Spanning trails in a 2-connected graph

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Abstract

In this article we prove the following: Let G be a 2-connected graph with circumference $c(G)$. If $c(G) \leqslant 5$, then G has a spanning trail starting from any vertex, if $c(G) \leqslant 7$, then G has a spanning trail.

As applications of this result, we obtain the following.

(1) Every 2-edge-connected graph of order at most 8 has a spanning trail starting from any vertex with the exception of six graphs.

(2) Let G be a 2-edge-connected graph and S a subset of $V(G)$ such that $E(G - S) = \emptyset$ and $|S| \leqslant 6$. Then G has a trail traversing all vertices of S with the exception of two graphs, moreover, if $|S| \leqslant 4$, then G has a trail starting from any vertex of S and containing S.

(3) Every 2-connected claw-free graph G with order n and minimum degree $\delta(G) > \frac{n}{7} + 4 \geqslant 23$ is traceable or belongs to two exceptional families of well-defined graphs, and moreover, if $\delta(G) > \frac{n}{6} + 4 \geqslant 13$, then G is traceable.

All above results are sharp in a sense.

Mathematics Subject Classifications: 05C38, 05C45

1 Introduction

A graph G is *simple* if it has no loops or parallel edges, otherwise we say that G is a *multigraph*. We consider finite simple undirected graphs $G = (V(G), E(G))$, and for concepts and notations not defined here we refer to [1].

Let G be a graph, and let H_1, H_2 be two subgraphs of G. For a vertex $v \in V(G)$, we define $N_{H_1}(v) = \{u \in V(H_1) \mid uv \in E(G)\}$. We define $N_{H_2}(H_1) = \bigcup_{v \in V(H_1)} N_{H_2}(v)$. The *degree* v in H_1 is denoted $d_{H_1}(v) = |N_{H_1}(v)|$. The *circumference* of G, denoted by $c(G)$, is the length of a longest cycle of G. For $S \subset V(G)$ (or $S \subset E(G)$), we use $\langle S \rangle_G$ to denote the subgraph of G *induced by* S. A *pendant vertex* of a graph is a vertex of degree 1, and a *pendant edge* is an edge having a pendant vertex as an end vertex.

Let G be a graph and H a subgraph of G. For $u, v \in V(H)$, the *distance* between u and v in H, denoted $\text{dist}_H(u, v)$, the length of a shortest path between u and v in H. The *contraction* G/H is the graph obtained from G by replacing H by a vertex v_H such that the number of edges in G/H joining any $v \in V(G - V(H))$ to v_H in G/H equal to the number of edges joining v in G to H. A graph G is *contractible* to a graph G' if G contains pairwise vertex-disjoint connected subgraphs $H_1, \cdots, H_t$ with $\bigcup_{i=1}^{t} V(H_i) = V(G)$ such that G' is obtained from G by successive contracting each H_i ($1 \leqslant i \leqslant t$). Each subgraph $H \in \{H_1, \cdots, H_t\}$ is called the preimage of the vertex v_H of G'. A vertex v_H in G' is *nontrivial* if v_H is the contraction image of a nontrivial connected subgraph H of G.

A graph is called *hamiltonian* if it contains a Hamilton cycle, i.e., a cycle containing all its vertices. A graph is called *traceable* if it contains a Hamilton path, i.e., a path containing all its vertices. A *trail* in a graph G is a sequence $W := v_0 e_1 v_2 \cdots v_{l-1} e_l v_l$, whose terms are alternately vertices (not necessarily distinct) and distinct edges of G, such that v_{i-1} and v_i are ends of e_i for $1 \leqslant i \leqslant l$. For convenience, we sometimes abbreviate the term $v_0 e_1 v_1 \cdots v_{l-1} e_l v_l$ to $v_0 v_1 \cdots v_{l-1} v_l$. A *spanning trail* of a graph G is a trail that contains all vertices of G. For a subset $S \subseteq V(G)$, if a trail of G traverses all vertices of S, then we call it *S-trail*. A subgraph H of a graph G is *dominating* if every edge of G has at least one end in H. A a graph G is *even* if every vertex of G has even degree. If H is a graph, then the *line graph* of H, denoted $L(H)$, is the graph with $E(H)$ as vertex set, in which two vertices are adjacent if and only if the corresponding edges have a vertex in common.

A vertex $x \in V(G)$ is *locally connected* if the neighborhood of x induces a connected subgraph in G. For $x \in V(G)$, the graph G'_x obtained from G by adding the edges $\{yz : y, z \in N_G(x) \text{ and } yz \notin E(G)\}$ is called the *local completion* of G at x. The closure of a claw-free graph G, denoted by cl(G), is obtained from G by recursively performing local completions at any locally connected vertex with non-complete neighborhood, as long as it is possible. If G is a claw-free graph such that $G = \text{cl(G)}$, then we say that G is *closed*.

A well-known result on spanning closed trail was obtained by Jaeger, later proved independently by Catlin.

Theorem 1. *(Catlin [4] and Jaeger [12]) Every 4-edge-connected graph has a spanning closed trail.*

Lai et al. presented a sufficient condition for spanning closed trail of 3-edge-connected graphs involved circumference, which was used in hamiltonian claw-free graphs in the same paper.

Theorem 2. *(Lai et al. [16]) Every 3-edge-connected graph G with $c(G) \leqslant 8$ has a spanning closed trail.*

The above result is sharp because the Pertersen graph has circumference 9 without any spanning closed trail. Chen et al. studied the existence of a spanning closed trail H

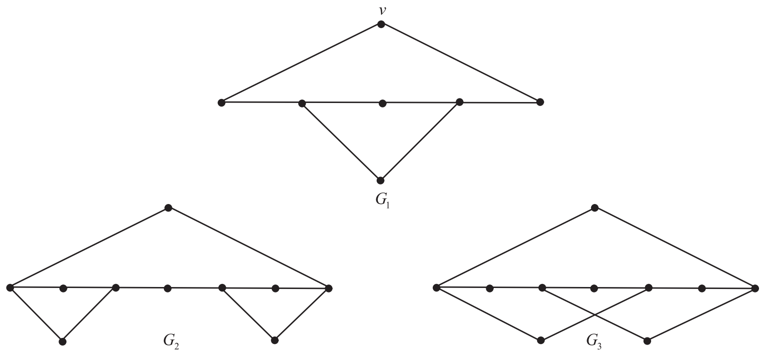

Figure 1: G_1 has no spanning trail starting from v, G_2 and G_3 have no spanning trail.

of 3-edge-connected graphs G such that H contains a given set of vertices of G, which was also used in the hamiltonian line graphs in their paper.

Theorem 3. *(Chen et al. [6]) Let G be a 3-edge-connected graph and let $S \subseteq V$ be a vertex subset such that $|S| \leqslant 12$. Then either G has a spanning closed trail H such that $S \subseteq V(H)$, or G can be contracted to the Petersen graph in such a way that the preimage of each vertex of the Petersen graph contains at least one vertex in S.*

Finding similar sufficient conditions for the existence of a spanning closed trail in a 2-edge-connected graph is somewhat trivial because $K_{2,3}$ is a counter-example even for the case $|S| = 3$. However, it is nontrivial to use the circumference condition to study the existence of a spanning trail of 2-connected graphs, we prove the following in this paper.

Theorem 4. *Let G be a 2-connected graph. Then*

(1) if $c(G) \leqslant 5$, then G has a spanning trail starting from any vertex,

(2) if $c(G) \leqslant 7$, then G has a spanning trail.

Theorem 5. *Let G be a 2-edge-connected graph and S a subset of $V(G)$ such that $E(G - S) = \emptyset$ and $|S| \leqslant 6$. Then G has an S-trail or $G \in \{G_2, G_3\}$, where G_2, G_3 are shown in Figure 1, moreover, if $|S| \leqslant 4$, then G has an S-trail starting from any vertex of S.*

Theorem 4 is sharp because of graphs G_1, G_2, G_3 shown in Figure 1. For more results involved spanning trail we refer to [3][5][8][15][18]. In [22], Tian et al. applied Theorem 4 to prove that every 2-edge-connected graph of order at most 11 has a spanning trail with the exception of six graphs. We also apply Theorem 4 to prove the similar result on special spanning trail.

