Spectral lower bounds for the orthogonal and projective ranks of a graph

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Abstract

The orthogonal rank of a graph $G = (V, E)$ is the smallest dimension ξ such that there exist non-zero column vectors $x_v \in \mathbb{C}^\xi$ for $v \in V$ satisfying the orthogonality condition $x_v^\dagger x_w = 0$ for all $vw \in E$. We prove that many spectral lower bounds for the chromatic number, χ, are also lower bounds for ξ. This result complements a previous result by the authors, in which they showed that spectral lower bounds for χ are also lower bounds for the quantum chromatic number χ_q. It is known that the quantum chromatic number and the orthogonal rank are incomparable.

We conclude by proving an inertial lower bound for the projective rank ξ_f, and conjecture that a stronger inertial lower bound for ξ is also a lower bound for ξ_f.

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1 Introduction

For any graph G, let V denote the set of vertices where $|V| = n$, E denote the set of edges where $|E| = m$, A denote the adjacency matrix, $\chi(G)$ denote the chromatic number, $\omega(G)$ denote the clique number, $\alpha(G)$ denote the independence number, and $\overline{G}$ denote the complement of G. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$ denote the eigenvalues of A. Then, the inertia of G is the ordered triple (n^+, n^0, n^-), where n^+, n^0 and n^- are the numbers of positive, zero and negative eigenvalues of A, including multiplicities. Note that $\text{rank}(A) = n^+ + n^-$ and $\text{null}(A) = n^0$. A graph is called non-singular if $n^0 = 0$.

Let D be the diagonal matrix of vertex degrees, and let $L = D - A$ denote the Laplacian of G and $Q = D + A$ denote the signless Laplacian of G. The eigenvalues of L are $\theta_1 \geqslant \cdots \geqslant \theta_n = 0$ and the eigenvalues of Q are $\delta_1 \geqslant \cdots \geqslant \delta_n$.

Let $\chi_v(G)$ and $\chi_{sv}(G)$ denote the vector and strict vector chromatic numbers as defined by Karger *et al* [14]. They proved that $\chi_{sv}(G) = \vartheta(\overline{G})$, where ϑ is the Lovász theta function [16]. Let ϑ^+ denote Szegedy's [22] variant of ϑ. Let $\chi_f(G)$ and $\chi_c(G)$ denote the fractional and circular chromatic numbers and let $\chi_q(G)$ and $\chi_q^{(r)}(G)$ denote the quantum and rank-r quantum chromatic numbers, as defined by Cameron *et al* [2].

Definition 1 (Orthogonal rank $\boldsymbol{\xi(G)}$)**.** The orthogonal rank of G is the smallest positive integer $\xi(G)$ such that there exists an orthogonal representation, that is a collection of non-zero column vectors $x_v \in \mathbb{C}^{\xi(G)}$ for $v \in V$ satisfying the orthogonality condition

$$x_v^\dagger x_w = 0 \tag{1}$$

for all $vw \in E$.

The normalized orthogonal rank of G is the smallest positive integer $\xi'(G)$ such that there exists an orthogonal representation, with the added restriction that the entries of each vector must all have the same modulus.

Let $\xi_f(G)$ denote the projective rank which was defined by Mančinska and Roberson [17], who showed that $\omega(G) \leqslant \xi_f(G) \leqslant \xi(G)$. We use the definition of the r-fold orthogonal rank $\xi^{[r]}(G)$ due to Hogben et al. in [11, Section 2.1.] and their results in [11, Section 2.2.] to provide an equivalent and simpler definition of the projective rank.

Definition 2 ($\boldsymbol{r}$-fold orthogonal rank $\boldsymbol{\xi^{[r]}(G)}$ and projective rank $\boldsymbol{\xi_f(G)}$)**.** A d/r-representation of $G = (V, E)$ is a collection of rank-r orthogonal projectors P_v for $v \in V$ such that $P_v P_w = 0_d$ for all $vw \in E$.

The r-fold orthogonal rank $\xi^{[r]}(G)$ is defined as follows:

$$\xi^{[r]}(G) = \min\Big\{d : G \text{ has a } d/r\text{-representation}\Big\}.$$

The projective rank, $\xi_f(G)$, is defined as follows:

$$\xi_f(G) = \lim_{r\to\infty} \frac{\xi^{[r]}(G)}{r}, \text{ and this limit exists.}$$

The projective rank is also called the fractional orthogonal rank.

