Classes of graphs with e-positive chromatic symmetric function

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Abstract

In the mid-1990s, Stanley and Stembridge conjectured that the chromatic symmetric functions of claw-free co-comparability (also called incomparability) graphs were e-positive. The quest for the proof of this conjecture has led to an examination of other, related graph classes. In 2013 Guay-Paquet proved that if unit interval graphs are e-positive, that implies claw-free incomparability graphs are as well. Inspired by this approach, we consider a related case and prove that unit interval graphs whose complement is also a unit interval graph are e-positive. We introduce the concept of strongly e-positive to denote a graph whose induced subgraphs are all e-positive, and conjecture that a graph is strongly e-positive if and only if it is (claw, net)-free.

Mathematics Subject Classifications: 05E05, 05C15

1 Introduction

A 1995 paper of Stanley [19] introduced the chromatic symmetric functions and proved a host of properties about them. A key element of this foundational paper was a conjecture due to Stanley and Stembridge (originally stated in other terms in [21]) that the chromatic symmetric functions of claw-free co-comparability (also called incomparability) graphs had the property known as e-positivity (defined in Section 2). As of this writing, this conjecture remains unproved, and work on it and on related results has fueled research

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in the area for over 20 years. A fundamental contribution to this endeavour was Guay-Paquet's result that if Stanley and Stembridge's conjecture holds for unit interval graphs, then it holds for claw-free co-comparability graphs [8]. This result has put a spotlight on unit interval graphs. In related work, Shareshian and Wachs [17] generalized the conjecture of Stanley and Stembridge, introducing an extra t parameter, and also conjectured a relation between the chromatic symmetric function and the natural representation of the symmetric group on the cohomology of an algebraic variety. In this setting, Harada and Precup [9] proved a special case of the Shareshian and Wachs generalization. Cho and Huh [1] and Harada and Precup [9] have proved e-positivity for several subclasses of unit interval graphs. The time is ripe for further investigations of subclasses and superclasses of unit interval graphs.

Graphs and their complements are natural pairs to study. Recall that the *complement* of a graph $G = (V, E)$ is the graph with vertices V and edges between any two vertices $u, v \in V$ precisely when there is no edge between u and v in G. We use notations $\overline{G}$ or *co-G* to refer to the complement of G. The (claw, co-claw)-free graphs hold particular interest. Two of the authors investigated them in [10], concluding they were not all e-positive. Here we revisit this result, showing that the particular graph called the *net* is the only exception. This result follows by careful consideration of the graph structure, and subsequent decomposition into constituent graphs. From this analysis, along with a number of powerful graph theory results, we derive a series of results, culminating in a theorem that states that if a graph G and its complement are both unit interval graphs, then G is e-positive. We present two proofs of this, one following along the lines we set out here, and one due to Timothy Y. Chow, using classical results to show that graphs that satisfy these conditions are e-positive.

The graph class universe we are working in is captured by Figure 1. The class of claw-free co-comparability graphs targeted by Stanley and Stembridge wholly contains the subclass of unit interval graphs. If we look at the larger picture we see that the superclass of claw-free, AT-free graphs (see definition of AT-free in Section 2) consists of co-triangle-free graphs (known to be e-positive [18], restated in Theorem 4) and claw-free co-comparability graphs. Thus proving the Stanley and Stembridge conjecture would prove all claw-free, AT-free graphs were e-positive.

Even farther beyond this is the class of (claw, net)-free graphs. The net (see Figure 2) is significant as this is the example originally given by Stanley [19] of a claw-free, non-e-positive, graph to show claw-free alone is not a property sufficient to guarantee e-positivity. We focus particularly on (claw, net)-free graphs (note that for $n = 4$ there is one non-e-positive graph (namely, the claw, $K_{1,3}$), for $n = 5$ there are 4 non-e-positive connected graphs (namely $K_{1,4}$, dart, cricket $= K_{1,4} + e$, co-$\{K_3 \cup 2K_1\}$), for $n = 6$, there are 44 non-e-positive connected graphs, and for $n = 7$ there are 374 non-e-positive connected graphs). To our knowledge, this paper is the first exploration of the (claw, net)-free e-positivity question. We conjecture these graphs are e-positive. We have verified our conjecture for graphs up to 9 vertices. We also introduce the term *strongly e-positive* to denote graphs whose induced subgraphs are also e-positive, and we conjecture a graph is strongly e-positive if and only if it is (claw, net)-free.
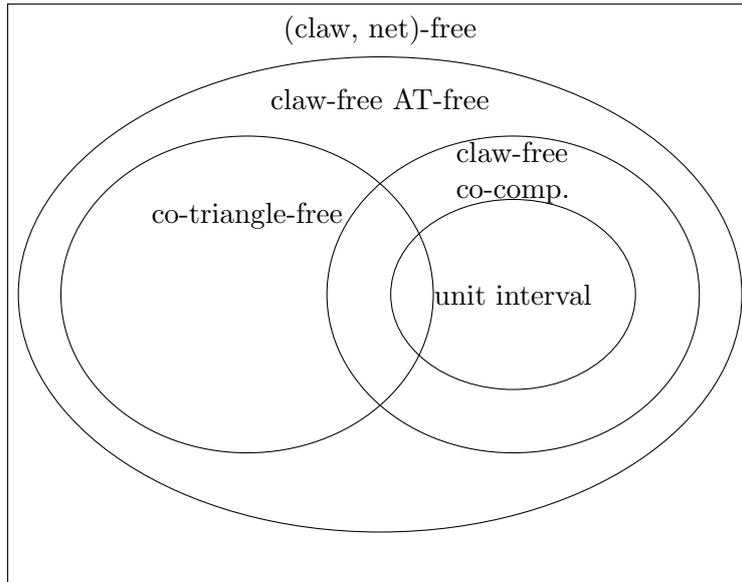
Figure 1: Classes of (claw, net)-free graphs. If the graphs are connected, then a claw-free AT-free graph is either a co-triangle-free graph or a claw-free co-comparability graph (or both).