Theorem 6. *Let G be a 2-edge-connected graph. If $|V(G)| \leqslant 8$, then either G has a spanning trail starting from any vertex or $G \in \{F_1, F_2, F_3, F_4, F_5, F_6\}$, where F_i is shown in Figure 2. Moreover, if $G \in \{F_1, F_2, F_3, F_4, F_5\}$, then G has no spanning trail starting from any vertex in $\{u, x_1, x_2\}$, but has a spanning trail starting from any vertex in $\{u_1, u_2, \cdots, u_7\}$.*

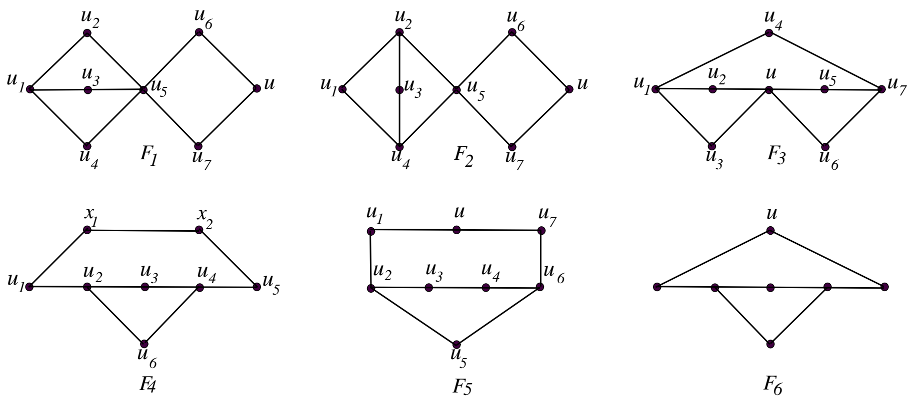

Figure 2: F_i ($1 \leqslant i \leqslant 6$) is a graph of order at most 8 that has no spanning trail starting from x_1, x_2 and u, but has a spanning trail starting from any vertex in $V(F_i)\backslash\{x_1, x_2\}$.

Actually, we use Theorems 4 and 6 to prove Theorem 5. As an application of Theorem 5, we obtain the following result (i.e., Theorem 7). We believe that Theorem 4 may have further applications in similar researches, as shown in the concluding remarks.

Before stating the following result, we need to define two families of graphs.

$\mathcal{C}_1 = \{H : H$ is obtained from G_2 shown in Figure 1, by adding at least one pendant edge to each vertex of degree two$\}$,

$\mathcal{C}_2 = \{H : H$ is obtained from G_3 shown in Figure 1, by adding at least one pendant edge to each vertex of degree two$\}$.

Theorem 7. *Let G be a 2-connected claw-free graph of order $n \geqslant 133$ such that $\delta(G) \geqslant \frac{n}{7} + 4$. Then G is traceable or* cl(G)= $L(H)$ *where* $H \in \mathcal{C}_1 \cup \mathcal{C}_2$.

As a corollary of Theorem 7, we prove the following result.

Corollary 8. *Let G be a 2-connected claw-free graph of order $n \geqslant 54$ with $\delta(G) > \frac{n}{6} + 4$. Then G is traceable.*

In fact, the results on claw-free graph are already known in [14], it is enough to check the exceptions for traceability. But only computer proofs were known so far.

In the next section, we will present some necessary results involved Ryjáček closure and Catlin reduction. In Section 3, we will complete the proof of Theorem 4. In Sections 4 and 5, we will prove Theorems 6 and 5, respectively. In Section 6, we will prove Theorem 7 and Corollary 8. In the final section, we will give some concluding remarks.

2 Preliminaries and basic results

2.1 Ryjáček closure

Ryjáček [20] first investigated the closure of a claw-free graph G, which becomes a useful tool in investigating hamiltonian properties of claw-free graphs. And in [20], he proved the following well-known theorem.

Theorem 9. *(Ryjáček [20]) Let G be a claw-free graph. Then*

(1) $\mathrm{cl(G)}$ *is uniquely determined,*

(2) $\mathrm{cl(G)}$ *is claw-free,*

(3) $\mathrm{cl(G)}$ *is the line graph of a triangle-free graph.*

Theorem 10. *(Brandt, Favaron and Ryjáček [2]) Let G be a claw-free graph. Then G is traceable if and only if* $\mathrm{cl(G)}$ *is traceable.*

2.2 Catlin reduction

For a graph G and a subgraph H of G, let $O(H)$ denote the set of all odd degree vertices in H. We say that G is *collapsible* if for every subset $X \subseteq V(G)$ with $|X|$ even, there is a spanning connected subgraph H_X of G such that $O(H_X) = X$. In [4], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, \cdots, H_t$ such that $\bigcup_{i=1}^{t} V(H_i) = V(G)$. The *reduction* G' of a graph G is the graph obtained from G by contracting each maximal collapsible subgraph H_i ($1 \leqslant i \leqslant t$) into a single vertex v_i.

Theorem 11. *(Catlin [4]) Let G be a connected graph. Then the reduction of G is a simple graph and has no cycle of length less than four.*

Theorem 12. *(Catlin [4]) Let G be a connected graph and H a collapsible subgraph of G. Then G has a spanning closed trail if and only if G/H has a spanning closed trail.*

Theorem 13. *(Xiong, et al. [25]) Let G be a connected graph and G' the reduction of G. Then G has a spanning trail if and only if G' has a spanning trail.*

We prove a similar result on special spanning trail.

Theorem 14. *Let G be a connected graph and G' the reduction of G. Then G has a spanning trail starting from any vertex of G if and only if G' has a spanning trail starting from any vertex of G'.*

Proof. G has a spanning trail starting from any vertex if and only if for any collapsible subgraph H of G, G/H has a spanning trail starting from any vertex in G/H. Note that a graph has a spanning trail T starting from v if and only if one can add at most one edge e_v to create a spanning closed trail. Also note that $G + e_v$ has a spanning closed trail if and only if $(G + e_v)/H$ has a spanning closed trail for any collapsible subgraph H. Thus Theorem 12 implies that Theorem 14 holds. □

3 The proof of Theorem 4

Let G be a 2-connected graph and C a cycle of G. Then every component D of $G-V(C)$ has at least two distinct neighbors on C. For any path P in D, if the two ends (possibly only one if P is a vertex) of P have two distinct neighbors x_1, x_2 on C, then P is called a 2-*attaching path* of C in D, and these two vertices x_1, x_2 are called a 2-*attaching pair* of P on C. We observe that if $D \cong K_1$ or K_2, then D is the 2-attaching path of C. Furthermore, if D has a longest 2-attaching path P of order k, then D is called a k-*component* of $G-V(C)$. For any cycle of a graph G, we can give C an orientation $\overrightarrow{C}$. We can bound any component of $G-V(C)$ by the circumference $c(G)$.

Lemma 15. *Let G be a 2-connected graph with circumference $c(G)$ and C a longest cycle of G. Then*

(1) every k-component D of $G-V(C)$ holds $k \leqslant \lfloor \frac{c(G)}{2} \rfloor - 1$,

(2) if $c(G) \leqslant 5$, then every vertex v of G lies on a longest cycle.

Proof. Suppose, by contradiction, that D is a k-component of $G-V(C)$ with $k \geqslant \lfloor \frac{c(G)}{2} \rfloor$. Then C has a 2-attaching path $P_k (k \geqslant \lfloor \frac{c(G)}{2} \rfloor)$ in D with a 2-attaching pair x_1, x_2, thus either $x_1 \overrightarrow{C} x_2 P_k x_1$ or $x_1 \overleftarrow{C} x_2 P_k x_1$ is a cycle of length at least $k + \lfloor \frac{c(G)}{2} \rfloor + 1 > c(G)$, a contradiction. This proves Lemma 15(1).

We now show (2), it suffices to consider the case when $v \notin V(C)$. Since $c(G) \leqslant 5$, by Lemma 15(1), v is a 1-component of $G-V(C)$ and let x_1, x_2 be a 2-attaching pair of v on C. Then either $x_1 \overrightarrow{C} x_2 v x_1$ or $x_1 \overleftarrow{C} x_2 v x_1$ is a cycle of length $c(G)$, we are done. □

We denote by $G[x; y_1, y_2, \cdots, y_t]$ a star with x as its center and $y_1, y_2, \cdots, y_t$ are its leaves.