Clearly, the vectors $x_v \in \mathbb{C}^{\xi(G)}$ of an orthogonal representation correspond to the rank-1 orthogonal projectors $P_v = x_v x_v^\dagger \in \mathbb{C}^{\xi(G)\times\xi(G)}$ of a $\xi(G)/1$-representation. It is also clear that $\xi^{[1]}(G) = \xi(G)$.

For $c \in \mathbb{N}$, we use the abbreviation $[c] = \{0, \dots, c-1\}$.

Definition 3 (Vectorial chromatic number $\boldsymbol{\chi_{\mathrm{vect}}(G)}$)**.** Paulsen and Todorov [20] defined the vectorial chromatic number, $\chi_{\mathrm{vect}}(G)$, as follows. Let $G = (V, E)$ be a graph and $c \in \mathbb{N}$. A vectorial c-coloring of G is a set of vectors $(x_{v,i} : v \in V, i \in [c])$ in a Hilbert space such that the following conditions are satisfied:

$$\langle x_{v,i}, x_{w,j}\rangle \geqslant 0\,, \quad v, w \in V,\ i, j \in [c] \tag{2}$$

$$\sum_{i\in[c]} x_{v,i} = \sum_{i\in[c]} x_{w,i}\,, \quad \Big\|\sum_{i\in[c]} x_{v,i}\Big\| = 1\,, \quad v, w \in V \tag{3}$$

$$\langle x_{v,i}, x_{v,j}\rangle = 0\,, \quad v \in V,\, i \neq j \in [c] \tag{4}$$

$$\langle x_{v,i}, x_{w,i}\rangle = 0\,, \quad vw \in E,\, i \in [c]. \tag{5}$$

The least integer c for which there exists a vectorial c-coloring will be denoted $\chi_{\text{vect}}(G)$ and called the vectorial chromatic number of G.

Note that χ_{vect} differs from χ_v. Cubitt *et al* [4] (Corollary 16) proved the following (unexpected) equality between a chromatic number and a theta function:

$$\chi_{\text{vect}}(G) = \lceil \vartheta^+(\overline{G}) \rceil,$$

and provided an example of a graph with $\chi_{\text{vect}} < \chi_q$. Roberson [21] (Lemma 6.14.1) proved that $\vartheta^+(\overline{G}) \leqslant \xi_f(G) \leqslant \xi(G)$, so $\chi_{\text{vect}}(G) = \lceil \vartheta^+(\overline{G}) \rceil \leqslant \xi(G)$.

2 Hierarchy of graph parameters

There are numerous graph parameters that lie between the clique number and the chromatic number. The following chains of inequalities summarise the relationships between many of them, and combine results in Cameron *et al* [2], Mančinska and Roberson ([18] and [17]), Paulsen *et al* [19] and Elphick and Wocjan [6]. The chains are broken into two parts so the rightmost ends of (6) and leftmost ends of (7) coincide.

$$\begin{tikzcd}
 & & & & \lceil \vartheta^+(\overline{G}) \rceil = \chi_{\text{vect}}(G) \\
\omega(G) \arrow[r] & \chi_v(G) \arrow[r] & \chi_{sv}(G) = \vartheta(\overline{G}) \arrow[r] & \vartheta^+(\overline{G}) \arrow[ur] \arrow[r] & \xi_f(G)
\end{tikzcd} \tag{6}$$

$$\begin{tikzcd}
\chi_{\text{vect}}(G) \arrow[r] \arrow[dr] & \xi(G) \arrow[dr] & & & \\
\xi_f(G) \arrow[r] \arrow[ur] \arrow[dr] & \chi_q(G) \arrow[r] & \chi_q^{(1)}(G) \arrow[r] & \xi'(G) \arrow[r] & \lceil \chi_c(G) \rceil = \chi(G) \\
 & \chi_f(G) \arrow[r] & \chi_c(G) \arrow[urr] & &
\end{tikzcd} \tag{7}$$

As illustrated above, Mančinska and Roberson ([18] and [17]) demonstrated that ξ and χ_q are incomparable, as are χ_f and χ_q; and also χ_f and ξ. They also proved that

ξ_f is a lower bound for ξ, χ_q and χ_f. Cubitt *et al* [4] demonstrated that χ_{vect} and ξ_f are incomparable. We can also demonstrate that χ_{vect} and χ_f are incomparable as follows. It is straightforward that for C_5, $\chi_{vect} > \chi_f$. However if we consider the disjunctive product $C_5 * K_3$, then from [4] we have $\chi_{\text{vect}}(C_5 * K_3) \leqslant 7$ but $\chi_f(C_5 * K_3) = 7.5$, because χ_f is multiplicative for the disjunctive product. Note that $\xi, \xi', \chi_{\text{vect}}, \chi_q, \chi_q^{(1)}$ are integers, χ_f is rational but it is unknown if ξ_f is necessarily always rational. These hierarchies of parameters resolve a question raised by Wocjan and Elphick (see Section 2.4 of [23]) of whether $\chi_v \leqslant \xi'$.

