \setminus L_1(C))$ with $y \in V^+$.

Corollary 9. *If C is a minimal vertex cover of D, then C is strong.*

Proof. By Proposition 6, we have $L_3(C) = \emptyset$, since C is minimal. Hence, C is strong. □

Remark 10. The vertex set $V = V(D)$ of D is a vertex cover. Also, if $z \in V$, then $N_D(z) \subseteq V \setminus z$. Hence, by Proposition 5, $L_3(V) = V$. Consequently, $L_1(V) = L_2(V) = \emptyset$. By Proposition 6, V is not a minimal vertex cover of D. Furthermore, by Remark 8, V is a strong vertex cover if and only if $N_D^-(x) \cap V^+ \neq \emptyset$ for each $x \in V$.

Definition 11. If G is a cycle with $E(D) = \{(x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$ and $V(D) = \{x_1, \ldots, x_n\}$, then D is called *oriented cycle*.

Definition 12. If G is a connected graph with exactly one cycle C, then D is called *oriented unicycle* when D satisfies the following conditions:

1) C is an oriented cycle in D. Furthermore, there is an oriented path from C to y in D, for each $y \in V(D) \setminus V(C)$.

2) $w(x) \neq 1$ if $\deg_D(x) > 1$.

Lemma 13. *If $V(D)$ is a strong vertex cover of D and D_1 is a maximal oriented subgraph of D such that D_1 is an oriented unicycle of D, then $V(D')$ is a strong vertex cover of $D' = D \setminus V(D_1)$.*

Proof. We take $x \in V(D')$. Thus, by Remark 10, there is $y \in N_D^-(x) \cap V^+(D)$. If $y \in D_1$, then we take D_2, where $V(D_2) = V(D_1) \cup \{x\}$ and $E(D_2) = E(D_1) \cup \{(y, x)\}$. Hence, if C is the oriented cycle of D_1, then C is the unique cycle of D_2, since $\deg_{D_2}(x) = 1$. If $y \in C$, then (y, x) is an oriented path from C to x in D_2. Now, if $y \notin C$, then there is an oriented path $\mathcal{L}$ from C to y in D_1. Consequently, $\mathcal{L} \cup \{(y, x)\}$ is an oriented path from C to x. Furthermore, $\deg_{D_2}(x) = 1$ and $w(y) \neq 1$, then D_2 is an oriented unicycle. A contradiction, since D_1 is maximal. This implies $y \in V(D')$.
Hence, $y \in N_{D'}^-(x) \cap V^+(D')$. Therefore, by Remark 10, $V(D')$ is a strong vertex cover of D'. □

Lemma 14. *If $V(D)$ is a strong vertex cover of D, then D has an oriented unicycle.*

Proof. Let y_1 be a vertex of D. Since $V = V(D)$ is a strong vertex cover, there is $y_2 \in V$ such that $y_2 \in N^-(y_1) \cap V^+$. Similarly, there is $y_3 \in N^-(y_2) \cap V^+$. Consequently, (y_3, y_2, y_1) is an oriented path. Continuing this process, we can assume there exists an oriented path $(y_k, y_{k-1}, \ldots, y_2, y_1)$ with $y_2, y_3, \ldots, y_k \in V^+$ and there is $1 \leqslant j \leqslant k-2$ such that $(y_j, y_k) \in E(D)$, since V is finite. Hence, $C = (y_k, y_{k-1}, \ldots, y_j, y_k)$ is an oriented cycle and $\mathcal{L} = (y_j, \ldots, y_1)$ is an oriented path from C to y_1. Furthermore, if $j = 1$, then $w(y_1) \neq 1$. Therefore, $C \cup \mathcal{L}$ is an oriented unicycle of D. □

Proposition 15. *Let D be a weighted oriented graph. Hence, $V(D)$ is a strong vertex cover of D if and only if there are $D_1, \ldots, D_s$ oriented unicycles of D such that $V(D_1), \ldots, V(D_s)$ is a partition of $V(D)$.*

Proof. $\Rightarrow$) By Lemma 14, D has an oriented unicycle. We take a maximal oriented unicycle D_1 of D. Hence, by Lemma 13, $V(D')$ is a strong vertex cover of $D' = D \setminus V(D_1)$. So, by Lemma 14, there is a maximal oriented unicycle D_2 of D'. Continuing this process we obtain oriented unicycles $D_1, \ldots, D_s$ such that $V(D_1), \ldots, V(D_s)$ is a partition of $V(D)$.

$\Leftarrow$) We take $x \in V = V(D)$. By hypothesis there is D_j such that $x \in V(D_j)$. We assume C is the oriented cycle of D_j. If $x \in V(C)$, then there is $y \in V(C)$ such that $(y, x) \in E(D_j)$ and $w(y) \neq 1$, since $\deg_{D_j}(y) \geqslant 2$ and D_j is an oriented unicycle. Now, we assume $x \notin V(C)$, then there is an oriented path $\mathcal{L} = (z_1, \ldots, z_r)$ from $z_1 \in V(C)$ to $z_r = x$. Thus, $(z_{r-1}, x) \in E(D)$. Furthermore, $w(z_{r-1}) \neq 1$, since $\deg_{D_j}(z_{r-1}) \geqslant 2$. Therefore, by Remark 10, V is a strong vertex cover. □

3 Edge ideals and their primary decomposition

As is usual if I is a monomial ideal of a polynomial ring R, we denote by $\mathcal{G}(I)$ the minimal monomial set of generators of I. Furthermore, there exists a unique decomposition, $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are irreducible monomial ideals such that $I \neq \bigcap_{i \neq j} \mathfrak{q}_i$ for each $j = 1, \ldots, r$. This is called the irredundant irreducible decomposition of I. Furthermore, $\mathfrak{q}_i$ is an irreducible monomial ideal if and only if $\mathfrak{q}_i = (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$ for some variables x_{i_j}. Irreducible ideals are primary, then a irreducible decomposition is a primary decomposition. For more details of primary decomposition of monomial ideals see [14, Chapter 6]. In this section, we define the edge ideal $I(D)$ of a weighted oriented graph D and we characterize its irredundant irreducible decomposition. In particular, we prove that this decomposition is an irreducible primary decomposition, i.e, the radicals of the elements of the irredundant irreducible decomposition of $I(D)$ are different.

Definition 16. Let D be a weighted oriented graph with $V(D) = \{x_1, \ldots, x_n\}$. The *edge ideal of* D, denote by $I(D)$, is the ideal of $R = K[x_1, \ldots, x_n]$ generated by

$$\{x_i x_j^{w(x_j)} \mid (x_i, x_j) \in E(D)\}.$$

Definition 17. A *source* of D is a vertex x, such that $N_D(x) = N_D^+(x)$. A *sink* of D is a vertex y such that $N_D(y) = N_D^-(y)$.

Remark 18. Let D be a weighted oriented graph. We take $D' = (V, E, w')$ a weighted oriented graph such that $w'(x) = w(x)$ if x is not a source and $w'(x) = 1$ if x is a source. Hence, $I(D) = I(D')$. For this reason in this paper, we will always assume that if x is a source, then $w(x) = 1$.

Definition 19. Let C be a vertex cover of D. The *irreducible ideal associated to* C is the ideal

$$I_C := \big(L_1(C) \cup \{x_j^{w(x_j)} \mid x_j \in L_2(C) \cup L_3(C)\}\big).$$

Lemma 20. *$I(D) \subseteq I_C$, for each vertex cover C of D.*

Proof. We take $I = I(D)$ and $m \in \mathcal{G}(I)$, then $m = xy^{w(y)}$, where $(x, y) \in E(D)$. Since C is a vertex cover, $x \in C$ or $y \in C$. If $y \in C$, then $y \in I_C$ or $y^{w(y)} \in I_C$. Thus, $m = xy^{w(y)} \in I_C$. Now, we assume $y \notin C$, then $x \in C$. Hence, $y \in N_D^+(x) \cap C^c$, so $x \in L_1(C)$. Consequently, $x \in I_C$ implying $m = xy^{w(y)} \in I_C$. Therefore $I \subseteq I_C$. □

Lemma 21. *Let D be a weighted oriented graph such that $I(D) \subseteq (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$. Then $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D.*

Proof. We take $J = (x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s})$. If $(a, b) \in E(D)$, then $ab^{w(b)} \in I(D) \subseteq J$. Thus, $x_{i_j}^{a_j} | ab^{w(b)}$ for some $1 \leqslant j \leqslant s$. Hence, $x_{i_j} \in \{a, b\}$ and $\{a, b\} \cap \{x_{i_1}, \ldots, x_{i_s}\} \neq \emptyset$. Therefore $\{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D. □

Definition 22. Let I be a monomial ideal. An irreducible monomial ideal $\mathfrak{q}$ containing I is called a *minimal irreducible monomial ideal of* I if for any irreducible monomial ideal $\mathfrak{p}$ such that $I \subseteq \mathfrak{p} \subseteq \mathfrak{q}$ one has that $\mathfrak{p} = \mathfrak{q}$.