The paper is structured as follows. Section 2 covers background and notation from both graph theory and symmetric function theory. It also summarizes much of what is already known about which graphs are e-positive. Section 3 proves our result on the e-positivity of unit interval graphs whose complement is also a unit interval graph. Along the way we consider the e-positivity question for (claw, co-claw)-free graphs. Section 4 contains our conjectures about strongly e-positive graphs and about (claw, net)-free graphs.

(a) The net   (b) The bull   (c) The chair
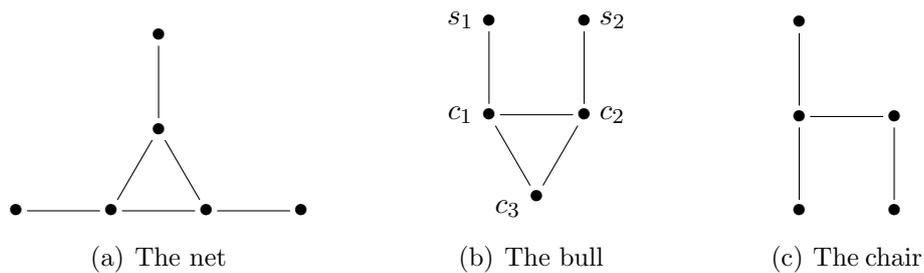
Figure 2: The net, bull and chair graphs.
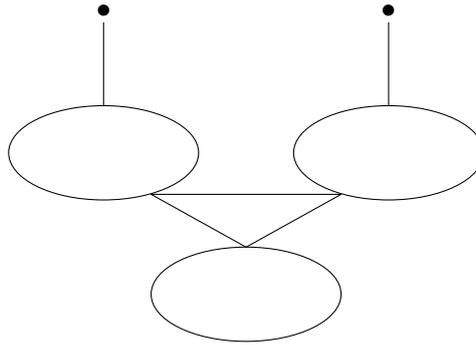
Figure 3: The generalized bull

2   Background and Notation

We begin by defining both graph theory and symmetric function terms and notation. Let $G = (V, E)$ be a finite, simple, undirected graph with vertex set V and edge set E. We assume all graphs are connected, an assumption necessary because of Lemma 2. For vertices $u, v \in V$, define $d(u, v)$ to be the length of the shortest path between u and v. For a vertex $v \in V$, the open neighbourhood of v is defined by $N(v) = \{u \in V : uv \in E\}$. For $U \subseteq V$, let $[U]$ denote the induced subgraph of G induced by U. For a set H of graphs, define H-*free* to be the class of graphs that do not contain any graph in H as an induced subgraph.

Let P_k be the chordless path on k vertices and C_k be the chordless cycle on k vertices. The *complete graph* (or *clique*) K_n is the graph on n vertices such that there is an edge between all pairs of vertices. A K-*chain* is a graph that is a sequence of complete graphs attached to one another sequentially at a single vertex, i.e. a vertex can belong to at most two maximal cliques. The graph K_3 is called the *triangle*, and its complement $3K_1$, is called the *co-triangle*. The *bull graph* is the graph on 5 vertices and 5 edges arranged as a triangle with two pendant edges. See Figure 2. The *generalized bull graphs* are the family of graphs that can be constructed from the bull graph where each vertex in the triangle of the bull is substituted by a clique (nonempty), i.e. every vertex is replaced by a clique and there are all edges between any two of these cliques. See Figure 3.

A *stable set* is a set S of vertices of a graph such that there are no edges between any of the vertices in S, e.g. a co-triangle is a stable set of size 3. Let $\alpha(G)$ denote the size of the largest stable set in G. An *astroidal triple (AT)* in a graph G is a stable set of three vertices in G such that for any pair of vertices in the set, there is a path between them that does not intersect the neighbourhood of the third. A graph is called *AT-free* exactly when it does not contain an astroidal triple.

A *coloring* of a graph G is a function κ from V to the positive integers $\mathbb{Z}^+$: $\kappa : V \to \mathbb{Z}^+$. A coloring κ is *proper* if $\kappa(u) \neq \kappa(v)$ whenever vertex u is adjacent to vertex v. Chromatic symmetric functions were defined by Stanley [19] as a generalization of the chromatic polynomial. Indeed, if we set $x_1 = x_2 = x_3 = \ldots = x_n = 1$ and any remaining variables

to zero, this expression reduces to the chromatic polynomial for a graph.

Definition 1. For a graph G with vertex set $V = \{v_1, v_2, \ldots, v_N\}$ and edge set E, the chromatic symmetric function is defined to be

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

where the sum is over all proper colorings κ of G.

A function is *symmetric* if a permutation of the variables does not change the function. In precise terms, using the wording of Stanley [18, p286], "it is a formal power series $\sum_{\alpha} c_{\alpha} x^{\alpha}$ where (a) α ranges over all weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ of n (of infinite length), (b) $c_{\alpha} \in R$, (c) x^{α} stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$, and (d) $f(x_{w(1)}, x_{w(2)}, \ldots) = f(x_1, x_2, \ldots)$ for every permutation w of the positive integers" (note that R is a commutative ring with identity). Full background details can be found in Macdonald [16] or Stanley [18]. It is well-known that certain sets of symmetric functions act as bases for the algebra of symmetric functions. One such set is the set of elementary symmetric functions. The *i*th *elementary symmetric function*, $e_i(x)$, is defined as

$$e_i(x) = \sum_{j_1 < j_2 < \cdots < j_i} x_{j_1} \cdots x_{j_i}.$$

We can extend this definition using partitions. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of a positive integer n is a nonincreasing sequence of positive integers: $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_\ell$, where λ_i is called the *i*th part of λ, $1 \leqslant i \leqslant \ell$. The *transpose*, λ', of λ, is defined by its parts: $\lambda'_i = |\{j : \lambda_j \geqslant i\}|$. The elementary symmetric function, $e_\lambda(x)$, is defined as $e_\lambda(x) = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_\ell}$. Note that we will use the notation e_i for $e_i(x)$ and e_λ for $e_\lambda(x)$.