3 Spectral lower bounds for the orthogonal and projective ranks

Wocjan and Elphick [24] proved that many spectral lower bounds for $\chi(G)$ are also lower bounds for $\chi_q(G)$. In this paper we prove that many spectral lower bounds for $\chi(G)$ are also lower bounds for $\xi(G)$. In Theorem 4 we prove an inertial lower bound for $\xi(G)$ by strengthening a proof in [6]. In Theorem 5 we prove several eigenvalue lower bounds for $\xi(G)$ by proving lower bounds for $\chi_{vect}(G)$. We *conjecture* that all of these bounds are also lower bounds for $\xi_f(G)$, and make limited progress in this direction in Theorem 6. (See Remark 8.)

Theorem 4 (Inertial lower bound for orthogonal rank)**.** *Let $\xi(G)$ be the orthogonal rank of a graph G with inertia (n^+, n^0, n^-). Then*

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \leqslant \xi(G).$$

Theorem 5 (Eigenvalue lower bounds for vectorial chromatic number)**.** *Let $\xi(G)$ be the orthogonal rank and $\chi_{vect}(G)$ be the vectorial chromatic number of a graph G. Then*

$$1 + \max\left(\frac{\mu_1}{|\mu_n|}, \frac{2m}{2m - n\delta_n}, \frac{\mu_1}{\mu_1 - \delta_1 + \vartheta_1}\right) \leqslant \chi_{vect}(G) \leqslant \xi(G). \tag{8}$$

These bounds, reading from left to right, have been proved to be lower bounds for $\chi(G)$ by Hoffman [12], Lima *et al* [15] and Kolotilina [13].

Theorem 6 (Inertial lower bound for projective rank)**.** *Let $\xi_f(G)$ be the projective rank of a graph G with inertia (n^+, n^0, n^-). Then,*

$$1 + \max\left(\frac{n^+}{n^- + n^0}, \frac{n^-}{n^+ + n^0}\right) \leqslant \xi_f(G)\,.$$

In particular, when the graph G is non-singular the lower bounds in Theorems 4 and 6 coincide.

Remark 7. All results also apply to weighted adjacency matrices $W \circ A$, where W is an arbitrary Hermitian matrix and $\circ$ denotes the Hadamard product (also called Schur product).

Remark 8. Bilu [1] (for the Hoffman bound) and the authors (for the other bounds) have subsequently strengthened Theorem 5 by proving that these three bounds are also lower bounds for the vector chromatic number $\chi_v(G)$ [25] provided that the weighted adjacency matrices are restricted to have only nonnegative entries. As a result the bounds in Theorem 5 are lower bounds for $\xi_f(G)$ with such weighted adjacency matrices. Theorem 5 in the present paper relies on techniques that are very different from those in [25].

4 Proof of the inertial lower bound on the orthogonal rank $\xi(G)$

Let $f_1, \ldots, f_n \in \mathbb{C}^n$ denote the eigenvectors of unit length corresponding to the eigenvalues $\mu_1 \geqslant \cdots \geqslant \mu_n$. Then, $A = B - C$, where

$$B = \sum_{i=1}^{n^+} \mu_i f_i f_i^\dagger \quad \text{and} \quad C = \sum_{i=n-n^-+1}^{n} (-\mu_i) f_i f_i^\dagger \,. \tag{9}$$

Note that B and C are positive semidefinite and that $\operatorname{rank}(B) = n^+$ and $\operatorname{rank}(C) = n^-$. Let

$$P^+ = \sum_{i=1}^{n+} f_i f_i^\dagger \,, \quad P^- = \sum_{i=n-n^-+1}^{n} f_i f_i^\dagger$$

denote the orthogonal projectors onto the subspaces spanned by the eigenvectors corresponding to the positive and negative eigenvalues respectively. Note that $B = P^+ A P^+$ and $C = -P^- A P^-$.