Lemma 23. *Let J be a minimal irreducible monomial ideal of $I(D)$ where $\mathcal{G}(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. If $a_j \neq 1$ for some $1 \leqslant j \leqslant s$, then:*

1) *There is $(x, x_{i_j}) \in E(D)$ where $x \notin \mathcal{G}(J)$.*

2) *$a_j = w(x_{i_j})$.*

Proof. We denote $M = \mathcal{G}(I)$. Since M is minimal, we have $x_{i_1}, \ldots, x_{i_s}$ are different.
1) By contradiction suppose there is $a_j \neq 1$ such that if $(x, x_{i_j}) \in E(D)$, then $x \in M$. We take the ideal $J' = (M \setminus \{x_{i_j}^{a_j}\})$, then $J' \subsetneq J$. If $(a,b) \in E(D)$, then $ab^{w(b)} \in I(D) \subseteq J$. Consequently, $x_{i_k}^{a_k} | ab^{w(b)}$ for some $1 \leqslant k \leqslant s$. If $k \neq j$, then $ab^{w(b)} \in J'$. Now, if $k = j$, then by hypothesis, $a_j \neq 1$. Hence, $x_{i_j}^{a_j} | b^{w(b)}$ implying $x_{i_j} = b$. Thus $(a, x_{i_j}) \in E(D)$. So, by hypothesis $a \in M \setminus \{x_{i_j}^{a_j}\}$, since $a \neq x_{i_j}$. This implies, $ab^{w(b)} \in J'$. Therefore, $I(D) \subseteq J' \subsetneq J$. A contradiction, since J is minimal.
2) By 1), there is $(x, x_{i_j}) \in E(D)$ with $x \notin M = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. Thus, $x x_{i_j}^{w(x_{i_j})} \in I(D) \subseteq J$, so $x_{i_k}^{a_k} | x x_{i_j}^{w(x_{i_j})}$ for some $1 \leqslant k \leqslant s$. Hence, $x_{i_k}^{a_k} | x_{i_j}^{w(x_{i_j})}$, since $x \notin M$. This implies, $k = j$ and $a_j \leqslant w(x_{i_j})$. If $a_j < w(x_{i_j})$, then we take $J_1 = (M')$ where $M' = \{M \setminus \{x_{i_j}^{a_j}\}\} \cup \{x_{i_j}^{w(x_{i_j})}\}$. So, $J_1 \subsetneq J$. Furthermore, if $(a,b) \in E(D)$, then $m = ab^{w(b)} \in I(D) \subseteq J$. Thus, $x_{i_\ell}^{a_\ell} | ab^{w(b)}$ for some $1 \leqslant \ell \leqslant s$. If $\ell \neq j$, then $x_{i_\ell}^{a_\ell} \in M'$ implying $ab^{w(b)} \in J'$. Now, if $\ell = j$ then $x_{i_j}^{a_j} | b^{w(b)}$, since $a_j > 1$. Consequently, $b = x_{i_j}$ so $x_{i_j}^{w(x_{i_j})} | m$, since $b^{w(b)} | m$. Then, $m \in J'$. Hence $I(D) \subseteq J' \subsetneq J$, a contradiction since J is minimal. Therefore, $a_j = w(x_{i_j})$. $\square$

Theorem 24. *The following conditions are equivalent:*

1) *J is a minimal irreducible monomial ideal of $I(D)$.*

2) *There is a strong vertex cover C of D such that $J = I_C$.*

Proof. 2) $\Rightarrow$ 1) By definition $J = I_C$ is a monomial irreducible ideal. By Lemma 20, $I(D) \subseteq J$. Now, suppose $I(D) \subseteq J' \subseteq J$, where J' is a monomial irreducible ideal. We can assume $\mathcal{G}(J') = \{x_{j_1}^{b_1}, \ldots, x_{j_s}^{b_s}\}$.
If $x \in L_1(C)$, then there is $(x,y) \in E(D)$ with $y \notin C$. Hence, $xy^{w(y)} \in I(D)$ and $y^r \notin J$ for each $r \in \mathbb{N}$. Consequently $y^r \notin J'$ for each r, implying $y \notin \{x_{j_1}, \ldots, x_{j_s}\}$. Furthermore $x_{j_i}^{b_i} | xy^{w(y)}$ for some $1 \leqslant i \leqslant s$, since $xy^{w(y)} \in I(D) \subseteq J'$. This implies, $x = x_{j_i}^{b_i} \in J'$.
Now, if $x \in L_2(C)$, then there is $(y,x) \in E(D)$ with $y \notin C$. Thus $y \notin J$, so $y \notin \{x_{j_1}^{b_1}, \ldots, x_{j_s}^{b_s}\}$. Also, $x^{w(x)} y \in I(D) \subseteq J'$, then $x_{j_i}^{b_i} | x^{w(x)} y$ for some $1 \leqslant i \leqslant s$. Consequently, $x_{j_i}^{b_i} | x^{w(x)}$ implies $x^{w(x)} \in J'$.
Finally if $x \in L_3(C)$, then there is $(y,x) \in E(D)$ where $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$, since C is a strong vertex cover. So, $x^{w(x)} y \in I(D) \subseteq J'$ implies $x_{j_i}^{b_i} | x^{w(x)} y$ for some i.

Furthermore $y \notin J = I_C$, since $y \in L_2(C) \cup L_3(C)$ and $w(y) \neq 1$. This implies $y \notin J'$ so, $x_{j_i}^{b_i} | x^{w(x)}$ then $x^{w(x)} \in J'$.
Hence, $J = I_C \subseteq J'$. Therefore, J is a minimal monomial irreducible ideal of $I(D)$.

1) $\Rightarrow$ 2) Since J is irreducible monomial ideal, we can suppose $\mathcal{G}(J) = \{x_{i_1}^{a_1}, \ldots, x_{i_s}^{a_s}\}$. By Lemma 23, we have $a_j = 1$ or $a_j = w(x_{i_j})$ for each $1 \leqslant j \leqslant s$. Also, by Lemma 21, $C = \{x_{i_1}, \ldots, x_{i_s}\}$ is a vertex cover of D. We can assume $\mathcal{G}(I_C) = \{x_{i_1}^{b_1}, \ldots, x_{i_s}^{b_s}\}$, then $b_j \in \{1, w(x_{i_j})\}$ for each $1 \leqslant j \leqslant s$.

Now, suppose $b_k = 1$ and $w(x_{i_k}) \neq 1$ for some $1 \leqslant k \leqslant s$. Consequently $x_{i_k} \in L_1(C)$. Thus, there is $(x_{i_k}, y) \in E(D)$ where $y \notin C$. So, $x_{i_k} y^{w(y)} \in I(D) \subseteq J$ and $x_{i_r}^{a_r} | x_{i_k} y^{w(y)}$ for some $1 \leqslant r \leqslant s$. Furthermore $y \notin C$, then $r = k$ and $a_k = a_r = 1$. Therefore, $a_k = 1$.

Hence, $\mathcal{G}(I_C) \cap V(D) \subseteq \mathcal{G}(J) \cap V(D)$. This implies, $I_C \subseteq J$, since $a_j, b_j \in \{1, w(x_{i_j})\}$ for each $1 \leqslant j \leqslant s$. Therefore $J = I_C$, since J is minimal. In particular $a_i = b_i$ for each $1 \leqslant i \leqslant s$.