Lemma 16. *Let G be a 2-connected graph and C a longest cycle of G, and let D be a 2-component of $G-V(C)$. Then*

(1) D is a star, denoted by $G[x; y_1, y_2, \cdots, y_t]$,

(2) if $6 \leqslant |V(C)| \leqslant 7$ and $t = 1$, then $|N_G(D) \cap V(C)| = 2$ and $2 \leqslant d_G(y_1) \leqslant 3$,

(3) if $6 \leqslant |V(C)| \leqslant 7$ and $t \geqslant 2$, then $|N_G(D) \cap V(C)| = 2$ and $d_G(y_i) = 2$, for $1 \leqslant i \leqslant t$.

Proof of Lemma 16. Let $C = v_0 v_1 \cdots v_{c(G)-1} v_0$ be a longest cycle of G. We have the following fact.

Claim 17. *D is a tree.*

Proof. Suppose, by contradiction, that there is a cycle C' in D. Since G is 2-connected, there exist two pairwise disjoint paths in G joining C' and C. This yields a 2-attaching path of C in D containing at least three vertices of C', which contradicts the fact that D is a 2-component of $G-V(C)$. This proves Claim 17. □

Claim 18. *$diam(D) \leqslant 2$.*

Proof. Suppose to the contrary that $diam(D) \geqslant 3$. Then there is a diameter path $P = x_1 \cdots x_k$ in D with $k \geqslant 4$. By Claim 17, D is a tree and hence x_1, x_k are two leaves of D. Since G is 2-connected, each x_i has a neighbor on C for $i = 1, k$. Since D is a 2-component of $G - V(C)$, P cannot be a 2-attaching path of C in D, implying that $|(N_G(x_1) \cup N_G(x_k)) \cap V(C)| = 1$, say $(N_G(x_1) \cup N_G(x_k)) \cap V(C) = \{v\}$. Since G is 2-connected, there exist two pairwise disjoint paths Q_1, Q_2 in G joining P and C, then one of them joins some internal vertex of P and some vertex of $V(C)\backslash\{v\}$. This yields a 2-attaching path of D containing at least three vertices, which contradicts the fact that D is a 2-component of $G - V(C)$. This proves Claim 18. □

By Claims 17 and 18, D is a star. This proves Lemma 16(1). Let $D \cong G[x; y_1, y_2, \cdots, y_t]$ be a star. Since $t = 1$, D is an edge xy_1. It is easy to see that xy_1 is the 2-attaching path of C. Let x_1, x_2 be a 2-attaching pair of xy_1 on C. Then $\text{dist}_C(x_1, x_2) \geqslant 3$, otherwise either $x_1\overrightarrow{C}x_2xy_1x_1$ or $x_1\overleftarrow{C}x_2xy_1x_1$ is a cycle of length at least $c(G) + 1$. Hence, by $6 \leqslant |V(C)| \leqslant 7$ we have $\text{dist}_C(x_1, x_2) = 3$. Therefore, since C is a longest cycle of G, $N_G(D) \cap V(C) = \{x_1, x_2\}$. This proves Lemma 16(2).

Since G is 2-connected and D is a star, $N_G(y_i) \cap V(C) \neq \emptyset$ for $1 \leqslant i \leqslant t$. By the definition of 2-component, $N_G(y_i) \cap V(C)$ and $N_G(y_j) \cap V(C)$ have the same vertex v_0 (say) for any pair of $\{i, j\} \subseteq \{1, 2, \cdots, t\}$, this implies that $d_G(y_i) = 2$ for $1 \leqslant i \leqslant t$. Since G is 2-connected, $(N_G(x) \cap V(C))\backslash\{v_0\} \neq \emptyset$, then xy_i is a 2-attaching path of C in D for $1 \leqslant i \leqslant t$. Therefore, since C is a longest cycle of G, $|(N_G(x) \cap V(C))\backslash\{v_0\}| = 1$. Then, since $N_G(y_i) \cap V(C) = N_G(y_j) \cap V(C) = \{v_0\}$ for any pair of $\{i, j\} \subseteq \{1, 2, \cdots, t\}$, we have $|N_G(D) \cap V(C)| = 2$. This proves Lemma 16(3). □

Let G be a 2-connected graph and C a cycle of G. Let D_1 and D_2 be two components of $G - V(C)$ and let P, P' be two 2-attaching paths of C in D_1 and D_2 with two 2-attaching pairs $\{x_1, x_2\}$ and $\{x_3, x_4\}$, respectively. If x_1, x_3, x_2, x_4 are four distinct vertices that lie along the direction of $\overrightarrow{C}$, then we say that D_1 *overlaps* D_2 on C. If $\{x_1, x_2\} = \{x_3, x_4\}$, then we say that D_1 is *equivalent* to D_2.

Proof of Theorem 4. Let $C = v_0v_1v_2 \cdots v_{c(G)-1}v_0$ be a longest cycle of G. By deleting all chords of C, the resulting 2-connected graph G_1 is a spanning subgraph of G. It suffices to show that G_1 has a spanning trail. If $V(G_1) \setminus V(C) = \emptyset$, then C is our desired spanning connected even subgraph. Hence we assume that $V(G_1) \setminus V(C) \neq \emptyset$ and we have the following fact.

Claim 19. *Any pair of components of $G_1 - V(C)$ cannot overlap on C.*

Proof. Suppose, by contradiction, that there is a pair of components D_1, D_2 in $G_1 - V(C)$ such that they overlap each other. Then there exist two 2-attaching paths P and P' of C in D_1 and D_2 with two 2-attaching pairs $\{x_1, x_2\}, \{x_3, x_4\}$ on C, respectively, such that x_1, x_3, x_2, x_4 are four distinct vertices along the orientation $\overrightarrow{C}$. Therefore, either

$x_1Px_2\overleftarrow{C}x_3P'x_4\overrightarrow{C}x_1$ or $x_1Px_2\overrightarrow{C}x_4P'x_3\overleftarrow{C}x_1$ is a cycle of length at least $\lceil\frac{|V(C)|}{2}\rceil+4$, then $\lceil\frac{|V(C)|}{2}\rceil+4>|V(C)|$ for $|V(C)|\leqslant 7$, a contradiction. This proves Claim 19. $\square$

Since C is a longest cycle of G, we clearly have the following two facts.

Claim 20. *For any 2-attaching path P of C and for each 2-attaching pair x_1,x_2 of P, it holds* $\mathrm{dist}_C(x_1,x_2)\geqslant 2$.

Claim 21. *For any 2-component D of $G_1-V(C)$ and for any longest 2-attaching path P of C in D, it holds* $\mathrm{dist}_C(x_1,x_2)\geqslant 3$ *for each 2-attaching pair x_1,x_2 of P.*

We first show Theorem 4(1). Fix a vertex v in G, it suffices to show that G_1 has a spanning trail starting from v. By Lemma 15(2), v lies on a longest cycle of G. Without loss of generality, we may assume that $v\in V(C)$. Let X be the set of all vertices in G_1 with degree odd. Since $c(G_1)\leqslant 5$ and C is a longest cycle of G_1, every component of $G_1-V(C)$ is a 1-component by Lemma 15(1). Again, since C is a longest cycle of G, every component of $G_1-V(C)$ has exactly two neighbors on C, implying that $X\subseteq V(C)$. If $|X|=4$, then there exist four vertices of $V(C)$ that are consecutive on C. Without loss of generality, we may assume that say $X=\{v_0,v_1,v_2,v_3\}$. Since C has no chord and every component of $G_1-V(C)$ has exactly two neighbors on C and by Claim 20, there exist two components w_1,w_2 of $G_1-V(C)$ such that $N_G(w_1)\cap\{v_0,v_1,v_2,v_3\}=\{v_0,v_2\}$ and $N_G(w_2)\cap\{v_0,v_1,v_2,v_3\}=\{v_1,v_3\}$, then w_1 overlaps w_2, contradicting Claim 19. Hence we have $|X|<4$, then $|X|=0,2$, it suffices to consider the case when $|X|=2$ (if $|X|=0$, then G_1 is a connected even graph, clearly G_1 has a spanning trail starting from v). If $v\in X$, then G_1 obviously has a spanning trail starting from v.