Lemma 9. *Let X and $Y \in \mathbb{C}^{n\times n}$ be two positive semidefinite matrices satisfying $X \succeq Y$, that is, their difference $X - Y$ is positive semidefinite. Then,*

$$\operatorname{rank}(X) \geqslant \operatorname{rank}(Y)\,. \tag{10}$$

Proof. Assume to the contrary that $\operatorname{rank}(X) < \operatorname{rank}(Y)$. Then, there exists a non-trival vector v in the range of Y that is orthogonal to the range of X. Consquently,

$$v^\dagger (X - Y) v = -v^\dagger Y v < 0$$

contradicting that $X - Y$ is positive semidefinite. □

Remark 10. Let $x_v = (x_v^1, \ldots, x_v^\xi)^T \in \mathbb{C}^\xi$ for $v \in V$ be an orthogonal representation. Note that we may assume that the first entries of these vectors are all equal to 1, that is,

$$x_v^1 = 1$$

for all $v \in V$ for the following reason. If we apply any unitary transformation $U \in \mathbb{C}^{\xi\times\xi}$ to x_v we obtain an equivalent orthogonal representation $y_v = U x_v$. Clearly, there must exist a unitary matrix U such that the resulting orthogonal representation $y_v = (y_v^1, \ldots, y_v^\xi)^T$ satisfies the condition $y_v^1 \neq 0$ for all $v \in V$ due to a simple parameter counting argument. We can now rescale each vector to additionally achieve $y_v^1 = 1$.

We now have all the tools to prove Theorem 4.

Proof. Let $x_v = (x_v^1, \ldots, x_v^\xi)^T$ for $v \in V$ be an orthogonal representation satisfying the additional condition $x_v^1 = 1$ as in the remark above. We define ξ diagonal matrices

$$D_i = \operatorname{diag}(x_v^i : v \in V) \in \mathbb{C}^{n\times n}$$

for $i = 1, \ldots, \xi$. Due to this construction, we have

$$\sum_{i=1}^{\xi} D_i^\dagger A D_i = (s_{vw}), \text{ with } s_{vw} = a_{vw} \cdot x_v^\dagger x_w \text{ for } v, w \in V.$$

We see that this sum is the zero matrix because all its entries s_{vw} are zero either due to the orthogonality condition of the orthogonal representation $x_v^\dagger x_w = 0$ for $vw \in E$ or due to $a_{vw} = 0$ for $vw \notin E$. Observe that $D_1 = I$ due to the above remark. We obtain

$$\sum_{i=2}^{\xi} D_i^\dagger A D_i = -A. \tag{11}$$

Equation (11) can be rewritten as

$$\sum_{i=2}^{\xi} D_i^\dagger (B - C) D_i = C - B.$$

Multiplying both sides by P^- from left and right yields:

$$P^- \left(\sum_{i=2}^{\xi} D_i^\dagger (B - C) D_i \right) P^- = C.$$

Using that

$$P^- \left(\sum_{i=2}^{\xi} D_i^\dagger C D_i \right) P^-$$

is positive semidefinite, it follows that

$$P^- \left(\sum_{i=2}^{\xi} D_i^\dagger B D_i \right) P^- \succeq C.$$

Then using that the rank of a sum is less than or equal to the sum of the ranks of the summands, that the rank of a product is less than or equal to the minimum of the ranks of the factors, and Lemma 9, we have that $(\xi - 1)n^+ \geqslant n^-$. Similarly, $(\xi - 1)n^- \geqslant n^+$ is obtained by multiplying (11) by -1 and repeating the arguments (but multiplying by P^+ instead of P^- from the left and right). $\square$

5 Proof of eigenvalue lower bounds on the orthogonal rank $\boldsymbol{\xi(G)}$

We now present a generalization of [24, Theorem 1].

Theorem 11. *Assume that there exists a vectorial c-coloring of G. Then, there exists a collection of orthogonal projectors $(P_{v,i} \in \mathbb{C}^{d\times d}, v \in V, i \in [c])$ and a unit (column) vector $y \in \mathbb{C}^d$ such that the block-diagonal orthogonal projectors*

$$P_i = \sum_{v\in V} e_v e_v^\dagger \otimes P_{v,i} \in \mathbb{C}^{n\times n} \otimes \mathbb{C}^{d\times d}$$

satisfy the following three conditions:

$$\sum_{i\in[c]} P_i = I_n \otimes I_d, \tag{12}$$

$$\Big(I_n \otimes yy^\dagger\Big) \sum_{i\in[c]} P_i(A\otimes I_d)P_i \Big(I_n \otimes yy^\dagger\Big) = 0_n \otimes 0_d, \tag{13}$$

$$\Big(I_n \otimes yy^\dagger\Big) \sum_{i\in[c]} P_i(E\otimes I_d)P_i \Big(I_n \otimes yy^\dagger\Big) = E \otimes yy^\dagger, \tag{14}$$

where $E \in \mathbb{C}^{d\times d}$ is an arbitrary diagonal matrix.