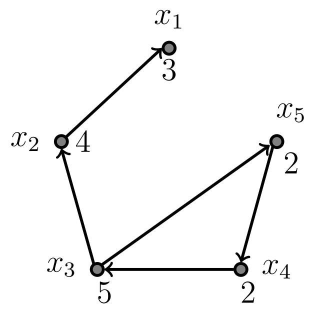
Now, assume C is not strong, then there is $x \in L_3(C)$ such that if $(y, x) \in E(D)$, then $w(y) = 1$ or $y \in L_1(C)$. We can suppose $x = x_{i_1}$, and we take J' the monomial ideal with $\mathcal{G}(J') = \{x_{i_2}^{a_2}, \ldots, x_{i_s}^{a_s}\}$. We take $(z_1, z_2) \in E(D)$. If $x_{i_j}^{a_j} | z_1 z_2^{w(z_2)}$ for some $2 \leqslant j \leqslant s$, then $z_1 z_2^{w(z_2)} \in J'$. Now, assume $x_{i_j}^{a_j} \nmid z_1 z_2^{w(z_2)}$ for each $2 \leqslant j \leqslant s$. Consequently $z_2 \notin \{x_{i_2}, \ldots, x_{i_s}\}$, since $a_j \in \{1, w(x_{i_j})\}$. Also $z_1 z_2^{w(z_2)} \in I(D) \subseteq J$, then $x_{i_1}^{a_1} | z_1 z_2^{w(z_2)}$. But $x_{i_1} \in L_3(C)$, so $z_1, z_2 \in N_D[x_{i_1}] \subseteq C = \{x_{i_1}, \ldots, x_{i_s}\}$. Furthermore $z_1 \neq z_2$, since $(z_1, z_2) \in E(D)$. But $z_2 \notin \{x_{i_2}, \ldots, x_{i_s}\}$, then $z_2 = x_{i_1}$, $z_1 \in C$ and $(z_1, x_{i_1}) \in E(D)$. Then, $w(z_1) = 1$ or $z_1 \in L_1(C)$. In both cases $z_1 \in \mathcal{G}(I_C)$. This implies $z_1 \in \mathcal{G}(J')$, since $z_1 \neq z_2$. So, $z_1 z_2^{w(z_2)} \in J'$. Hence, $I(D) \subseteq J'$. This is a contradiction, since J is minimal and $J' \subsetneq J$. Therefore C is strong. □

Theorem 25. *If $\mathcal{C}_s$ is the set of strong vertex covers of D, then the irredundant irreducible decomposition of $I(D)$ is given by $I(D) = \bigcap_{C \in \mathcal{C}_s} I_C$.*

Proof. By [8, Theorem 1.3.1], there is a unique irredundant irreducible decomposition $I(D) = \bigcap_{i=1}^{m} I_i$. Now, if there is an irreducible ideal I_j' such that $I(D) \subseteq I_j' \subseteq I_j$ for some $j \in \{1, \ldots, m\}$, then $I(D) = (\bigcap_{i \neq j} I_i) \cap I_j'$ is an irreducible decomposition. Furthermore this decomposition is irredundant, since $\bigcap_{i=1}^{m} I_i$ is irredundant. Thus, $I_j' = I_j$, since the irredundant irreducible decomposition is unique. Hence, $I_1, \ldots, I_m$ are minimal irreducible ideals of $I(D)$. Then, by Theorem 24, $\{I_1, \ldots, I_m\} \subseteq \{I_C \mid C \in \mathcal{C}_s\}$.
Now, if there is $C \in \mathcal{C}_s$ such that $I_C \notin \{I_1, \ldots, I_m\}$, then there is $x_{j_i}^{\alpha_i} \in I_i \setminus I_C$ for each $i \in \{1, \ldots, m\}$. Consequently, $\bar{m} = \operatorname{lcm}(x_{j_1}^{\alpha_1}, \ldots, x_{j_m}^{\alpha_m}) \in \bigcap_{i=1}^{m} I_i = I(D) \subseteq I_C$. Furthermore, if $C = \{x_{i_1}, \ldots, x_{i_k}\}$, then $I_C = (x_{i_1}^{\beta_1}, \ldots, x_{i_k}^{\beta_k})$ where $\beta_j \in \{1, w(x_{i_j})\}$. Hence, there is $j \in \{1, \ldots, k\}$ such that $x_{i_j}^{\beta_j} | \bar{m}$. So, there is $1 \leqslant u \leqslant m$ such that $x_{i_j}^{\beta_j} \mid x_{j_u}^{\alpha_u}$. A contradiction, since $x_{j_u}^{\alpha_u} \notin I_C$. Therefore $\bigcap_{i=1}^{m} I_i = \bigcap_{C \in \mathcal{C}_s} I_C$ is the irredundant irreducible decomposition of $I(D)$. □

Remark 26. If $C_1, \ldots, C_s$ are the strong vertex covers of D, then by Theorem 25, $I_{C_1} \cap \cdots \cap I_{C_s}$ is the irredundant irreducible decomposition of $I(D)$. Furthermore, if $P_i = \operatorname{rad}(I_{C_i})$, then $P_i = (C_i)$. So, $P_i \neq P_j$ for $1 \leqslant i < j \leqslant s$. Thus, $I_{C_1} \cap \cdots \cap I_{C_s}$ is an irredundant primary decomposition of $I(D)$. In particular we have $\mathrm{Ass}(I(D)) = \{P_1, \ldots, P_s\}$.

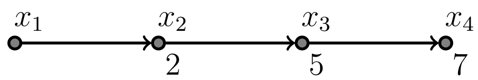
Example 27. Let D be the following oriented weighted graph



whose edge ideal is $I(D) = (x_1^3x_2, x_2^4x_3, x_3^5x_4, x_3x_5^2, x_4^2x_5)$. From Theorem 25, the irreducible decomposition of $I(D)$ is:

$$I(D) = (x_1^3, x_3, x_4^2) \cap (x_1^3, x_3, x_5) \cap (x_2, x_3, x_4^2) \cap (x_2, x_3^5, x_5) \cap (x_2, x_4, x_5^2) \cap (x_1^3, x_2^4, x_3^5, x_5) \cap (x_1^3, x_2^4, x_4, x_5^2) \cap (x_2, x_3^5, x_4^2, x_5^2) \cap (x_1^3, x_2^4, x_3^5, x_4^2, x_5^2).$$

Example 28. Let D be the following oriented weighted graph



Hence, $I(D) = (x_1x_2^2, x_2x_3^5, x_3x_4^7)$. By Theorem 25, the irreducible decomposition of $I(D)$ is:

$$I(D) = (x_1, x_3) \cap (x_2^2, x_3) \cap (x_2, x_4^7) \cap (x_1, x_3^5, x_4^7) \cap (x_2^2, x_3^5, x_4^7).$$

In Example 27 and Example 28, $I(D)$ has embedded primes. Furthermore the monomial ideal $(V(D))$ is an associated prime of $I(D)$ in Example 27. Proposition 15 and Remark 26 give a combinatorial criterion for to decide when $(V(D)) \in \mathrm{Ass}(I(D))$.

4 Unmixed weighted oriented graphs

In this section we characterize the unmixed property of $I(D)$ and we prove that this property is closed under c-minors. In particular if G is a bipartite graph or G is a graph with whiskers or G is a cycle, we give an effective (combinatorial) characterization of this property.

Definition 29. Let I be an ideal of R. We say I is *unmixed* if each one of its associated primes has the same height. In other case I is called *mixed*.

Remark 30. [7, 12] $I(G)$ is unmixed if and only if each minimal vertex cover of G has the same cardinality.

Theorem 31. *The following conditions are equivalent:*

1) $I(D)$ *is unmixed.*

2) *Each strong vertex cover of D has the same cardinality.*

3) *$I(G)$ is unmixed and $L_3(C) = \emptyset$ for each strong vertex cover C of D.*

Proof. Let $C_1, \ldots, C_\ell$ be the strong vertex covers of D. By Remark 26, the associated primes of $I(D)$ are $P_1, \ldots, P_\ell$, where $P_i = \mathrm{rad}(I_{C_i}) = (C_i)$ for $1 \leqslant i \leqslant \ell$.

1) $\Rightarrow$ 2) Since $I(D)$ is unmixed, $|C_i| = \mathrm{ht}(P_i) = \mathrm{ht}(P_j) = |C_j|$ for $1 \leqslant i < j \leqslant \ell$.

2) $\Rightarrow$ 3) By hypothesis, $|C_i| = |C_j|$ for each $1 \leqslant i \leqslant j \leqslant \ell$, then C_i is a minimal vertex cover of D. Thus, by Corollary 9, $C_1, \ldots, C_\ell$ are the minimal vertex covers of G. Hence, by Remark 30, $I(G)$ is unmixed. Now, if $C \in \{C_1, \ldots, C_\ell\}$, then, by Proposition 6, $L_3(C_i) = \emptyset$, since C is minimal.