Hence we assume that $v\notin X$ and let $X=\{v_i,v_j\}$ with $i\neq j$. If $v_iv_j\in E(C)$, then $G_1\backslash v_iv_j$ is a connected even graph, clearly G_1 has a spanning trail starting from v. Hence we assume that $v_iv_j\notin E(C)$, then by $|V(C)|\leqslant 5$, there is a vertex in $\{v_i,v_j\}$, say v_i, that is adjacent to v, then $G_1\backslash vv_i$ is a connected spanning subgraph of G_1 with exactly two odd degree vertices v,v_i, and hence G has a spanning trail starting from v. This proves Theorem 4(1).

We now show Theorem 4(2). By Theorem 4(1), it suffices to consider the case $6\leqslant c(G)\leqslant 7$. Since C is a longest cycle of G_1 and by Lemma 16(2)-(3), we have the following.

Claim 22. *Every 2-component of $G_1-V(C)$ has exactly two neighbors on C.*

Claim 23. *For any pair of 2-components D_1 and D_2 of $G_1-V(C)$, it holds that $N_{G_1}(D_1)\cap N_{G_1}(D_2)\cap V(C)\neq\emptyset$.*

Proof. Suppose, by contradiction, that there is a pair of 2-components D_1,D_2 of $G_1-V(C)$ such that $N_{G_1}(D_1)\cap N_{G_1}(D_2)\cap V(C)=\emptyset$. Then by Claim 22, there exist four distinct vertices v_i,v_j,v_k,v_l on C such that v_i,v_j and v_k,v_l are two neighbors of D_1 and D_2, respectively. By Claim 19, we may, without loss of generality, assume that $0\leqslant i<j<k<l\leqslant c(G)-1$. Since C is a longest cycle of G and by the definition of 2-component, $\mathrm{dist}_C(v_i,v_j)\geqslant 3$ and $\mathrm{dist}_C(v_k,v_l)\geqslant 3$. This implies that $j-i\geqslant 3$ and $l-k\geqslant 3$, hence $|V(C)|\geqslant 8$, a contradiction. This proves Claim 23. $\square$

Claim 24. *The neighbors of all* 2*-components of* $G_1 - V(C)$ *on* C *are at most three.*

Proof. If all 2-components of $G_1 - V(C)$ are equivalent each other, then by Claim 22, the neighbors of all 2-components of $G_1 - V(C)$ on C are exactly two, we are done.

Hence we assume that there is a pair of 2-components D_1 and D_2 of $G_1 - V(C)$ such that they are not equivalent. By Claims 22 and 23, $|N_{G_1}(D_1) \cap N_{G_1}(D_2) \cap V(C)| = 1$, say $N_{G_1}(D_1) \cap V(C) = \{v_0, v_i\}$ and $N_{G_2}(D_1) \cap V(C) = \{v_0, v_j\}$ with $0 < i < j$. By Claim 21, we have $\text{dist}_C(v_0, v_i) \geqslant 3$ and $\text{dist}_C(v_0, v_j) \geqslant 3$. Hence, by $6 \leqslant c(G) \leqslant 7$ and $i < j$, we have $c(G) = 7$ and then $i = 3, j = 4$.

We will show that the neighbors of all 2-components of $G_1 - V(C)$ on C belong to $\{v_0, v_3, v_4\}$ to complete the remaining part of the claim, it is suffices to show that any 2-component of $G_1 - V(C)$ is equivalent to D_1 or D_2. Suppose not. Then $G_1 - V(C)$ has a 2-component D_3 is neither equivalent to D_1 nor to D_2, thus $(N_G(D_3) \cap V(C)) \backslash \{v_0, v_3, v_4\} \neq \emptyset$, say $v_k \in (N_G(D_3) \cap V(C)) \backslash \{v_0, v_3, v_4\}$. This implies that $v_k \in \{v_1, v_2\}$ or $v_k \in \{v_5, v_6\}$. Up to symmetry, we may assume that $v_k \in \{v_1, v_2\}$, then by Claims 19 and 22, there is only one vertex $v_l \in (N_{G_1}(D_3) \cap V(C)) \backslash \{v_k\}$ such that $v_l \notin \{v_4, v_5, v_6\}$ and hence $v_l \in \{v_0, v_1, v_2, v_3\} \backslash \{v_k\}$. Since $v_k, v_l \in N_{G_1}(D_3) \cap V(C)$, by Claim 22, there is a 2-attaching path P of C in D_3 with 2-attaching pair v_k, v_l, then either $v_k P v_l \overrightarrow{C} v_k$ or $v_k P v_l \overleftarrow{C} v_k$ is a cycle of length at least 8, a contradiction. This proves Claim 24. $\square$

By Lemma 16(1), every 2-component of $G_1 - V(C)$ is a star. By Claim 24, we may assume that $v_{i'}, v_{i''}, v_{i'''}$ are neighbors of all 2-components of $G_1 - V(C)$ on C and $2 \leqslant |\{v_{i'}, v_{i''}, v_{i'''}\}| \leqslant 3$. By Claim 22, every 2-component of $G_1 - V(C)$ has exactly two neighbors on C. By Claim 21 and $|V(C)| \leqslant 7$, no triple of 2-components F_1, F_2, F_3 of $G_1 - V(C)$ holds that $N_{G_1}(F_1) \cap V(C) = \{v_{i'}, v_{i''}\}$ and $N_{G_1}(F_2) \cap V(C) = \{v_{i''}, v_{i'''}\}$ and $N_{G_1}(F_3) \cap V(C) = \{v_{i'}, v_{i'''}\}$. Without loss of generality, we may let $\mathcal{H}_1 = \{D_1, D_2, \cdots, D_{s_1}\}$ be the set of all 2-components of $G_1 - V(C)$ such that $N_{G_1}(D_j) \cap V(C) = \{v_{i'}, v_{i''}\}$ and let $\mathcal{H}_2 = \{D_{s_1+1}, D_{s_1+2}, \cdots, D_{s_2}\}$ be the set of all 2-components of $G_1 - V(C)$ such that $N_{G_1}(D_j) \cap V(C) = \{v_{i''}, v_{i'''}\}$, where $D_j \cong G[x_j; y_{j,1}, y_{j,2}, \cdots, y_{j,t_j}]$. Let $\mathcal{H}_3 = \{D_{s_2+1}, D_{s_2+2}, \cdots, D_{s_3}\}$ be the set of all 1-components of $G_1 - V(C)$. Since $6 \leqslant c(G_1) \leqslant 7$ and C is a longest cycle of G_1, $G_1 - V(C)$ has no i-component with $i \geqslant 3$ by Lemma 15(1). This implies that $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$ is the set of all components of $G_1 - V(C)$.

Without lose of generality, let $\mathcal{H} = \{D_1, \cdots, D_s\}$ be the set of all 2-components of $G_1 - V(C)$ such that D_j is an edge $x_j y_{j,1}$. Clearly $\mathcal{H} \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$. By Lemma 16(2), $2 \leqslant d_{G_1}(y_{j,1}) \leqslant 3$ for all $j \in \{1, 2, \cdots, s\}$. Let G_2 be a spanning subgraph of G_1 obtained from G_1 by deleting all edges $y_{j,1}z$ such that $d_{G_1}(y_{j,1}) = 3$ for all $j \in \{1, \cdots, s\}$, where $z \in \{v_{i'}, v_{i''}, v_{i'''}\}$. Then $d_{G_2}(y_{j,1}) = 2$ for all $j \in \{1, 2, \cdots, s\}$. For $s + 1 \leqslant i \leqslant s_2, j \in \{1, 2, \cdots, t_i\}$, we have $d_{G_2}(y_{i,j}) = 2$ by Lemma 16(3). Again, by Lemma 16(2)-(3), each x_j in D_j such that $2 \leqslant d_{G_2}(x_j) \leqslant 3$ with $1 \leqslant j \leqslant s_2$. Let G_3 be the resulting graph obtained from G_2 by deleting exactly one edge $x_j z$ such that $d_{G_2}(x_j) = 3$ where z is a neighbor of x_j in $\{v_{i'}, v_{i''}, v_{i'''}\}$ for $1 \leqslant j \leqslant s_2$. Clearly, G_3 is a spanning subgraph of G_1 and $d_{G_3}(w)$ is even for any $w \in V(\bigcup_{i=1}^{s_2} D_i)$. It suffices to show that G_3 has a spanning trail.