Proof. We now prove condition (12). Let $(x_{v,i} : v \in V, i \in [c])$ be a vectorial c-coloring of G. Conditions (3) and (4) in Definition 3 imply that there exist orthogonal projectors $P_{v,i} \in \mathbb{C}^{d\times d}$ and a unit (column) vector $y \in \mathbb{C}^d$ such that the $P_{v,i}$ form a resolution of the identity I_d

$$\sum_{i\in[c]} P_{v,i} = I_d \tag{15}$$

for all $v \in V$ and

$$x_{v,i} = P_{v,i} y$$

for all $v \in V$ and $i \in [c]$. The unit vector y is simply equal to the sum $\sum_{i\in[c]} x_{v,i}$. For $i \in [c]$, $P_{v,i}$ is equal to the orthogonal projector onto the subspace spanned by $x_{v,i}$. Note that if their sum $\sum_{i\in[c]} P_{v,i}$ is not equal to the identity I_d, then we can add the projector onto the missing orthogonal complement to, say, $P_{v,0}$.

We now prove condition (13). Let e_v denote the standard basis (column) vectors of $\mathbb{C}^n$ corresponding to the vertices $v \in V$ so that $A = \sum_{v,w\in V} a_{vw} e_v e_w^\dagger$. For $v \in V, i \in [c]$, the block-diagonal projectors $P_i \in \mathbb{C}^{n\times n} \otimes \mathbb{C}^{d\times d}$ form a resolution of the identity $I_n \otimes I_d$, which follows by applying condition (15) to each block of these projectors. For $v, w \in V$, we use $v \sim w$ to denote that these vertices are connected. When used in a summation symbol, it means that the sum is taken over all pairs of connected vertices. To abbreviate, we define the projector $\Upsilon = yy^\dagger$.

$$\Big(I_n \otimes \Upsilon\Big) \sum_{i\in[c]} P_i \Big(A \otimes I_d\Big) P_i \Big(I_n \otimes \Upsilon\Big) \tag{16}$$

$$= \Big(I_n \otimes \Upsilon\Big) \sum_{i \in [c]} \left(\sum_{v \in V} e_v e_v^\dagger \otimes P_{v,i} \right) \Big(A \otimes I_d\Big) \left(\sum_{w \in V} e_w e_w^\dagger \otimes P_{w,i} \right) \Big(I_n \otimes \Upsilon\Big) \tag{17}$$

$$= \Big(I_n \otimes \Upsilon\Big) \sum_{i \in [c]} \left(\sum_{v,w \in V} a_{v,w} \cdot e_v e_w^\dagger \otimes P_{v,i} P_{w,i} \right) \Big(I_n \otimes \Upsilon\Big) \tag{18}$$

$$= \Big(I_n \otimes \Upsilon\Big) \sum_{i \in [c]} \left(\sum_{v \sim w} e_v e_w^\dagger \otimes P_{v,i} P_{w,i} \right) \Big(I_n \otimes \Upsilon\Big) \tag{19}$$

$$= \sum_{i \in [c]} \left(\sum_{v \sim w} e_v e_w^\dagger \otimes \Upsilon P_{v,i} P_{w,i} \Upsilon \right) \tag{20}$$

$$= \sum_{i \in [c]} \left(\sum_{v \sim w} x_{v,i}^\dagger x_{w,i} \cdot e_v e_w^\dagger \otimes \Upsilon \right) \tag{21}$$

$$= 0_n \otimes 0_d, \tag{22}$$

where we used

$$\Upsilon P_{v,i} P_{w,i} \Upsilon = y(y^\dagger P_{v,i})(P_{w,i} y) y^\dagger = y(x_{v,i}^\dagger x_{w,i}) y^\dagger = x_{v,i}^\dagger x_{w,i} \cdot \Upsilon \tag{23}$$

and (5), which states that $x_{v,i}^\dagger x_{w,i} = 0$ for all $i \in [c]$ and all $v \sim w$.

Finally, condition (14) is proved similarly. □

We need the following general lemma [24, Lemma 1], which allows us to replace pinching in Theorem 11 by twirling.

Lemma 12. *Let $(P_i \in \mathbb{C}^{m \times m}, i \in [c])$ be any collection of orthogonal projectors that form a resolution of the identity I_m, where the dimension m and number of projectors c are arbitrary. Then, there exists a unitary matrix $U \in \mathbb{C}^{m \times m}$ such that*

$$\frac{1}{c} \sum_{\ell \in [c]} U^\ell X (U^\dagger)^\ell = \sum_{i \in [c]} P_i X P_i$$

for any matrix $X \in \mathbb{C}^{m \times m}$. The left hand side of this equation defines a so-called twirling of the matrix X, whereas the right hand side defines a pinching.