3) $\Rightarrow$ 1) By Proposition 6, C_i is a minimal vertex cover, since $L_3(C_i) = \emptyset$ for each $1 \leqslant i \leqslant \ell$. So, by Corollary 9, $C_1, \ldots, C_\ell$ are the minimal vertex covers of G. By Remark 30, $\mathrm{ht}(P_i) = |C_i| = |C_j| = \mathrm{ht}(P_j)$ for $1 \leqslant i < j \leqslant \ell$, since $I(G)$ is unmixed. Therefore $I(D)$ is unmixed. □

Definition 32. A weighted oriented graph D has the *minimal-strong property* if each strong vertex cover is a minimal vertex cover.

Remark 33. Using Proposition 6, we have that D has the minimal-strong property if and only if $L_3(C) = \emptyset$ for each strong vertex cover C of D.

Definition 34. D' is a *c-minor* of D if there is a stable set S of D, such that $D' = D \setminus N_D[S]$.

Lemma 35. *If D has the minimal-strong property, then $D' = D \setminus N_D[x]$ has the minimal-strong property, for each $x \in V(D)$.*

Proof. We take a vertex cover C' of $D' = D \setminus N_D[x]$ where $x \in V(D)$. Thus, $C = C' \cup N_D(x)$ is a vertex cover of D. If $y' \in L_3(C')$, then by Proposition 5, $N_{D'}(y') \subseteq C'$. Furthermore, $y' \notin N_D[x]$, so $N_D(y') \subseteq C' \cup N_D(x) = C$. This implies, $y' \in L_3(C)$. Hence, $L_3(C') \subseteq L_3(C)$. Now, we take $y \in L_3(C)$, then $N_D(y) \subseteq C$. Also, $x \notin C = C' \cup N_D(x)$, then $x \notin N_D(y)$. Thus $y \notin N_D(x)$, implying $y \in C'$, since $y \in C$. Then, $N_{D'}(y) \cup (N_D(y) \cap N_D(x)) = N_D(y) \subseteq C = C' \cup N_D(x)$. So, $N_{D'}(y) \subseteq C'$ implies $y \in L_3(C')$. Therefore $L_3(C) = L_3(C')$.
Now, we will prove that if C' is strong, then C is strong. We assume C' is strong. We take $y \in L_3(C) = L_3(C')$, then there is $z \in C' \setminus L_1(C')$ with $w(z) \neq 1$, such that $(z, y) \in E(D')$. If $z \in L_1(C)$, then there exist $z' \notin C$ such that $(z, z') \in E(D)$. Also, $z' \neq x$, since $z \in C'$. Since $z' \notin C = C' \cup N_D(x)$, then $z' \in V(D')$ and $z \in L_1(C')$. A contradiction, implies $z \in C \setminus L_1(C)$. Therefore, C is strong.
Hence, if C' is a strong, then C is strong and $L_3(C) = \emptyset$, since D has the minimal-strong property. Thus, $L_3(C') = L_3(C) = \emptyset$. Therefore D' has the minimal-strong property. □

We say that D is unmixed if and only if $I(D)$ is unmixed.

Proposition 36. *If D is unmixed and $x \in V(D)$, then $D' = D \setminus N_D[x]$ is unmixed.*

Proof. By Theorem 31, $I(G)$ is unmixed and D has the minimal-strong property. Hence, by [14], $I(G')$ is unmixed, where $G' = G \setminus N_D[x]$. Also, by Lemma 35, D' has the minimal-strong property. Therefore, by Theorem 31, D' is unmixed. $\square$

Theorem 37. *If D is unmixed, then a c-minor of D is unmixed.*

Proof. If D' is a c-minor of D, then there is a stable $S = \{a_1, \ldots, a_s\}$ such that $D' = D \setminus N_D[S]$. Since S is stable,

$$D' = (\cdots((D \setminus N_D[a_1]) \setminus N_D[a_2]) \setminus \cdots) \setminus N_D[a_s].$$

Hence, by induction and Proposition 36, D' is unmixed. $\square$

Proposition 38. *If $V(D)$ is a strong vertex cover of D, then $I(D)$ is mixed.*

Proof. By Remark 10, $V(D)$ is not minimal. Therefore, by Theorem 31, $I(D)$ is mixed. $\square$

Corollary 39. If $V(D) = V^+$, then $I(D)$ is mixed.

Proof. We take $x \in V = V(D)$. By Remark 18, there is $y \in V$ such that $(y, x) \in E(D)$, since $V = V^+$. Also, $w(y) \neq 1$ and $y \in V = L_3(V)$. So, V is a strong vertex cover. Hence, by Proposition 38, $I(D)$ is mixed. $\square$

In the following three results we assume that $D_1, \ldots, D_r$ are the connected components of D, where, G_i is the underlying graph of D_i. Furthermore, we have that, if C is a vertex cover of D, then $C \cap V(D_i)$ is a vertex cover of D_i.

Lemma 40. *Let C be a vertex cover of D, then $L_1(C) = \bigcup_{i=1}^{r} L_1(C_i)$ and $L_3(C) = \bigcup_{i=1}^{r} L_3(C_i)$, where $C_i = C \cap V(D_i)$.*

Proof. We take $x \in C$, then $x \in C_j$ for some $1 \leqslant j \leqslant r$. Thus, $N_D(x) = N_{D_j}(x)$. In particular $N_D^+(x) = N_{D_j}^+(x)$, so $C \cap N_D^+(x) = C_j \cap N_{D_j}^+(x)$. Hence, $L_1(C) = \bigcup_{i=1}^{r} L_1(C_i)$. On the other hand,

$$x \in L_3(C) \Leftrightarrow N_D(x) \subseteq C \Leftrightarrow N_{D_j}(x) \subseteq C_j \Leftrightarrow x \in L_3(C_j).$$

Therefore, $L_3(C) = \bigcup_{i=1}^{r} L_3(C_i)$. $\square$

Lemma 41. *Let C be a vertex cover of D, then C is strong if and only if $C_i = C \cap V(D_i)$ is a strong vertex cover of D_i for each $i \in \{1, \ldots, r\}$.*

Proof. $\Rightarrow$) We take $x \in L_3(C_j)$. By Lemma 40, $x \in L_3(C)$ and there is $z \in N_D^-(x) \cap V^+$ with $z \in C \setminus L_1(C)$, since C is strong. So, $z \in V(D_j)$ and $z \in N_{D_j}^-(x)$, since $x \in D_j$. Consequently, by Lemma 40, $z \in C_j \setminus L_1(C_j)$. Therefore C_j is strong.

$\Leftarrow$) We take $x \in L_3(C)$, then $x \in C_i$ for some $1 \leqslant i \leqslant r$. Then, by Lemma 40, $x \in L_3(C_i)$. Thus, there is $a \in N_{D_i}^-(x)$ such that $w(a) \neq 1$ and $a \in C_i \setminus L_1(C_i)$, since C_i is strong. Hence, by Lemma 40, $a \in C \setminus L_1(C)$. Therefore C is strong. $\square$

Corollary 42. *$I(D)$ is unmixed if and only if $I(D_i)$ is unmixed for each $1 \leqslant i \leqslant r$.*

Proof. $\Rightarrow$) By Theorem 37, since D_i is a c-minor of D.

$\Leftarrow$) By Theorem 31, G_i is unmixed thus G is unmixed. Now, if C is a strong vertex cover, then by Lemma 41, $C_i = C \cap V(D_i)$ is a strong vertex cover. Consequently, $L_3(C_i) = \emptyset$, since $I(D_i)$ is unmixed. Hence, by Lemma 40, $L_3(C) = \bigcup_{i=1}^{r} L_3(C_i) = \emptyset$. Therefore, by Theorem 31, $I(D)$ is unmixed. $\square$

Definition 43. Let H be a simple graph whose vertex set is $V(H) = \{x_1, \ldots, x_k\}$ and edge set $E(H)$. The *graph with whiskers* of H is the graph G whose vertex set is $V(G) = V(H) \cup \{y_1, \ldots, y_k\}$ and whose edge set is $E(G) = E(H) \cup \{\{x_1, y_1\}, \ldots, \{x_k, y_k\}\}$.

Definition 44. Let D and H be weighted oriented graphs. D is a *weighted oriented graph with whiskers* of H if $H \subseteq D$ and the underlying graph of D is the graph with whiskers of the underlying graph of H.