If a given symmetric function can be written as a nonnegative linear combination of elementary symmetric functions we say the symmetric function is *e-positive*. By abuse of notation we say a graph is *e*-positive if its chromatic symmetric function is *e*-positive. Furthermore, we say that a class of graphs is *e*-positive if every graph in the class is *e*-positive.

Note that the property of a graph being *e*-positive is not hereditary. That is, if a graph is *e*-positive, all of its induced subgraphs are not necessarily *e*-positive. For example, the chair (or fork) graph (see Figure 2) is *e*–positive with chromatic symmetric function $X_F = e_{2,2,1} + 2e_{3,1,1} + e_{3,2} + 7e_{4,1} + 5e_5$, but contains an induced claw $K_{1,3}$ which is not *e*-positive, as $X_{K_{1,3}} = e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}$. In Section 4 we consider graphs whose induced subgraphs are all *e*-positive and dub these graphs *strongly e-positive*.

The following lemmas from Stanley [19] is useful in constructing new classes of graphs:

Lemma 2 ([19], Proposition 2.3). *If a graph G is a disjoint union of subgraphs $G_1 \cup G_2$, then $X_G = X_{G_1} X_{G_2}$.*

Lemma 3 ([19], Corollary 3.6). *If the vertices can be partitioned into disjoint cliques, then the graph is e-positive.*
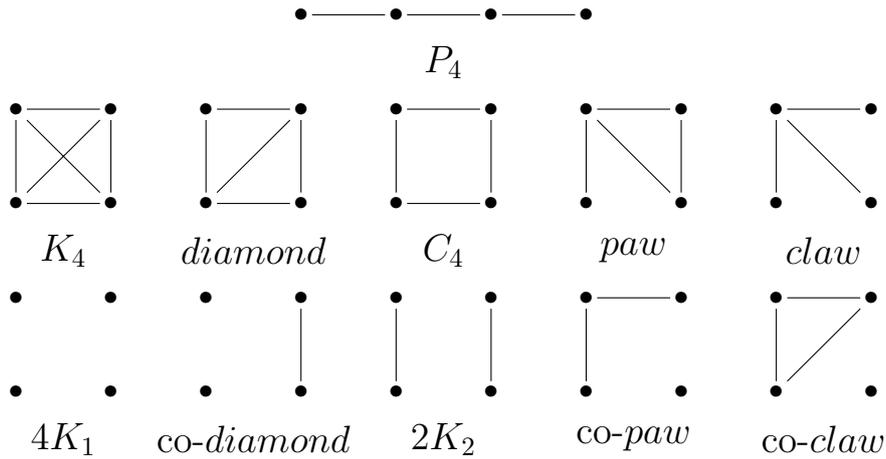
Figure 4: All four-vertex graphs.

Set H	Positivity	Reference
P_3	e-positive	Theorem 5
$3K_1$	e-positive	[18]
claw, K_3	e-positive	Theorem 5
claw, co-P_3	e-positive	Theorem 5

Figure 5: Table of e-positivity for H-free graphs where H contains a three-vertex graph.

Set H	Positivity	Reference
claw, P_4	e-positive	[22]
claw, paw	e-positive	[10]
claw, co-paw	e-positive	[10]
claw, co-claw (excluding the net)	e-positive	Corollary 17
claw, co-diamond	conjectured e-positive	Conjecture 22
claw, diamond	not necessarily e-positive	[10]
claw, K_4	not necessarily e-positive	[10]
claw, $4K_1$	not necessarily e-positive	[10]
claw, C_4	not necessarily e-positive	[10]
claw, $2K_2$	not necessarily e-positive	[10]

Figure 6: Table of e-positivity for H-free graphs where H contains four-vertex graphs. Note that the exceptional case, the net, is not e-positive.

The following theorem summarizes the e-positive status of a number of graph classes:

Theorem 4. *The following graph classes are known to be e-positive (proofs in the individual references given):*

1. P_k [19]

2. C_k [19]

3. K_n [2]; implicit in [19]

4. co-triangle-free [18, p514]

5. K-chains [7]

6. generalized bull [1]

7. (claw, P_4-free) [22]

8. (claw, paw)-free [10]

9. (claw, co-paw)-free [10]

Theorem 5. *The following graph classes are e-positive:*

1. P_3-free

2. (claw, triangle)-free

3. (claw, co-P_3)-free

Proof. We have the following arguments:

Item 1: If G is P_3-free then the components of G are cliques. By [2], restated in Theorem 4, and Lemma 2, G is e-positive.

Item 2: If G is (claw, triangle)-free, then each component of G is a chordless path or cycle. Then together with [19], restated in Theorem 4, and Lemma 2, G is e-positive.

Item 3: The class of (claw, co-P_3)-free graphs is a subclass of (claw, co-paw)-free graphs which was shown to be e-positive in [10]. □

Finally, this is Stanley and Stembridge's celebrated conjecture. They expressed it in terms of incomparability graphs of (3+1)-free posets.

Conjecture 6 ([21])**.** A claw-free, co-comparability graph is e-positive.