By combining Theorem 11 and Lemma 12, we obtain the following lemma. It allows us to prove the lower bounds in Theorem 5.

Lemma 13. *Assume that there exists a vectorial c-coloring of G. Then, there exists a unitary matrix $U \in \mathbb{C}^{n \times n} \otimes \mathbb{C}^{d \times d}$ and a unit (column) vector $y \in \mathbb{C}^d$ such that*

$$\Big(I_n \otimes yy^\dagger\Big) \sum_{\ell \in [c]} U^\ell (A \otimes I_d)(U^\dagger)^\ell \Big(I_n \otimes yy^\dagger\Big) = 0_n \otimes 0_d \tag{24}$$

$$\Big(I_n \otimes yy^\dagger\Big) \sum_{\ell \in [c]} U^\ell (E \otimes I_d)(U^\dagger)^\ell \Big(I_n \otimes yy^\dagger\Big) = c\, E \otimes yy^\dagger \tag{25}$$

for any diagonal matrix $E \in \mathbb{C}^{n \times n}$.

We note that we did not make use of condition (2).

5.1 Proof of the Lima bound in Theorem 5

Proof. The proof is almost identical to the proof for the chromatic number. We use the identity $D - Q = -A$. To abbreviate, we set $P = I_n \otimes \Upsilon = I_n \otimes yy^\dagger$. We have:

$$\begin{aligned}
A \otimes yy^\dagger &= P(A \otimes I_d)P \\
&= \sum_{\ell=1}^{c-1} PU^\ell(-A \otimes I_d)(U^\dagger)^\ell P \\
&= \sum_{\ell=1}^{c-1} PU^\ell\left((D-Q) \otimes I_d\right)(U^\dagger)^\ell P \\
&= (c-1)(D \otimes yy^\dagger) - \sum_{\ell=1}^{c-1} PU^\ell(Q \otimes I_d)(U^\dagger)^\ell P
\end{aligned}$$

Define the column vector $v = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^\dagger \otimes y$. Multiply the left and right most sides of the above matrix equation by $v^\dagger$ from the left and by v from the right to obtain

$$\frac{2m}{n} = v^\dagger(A \otimes yy^\dagger)v = (c-1)\frac{2m}{n} - \sum_{\ell=1}^{c-1} v^\dagger PU^\ell(Q \otimes I_d)(U^\dagger)^\ell Pv \leqslant (c-1)\frac{2m}{n} - (c-1)\delta_n \,.$$

This uses that $v^\dagger(A \otimes yy^\dagger)v = v^\dagger(D \otimes yy^\dagger)v = 2m/n$, which is equal to the sum of all entries of respectively A and D divided by n due to the special form of the vector v, and that $v^\dagger PU^\ell(Q \otimes I_d)(U^\dagger)^\ell Pv = v^\dagger U^\ell(Q \otimes I_d)(U^\dagger)^\ell v \geqslant \lambda_{\min}(Q) = \delta_n$. □

5.2 Proof of the Hoffman and Kolotilina bounds in Theorem 5

Proof. Let $E \in \mathbb{C}^{n \times n}$ be an arbitrary diagonal matrix. Using (24) and (25), we obtain

$$\sum_{\ell=1}^{c-1} PU^\ell(E \otimes I_d - A \otimes I_d)(U^\dagger)^\ell P = (c-1)E \otimes yy^\dagger + A \otimes yy^\dagger \,.$$

Using that $\lambda_{\max}(X) \geqslant \lambda_{\max}(PXP)$ and $\lambda_{\max}(X) + \lambda_{\max}(Y) \geqslant \lambda_{\max}(X+Y)$ for arbitrary Hermitian matrices X and Y, we obtain

$$\begin{aligned}
\lambda_{\max}(E - A) &= \lambda_{\max}(E \otimes I_d - A \otimes I_d) \\
&\geqslant \lambda_{\max}\left(E \otimes yy^\dagger + \frac{1}{c-1}A \otimes yy^\dagger\right) \\
&= \lambda_{\max}\left(E + \frac{1}{c-1}A\right) .
\end{aligned}$$