Theorem 45. *Let D be a weighted oriented graph with whiskers of H, where $V(H) = \{x_1, \ldots, x_k\}$ and $V(D) = V(H) \cup \{y_1, \ldots, y_k\}$, then the following conditions are equivalents:*

1) *$I(D)$ is unmixed.*

2) *If $(x_i, y_i) \in E(D)$ for some $1 \leqslant i \leqslant k$, then $w(x_i) = 1$.*

Proof. 2) $\Rightarrow$ 1) We take a strong vertex cover C of D. Suppose $x_j, y_j \in C$, then $y_j \in L_3(C)$, since $N_D(y_j) = \{x_j\} \subseteq C$. Consequently, $(x_j, y_j) \in E(G)$ and $w(x_j) \neq 1$, since C is strong. This is a contradiction by condition 2). This implies, $|C \cap \{x_i, y_i\}| = 1$ for each $1 \leqslant i \leqslant k$. So, $|C| = k$. Therefore, by Theorem 31, $I(D)$ is unmixed.

1) $\Rightarrow$ 2) By contradiction suppose $(x_i, y_i) \in E(D)$ and $w(x_i) \neq 1$ for some i. By Remark 18, there is $x_j \in V(D)$ such that $(x_j, x_i) \in E(D)$, since $w(x_i) \neq 1$. We take the vertex cover $C = \{V(D) \setminus x_j\} \cup \{y_j, y_i\}$, then by Proposition 5, $L_3(C) = \{y_i\}$. Furthermore $N_D(x_i) \setminus C = \{x_j\}$ and $(x_j, x_i) \in E(H)$, then $x_i \in L_2(C)$. Hence C is strong, since $L_3(C) = \{y_i\}$, $(x_i, y_i) \in E(G)$ and $x_i \in L_2(C) \cap V^+$. On the other hand, $V(H)$ is a minimal vertex cover of D. By Corollary 9, $V(H)$ is strong. A contradiction by Theorem 31, since $I(D)$ is unmixed and $|C| \neq |V(H)|$. $\square$

Theorem 46. *Let D be a bipartite weighted oriented graph, then $I(D)$ is unmixed if and only if D satisfies the following conditions:*

1) *G has a perfect matching $\{\{x_1^1, x_1^2\}, \ldots, \{x_s^1, x_s^2\}\}$ where $\{x_1^1, \ldots, x_s^1\}$ and $\{x_1^2, \ldots, x_s^2\}$ are stable sets. Furthermore if $\{x_j^1, x_i^2\}$, $\{x_i^1, x_k^2\} \in E(G)$ then $\{x_j^1, x_k^2\} \in E(G)$.*

2) *If $w(x_j^k) \neq 1$ and $N_D^+(x_j^k) = \{x_{i_1}^{k'}, \ldots, x_{i_r}^{k'}\}$ where $\{k, k'\} = \{1, 2\}$, then $N_D(x_{i_\ell}^k) \subseteq N_D^+(x_j^k)$ and $N_D^-(x_{i_\ell}^k) \cap V^+ = \emptyset$ for each $1 \leqslant \ell \leqslant r$.*

Proof. $\Leftarrow$) By 1) and [4, Theorem 2.5.7], $I(G)$ is unmixed. Now, we take a strong vertex cover C of D. Suppose $L_3(C) \neq \emptyset$, thus there exist $x_i^k \in L_3(C)$. Then, $x_i^{k'} \in N_D(x_i^k) \subseteq C$. Since C is strong, there is $x_j^{k'} \in V^+$ such that $(x_j^{k'}, x_i^k) \in E(D)$, $x_j^{k'} \in C \setminus L_1(C)$ and $\{k, k'\} = \{1, 2\}$. Thus, $N_D^+(x_j^{k'}) \subseteq C$, since $x_j^{k'} \notin L_1(C)$. Furthermore, by 2),

$N_D(x_i^{k'}) \subseteq N_D^+(x_j^{k'}) \subseteq C$. So, $x_i^{k'} \in L_3(C)$ implies there is $(z, x_i^{k'}) \in E(D)$ with $z \in V^+$. A contradiction, since by 2) $N_D^-(x_i^{k'}) \cap V^+ = \emptyset$. Hence, $L_3(C) = \emptyset$. Therefore, by Theorem 31, $I(D)$ is unmixed.

$\Rightarrow$) By Theorem 31, $I(G)$ is unmixed. Hence, by [4, Theorem 2.5.7], G satisfies 1).

Now, assume $w(x_j^k) \neq 1$. We take $C = N_D^+(x_j^k) \cup \{x_i^k \mid N_D(x_i^k) \not\subseteq N_D^+(x_j^k)\}$ and k' such that $\{k, k'\} = \{1, 2\}$. Thus, $C \cap \{x_1^{k'}, \ldots, x_s^{k'}\} = N_D^+(x_j^k)$.
If $\{x_i^k, x_{i'}^{k'}\} \in E(G)$ with $x_i^k \notin C$, then $N_D(x_i^k) \subseteq N_D^+(x_j^k)$. Furthermore, $x_{i'}^{k'} \in N_D(x_i^k)$ and $N_D^+(x_j^k) \subseteq C$. Hence, $x_{i'}^{k'} \in C$. This implies, C is a vertex cover of D.
Now, if $x_{i_1}^k \in L_3(C)$, then $N_D(x_{i_1}^k) \subseteq C$. Consequently $N_D(x_{i_1}^k) \subseteq N_D^+(x_j^k)$, since $N_D(x_{i_1}^k) \subseteq \{x_1^{k'}, \ldots, x_s^{k'}\}$. This implies, $x_{i_1}^k \notin C$. A contradiction, then $L_3(C) \subseteq N_D^+(x_j^k)$. By Remark 18, $N_D^-(x_j^k) \neq \emptyset$, since $w(x_j^k) \neq 1$. Then, $N_D(x_j^k) \not\subseteq N_D^+(x_j^k)$. Thus, $x_j^k \in C \setminus L_1(C)$, since $N_D^+(x_j^k) \subseteq C$. Hence C is strong, since $x_j^k \in V^+$.
On the other hand $\{x_1^{k'}, \ldots, x_s^{k'}\}$ is a minimal vertex cover, then by Theorem 31, $|C| = s$, since D is unmixed. Also, $N_D^+(x_j^k) = \{x_{i_1}^{k'}, \ldots, x_{i_r}^{k'}\}$, then

$$\{x_i^k \mid i \notin \{i_1, \ldots, i_r\}\} \subseteq C,$$

since $C \cap \{x_{i_1}^{k'}, \ldots, x_{i_s}^{k'}\} = N_D^+(x_j^k)$ and $\{x_i^k, x_i^{k'}\} \in E(D)$ for $1 \leqslant i \leqslant s$. So, $x_{i_m}^k \notin C$ for each $1 \leqslant m \leqslant r$, since $|C| = s$. This implies $N_D(x_{i_m}^k) \subseteq N_D^+(x_j^k)$.

Now, suppose $z \in N_D^-(x_{i_\ell}^k) \cap V^+$ with $1 \leqslant \ell \leqslant r$. So $z = x_{i_{\ell'}}^{k'}$ for some $1 \leqslant \ell' \leqslant r$, since $N_D(x_{i_\ell}^k) \subseteq N_D^+(x_j^k)$. We take

$$C' = N_D^+(x_j^k) \cup \{x_i^k \mid i \notin \{i_1, \ldots, i_r\}\} \cup N_D^+(x_{i_{\ell'}}^{k'}),$$

then $x_{i_{\ell'}}^{k'} \in N_D^+(x_j^k) \subseteq C'$. Hence, C' is a vertex cover, since $\{x_1^k, \ldots, x_s^k\}$ is a vertex cover and $N_D(x_i^k) \subseteq N_D^+(x_j^k)$ for each $i \in \{i_1, \ldots, i_r\}$. Furthermore, by Remark 18, $N_D^-(x_j^k) \neq \emptyset$ implying $N_D(x_j^k) \not\subseteq N_D^+(x_j^k)$. Thus, $j \notin \{i_1, \ldots, i_k\}$ then $x_j^k \in C'$.
Now, if $\{x_q^k, x_q^{k'}\} \cap L_3(C') \neq \emptyset$, then $\{x_q^k, x_q^{k'}\} \subseteq C'$. So $x_q^{k'} \in N_D^+(x_j^k)$ implies $q \in \{i_1, \ldots, i_r\}$. Consequently, $x_q^k \in N_D^+(x_{i_{\ell'}}^{k'})$, since $x_q^k \in C'$. This implies, $(x_j^k, x_q^{k'}), (x_{i_{\ell'}}^{k'}, x_q^k) \in E(D)$. Furthermore, $N_D^+(x_{i_{\ell'}}^{k'}) \cup N_D^+(x_j^k) \subseteq C'$, then $x_{i_{\ell'}}^{k'}, x_j^k \in C' \setminus L_1(C')$. Thus, C' is strong, since $x_j^k, x_{i_{\ell'}}^{k'} \in V^+$.
Hence, by Theorem 31, $|C'| = s$. But $|N_D^+(x_j^k) \cup \{x_i^k \mid i \notin \{i_1, \ldots, i_r\}\}| = s$ and $x_{i_\ell}^k \in N_D^+(x_{i_{\ell'}}^{k'})$. A contradiction, so $N_D^-(x_{i_\ell}^k) \cap V^+ = \emptyset$. Therefore D satisfies 2). □