In [10] classes of (claw, H)-free graphs were analyzed, where H is a graph on 4 vertices. However several classes of graphs, including (claw, diamond)-free graphs and (claw, co-claw)-free graphs are not necessarily e-positive, as was demonstrated using a counter-example. The counter-example that is used is a six-vertex graph called the net (Figure 2). Here we extend the results of [10] to remark that there are infinite families of non-e-positive graphs:

Theorem 7. *There are infinitely many (claw, diamond)-free graphs, (claw, C_4)-free graphs, (claw, K_4)-free graphs, and (claw, $2K_2$)-free graphs that are not e-positive.*

Proof. The family of triangle tower graphs, described in [4] are not e-positive, but are (claw, diamond, C_4, K_4)-free. The family of generalized nets, described in [5] are not e-positive, but are (claw, $2K_2$)-free. $\square$

Returning to Stanley's singular counter-example—the net—we focus on this special graph. In the next section we will show it is the only (claw, co-claw)-free graph that is not e-positive. We also note that the net contains an asteroidal triple and this causes us to turn our focus to AT-free graphs. In particular, we note this significant result of Kloks, Kratsch, and Müller which shows that the claw-free, co-comparability graphs are one half of the set of (claw, AT)-free graphs.

Theorem 8 ([14]). *A connected graph G is claw-free and AT-free if and only if at least one of the following holds:*

1. *G is a claw-free co-comparability graph.*

2. *G is co-triangle-free.*

Together with a result from [18] (restated in Theorem 4), Conjecture 6 would imply that the class of claw-free AT-free graphs is e-positive.

3   Unit interval graphs

An *interval graph G* is a graph whose vertices can be represented by intervals on a straight line where two vertices in G are adjacent if and only if their corresponding intervals intersect. A *unit interval graph* is an interval graph whose intervals are given by unit lengths. It has been shown in [15] that interval graphs are exactly the class of chordal *AT*-free graphs, where a graph is a *chordal graph* if each of its cycles of length at least four has a chord, i.e. an edge that is not part of the cycle but connects two vertices of the cycle. At the same time, unit interval graphs have been shown to be exactly the class of claw-free interval graphs [6]. Guay-Paquet [8] has proved that Conjecture 6 can be reduced to the statement that the chromatic symmetric function of unit interval graphs are e-positive. So proving certain classes of unit interval graphs are e-positive will support this conjecture. This gives the motivation to study H-free unit interval graphs or equivalently (claw, H)-free *AT*-free chordal graphs.

Graph H	Positivity	Reference
P_4	e-positive	[22]
paw	e-positive	[10]
co-paw	e-positive	[10]
co-claw	e-positive	Theorem 18
diamond	e-positive	Theorem 10
co-diamond	unknown	unknown
K_4	unknown	unknown
$4K_1$	unknown	unknown
$2K_2$	unknown	unknown

Figure 7: Table of e-positivity results for H-free unit interval graphs where H is a four-vertex graph.

We investigate two different angles on the e-positivity question on unit interval graphs: 1) unit interval graphs that are H-free for H a four vertex graph; and, 2) co-claw-free unit interval graphs.

3.1 H-free unit interval graphs

The table in Figure 7 summarizes what is known about H-free unit interval graphs, where H is a four vertex graph. This lemma from Hempel and Kratsch is required in what follows:

Lemma 9 ([11]). *Let G=(V,E) be a claw-free, AT-free graph. Let $N_0 = \{w\}$, $N_1 = N(w)$, $N_2, \ldots, N_i = \{x \in V | d(x,w) = i\}$ for $i \geqslant 2$. Then the following holds:*

1. *N_i is a clique for all $i = 0, 2, 3, \ldots$*

2. *$\alpha([N_1]) \leqslant 2$.*

Theorem 10. *If G is a diamond-free unit interval graph, then G is e-positive.*

Proof. Let $G = (V, E)$ be a diamond-free unit interval graph. From Lemma 9, fix $w \in V$ and define $N_0 = \{w\}$, $N_1 = N(w)$, $N_2, \ldots, N_i = \{x \in V | d(x,w) = i\}$ for $i \geqslant 2$. By Lemma 9, N_i is a clique for all $i \neq 1$. Since G is diamond-free, $[N_1]$ must be P_3-free. Then since $\alpha([N_1]) \leqslant 2$, either $[N_1]$ is a complete graph or the disjoint union of two complete graphs.

Case 1. Suppose $[N_1]$ is a complete graph.

Every vertex in N_2 must have exactly one neighbour in N_1. If $y \in N_2$ has two neighbours $x_1, x_2 \in N_1$ then $\{w, x_1, x_2, y\}$ induces a diamond. Then every vertex in N_2 must be adjacent to the same vertex in N_1 say x, or G will contain an induced C_4, which contradicts G being chordal. For $i \geqslant 0$, this arguement can be continuely applied to N_{i+2} since there is a vertex in N_i that every vertex in N_{i+1} is adjacent to. Therefore G is a K-chain and by [7], restated in Theorem 4, G is e-positive.

Figure 8: A generalized pyramid

Case 2. Suppose $[N_1]$ is the disjoint union of two complete graphs.

Note that no vertex in N_2 can have more than one neighbour in N_1. If a vertex in N_2 has two neighbours in the same component of $[N_1]$, then G will contain an induced diamond. If a vertex in N_2 has a neighbour in each component of $[N_1]$, then G will contain an induced C_4. Thus every vertex in N_2 has exactly one neighbour in N_1. Every vertex in N_2 has the same neighbour in N_1 or G will contain an induced C_4 or C_5. Then one component of $[N_1]$ has no neighbours in N_2. For $i \geqslant 0$, a similar argument can be continually applied to N_{i+2} since there is a vertex in N_i that every vertex in N_{i+1} is adjacent to. Therefore G is a K-chain and by [7], restated in Theorem 4, G is e-positive. □

We also remark that it can be shown that a graph is a K-chain graph if and only if it is diamond-free unit interval graph.