[5, Corollary 5] shows that the above eigenvalue bound implies

$$\lambda_{\max}(E - A) \geqslant \lambda_{\max}(E + A) - \frac{c-2}{c-1}\lambda_{\max}(A) \,,$$

or equivalently

$$c \geqslant 1 + \frac{\lambda_{\max}(A)}{\lambda_{\max}(A) - \lambda_{\max}(E+A) + \lambda_{\max}(E-A)}\,,$$

from which the Hoffman and Kolotilina bounds are obtained by setting $E = 0$ and $E = D$, respectively. □

5.3 Inertial and generalized Hoffman and Kolotilina bounds

We do not know whether the inertial bound in Theorem 4 or the generalized (multi-eigenvalue) bounds in [5] are also lower bounds for the vectorial chromatic number. The difficulty seems to be in determining what happens to the *entire* spectrum of the various matrices when they are compressed by $P = I_n \otimes yy^\dagger$. The Kolotilina and Lima bounds only use the maximal and/or minimal eigenvalues.

6 Proof of the inertial lower bound on the projective rank $\xi(G)$

We conjecture that for all graphs G the projective rank $\xi_f(G)$ is lower bounded by

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \leqslant \xi_f(G)\,.$$

Unfortunately, we are not able to settle this question by either providing a counterexample or proving this bound for all graphs. However, we are able to prove the weaker lower bound in Theorem 6.

We derive two lemmas to better organize the proof of Theorem 6.

Lemma 14. *Let P be an orthogonal projector and X a positive semidefinite matrix in $\mathbb{C}^{m\times m}$. Then, we have*

$$\mathrm{rank}(PXP) \geqslant \mathrm{rank}(P) - \mathrm{null}(X)\,.$$

Proof. There exist positive semidefinite matrices Y and Δ such that Y has full rank, Δ has rank $\mathrm{null}(X)$, and $X + \Delta = Y$. Using that $\mathrm{rank}(M+N) \leqslant \mathrm{rank}(M) + \mathrm{rank}(N)$ for arbitrary matrices, we obtain

$$\mathrm{rank}(PXP) + \mathrm{rank}(P\Delta P) \geqslant \mathrm{rank}(PYP)\,.$$

Using that $\mathrm{rank}(MN) \leqslant \mathrm{rank}(M)$ for arbitrary matrices M and N, we obtain

$$\mathrm{rank}(PXP) \geqslant \mathrm{rank}(PYP) - \mathrm{rank}(\Delta)\,.$$

We can write

$$PYP = (Y^{1/2}P)^\dagger (Y^{1/2}P)\,.$$

Using that $\mathrm{rank}(M^\dagger M) = \mathrm{rank}(M)$ for arbitrary matrices, we obtain

$$\mathrm{rank}(PYP) = \mathrm{rank}(Y^{1/2}P) = \mathrm{rank}(P)$$

because $Y^{1/2}$ has full rank. □

Lemma 15. *Let P_v be the projectors of a (d/r)-orthogonal representation. Define the block diagonal projector*

$$P = \sum_{v \in V} e_v e_v^\dagger \otimes P_v \,.$$

Then, we have

$$P(A \otimes I_d)P = 0_n \otimes 0_d$$

and

$$\mathrm{rank}(P) = nr \,.$$

Proof. This follows directly from the orthogonality condition $P_v P_w = 0_d$ for all $vw \in E$. The proof is very similar to the proof for the vectorial chromatic number in the previous section. The projectors P_v have rank r for all $v \in V$ so $\mathrm{rank}(P) = nr$. $\square$

We are now ready to prove Theorem 6.

Proof. Let $A = B - C$, defined as in Section 4, so $\mathrm{rank}(B) = n^+$ and $\mathrm{rank}(C) = n^-$. Note that Lemma 15 implies

$$P(B \otimes I_d)P = P(C \otimes I_d)P \,.$$

so that

$$P(B \otimes I_d)P = \frac{1}{2} P((B + C) \otimes I_d)P \,. \tag{26}$$

Clearly, the rank of the left hand side of (26) is bounded from above by $n^+ d = \mathrm{rank}(B \otimes I_d)$.