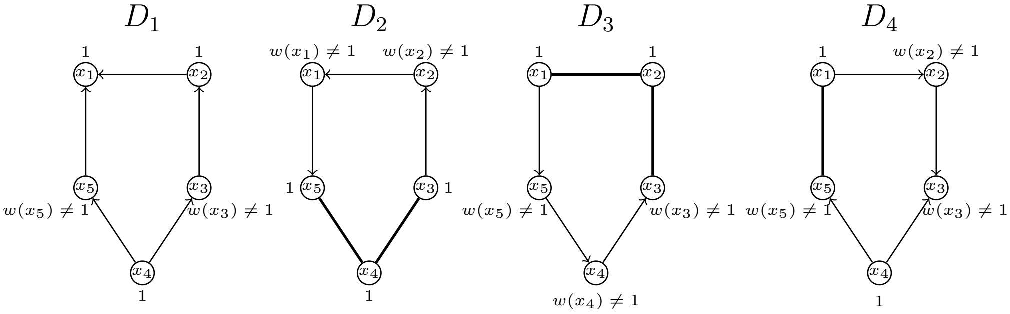
Lemma 47. *If the vertices of V^+ are sinks, then D has the minimal-strong property.*

Proof. We take a strong vertex cover C of D. Hence, if $y \in L_3(C)$, then there is $(z, y) \in E(D)$ with $z \in V^+$. Consequently, by hypothesis, z is a sink. A contradiction, since $(z, y) \in E(D)$. Therefore, $L_3(C) = \emptyset$ and C is a minimal vertex cover. □

Lemma 48. *Let D be a weighted oriented graph, where $G \simeq C_n$ with $n \geqslant 6$. Hence, D has the minimal-strong property if and only if the vertices of V^+ are sinks.*

Proof. $\Leftarrow$) By Lemma 47.

$\Rightarrow$) By contradiction, suppose there is $(z, y) \in E(D)$, with $z \in V^+$. We can assume $G = (x_1, x_2, \ldots, x_n, x_1) \simeq C_n$, with $x_2 = y$ and $x_3 = z$. We take a vertex cover C in the following form: $C = \{x_1, x_3, \ldots, x_{n-1}\} \cup \{x_2\}$ if n is even or $C = \{x_1, x_3, \ldots, x_{n-2}\} \cup \{x_2, x_{n-1}\}$ if n is odd. If $x \in C$ and $N_D(x) \subseteq C$, then $x = x_2$. Hence, $L_3(C) = \{x_2\}$. Furthermore, $(x_3, x_2) \in E(D)$. Since $x_3 \in V^+$, by Remark 18, x_3 is not a source. So, $(x_4, x_3) \in E(D)$ and $x_3 \in L_2(C)$. This implies C is a strong vertex cover. But $L_3(C) \neq \emptyset$. A contradiction, since D has the minimal-strong property. $\square$



Theorem 49. *If $G \simeq C_n$, then $I(D)$ is unmixed if and only if one of the following conditions hold:*

1) $n = 3$ *and there is $x \in V(D)$ such that $w(x) = 1$.*

2) $n \in \{4, 7\}$ *and the vertices of V^+ are sinks.*

3) $n = 5$, *there is $(x, y) \in E(D)$ with $w(x) = w(y) = 1$ and $D \notin \{D_1, D_2, D_3\}$.*

4) $D \simeq D_4$.

Proof. $\Rightarrow$) By Theorem 31, $I(G)$ is unmixed. Then, by [4, Exercise 2.4.22], $n \in \{3, 4, 5, 7\}$. If $n = 3$, then by Remark 39, D satisfies 1). If $n = 7$, then by Lemma 48, D satisfies 2). Now suppose $n = 4$ and D does not satisfies 2), then we can assume $x_1 \in V^+$ and $(x_1, x_2) \in E(D)$. Consequently, by Remark 18, $(x_4, x_1) \in E(G)$, since $w(x_1) \neq 1$. So, $\mathcal{C} = \{x_1, x_2, x_3\}$ is a vertex cover with $L_3(\mathcal{C}) = \{x_2\}$. Also, $(x_1, x_2) \in E(D)$ and $x_1 \in L_2(\mathcal{C}) \cap V^+$. Thus, $\mathcal{C}$ is strong. A contradiction, since $\mathcal{C}$ is not minimal. This implies D satisfies 2).

Now, assume $n = 5$. First, we will prove D_1, D_2 and D_3 are mixed. We take $\mathcal{C}_1 = \{x_1, x_2, x_3, x_5\}$, then $\mathcal{C}_1$ is a vertex cover of D_1 with $L_3(\mathcal{C}_1) = \{x_1, x_2\}$. Also (x_5, x_1), $(x_3, x_2) \in E(D_1)$ and $x_5, x_3 \in L_2(\mathcal{C}_1) \cap V^+$. Then, $\mathcal{C}_1$ is strong. Consequently, D_1 is mixed, since $L_3(\mathcal{C}_1) \neq \emptyset$. Now, $\mathcal{C}_2 = \{x_1, x_2, x_4, x_5\}$ is a vertex cover of D_2 where $L_3(\mathcal{C}_2) = \{x_1, x_5\}$ and $(x_2, x_1), (x_1, x_5) \in E(D_2)$ with $x_2, x_1 \in V^+$. Hence, $\mathcal{C}_2$ is strong, since $x_2, x_1 \notin L_1(\mathcal{C}_2)$. So, D_2 is mixed, since $L_3(\mathcal{C}_2) \neq \emptyset$. On the other hand, $\mathcal{C}_3 = \{x_2, x_3, x_4, x_5\}$ is a vertex

cover of D_3 where $L_3(\mathcal{C}_3) = \{x_3, x_4\}$ and $(x_4, x_3), (x_5, x_4) \in E(D)$ with $x_4, x_5 \in V^+$. Thus, $\mathcal{C}_3$ is strong, since $x_4, x_5 \notin L_1(\mathcal{C}_3)$. Hence, D_3 is mixed, since $L_3(\mathcal{C}_3) \neq \emptyset$. Therefore, $D \notin \{D_1, D_2, D_3\}$ since D is unmixed.
Now, suppose D does not satisfies 3), i.e. there are not two adjacent vertices with weight 1. Consequently, there are two adjacent vertices in V^+, since $n = 5$. Without loss of generality, we can assume $(x_2, x_3) \in E(D)$ with $x_2, x_3 \in V^+$. By Remark 18, $(x_1, x_2) \in E(D)$, since $x_2 \in V^+$. Now, we take two cases:
Case 1. Suppose there are not 3 vertices z_1, z_2, z_3 in V^+ such that (z_1, z_2, z_3) is a path in G. Then $w(x_4) = w(x_1) = 1$. Furthermore, $w(x_5) \neq 1$, since there are not adjacent vertices with weight 1. So, $\mathcal{C}_4 = \{x_2, x_3, x_4, x_5\}$ is a vertex cover of D, where $L_3(\mathcal{C}_4) = \{x_3, x_4\}$. Also $(x_2, x_3) \in E(G)$ and $x_2 \in L_2(\mathcal{C}_4) \cap V^+$. Assume, $(x_3, x_4) \in E(D)$ or $(x_5, x_4) \in E(D)$. If $(x_5, x_4) \in E(D)$, then by Remark 18, $(x_1, x_5) \in E(D)$ and $x_5 \in L_2(\mathcal{C}_4)$, since $x_5 \in V^+$. This implies, $\mathcal{C}_4$ is strong, since $x_3, x_5 \in V^+$ and $x_3 \in L_3(\mathcal{C}_4)$ or $x_5 \in L_2(\mathcal{C}_4)$. A contradiction, since $\mathcal{C}_4$ is not minimal. Hence, $(x_4, x_3), (x_4, x_5) \in E(D)$ and $D \simeq D_4$.
Case 2. Assume there is a path (z_1, z_2, z_3) in G such that $z_1, z_2, z_3 \in V^+$. Since there are not adjacent vertices with weight 1, we can suppose there is $z_4 \in V^+$ such that $\mathcal{L} = (z_1, z_2, z_3, z_4)$ is a path. We take $\{z_5\} = V(D) \setminus V(\mathcal{L})$ and we can assume $(z_2, z_3) \in E(D)$. So, by Remark 18, (z_1, z_2), $(z_5, z_1) \in E(D)$, since $z_1, z_2 \in V^+$. Thus, $\mathcal{C}_5 = \{z_1, z_2, z_3, z_4\}$ is a vertex cover with $L_3(\mathcal{C}_5) = \{z_2, z_3\}$. Then, $\mathcal{C}_5$ is strong, since $(z_1, z_2), (z_2, z_3) \in E(D)$ with $z_1 \in L_2(\mathcal{C}_5) \cap V^+$ and $z_2 \in L_3(\mathcal{C}_5) \cap V^+$. A contradiction, since $\mathcal{C}_5$ is not minimal.