In the next theorem we will make use of the following term: G is a *generalized pyramid* if every oval is a clique and there are all edges between any two ovals if there was an edge between them in the original graph (see Figure 8).

Theorem 11. *If G is a $(2K_2$, co-diamond)-free unit interval graph, then G is e-positive.*

Proof. From [10], it was determined that (claw, co-diamond, $2K_2$)-free graphs that are not known to be e-positive are the generalized pyramid graphs. Note that if all 3 ovals of G are non-empty, then G contains an astroidal triple $\{a, b, c\}$. Since unit interval graphs are AT-free, it must be the case that one or two ovals of G are empty. In either case, G is a generalized bull graph, and from [1], restated in Theorem 4, G is e-positive. □

As a side issue we can consider the families of $2K_2$-free unit interval graphs that are not known to be e-positive. Let $G = (V, E)$ be a $2K_2$-free unit interval graph. We can assume G is connected since if not G has at most one component that is not an isolated vertex. From Lemma 9, fix $w \in V$ and define $N_0 = \{w\}$, $N_1 = N(w)$, $N_2, \ldots, N_i = \{x \in V | d(x, w) = i\}$ for $i \geqslant 2$. Since G is $2K_2$-free, $N_i = \varnothing$ for all $i \geqslant 4$. Then by Lemma 9, N_i is a clique for $i = 0, 2, 3$ and $\alpha([N_1]) \leqslant 2$. If N_3 has an edge $\{z_1 z_2\}$ then for any vertex $x \in N_1$, $\{w, x, z_1, z_2\}$ induces a $2K_2$ in G. Then either N_3 is empty or N_3 has a single vertex.

Case 1. Suppose $[N_1]$ is not connected.

Then $[N_1]$ has two components, say $N_{1,0}$ and $N_{1,1}$, otherwise G contains an induced claw. At least one component of $[N_1]$ is a single vertex, say $N_{1,1}$, and the other is a clique. If N_2 is empty, then G can be partitioned into 2 cliques, $N_0 \cup N_{1,0}$ and $N_{1,1}$, and by [19], restated in Lemma 3, G is e-positive. Now suppose $y_1 \in N_2$. Then vertices in N_2 can only have neighbours in one component of $[N_1]$ or G will have a chordless cycle on at least 4 vertices. Then if $N_3 \neq \varnothing$ or $[N_2]$ has an edge, this edge together with the edge formed by w and a vertex from the component of $[N_1]$ with no neighbours in N_2, will induce a $2K_2$ in G. Then N_2 has exactly one vertex and N_3 is empty. In this case G is a generalized bull, and by [1], restated in Theorem 4, G is e-positive.

Case 2. Suppose $[N_1]$ is connected.

Suppose N_3 has a single vertex, z. Any vertex $y_1 \in N_2$ that is adjacent to z must be adjacent to every vertex in N_1. If y_1 is not adjacent to $x \in N_1$, then $\{w, x, y_1, z\}$ induces a $2K_2$ in G. Also any vertex $y_2 \in N_2$ that is not adjacent to z must be adjacent to every vertex in N_1. If y_2 is not adjacent to $x \in N_1$ then $\{y_1, y_2, z, x\}$ induces a claw in G. It must also be the case that N_1 is a clique. If $x_1, x_2 \in N_1$ are not adjacent then $\{x_1, x_2, y_1, z\}$ induces a claw in G. Then G is a generalized bull and by [1], restated in Theorem 4, G is e-positive.

Now suppose N_3 is empty. Then if N_1 is a clique, G can be partitioned into two cliques, $N_0 \cup N_1$ and N_2, and by [19], restated in Lemma 3, G is e-positive. Then N_1 must contain an induced P_3.

The family of $2K_2$-free unit interval graphs that are not known to be e-positive have $[N_1]$ connected, $[N_1]$ contains an induced P_3, $\alpha([N_1]) = 2$, $N_2 \neq \varnothing$, and all $N_i = \varnothing$ for $i \geqslant 3$.

3.2 The structure of (claw, co-claw)-free graphs

As a preliminary to considering co-claw, unit interval graphs, we investigate (claw, co-claw)-free graphs. First observe that the net is a (claw, co-claw)-free graph that is not e-positive. Here we will show this is the only graph in this class that is not e-positive. Theorems 12 and 13 below are implicitly implied by the proof of Theorem 3 in [13]. For the sake of completeness, we will give a proof of both theorems here. Note that the complement of the net is called the 3-sun.

Theorem 12 ([13]). *Let G be a (claw, co-claw)-free graph. If G contains a triangle and a co-triangle that are vertex-disjoint, then G contains a net or a 3-sun as an induced subgraph.*

Proof. Let G be a (claw, co-claw)-free graph. Suppose G contains a triangle T with vertices c_1, c_2, c_3 and a co-triangle C with vertices s_1, s_2, s_3 such that the triangle and co-triangle are vertex-disjoint. We claim that

$$\text{For a triangle } R, \text{ every vertex in } G - R \text{ is adjacent to at least one vertex in } R. \tag{1}$$

If (1) failed then R and a vertex of $G - R$ with no neighbors in R would form a co-claw with R, a contradiction.