For any 1-component x of $G_3 - V(C)$, it holds that $|N_{G_3}(x) \cap V(C)| \geqslant 2$, and we take exactly two edges e_x, e'_x incident with x. Let $E_1 = \{e_x, e'_x : x$ is a 1-component of $G_3 - V(C)\}$, and let $G_4 \cong \langle E(G_3 - \bigcup_{i=s_2+1}^{s_3} V(D_i)) \cup E_1 \rangle_{G_3}$. Clearly, G_4 is a spanning subgraph of G_3, it suffices to show that G_4 has a spanning trail. Let Y be the set of all odd degree vertices in G_4. Then $Y \subseteq V(C)$. Since $|Y| \leqslant 7$, we have $|Y| \in \{0, 2, 4, 6\}$, it suffices to consider the case when $|Y| = 4$ or 6. (if $|Y| = 0$ or 2, then G_4 obviously has a spanning trail). Suppose first that $|Y| = 6$. Without loss of generality, we may assume that $Y = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, then $G_4 \backslash \{v_0v_1, v_3v_4\}$ is a connected spanning subgraph of G_4 with exactly two odd degree vertices; otherwise $G_4 \backslash \{v_0v_1, v_3v_4\}$ has two components such that one of them has only one odd degree vertex v_2, this contradicts the fact that every graph has even number of vertices with degree odd. Therefore, $G_4 \backslash \{v_0v_1, v_3v_4\}$ has a spanning trail, implying that G_4 has a spanning trail.

Now suppose that $|Y| = 4$. Then, by $|V(C)| \leqslant 7$, there is a pair of vertices v_i, v_j in Y such that $v_iv_j \in E(C)$. Therefore, $G_4 \backslash v_iv_j$ is a connected spanning subgraph of G_4 with exactly two odd degree vertices, then $G_4 \backslash v_iv_j$ has a spanning trail, therefore, G_4 has a spanning trail. This completes the proof. □

4 The proof of Theorem 6

Suppose that there is a vertex u in G that has no spanning trail starting from u. It suffices to show that $G \in \{F_1, F_2, F_3, F_4, F_5, F_6\}$, where F_i is shown in Figure 2. Let G' be the reduction of G. By Theorem 14, G' has no spanning trail starting from u. By Theorem 11, G' is triangle-free.

Suppose first that $\kappa(G') = 1$. Since G' is triangle-free, every block of G' has at least four vertices. Since $|V(G')| \leqslant 8$, G' has exactly two blocks H_1, H_2 with $4 \leqslant |V(H_i)| \leqslant 5$. If $|V(H_1)| = |V(H_2)| = 4$, then clearly $G' \cong C_4 \cup C_4$ is even, contradicting our hypothesis. Hence one of H_1, H_2 has exactly five vertices. Without loss of generality, we may assume that $|V(H_1)| = 5$. Since $|V(G')| \leqslant 8$, we have $|V(H_2)| = 4$, then $H_2 \cong C_4$. Recall that G' has no spanning trail starting from u, so $H_1 \ncong C_5$, otherwise $G' \cong C_5 \cup C_4$ is even. Therefore, since G' is triangle-free, $H_1 \cong K_{2,3}$ and then $G' \in \{F_1, F_2\}$, where F_1, F_2 are shown in Figure 2.

Let us in the following consider the case when $\kappa(G') \geqslant 2$. Let $C = v_0v_1 \cdots v_{c(G)-1}v_0$ be a longest cycle of G'. Then $c(G') \leqslant 7$, otherwise C is a Hamilton cycle of G'. Since G' has no spanning trail starting from u, by Theorem 4(1) we have $c(G') \geqslant 6$. Since $|V(G')| \leqslant 8$ and $6 \leqslant |V(C)| \leqslant 7$, $G' - V(C)$ has at most two vertices. Possibly $G' - V(C) \cong K_1, K_2$ or $2K_1$. Let $\mathcal{H}$ be the set of all components of $G' - V(C)$. Then clearly $|\mathcal{H}| \leqslant 2$. For any $F \in \mathcal{H}$, there is a path P_F in G' joining two distinct vertices of C and containing $V(F)$. Let $G_1 = \langle E(C) \bigcup_{F \in \mathcal{H}} E(P_F) \rangle_G$. Then G_1 is a 2-connected spanning subgraph of G'. Let X be the set of all vertices in G_1 with degree odd. Then $X \subset V(C)$ and $|X| = 2, 4$. (If $|X| = 0$, then G_1 is even).

We now distinguish the following two cases.

Case 1. $|V(C)| = 6$.

Case 1.1. $G_1 - V(C) \cong K_2$.

Let $V(G_1) \setminus V(C) = \{x_1, x_2\}$. By the definition of G_1, $|X| = 2$ and let $N_{G_1}(x_1) \cap V(C) = \{v_i\}$ and $N_{G_1}(x_2) \cap V(C) = \{v_j\}$. Since C is a longest cycle of G_1, we have $\text{dist}_C(v_i, v_j) = 3$. Without loss of generality, we may assume that $v_i = v_0, v_j = v_3$, then $E(G_1) = \{x_1v_0, x_2v_3, x_1x_2\} \cup E(C)$, hence G_1 has a spanning trail starting from any vertex of $\{v_0, v_3\}$. Note that for any vertex $v \in V(G_1)\setminus\{v_0, v_3\}$, there is an edge $e \in E(G_1)$ incident with v such that $G_1\setminus e$ has exactly two vertices with degree odd. Since $G_1\setminus e$ is connected, G_1 has a spanning trail starting from any vertex in $V(G_1)\setminus\{v_0, v_3\}$. Therefore, the above facts imply that G_1 has a spanning trail starting from any vertex, a contradiction.

Case 1.2. $G_1 - V(C) \cong 2K_1$.

Let $V(G_1) \setminus V(C) = \{y_1, y_2\}$. If $|X| = 4$, then G_1 has a cycle of length at least 7. Hence we have $|X| = 2$ and let $X = \{w_1, w_2\}$. By the definition of G_1, $|N_{G_1}(y_1) \cap X| = |N_{G_1}(y_2) \cap X| = 1$ and $N_{G_1}(y_1) \cap X \cap N_{G_1}(y_2) = \emptyset$. Without loss of generality, we may assume that $w_1 \in N_{G_1}(y_1) \cap X$ and $w_2 \in N_{G_1}(y_2) \cap X$, then by the definition of G_1, there is a vertex w in $V(C)\setminus\{w_1, w_2\}$ such that $w \in N_{G_1}(y_1) \cap N_{G_1}(y_2)$. Since C is a longest cycle of G_1, we have $2 \leqslant \text{dist}_C(w_i, w) \leqslant 3$ for each $i \in \{1, 2\}$. Suppose that $\text{dist}_C(w_i, w) = 3$ for some $i \in \{1, 2\}$, say $i = 1$. Without loss of generality, we may assume that $w_1 = v_0, w = v_3$, then, since C is a longest cycle of G_1, $w_2 \notin \{v_2, v_4\}$ and thus $w_2 \in \{v_1, v_5\}$. Without loss of generality, we may assume that $w_2 = v_5$, then $E(G_1) = \{y_1v_0, y_1v_3, y_2v_3, y_2v_5\} \cup E(C)$, thus $G_1\setminus v_0v_5$ is even, a contradiction.

Hence $\text{dist}_C(w_i, w) = 2$ for all $i \in \{1, 2\}$. Without loss of generality, we may assume that $w_1 = v_0, w = v_2$, then $w_2 = v_4$. Hence $E(G_1) = \{y_1v_0, y_1v_2, y_2v_2, y_2v_4\} \cup E(C)$, then G_1 has a spanning trail starting from any vertex of $\{v_0, v_4\}$. Note that for any vertex $v \in \{y_1, y_2, v_1, v_3, v_5\}$, there is an edge $e \in E(G_1)$ incident with v such that $G_1\setminus e$ has exactly two vertices with degree odd. Since $G_1\setminus e$ is connected, G_1 has a spanning trail starting from any vertex of $\{y_1, y_2, v_1, v_3, v_5\}$. Therefore, the above facts imply that G_1 has a spanning trail starting from any vertex in $\{y_1, y_2, v_0, v_1, v_3, v_4, v_5\}$. However, G_1 has no spanning trail starting from v_2 and thus $u = v_2$, hence $G_1 \cong F_3$, where F_3 is shown in Figure 2.