$\Leftarrow$) If $n \in \{3, 4, 5, 7\}$, then by [4, Exercise 2.4.22] G is unmixed. By Theorem 31, we will only prove that D has the minimal-strong property.

If D satisfies 2), then by Lemma 47, D has the minimal-strong property. If D satisfies 1) and $\mathcal{C}$ is a strong vertex cover, then by Proposition 15, $|\mathcal{C}| \leqslant 2$. This implies $\mathcal{C}$ is minimal.

Now, assume $n = 5$. By contradiction, there is a strong vertex cover $\mathcal{C}'$ of D with $|\mathcal{C}'| \geqslant 4$. If $D \simeq D_4$, then $x_2, x_5 \notin L_3(\mathcal{C}')$, since $(N_D^-(x_2) \cup N_D^-(x_5)) \cap V^+ = \emptyset$. So, $N_D(x_2) \not\subseteq \mathcal{C}'$ and $N_D(x_5) \not\subseteq \mathcal{C}'$. Consequently, $x_1 \notin \mathcal{C}'$, since $|\mathcal{C}'| \geqslant 4$. Hence, $\mathcal{C}' = \{x_2, x_3, x_4, x_5\}$. But $x_4 \in L_3(\mathcal{C}')$ and $N_D^-(x_4) = \emptyset$. A contradiction, since $\mathcal{C}'$ is strong.
Now suppose D satisfies 3). We take two cases:
Case 1. Suppose there is a path $\mathcal{L} = (x_1, x_2, x_3)$ in G such that $w(x_1) = w(x_2) = w(x_3) = 1$. Since $w(x_1) = w(x_3) = 1$, we have that $x_2 \notin L_3(\mathcal{C}')$ and $|\mathcal{C}'| = 4$. If $x_2 \notin \mathcal{C}'$, then $\mathcal{C}' = \{x_1, x_3, x_4, x_5\}$. We can assume $(x_4, x_5) \in E(D)$, then $N_D^-(x_4) \cap V^+ = \emptyset$. But $x_4 \in L_3(\mathcal{C}')$, a contradiction, since $\mathcal{C}'$ is strong. Hence $x_2 \in \mathcal{C}'$. We can suppose $x_3 \notin \mathcal{C}'$, since $x_2 \notin L_3(\mathcal{C}')$. This implies $\mathcal{C}' = \{x_1, x_2, x_4, x_5\}$ and $L_3(\mathcal{C}') = \{x_1, x_5\}$. Thus, $(x_5, x_1), (x_4, x_5) \in E(D)$ and $x_5, x_4 \in V^+$, since $\mathcal{C}'$ is strong. By Remark 18, $(x_3, x_4) \in E(D)$, since $x_4 \in V^+$. A contradiction, since $D \not\simeq D_2$.
Case 2. There are not three consecutive vertices whose weights are 1. Since D satisfies 3), we can assume $w(x_1) = w(x_2) = 1$, $w(x_3) \neq 1$ and $w(x_5) \neq 1$. If $w(x_4) = 1$, then $x_3, x_5 \notin L_3(\mathcal{C}')$ since $N_D(x_3, x_5) \cap V^+ = \emptyset$. This implies $N_D(x_3) \not\subseteq \mathcal{C}'$ and $N_D(x_5) \not\subseteq \mathcal{C}'$. Then, $x_4 \notin \mathcal{C}'$ and $\mathcal{C}' = \{x_1, x_2, x_3, x_5\}$, since $|\mathcal{C}'| \geqslant 4$. Thus, $(x_5, x_1), (x_3, x_2) \in E(D)$, since $L_3(\mathcal{C}') = \{x_1, x_2\}$. By Remark 18, $(x_4, x_5), (x_4, x_3) \in E(D)$, since $x_5, x_3 \in V^+$. A contradiction, since $D \not\simeq D_1$. Hence, $w(x_4) \neq 1$ and by Remark 18, we can assume

$(x_5, x_4) \in E(D)$, since $x_4 \in V^+$. Similarly $(x_1, x_5) \in E(D)$, since $x_5 \in V^+$. Consequently, $(x_3, x_4) \in E(D)$, since $D \not\simeq D_3$. Then $(x_2, x_3) \in E(D)$, since $x_3 \in V^+$. This implies $x_1, x_2, x_3, x_5 \notin L_3(\mathcal{C}')$, since $N_D^-(x_i) \cap V^+ = \emptyset$ for $i \in \{1, 2, 3, 5\}$. A contradiction, since $|\mathcal{C}'| \geqslant 4$. □

5 Cohen–Macaulay weighted oriented graphs

In this section we study the Cohen–Macaulayness of $I(D)$. In particular, we give a combinatorial characterization of this property when G is a path or G is a complete graph. In this case we prove that Cohen–Macaulay property and unmixed property are equivalent. On the other hand, we show the Cohen–Macaulay property of $I(D)$ depends of the characteristic of K.

Remark 50. If G is the underlying graph of D, then $\text{rad}(I(D)) = I(G)$.

Proposition 51. *If $I(D)$ is Cohen–Macaulay, then $I(G)$ is Cohen–Macaulay and D has the minimal-strong property.*

Proof. By Remark 50, $I(G) = \text{rad}(I(D))$. Then, by [9, Theorem 2.6], $I(G)$ is Cohen–Macaulay. Furthermore, by [1, Theorem 2.1.2], $I(D)$ is unmixed, since $I(D)$ is Cohen–Macaulay. Hence, by Theorem 31, D has the minimal-strong property. □

Example 52. In Example 27 and Example 28, $I(D)$ is mixed. Hence, $I(D)$ is not Cohen–Macaulay, but $I(G)$ is Cohen–Macaulay.

Conjecture 53. $I(D)$ is Cohen–Macaulay if and only if $I(G)$ is Cohen–Macaulay and D has the minimal-strong property. Equivalently, $I(D)$ is Cohen–Macaulay if and only if $I(D)$ is unmixed and $I(G)$ is Cohen–Macaulay.

Proposition 54. *If G is a path, then, the following conditions are equivalent:*

1) *$R/I(D)$ is Cohen–Macaulay.*

2) *$I(D)$ is unmixed.*

3) *$|V(G)| = 2$ or $|V(G)| = 4$. In the second case, if $(x_2, x_1) \in E(D)$ or $(x_3, x_4) \in E(D)$, then $w(x_2) = 1$ or $w(x_3) = 1$ respectively.*

Proof. We assume G is the path $(x_1, \ldots, x_n)$.

1) $\Rightarrow$ 2) By [1, Theorem 2.1.2].

2) $\Rightarrow$ 3) By Theorem 46, G has a perfect matching, since D is bipartite. Consequently n is even and $\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{n-1}, x_n\}$ is the perfect matching of D. If $n \geqslant 6$, then by Theorem 46, we have $\{x_2, x_5\} \in E(G)$, since $\{x_2, x_3\}$ and $\{x_4, x_5\} \in E(G)$. A contradiction, since $\{x_2, x_5\} \notin E(G)$. Hence, $n \in \{2, 4\}$. Furthermore if $n = 4$, then G is a graph with whiskers. Hence, by Theorem 45, $w(x_2) = 1$ or $w(x_3) = 1$ when $(x_2, x_1) \in E(D)$ or $(x_3, x_4) \in E(D)$, respectively.