Suppose s_1 is adjacent to all vertices of the triangle T. Consider the triangle $\{s_1, c_1, c_2\}$. By (1), vertex s_2 is adjacent to c_1, or c_2. We may assume s_2 is adjacent to c_2. We have the following implications:

- the vertex c_2 is not adjacent to s_3, for otherwise $\{c_2, s_1, s_2, s_3\}$ induces a claw, a contradiction

- the vertex s_3 is adjacent to c_1, for otherwise $\{s_3, c_1, c_2, s_1\}$ induces a co-claw, a contradiction

- the vertex s_3 is adjacent to c_3, for otherwise $\{s_3, c_2, c_3, s_1\}$ induces a co-claw, a contradiction

- the vertex s_2 is not adjacent to c_1, for otherwise $\{c_1, s_1, s_2, s_3\}$ induces a claw, a contradiction

- the vertex s_2 is not adjacent to c_3, for otherwise $\{c_3, s_1, s_2, s_3\}$ induces a claw, a contradiction

- the vertex $\{s_2, c_1, c_3, s_1\}$ induces a co-claw, a contradiction.

Thus, we may assume that every s_i ($i = 1, 2, 3$) is non-adjacent to at least one c_j ($j = 1, 2, 3$). Consider the vertex s_1 and suppose s_1 is adjacent to two vertices of $\{c_1, c_2, c_3\}$. We may assume s_1 is adjacent to c_1, c_2 and non-adjacent to c_3. If c_1 is adjacent to both s_2, s_3, then $\{c_1, s_1, s_2, s_3\}$ induces a claw, a contradiction. Thus, we may assume c_1 is not adjacent to s_2. We have the following implications:

- the vertex s_2 is adjacent to c_2, for otherwise, $\{s_2, c_1, c_2, s_1\}$ induces a co-claw

- the vertex s_3 is not adjacent to c_2, for otherwise, $\{c_2, s_1, s_2, s_3\}$ induces a claw

- the vertex s_3 is adjacent to c_1, for otherwise $\{s_3, c_1, c_2, s_1\}$ induces a co-claw

- the vertex s_3 is adjacent to c_3, for otherwise $\{c_1, s_1, c_3, s_3\}$ induces a claw

- the vertex s_2 is adjacent to c_3, for otherwise $\{s_2, c_1, c_3, s_3\}$ induces a co-claw.

Now, $T \cup C$ induces a 3-sun in G. So, we may assume every vertex in C is adjacent to exactly one vertex in T. If two vertices of C are adjacent to the same vertex in T, then G contains a claw. It is now easily to see that $T \cup C$ induces a net in G. $\square$

Theorem 13 ([13]). *Let G be a (claw, co-claw)-free graph. If G contains a net or a 3-sun as an induced subgraph, then G is a net, or a 3-sun.*
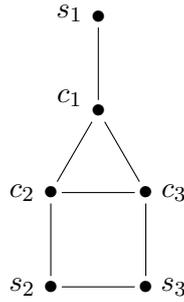
Figure 9: The antenna

Proof. Let G be a (claw, co-claw)-free graph. Suppose that G contains a net with vertices $c_1, c_2, c_3, s_1, s_2, s_3$ such that $C = \{c_1, c_2, c_3\}$ induces a triangle, $S = \{s_1, s_2, s_3\}$ induces a co-triangle, and c_i is adjacent to s_i for $i = 1, 2, 3$. Consider a vertex t not belonging to the net. Vertex t cannot be adjacent to all vertices of S, for otherwise $S \cup \{t\}$ induces a claw. So we may assume t is not adjacent to s_1. Suppose t is adjacent to c_1. Since $\{c_1, s_1, c_2, t\}$ cannot induce a claw, t must be adjacent to c_2. Similarly, since $\{c_1, s_1, c_3, t\}$ cannot induce a claw, t must be adjacent to c_3. Now $\{s_1, t, c_2, c_3\}$ induces a co-claw, a contradiction.

So we know t is not adjacent to c_1. Vertex t must be adjacent to c_2, or c_3 (or both), for otherwise, $\{t, c_1, c_2, c_3\}$ induces a co-claw, a contradiction. Without loss of generality, assume t is adjacent to c_2. Since $\{c_2, c_1, s_2, t\}$ cannot induce a claw, t must be adjacent to s_2. But now $\{s_1, t, c_2, s_2\}$ induces a co-claw. We now can conclude that if G contains a net, the G is a net. By considering the complement of G, it follows that if G contains a 3-sun, then G is a 3-sun. $\qquad\square$

Theorem 14. *Let G be a (claw, co-claw)-free graph. If G contains an antenna, then G is an induced subgraph of the graph F_1.*

Proof. Let G be a (claw, co-claw)-free graph. Suppose G contains an antenna A with vertices $c_1, c_2, c_3, s_1, s_2, s_3$ as indicated by Figure 9 (the vertices c_i's form a clique, the set $\{s_1, s_2, s_3\}$ contains an unique edge $s_2 s_3$, and s_i is adjacent to c_i for $i = 1, 2, 3$.). Let D_1 be the set of vertices in $G - A$ that are adjacent to s_2, s_3, c_1, s_1 and no other vertices in A. Let D_2 be the set of vertices in $G - A$ that are adjacent to s_1, c_2, s_2 and no other vertices in A. Let D_3 be the set of vertices in $G - A$ that are adjacent to s_1, c_3, s_3 and no other vertices in A. We claim that

Every vertex in $G - A$ belong to $D_1 \cup D_2 \cup D_3$. $\qquad\qquad(2)$

Let v be a vertex in $G - A$. Consider the triangle T with vertices c_1, c_2, c_3. Then v must be adjacent to at least one vertex in T, for otherwise v and T form a co-triangle. Suppose that v is adjacent to both c_2 and c_3. We have the following implications:

- the vertex v is adjacent to s_1, for otherwise, $\{s_1, v, c_2, c_3\}$ induces a co-claw

- the vertex v is not adjacent to s_3, for otherwise, $\{v, s_1, c_2, s_3\}$ induces a claw

- the vertex v is adjacent to c_1, for otherwise, $\{c_3, c_1, v, s_3\}$ induces a claw

- the vertex $\{s_3, v, c_1, s_1\}$ induces a co-claw, a contradiction.