We claim that $G' \cong G_1 \cong F_3$. Otherwise, there is an edge $f \in E(G')\setminus E(G_1)$. Since G' is triangle-free and C is a longest cycle of G_1, $f \in \{v_0v_3, v_2v_5, v_1v_4, y_1v_4, y_2v_0\}$, thus $G_1 \cup \{f\}$ has a spanning trail starting from v_2. This implies that G' has a spanning trail starting from any vertex, a contradiction.

Case 1.3. $G_1 - V(C) \cong K_1$.

Let $V(G_1) \setminus V(C) = \{z\}$. By the definition of G_1, $N_{G_1}(z) = X$ and $|X| = 2$, and let $X = \{v_i, v_j\}$. Since C is a longest cycle of G_1, we have $2 \leqslant \text{dist}_C(v_i, v_j) \leqslant 3$. Suppose that $\text{dist}_C(v_i, v_j) = 3$. Without loss of generality, we may assume that $v_i = v_0, v_j = v_3$, then $E(G_1) = \{zv_0, zv_3\} \cup E(C)$, hence G_1 has a spanning trail starting from any vertex in $\{v_0, v_3\}$. Note that for any vertex v in $V(G_1) \setminus \{v_0, v_3\}$, there is an edge $e \in E(G_1)$

incident with v such that $G_1 \backslash e$ has exactly two vertices with degree odd. Since $G_1 \backslash e$ is connected, G_1 has a spanning trail starting from any vertex in $V(G_1) \setminus \{v_0, v_3\}$. The above facts imply that G_1 has a spanning trail starting from any vertex, a contradiction.

Hence we have $\mathrm{dist}_C(v_i, v_j) = 2$. Without loss of generality, we may assume that $v_i = v_0, v_j = v_2$, then $E(G_1) = \{zv_0, zv_2\} \cup E(C)$, hence G_1 has a spanning trail starting from any vertex of $\{v_0, v_2\}$. Note that for any $v \in \{z, v_1, v_3, v_5\}$, there is an edge $e \in E(G_1)$ incident with v such that $G_1 \backslash e$ has exactly two vertices with degree odd. Since $G_1 \backslash e$ is connected, G_1 has a spanning trail starting from any vertex of $\{z, v_1, v_3, v_5\}$. The above facts imply that G_1 has a spanning trail starting from any vertex of $\{z, v_0, v_1, v_2, v_3, v_5\}$. However, G_1 has no spanning trail starting from v_4 and then $u = v_4$, hence $G_1 \cong F_6$, where F_6 is shown in Figure 2.

We claim that $G' \cong G_1 \cong F_6$. Suppose not. Then there is an edge $f \in E(G') \backslash E(G_1)$. Since G' is triangle-free and C is a longest cycle of G', $f \in \{v_0v_3, v_1v_4, v_2v_5, zv_4\}$, thus $G_1 \cup \{f\}$ has a spanning trail starting from v_4. This implies that G' has a spanning trail starting from any vertex, a contradiction.

Case 2. $|V(C)| = 7$.

Then $G_1 - V(C)$ has only one vertex z. By the definition of G_1, $N_{G_1}(z) = X$ and $|X| = 2$, and let $X = \{v_i, v_j\}$. Since C is a longest cycle of G_1, we have $2 \leqslant \mathrm{dist}_C(v_i, v_j) \leqslant 3$. Suppose that $\mathrm{dist}_C(v_i, v_j) = 2$. Without loss of generality, we may assume that $v_i = v_0, v_j = v_2$, then $E(G_1) = \{zv_0, zv_2\} \cup E(C)$, hence G_1 has a spanning trail starting from any vertex of $\{v_0, v_2\}$. Note that for any vertex $v \in \{v_1, v_3, v_6, z\}$, there is an edge $e \in E(G_1)$ incident with v such that $G_1 \backslash e$ has exactly two vertices with degree odd. Since $G_1 \backslash e$ is connected, G_1 has a spanning trail starting from any vertex of $\{v_1, v_3, v_6, z\}$, hence G_1 has a spanning trail starting from any vertex of $\{v_0, v_1, v_2, v_3, v_6, z\}$. However, G_1 has no spanning trail starting from any vertex of $\{v_4, v_5\}$ and then $u \in \{v_4, v_5\}$, hence $G_1 \cong F_4$ where F_4 is shown in Figure 2. We claim that $G' \cong G_1 \cong F_4$. Suppose, to the contrary, and let f be an edge in $E(G') \backslash E(G_1)$. Since G' is triangle-free, $f \in \{v_0v_3, v_0v_4, v_1v_4, v_1v_5, zv_4, zv_5, v_2v_5, v_2v_6, v_3v_6\}$, then $G_1 \cup \{f\}$ has a spanning trail starting from v_4. This implies that G' has a spanning trail starting from any vertex, a contradiction.

Hence we have $\mathrm{dist}_C(v_i, v_j) = 3$. Without loss of generality, we may assume that $v_i = v_0, v_j = v_3$, then $E(G_1) = \{zv_0, zv_3\} \cup E(C)$. Hence G_1 has a spanning trail starting from any vertex of $\{v_0, v_3\}$. Note that for any vertex $v \in \{z, v_1, v_2, v_4, v_6\}$, there is an edge $e \in E(G_1)$ incident with v such that $G_1 \backslash e$ has exactly two vertices with degree odd. Since $G_1 \backslash e$ is connected, G_1 has a spanning trail starting from any vertex of $\{z, v_1, v_2, v_4, v_6\}$. Hence G_1 has a spanning trail starting from any vertex of $\{z, v_0, v_1, v_2, v_3, v_4, v_6\}$. However, G_1 has no spanning trail starting from v_5 and then $u = v_5$, hence $G_1 \cong F_5$, where F_5 is shown in Figure 2. We claim that $G' \cong G_1 \cong F_5$. Suppose not, and let f be an edge in $E(G') \backslash E(G_1)$. Since G is triangle-free, $f \in \{v_0v_3, v_0v_4, v_1v_5, zv_5, v_2v_5, v_3v_6\}$, thus $G_1 \cup \{f\}$ has a spanning trail starting from v_5. This implies that G' has a spanning trail starting from any vertex, a contradiction.

Summarizing all possible cases, we obtain that $G' \in \{F_1, F_2, F_3, F_4, F_5, F_6\}$, where F_i is shown in Figure 2. We claim that $G \cong G'$. Suppose not. Then there is a vertex

v in G' which is the contraction image of some nontrivial collapsible graph H of G, $|V(G)| \geqslant V(G') + |V(H)| - 1 \geqslant 7 + 3 - 1 = 9$, because the nontrivial subgraph with smallest number of vertices is the triangle, a contradiction. This proves Theorem 6.

5 The proof of Theorem 5

The proof of Theorem 5 can be separated the following two results.

Theorem 25. *Let G be a 2-edge-connected graph and S a subset of $V(G)$ such that $E(G-S) = \emptyset$ and $|S| \leqslant 4$. Then G has an S-trail starting from any vertex of S.*

Theorem 26. *Let G be a 2-edge-connected graph and S a subset of $V(G)$ such that $E(G-S) = \emptyset$ and $|S| \leqslant 6$. Then G has an S-trail or $G \in \{G_2, G_3\}$, where G_2, G_3 are shown in Figure 1.*

Proof of Theorem 25. Suppose, by contradiction, that G is a counterexample to the theorem such that the number of blocks of G is minimized. We now distinguish the following two cases to obtain our desired contradiction.

Case 1. $\kappa(G) \geqslant 2$.