We now bound the rank of the right hand side of (26) from below. Observe that $B + C = |A|$, where $|A| = \sum_{i=1}^n |\mu_i| e_i e_i^\dagger$ and μ_i and e_i are the eigenvalues and eigenvectors of A, respectively. Clearly, $|A|$ is positive semidefinite, its rank is equal to $\mathrm{rank}(A) = n^+ + n^-$ and its nullity is equal to $\mathrm{null}(A) = n_0$. Therefore, $|A| \otimes I_d$ is positive semidefinite, its rank is equal to $(n^+ + n^-)d$ and its nullity is equal to $n_0 d$. We can now apply Lemma 14 to obtain

$$\mathrm{rank}\Big(P\big(|A| \otimes I_d\big)P\Big) \geqslant \mathrm{rank}(P) - n_0 d = nr - n_0 d \,.$$

Combining the upper and lower bounds on the ranks, we obtain

$$n^+ d \geqslant nr - n_0 d \quad \iff \quad \frac{d}{r} \geqslant 1 + \frac{n^-}{n^+ + n^0} \,.$$

The result

$$\frac{d}{r} \geqslant 1 + \frac{n^+}{n^- + n^0}$$

is obtained by considering $P(C \otimes I_d)P$ on the left hand side of (26). $\square$

7 Implications for the projective rank

The following examples demonstrate that the inertial bound is exact for ξ_f for various classes of graphs. We also use Theorem 6 to derive the value of ξ_f for some graphs.

- For *odd cycles*, C_{2k+1} (see [6] and [17]):

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = \chi_f = \xi_f = 2 + \frac{1}{k}; \text{ but } \chi_{vect} = \chi_q = \xi = \chi = 3.$$

- For *Kneser* graphs, $K_{p,k}$ (see [6], [9], [19], and [10]):

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = \chi_v = \chi_f = \xi_f = \frac{p}{k}; \chi_{vect} = \left\lceil \frac{p}{k} \right\rceil; \text{ but } \xi_{\mathbb{R}} = \chi = p - 2k + 2,$$

  $\xi_{\mathbb{R}}$ is the orthogonal rank over the reals.

- The *orthogonality* graph, $\Omega(n)$, has vertex set the set of $\pm 1-$vectors of length n, with two vertices adjacent if they are orthogonal. With n a multiple of 4 (see [17, Lemma 4.2 and Theorem 6.4]):

$$\chi_{sv} = \chi_{vect} = \xi_f = \xi = \chi_q = n; \text{ but } \chi_f \text{ and } \chi \text{ are exponential in } n \text{ for large } n.$$

  $\Omega(4)$ has spectrum $(6^2, 0^8, -2^6)$ so when $n = 4$:

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = 1 + \frac{n^-}{n^+} = 4 = \xi_f = \chi_q = \chi$$

  but for $n > 4$ this inertial bound is less than ξ_f.

- The *Andrásfai* graphs, And(k), are k-regular with $(3k-1)$ vertices. It is known ([8] and [7]) that

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = 1 + \frac{n^+}{n^-} = 1 + \frac{2k-1}{k} = 3 - \frac{1}{k} = \chi_f$$

  but

$$\chi = \chi_q = \xi = 3.$$

  The Andrásfai graphs are non-singular, so using Theorem 6 and that $\xi_f \leqslant \chi_f$ it follows that $\xi_f = 3 - 1/k$.

- The *Clebsch* graph on 16 vertices has spectrum $(5^1, 1^{10}, -3^5)$ and $\chi_f = 3.2$ (see [7]). Therefore

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) = 1 + \frac{n^+}{n^-} = 3.2 = \chi_f; \text{ but } \chi = \xi = 4.$$

  The Clebsch graph is non-singular, so using Theorem 6 and that $\xi_f \leqslant \chi_f$, it follows that $\xi_f = 3.2$.

  The Clebsch graph is the folded 5-cube. The folded 7-cube on 64 vertices has spectrum 7^1, 3^{21}, -1^{35}, -5^7, so ξ_f for the folded 7-cube is greater than or equal to 32/11.

More generally, if the inertial bound is exact for the fractional chromatic number of a non-singular graph, then it is also exact for the projective rank. Vertex transitive graphs have $\xi_f \leqslant \chi_f = n/\alpha$, so if a non-singular vertex transitive graph has

$$\alpha = \min(n^+, n^-), \text{ then } \xi_f = \chi_f = \frac{n}{\alpha}.$$

8 Conclusion

We have proved that many lower bounds for $\chi(G)$ are also lower bounds for $\xi(G)$. We have also proved that for non-singular graphs

$$1 + \max\left(\frac{n^+}{n^-}, \frac{n^-}{n^+}\right) \leqslant \xi_f(G).$$

Elphick and Wocjan [6] proved this lower bound for χ_f for non-singular graphs, using a simpler proof technique.

Costello *et al* [3] proved that almost all (random) graphs with no isolated vertices are non-singular. This provides limited support for our conjecture that the inertial lower bound for $\xi(G)$ is also a lower bound for $\xi_f(G)$ and consequently for $\chi_f(G)$.

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