So v must be non-adjacent to c_2, or c_3. Suppose that v is non-adjacent to both c_2 and c_3. We will show that v must be in D_1. Note that v is adjacent to c_1 for otherwise $\{v, c_1, c_2, c_3\}$ induces a co-claw. We have the following implications:

- the vertex v is adjacent to s_1, for otherwise $\{c_1, s_1, v, c_3\}$ induces a claw

- the vertex v is adjacent to s_3, for otherwise $\{s_3, v, c_1, s_1\}$ induces a co-claw

- the vertex v is adjacent to s_2, for otherwise $\{s_2, v, c_1, s_1\}$ induces a co-claw

Thus, v belongs to D_1. So we may assume v is adjacent to exactly one vertex of $\{c_2, c_3\}$. Suppose that v is adjacent to c_3 but not to c_2. We will show that v belongs to D_3. We have the following implications:

- the vertex v is adjacent to s_3, for otherwise, $\{c_3, c_2, v, s_3\}$ induces a claw

- the vertex v is adjacent to s_1, for otherwise, $\{s_1, v, c_3, s_3\}$ induces a co-claw

- the vertex v is non-adjacent to s_2, for otherwise, $\{v, s_1, c_3, s_2\}$ induces a claw

- the vertex v is non-adjacent to c_1, for otherwise, $\{s_2, v, c_1, s_1\}$ induces a co-claw

Thus, v belongs to D_3. By symmetry, if v is adjacent to c_2 but not to c_3, then v belongs to D_2. We have established (2). If some D_i contains at least two vertices, then it is easy to see that G contains a claw, or co-claw. Thus G has at most 9 vertices and is an induced subgraph of F_1. $\qquad\square$

Theorem 15. *Let G be a (claw, co-claw)-free graph. If G does not contain a net, a 3-sun, or an antenna, and contains a bull, then G is an induced subgraph of the graph F_2.*

Proof. Let G be a (claw, co-claw)-free graph that does not contain a net, or an antenna, but contains a bull B. Let the vertices of the bull B be c_1, c_2, c_3, s_1, s_2 as indicated by Figure 2 (the set $\{c_1, c_2, c_3\}$ forms a triangle, s_1 is adjacent to c_i for $i = 1, 2$). We may assume G is not the bull, for otherwise we are done. Let X be the set of vertices of $G - B$ that are adjacent to s_1, c_1, s_2 and non-adjacent to c_2, c_3. Let Y be the set of vertices of $G - B$ that are adjacent to s_1, c_2, s_2 and non-adjacent to c_1, c_3. We are going to show that every vertex in $G - B$ belongs to $X \cup Y$.

Consider a vertex $v \in G - B$. Suppose that v is adjacent to c_3. Then v is non-adjacent to at least one vertex of $\{s_1, s_2\}$, for otherwise, $\{v, s_1, s_2, c_3\}$ induces a claw, a contradiction. Without loss of generality, assume v is non-adjacent to s_1. Then v is non-adjacent to c_2, for otherwise, $\{s_1, v, c_2, c_3\}$ induces a co-claw. Now, v is non-adjacent

(a) F_1 (b) F_2
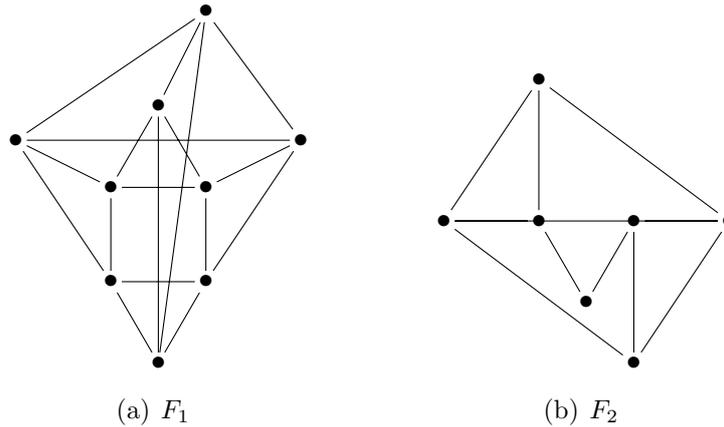
Figure 10: The graphs F_1 and F_2

to c_1, for otherwise, $\{c_1, s_1, v, c_2\}$ induces a claw. But then the set $\{c_1, c_2, c_3, s_1, s_2, v\}$ induces a net or an antenna in G, a contradiction. So, v is non-adjacent to c_3.

Consider the triangle with vertices c_1, c_2, c_3. Vertex v must be adjacent to c_1, or c_2, for otherwise, the triangle and v induce a co-claw. Suppose that v is adjacent to c_1. Now v must be adjacent to s_1, for otherwise, $\{c_1, s_1, v, c_3\}$ induces a claw. Then v must be adjacent to s_2, for otherwise, $\{s_2, s_1, c_1, v\}$ induces a co-claw. Vertex v is non-adjacent to c_2, for otherwise, B and v induce a 3-sun, and we are done. Now we know $v \in X$. Similarly, if v is adjacent to c_2, the $v \in Y$. If X or Y contains two vertices, then it is easy to see G contains a claw, or co-claw. So, G has at most seven vertices and is an induced subgraph of the graph F_2. □

Let $\mathcal{F}_1$ (respectively, $\mathcal{F}_2$) be the class of graphs G such that G or $\overline{G}$ contains an antenna (respectively, bull) and is an induced subgraph of F_1 (respectively F_2).