Let $C = v_1 v_2 \cdots v_{c(G)} v_1$ be a longest cycle of G. Then $G - V(C)$ contains at least one vertex of S, otherwise C is a Hamilton cycle. Therefore, since $|S| \leqslant 4$, we have $|V(C) \cap S| \leqslant 3$. Since C is a subgraph of G and $E(G-S) = \emptyset$, we have $E(C-S) = \emptyset$. Note that G has no spanning trail starting from some vertex of S. By Theorem 4(1) we have $c(G) \geqslant 6$. If $c(G) \geqslant 7$, then $|V(C) \cap S| \geqslant 4$ since $E(C-S) = \emptyset$, contradicting $|V(C) \cap S| \leqslant 3$. Hence we have $c(G) = 6$, then $|V(C) \cap S| \geqslant 3$ since $E(C-S) = \emptyset$, thus $|V(C) \cap S| = 3$. We therefore assume that $V(C) \cap S = \{v_1, v_3, v_5\}$ and $V(C) \backslash S = \{v_2, v_4, v_6\}$. Since $|S| \leqslant 4$ and $G - V(C)$ contains at least one vertex of S, $G - V(C)$ contains exactly one vertex of S, say b. Let D be a component of $G - V(C)$ such that $b \in V(D)$. Then $S \subset V(C) \cup V(D)$. Let $G' \cong \langle V(C) \cup V(G/D) \rangle_G$. Clearly, G' has exactly seven vertices. Since G has no spanning trail starting from some vertex of S and by the the definition of G', G' also has no spanning trail starting from some vertex of S. By Theorem 6, G' is isomorphic to F_6 shown in Figure 2. Let U_1 be the set of vertices of degree two in F_6 and U_2 the set of vertices of degree three in F_6. Then $\{v_2, v_4, v_6\} \subset U_1$ and $U_2 \subset \{v_1, v_3, v_5\}$, one can easily check that F_6 has an S-trail starting from any vertex of S, a contradiction.

Case 2. $\kappa(G) = 1$.

Since $E(G-S) = \emptyset$, each cycle of G contains at least two vertices of S.

Claim 27. $|V(F) \cap S| \geqslant 3$ *for any end block F of G.*

Proof. Suppose, by contradiction, that there is an end block F of G such that $|V(F) \cap S| \leqslant 2$. Let C be a cycle in F that contains the only cut vertex of G in F. Since $E(G-S) = \emptyset$, each cycle of G contains at least two vertices of S, then C contains all vertices in $V(F) \cap S$ since $|V(F) \cap S| \leqslant 2$.

Consider the graph G/F such that the contraction image v of F is in S. Then $|V(G/F) \cap S| \leqslant 4 - 2 + 1 = 3$. Note that G/F has blocks less than G and G/F is also 2-edge-connected, by the choice of G, G/F has an $S \cap V(G/F)$-trail T starting from any vertex of S, then $T \cup C$ is an S-trail of G since C contains all vertices in $V(F) \cap S$, a contradiction. This proves Claim 27. □

Let F_1 and F_2 be two end blocks of G. By Claim 27, $|S \cap V(F_i)| \geqslant 3$ for $1 \leqslant i \leqslant 2$, then $|S| \geqslant |S \cap V(F_1 \cup F_2)| = |S \cap V(F_1)| + |S \cap V(F_2)| - |S \cap V(F_1 \cap F_2)| \geqslant 3 + 3 - 1 = 5$, a contradiction. This proves Theorem 25. □

We will apply Theorem 25 and the following result to prove Theorem 26.

Theorem 28. *(Niu. et. al [19]) Let G be a 2-edge-connected graph of order at most ten. Then G has a spanning trail or $G \in \{G_2, G_3\}$, where G_2, G_3 are shown in Figure 1.*

Proof of Theorem 26. Suppose, by contradiction, that G is a counter-example to the theorem such that the number of blocks of G is minimized. Then G has no spanning trail. Let S be a subset of $V(G)$ such that $E(G - S) = \emptyset$ and $|S| \leqslant 6$. We now consider two cases to obtain our desired contradiction.

Case 1. $\kappa(G) \geqslant 2$.

Let $C = v_1 v_2 \cdots v_{c(G)} v_1$ be a longest cycle of G. Then $G - V(C)$ contains at least two vertices of S, otherwise G obviously has a spanning trail. Then, by $|S| \leqslant 6$, we have $|V(C) \cap S| \leqslant 4$. Since C is a subgraph of G and $E(G - S) = \emptyset$, we have $E(C - S) = \emptyset$. If $c(G) \leqslant 7$, then by Theorem 4(2), G has a spanning trail, a contradiction. If $c(G) \geqslant 9$, then $|V(C) \cap S| \geqslant 5$ since $E(C - S) = \emptyset$, contradicting $|V(C) \cap S| \leqslant 4$. Hence we have $c(G) = 8$, then $|V(C) \cap S| = 4$. We therefore assume that $V(C) \cap S = \{v_1, v_3, v_5, v_7\}$ and $V(C) \backslash S = \{v_2, v_4, v_6, v_8\}$. Therefore, since $G - V(C)$ contains at least two vertices of S and $|S| \leqslant 6$, $G - V(C)$ contains exactly two vertices of S, say b_1, b_2. Then b_1 and b_2 cannot lie in a same component of $G - V(C)$; otherwise, one can easily find an S-trail of G. Let D_1 and D_2 be two components of $G - V(C)$ such that $b_i \in V(D_i)$. Then $S \subset V(C) \cup V(D_1) \cup V(D_2)$. Let $G' = \langle V(C) \cup V(G/D_1) \cup V(G/D_2) \rangle_G$. Clearly, G' has exactly ten vertices. Since G has no S-trail and by the definition of G', G' has no S-trail, implying that G' has no spanning trail. Note that G' is also 2-connected. By Theorem 28, $G' \in \{G_2, G_3\}$, where G_2, G_3 are shown in Figure 1.

Let U_1 be the set of vertices of degree two in G' and U_2 the set of vertices of degree three in G'. Then $|U_1| = 6$ and $|U_2| = 4$. Note that $S = \{v_1, v_3, v_5, v_7, b_1, b_2\}$. Then $U_1 = S$, otherwise, $|U_2 \cap S| \neq \emptyset$, one can easily find an S-trail of G'. We furthermore have $V(D_i) = \{b_i\}$ for $1 \leqslant i \leqslant 2$, otherwise contradicting $E(G - S) = \emptyset$. This implies that G' is a subgraph of G.

Claim 29. $V(G) = V(G')$.

Proof. Suppose to the contrary that there is a vertex $w \in V(G) \setminus V(G')$. Then there exist two edges between w and C in G since G is 2-connected. Therefore, since G' is a subgraph of G and $U_1 = S$, one can easily find an S-trail of G, a contradiction. This proves Claim 29. □

Since $G' \subseteq G$ and by Claim 29, G' is a spanning subgraph of G. Since $G' \in \{G_2, G_3\}$, it is easy to check that adding any edge to G' can make it have an S-trail, this implies that $G \cong G'$, contradicting the hypothesis.

Case 2. $\kappa(G) = 1$.
Then G has at least two blocks.

Claim 30. *$|V(F) \cap S| \geqslant 3$ for any end block F of G.*

Proof. Suppose, by contradiction, that there is an end block F of G such that $|V(F) \cap S| \leqslant 2$. Let C be a cycle in F that contains the only cut vertex of G in F. Since $E(G-S) = \emptyset$, each cycle of G contains at least two vertices of S, then C contains all vertices in $V(F) \cap S$ since $|V(F) \cap S| \leqslant 2$.