3) $\Rightarrow$ 1) We take $I = I(D)$. If $n = 2$, then $\mathrm{ht}(I) = \mathrm{ht}(\mathrm{rad}(I)) = \mathrm{ht}(I(G)) = 1$. Thus, $\dim(R/I) = n-1 = 1$. By Proposition 15, $V(D)$ is not a strong vertex cover of D. So, by Remark 26, $(x_1, x_2) \notin \mathrm{Ass}(I)$. Then, $\mathrm{depth}(R/I) \geqslant 1$. Hence, R/I is Cohen–Macaulay. Now, if $n = 4$, then $\mathrm{ht}(I) = \mathrm{ht}(I(G)) = 2$. Consequently, $\dim(R/I) = n - 2 = 2$.We take $s = |E(D) \cap \{(x_2, x_1), (x_3, x_4)\}|$. If $s = 0$ or $s = 2$, then we can assume $(x_2, x_3) \in E(D)$. If $s = 1$, then we can suppose $(x_2, x_1), (x_4, x_3) \in E(D)$. This implies,

1) $I(D) = (x_1 x_2^{w(x_2)}, x_2 x_3^{w(x_3)}, x_3^{w(x_3)} x_4)$, $X = \{x_1 - x_2^{w(x_2)}, x_4 - x_3^{w(x_3)}\}$.
2) $I(D) = (x_1^{w(x_1)} x_2, x_2 x_3, x_3 x_4^{w(x_4)})$, $X = \{x_2 - x_1^{w(x_1)}, x_3 - x_4^{w(x_4)}\}$.
3) $I(D) = (x_1^{w(x_1)} x_2, x_2 x_3^{w(x_3)}, x_3^{w(x_3)} x_4)$, $X = \{x_2 - x_1^{w(x_1)}, x_4 - x_3^{w(x_3)}\}$.
4) $I(D) = (x_1^{w(x_1)} x_2, x_2 x_3, x_3^{w(x_3)} x_4)$, $X = \{x_2 - x_1^{w(x_1)}, x_4 - x_3^{w(x_3)}\}$.

The first two cases when $s = 0$ and $s = 2$, respectively. The two last cases when $s = 1$. Using *Macaulay2* [5], we show that X is a regular sequence of R/I, in each case. Hence, $\mathrm{depth}(R/I) \geqslant 2$. Therefore, I is Cohen–Macaulay. □

Theorem 55. *If G is a complete graph, then the following conditions are equivalent:*

1) *$I(D)$ is unmixed.*

2) *$I(D)$ is Cohen–Macaulay.*

3) *There are not $D_1, \ldots, D_s$ oriented unicycles of D such that $V(D_1), \ldots, V(D_s)$ is a partition of $V(D)$*

Proof. We take $I = I(D)$. Since $I(G) = \mathrm{rad}(I)$ and G is a complete graph, we have that $\mathrm{ht}(I) = \mathrm{ht}(I(G)) = n - 1$, where $n = |V(G)|$.

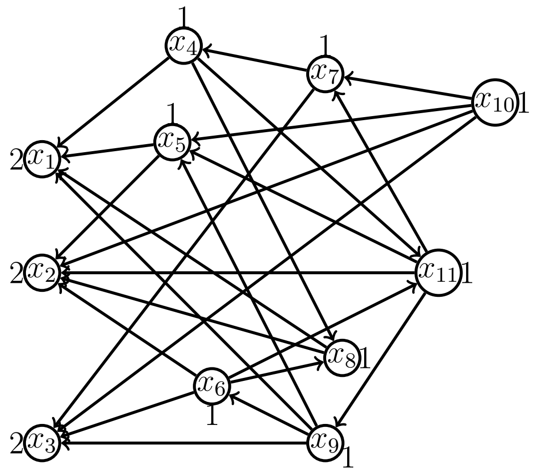
1) $\Rightarrow$ 3) Since I is unmixed and $\mathrm{ht}(I) = n-1$, we have that $(x_1, \ldots, x_n) \notin \mathrm{Ass}(I)$. Thus, by Remark 26, $V(D)$ is not a strong vertex cover of D. Therefore, by Proposition 15, D satisfies 3).

3) $\Rightarrow$ 2) By Proposition 15, $V(D)$ is not a strong vertex cover of D. Consequently, by Remark 26, $(x_1, \ldots, x_n) \notin \mathrm{Ass}(I)$. This implies, $\mathrm{depth}(R/I) \geqslant 1$. Furthermore, $\dim(R/I) = 1$, since $\mathrm{ht}(I) = n - 1$. Therefore I is Cohen–Macaulay.

2) $\Rightarrow$ 1) By [1, Theorem 2.1.2]. □

Hence, if G is a complete graph or G is a path, then unmixed and Cohen–Macaulay are equivalent properties in $I(D)$. Furthermore, in these cases Cohen–Macaulay property not depend of the field K. It is not true in general, see the following example.

Example 56. Let D be the following weighted oriented graph:



Hence,

$$I(D) = (x_1^2x_4, x_1^2x_5, x_1^2x_8, x_1^2x_9, x_2^2x_5, x_2^2x_6, x_2^2x_8, x_2^2x_{10}, x_2^2x_{11}, x_3^2x_6, x_3^2x_7, x_3^2x_9,$$
$$x_3^2x_{10}, x_4x_7, x_4x_8, x_4x_{11}, x_5x_9, x_5x_{10}, x_5x_{11}, x_6x_8, x_6x_9, x_6x_{11}, x_7x_{10}, x_7x_{11}, x_9x_{11}).$$

By [10, Example 2.3], $I(G)$ is Cohen–Macaulay when the characteristic of the field K is zero but it is not Cohen–Macaulay in characteristic 2. Consequently, $I(G)$ is unmixed and $I(D)$ is not Cohen–Macaulay when the characteristic of K is 2. Furthermore, by Lemma 47, $I(D)$ has the minimal-strong property. Thus, by Theorem 31, $I(D)$ is unmixed. Finall, using *Macaulay2* [5], we show that $I(D)$ is Cohen–Macaulay when the characteristic of K is zero.

Acknowledgements

The authors are grateful to the referees whose suggestions improved the presentation of this paper.

References

[1] A. Bruns, J. Herzog, *Cohen–Macaulay Rings*. Cambridge University Press, Cambridge, 1997.

[2] C. Carvalho, V. G. Lopez Neumann and H. H. López, *Projective nested cartesian codes*, Bull. Braz. Math. Soc. (N.S.) **48** (2017), no. 2, 283–302.

[3] I. D. Castrillón, R. Cruz, and E. Reyes. *On well-covered, vertex decomposable and Cohen–Macaulay graphs*. Electron. J. Combin., **23**, (2016), Paper 39, p 17.

[4] I. Gitler, R. H. Villarreal, *Graphs, Rings and Polyhedra*, Aportaciones Mat. Textos, **35**, Soc. Mat. Mexicana, México, 2011.

[5] D. Grayson and M. Stillman, *Macaulay2*, 1996. Available via anonymous ftp from `math.uiuc.edu.`

[6] H. T. Há, K. N. Lin, S. Morey, E. Reyes and R. H. Villarreal, *Edge ideals of oriented graphs*, Internat. J. Algebra Comput. **29** (2019) No. 03, 535–559.

[7] H. T. Há, S. Morey and R. H. Villarreal, *Cohen–Macaulay admissible clutters*. J. Commut. Algebra **1**(3)(2009), 463–480.

[8] J. Herzog and T. Hibi, *Monomial Ideals*, Graduate Texts in Mathematics. **260**, Springer, 2011.

[9] J. Herzog, Y. Takayama, N. Terai, *On the radical of a monomial ideal*, Arch. Math. **85** (2005), 397–408.

[10] A. Madadi and R. Zaare-Nahandi, *Cohen–Macaulay r-partite graphs with minimal clique cover*, Bull. Iranian Math. Soc. **40** (2014) No. 3, 609–617.

[11] J. Martínez-Bernal, Y. Pitones and R. H. Villarreal, *Minimum distance functions of graded ideals and Reed–Muller-type codes*, J. Pure Appl. Algebra **221** (2017), 251–275.

[12] S. Morey, E. Reyes, R.H. Villarreal, *Cohen–Macaulay, shellable and unmixed clutters with perfect matching of König type*, Journal of Pure and Applied Algebra **212** (2008), 1770–1786.

[13] A. Simis, W. Vasconcelos, R. H. Villarreal, On the ideal theory of graphs, J. Algebra **167**(1994) 389–416.

[14] R. H. Villarreal, *Monomial Algebras, Second Edition*, Monographs and Research Notes in Mathematics, Chapman and Hall/CRC, 2015.