Theorem 16. *Let G be a (claw, co-claw)-free graph. Then one of the following holds.*

(i) *G or $\overline{G}$ contain no triangle.*

(ii) *G or $\overline{G}$ is the net.*

(iii) *G or $\overline{G}$ belong to $\mathcal{F}_1$.*

(iv) *G or $\overline{G}$ belong to $\mathcal{F}_2$.*

Proof. Let G be a (claw, co-claw)-free graph. We may assume G contains a triangle T and a co-triangle C, for otherwise, (i) holds, and we are done. The sets T and C cannot be vertex-disjoint, for otherwise, (ii) holds by Theorems 12 and 13, and we are done. So, T and C must intersect at one vertex. It is easy to verify that $T \cup C$ induces a bull. We may assume G contains no antenna, for otherwise, by Theorem 14, (iii) holds and we are done. By Theorem 15, G belongs to $\mathcal{F}_2$, and so (iv) holds, and we are done. □

Corollary 17. *If G is a (claw, co-claw)-free graph that is not isomorphic to the net, then G is e-positive.*

Proof. This follows directly from the Theorem 16 and the facts that all graphs in $\mathcal{F}_1$ and $\mathcal{F}_2$ were verified by computer program to be e-positive, (claw, triangle)-free graphs and co-triangle-free graphs are e-positive, and the complement of the net is e-positive. $\square$

The following theorem is one of our main results:

Theorem 18. *If G is a co-claw-free unit interval graph, then G is e-positive.*

Proof. The only (claw, co-claw)-free graph that is not e-positive is the net and the net is not a unit interval graph (since it contains an astroidal triple). Then it follows from Corollary 17 that co-claw-free unit interval graphs are e-positive. $\square$

Corollary 19. *If G is a unit interval graph and the complement of G is a unit interval graph, then G is e-positive.*

Proof. This follows directly from the previous theorem and the fact that if the complement of G is a unit interval graph, then G is co-claw-free. $\square$

In fact, the class of graphs stated in Theorem 18 can be shown to be e-positive by other means, i.e. this theorem does not provide any new examples of e-positive graphs. The following proof is due to Timothy Y. Chow [3]:

Proof. (Alternative proof of Theorem 18) If the graph is triangle-free, then since it is a unit interval graph, it must be a disjoint union of paths which, by [19] are e-positive. So assume G contains a triangle. Since G is a unit interval graph, assume G is represented as a set of unit intervals with vertices arranged left to right and that there is a triangle formed by three consecutive vertices a, b, and c. Moreover, since all unit interval graphs with at most five vertices are known to be e-positive, we may assume G has at least 6 vertices. This means there are two vertices we will call d and e either to the right of c or to the left of a. Without loss of generality, assume they are to the right of c. But then all vertices to the right of c, and, in particular d and e, must be adjacent to c and to each other (since G is co-claw-free) so these vertices all form a clique. By a similar argument, all vertices to the left of d form a clique. Since G is covered by two cliques, its complement cannot contain a triangle, and by Theorem 4, property 4, we know that complements of triangle-free graphs are e-positive. $\square$

4 Strongly e-positive graphs

As discussed in Section 1, the property of a graph being e-positive is not hereditary. This gives motivation to seek out which graphs have this special property. We coin the term *strongly e-positive* graphs for graphs with this property.

Definition 20. A graph G is strongly e-positive if for all induced subgraphs H of G, H is e-positive.

Note that the classes of claw-free co-comparability graphs, unit interval graphs, and (claw, co-diamond)-free graphs are all subclasses of (claw, net)-free graphs. See Figure 1. We conjecture that this class of graphs is exactly the class of strongly e-positive graphs.

Conjecture 21. A graph is strongly e-positive if and only if it is (claw, net)-free.

Clearly if a graph is strongly e-positive, then it is (claw, net)-free since both the claw and net are not e-positive. However, proving the other direction seems to be quite challenging. All (claw, net)-free graphs up to and including 9 vertices were verified by computer to be e-positive. This provides strong evidence in support of the conjecture.

The contrapositive of part of this conjecture is:

Conjecture 22. If G is not e-positive, then G contains an induced claw or an induced net.

Note a kinship between strongly e-positive and the *nice* property of Stanley [20] where a graph G is nice if whenever there is a stable partition of G of type λ (i.e. a partition into stable sets of size $\lambda_1, \lambda_2, \dots$) and whenever $\mu \leqslant \lambda$ in dominance order, there exists a stable partition of type μ. Then Proposition 1.6 of [20] states that a graph G and all its induced subgraphs are nice if and only if G is claw-free.

We introduce a further conjecture that, in conjunction with a result from [12], would show that (claw, bull)-free graphs are e-positive. The work of [12] shows that every connected (claw, bull)-free graph is either the generalized graph of a path or cycle, or the complement of a triangle-free graph. As we know from Theorem 4, the complement of a triangle-free graph is e-positive, so the only remaining type of (claw, bull)-free graph to consider are those that take the form of the generalized graph of a path or cycle. Again, by Theorem 4 we know paths and cycles are e-positive. Creating the generalized graphs of paths or cycles is akin to adding what are called twin vertices (defined below). Conjecture 23 proposes that this twinning preserves e-positivity.

Two vertices x and y are *twins* (also called *true twins*) if they are adjacent and any vertex z is adjacent to both x and y or non adjacent to both x and y. Let G be a graph, define G' to be the graph obtained from G by adding a vertex x that is a twin with some vertex y in G. Then the following conjecture would be implied by Conjecture 22:

Conjecture 23. If G is e-positive, then G' is e-positive.

Acknowledgements

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