Consider the graph G/F such that the contraction image v of F is in S. Then $|V(G/F) \cap S| \leqslant 6 - 2 + 1 = 5$. Note that G/F has blocks less than G, G/F has an $S \cap V(G/F)$-trail T; otherwise, by the choice of G, the connectivity of G/F is at least two, by Case 1 we have $G/F \in \{G_2, G_3\}$, where G_2, G_3 are shown in Figure 1. Let U be the set of vertices of degree two in G/F. Then $|U| = 6$. Since G/F has no $S \cap V(G/F)$-trail, this implies that $U \subseteq (S \cap V(G/F))$, then $|V(G/F) \cap S| \geqslant 6$, contradicting $|V(G/F) \cap S| \leqslant 5$. Therefore, $v \in V(T)$, $T \cup C$ is an S-trail of G since C contains all vertices in $V(F) \cap S$, a contradiction. This proves Claim 30. □

Claim 31. *G has exactly two end blocks.*

Proof. Suppose, to the contrary, that G has three end blocks F_1, F_2 and F_3. By Claim 30, $|S \cap V(F_1 \cup F_2 \cup F_3)| = |S \cap V(F_1)| + |S \cap V(F_2)| + |S \cap V(F_3)| - |S \cap V(F_1 \cap F_2)| - |S \cap V(F_1 \cap F_3)| - |S \cap V(F_2 \cap F_3)| + |S \cap V(F_1 \cap F_2 \cap F_3)| \geqslant 3 + 3 + 3 - 1 - 1 - 1 + 1 = 7$, a contradiction. This proves Claim 31. □

By Claim 31, let F_1 and F_2 be all end blocks of G such that v_i is the cut vertex of G in F_i. By Claim 30, $|V(F_i) \cap S| \geqslant 3$ for $1 \leqslant i \leqslant 2$. Since $|S| \leqslant 6$, we have $3 \leqslant |V(F_i) \cap S| \leqslant 4$ for $1 \leqslant i \leqslant 2$. Indeed, $F_1 \cup F_2$ contains all vertices of S. Let P be a path in G connecting v_1 and v_2 (possibly P is a vertex). For $i \in \{1, 2\}$, F_i satisfies the conditions of Theorem 25, F_i has an $S \cap V(F_i)$-trail starting from any vertex of S. Note that $v_i \in S$ or v_i is adjacent to some vertex in S for each $i \in \{1, 2\}$. This implies that F_i has an $S \cap V(F_i)$-trail T_i starting from v_i for $i = 1, 2$, hence $T_1 \cup P \cup T_2$ is an S-trail of G, a contradiction. This proves Theorem 26. □

6 Proofs of Theorem 7 and Corollary 8

The *core* of a graph G, denoted by G_0, is obtained by recursively deleting all pendant vertices in G. The *clique covering number* of a graph G, denoted by $\theta(G)$, is the minimum number of cliques necessarily that cover $V(G)$. Color the vertices of a graph G black and white, we use $B(G)$ and $W(G)$ to denote the set of black vertices and whites vertices in G, respectively. A *star decomposition* of a graph G is a family $\mathcal{F}$ of edge-disjoint stars F of G such that $\bigcup_{F \in \mathcal{F}} E(F) = E(G)$.

Theorem 32. *(Li, et al. [17]) Let G be a graph with $|E(G)| \geqslant 3$. Then the line graph $L(G)$ is traceable if and only if G has a dominating connected subgraph, i.e., dominating trail.*

Lemma 33. *(Chvátal-Erdős, [7]) Every connected graph G of order at least three with $\alpha(G) \leqslant \kappa(G)+1$ is traceable.*

Lemma 34. *(Kasier, et al. [13]) Every 5-connected claw-free graph with minimum degree at least 6 is hamiltonian.*

Lemma 35. *(Favaron, et al. [9]) Let $k \geqslant 2$ be an integer and let G be a claw-free graph of order $n \geqslant 2k^2-3k$ and minimum degree $\delta(G) > \frac{n}{k}+k-2$. Then $\theta(\mathrm{cl(G)}) \leqslant \mathrm{k}-1$.*

Lemma 36. *(Frončec, et al. [10]) Let $k \geqslant 2$ be an integer, let G be a claw-free graph of order n and let $\kappa = \kappa(\mathrm{cl(G)})$. Suppose that G is such that $n \geqslant 3k^2-k-\kappa-2$ and $\delta(G) > \frac{n+k^2-4k+2+\kappa}{k}$. Then $\theta(\mathrm{cl(G)}) \leqslant \mathrm{k}-1$, or $\alpha(\mathrm{cl(G)})) \leqslant \kappa$.*

Theorem 37. *Let G be a 2-connected closed claw-free graph with $\theta(G) \leqslant 6$. Then G is traceable or $G = L(H)$ where $H \in \mathcal{C}_1 \cup \mathcal{C}_2$.*

Proof. Let G be a closed claw-free graphs with clique covering number θ. Let $\mathcal{P}_G = \{B_1, \cdots, B_\theta\}$ be a clique covering of G such that each B_i is maximal. Then $H = L^{-1}(G)$ has a star decomposition $T_1, \cdots, T_\theta$ such that $L(T_i) \cong B_i$. Color black on those centers c_{B_i} of T_i and white on remaining vertices in $V(H)\backslash\{c_{B_1}, \cdots, c_{B_t}\}$. Then $B(H) = \{c_{B_1}, \cdots, c_{B_\theta}\}$. Note that $B(H)$ is a vertex covering of H (that is, every edge of H has at least one vertex in $B(H)$), $E(H-B(H)) = \emptyset$.

Suppose that G is non-traceable. It suffices to show that $H \in \mathcal{C}_1 \cup \mathcal{C}_2$. By the definition of the core H_0, the black vertices of H are alway black in H_0, i.e., $B(H) = B(H_0)$. Since G is non-traceable and by Theorem 32, H_0 has no $B(H_0)$-trail. Since $\theta(G) \leqslant 6$, we have $|B(H)| \leqslant 6$ and then $|B(H_0)| \leqslant 6$. Since H_0 is a subgraph of H and $E(H-B(H)) = \emptyset$, we have $E(H_0 - B(H_0)) = \emptyset$. Since H_0 has no $B(H_0)$-trail and by Theorem 5, $H_0 \in \{G_2, G_3\}$, where G_2, G_3 are shown in Figure 1. Let U_1 be the set of vertices of degree two in H_0 and U_2 the set of vertices of degree three in H_0. Then $|U_1| = 6$ and $|U_2| = 4$.

If $U_2 \cap B(H_0) \neq \emptyset$, then, since $|B(H_0)| \leqslant 6$ and $|U_1| = 6$, we have $U_1 \setminus B(H_0) \neq \emptyset$, say $u \in U_1 \setminus B(H_0)$, it is easy to see that $H_0 - u$ has a spanning trail, then H_0 has a $B(H_0)$-trail, a contradiction. Hence $U_2 \cap B(H_0) = \emptyset$ and then $B(H_0) \subseteq U_1$, hence $B(H_0) = U_1$; otherwise, one can easily check that H_0 has a $B(H_0)$-trail, a contradiction. Therefore, every vertex in U_1 has at least one pendant vertex in H, implying that $H \in \mathcal{C}_1 \cup \mathcal{C}_2$. This proves Theorem 37. □

The following result is a consequence of Theorem 37.

Corollary 38. *Let G be a 2-connected closed claw-free graph with $\theta(G) \leqslant 5$. Then G is traceable.*

Proof of Theorem 7. Suppose that G is non-traceable. Then by Theorem 10, cl(G) is also non-traceable. By Lemma 33, $\alpha(\mathrm{cl(G)}) > \kappa(\mathrm{cl(G)}) + 1$. Thus, Lemma 34 implies $1 \leqslant \kappa(\mathrm{cl(G)}) \leqslant 5$. The assumptions of theorem satisfy the conditions of Lemma 36 (for $k = 7$), we then obtain $\theta(\mathrm{cl(G)}) \leqslant 6$, by Theorem 37, cl(G) = L(H) where $H \in \mathcal{C}_1 \cup \mathcal{C}_2$. This proves Theorem 7. □

Proof of Corollary 8. Let G satisfy the assumptions of Corollary 8. Then clearly so is cl(G). Thus, suppose that G is closed. From Lemma 35 (for $k = 6$) we then obtain $\theta(G) \leqslant 5$. By Corollary 38, G is traceable. This proves Corollary 8. □

7 Concluding remarks

In this paper, we prove our main result (Theorem 4), we also use it to obtain Theorems 5 and 6. By using Theorem 5 and more detailed discussion as Theorem 7, Tian et al. [22] gave a sharp slight weaker minimum degree sum of a pair of adjacent vertices for those 2-connected claw-free graphs with minimun degree at least three to be traceable, and Tian et al. [23] gave also a sharp minimun degree sum of t independent vertex set, however, these 2-connected claw-free traceable graphs deduced by these conditions have some exceptional graphs obtained from G_2, G_3 shown in Figure 1.

We believe that Theorem 4 would have more applications. As Theorem 2 was used in the forbidden subgraph condition for a 3-connected claw-free H-free graph to be hamiltonian, it will be also used in the similar forbidden subgraph condition for a 2-connected claw-free graph to be traceable in our futher work [24].

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