On the Sweep Map for $\vec{k}$-Dyck Paths

Guoce Xin
School of Mathematical Sciences
Capital Normal University
Beijing 100048, PR China
guoce.xin@gmail.com

Yingrui Zhang
School of Mathematical Sciences
Capital Normal University
Beijing 100048, PR China
zyrzuhe@126.com

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Abstract

Garsia and Xin gave a linear algorithm for inverting the sweep map for Fuss rational Dyck paths in $D_{m,n}$ where $m = kn \pm 1$. They introduced an intermediate family $\mathcal{T}_n^k$ of certain standard Young tableaux. Then inverting the sweep map is done by a simple walking algorithm on a $T \in \mathcal{T}_n^k$. We find their idea naturally extends for $\mathbf{k}^{\pm}$-Dyck paths, and also for $\mathbf{k}$-Dyck paths (reducing to k-Dyck paths for the equal parameter case). The intermediate object becomes a similar type of tableau in $\mathcal{T}_{\mathbf{k}}$ of different column lengths. This approach is independent of the Thomas-Williams algorithm for inverting the general modular sweep map.

Mathematics Subject Classifications: 05A19, 05E99

1 Introduction

The sweep map is a mysterious simple sorting algorithm that is also invertible. The best way to introduce the sweep map is by using rational Dyck paths, because it already raises complicated enough problems and it has natural generalizations. We will use three models, each having its own advantages.

Model 1: Classical path model. For a pair of positive integers (m, n), the so called *rational* (m, n)*-Dyck paths* are paths proceed by North and East unit steps from $(0, 0)$ to (m, n) remaining always above the main diagonal (of slope n/m). A (kn, n)-Dyck path is also called a k-Dyck path of length n. Each vertex is assigned a rank as follows. We start by assigning 0 to $(0, 0)$. This is done we add an m as we go North and subtract an n as we go East. Figure 1 illustrates an example of an $(12, 4)$-Dyck path, or a 3-Dyck path of length 4.

The *sweep* map is a bijection of the family $\mathcal{D}_{m,n}$ of (m, n)-Dyck paths onto itself. The construction of the sweep map is deceptively simple. Geometrically in model 1, we sweep a path $\overline{D} \in \mathcal{D}_{m,n}$ by letting lines of slope $n/m + \epsilon$, where $\epsilon > 0$ is sufficiently small, move

$\overline{D} =$

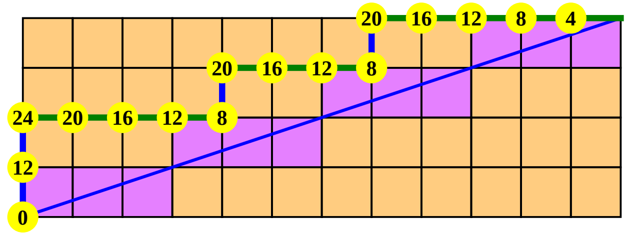


Figure 1: An example of $(12, 4)$-Dyck path.

$D =$

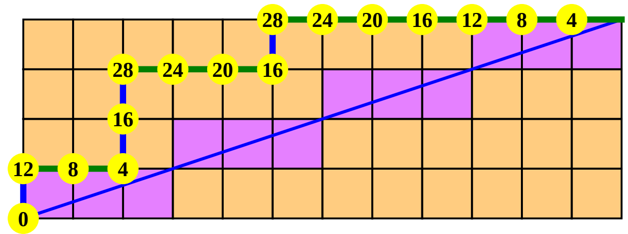


Figure 2: The sweep map image of the Dyck path $\overline{D}$ in Figure 1.

from right to left, and draw a North step when we sweep the South end of a North step of $\overline{D}$ and draw an East step when we sweep the West end of an East step of $\overline{D}$. The resulting path will be denoted by $D = \Phi(\overline{D})$ and can be shown to be in $\mathcal{D}_{m,n}$. For instance, Figure 2 illustrates the sweep map image of the Dyck path $\overline{D}$ in our running example.

Model 2: Word model. The SW-word $\mathtt{SW}(\overline{D}) = \sigma_1\sigma_2 \cdots \sigma_{m+n}$ is a natural encoding of $\overline{D}$, where σ_i is either an S or a W depending on whether the i-th vertex of $\overline{D}$ is a South end (of a North step) or a West end (of an East step). The rank is then associated to each letter of $\mathtt{SW}(\overline{D})$ by assigning $r_1 = 0$ to the first letter $\sigma_1 = S$ and for $1 \leqslant i \leqslant m + n - 1$, recursively assigning r_{i+1} to be either $r_i + m$ if the i-th letter $\sigma_i = S$, or $r_i - n$ if otherwise $\sigma_i = W$. We can then form the two line array $\binom{\mathtt{SW}(\overline{D})}{r(\overline{D})}$. For instance for the path $\overline{D}$ in Figure 1 this gives

$$\begin{pmatrix} \mathtt{SW}(\overline{D}) \\ r(\overline{D}) \end{pmatrix} = \begin{pmatrix} S & S & W & W & W & W & S & W & W & W & S & W & W & W & W & W \\ 0 & 12 & 24 & 20 & 16 & 12 & 8 & 20 & 16 & 12 & 8 & 20 & 16 & 12 & 8 & 4 \end{pmatrix}. \tag{1}$$

Now sort the columns of (1) according to the second row, and let the right one comes first for equal ranks. Then the top row is just the SW-word $\mathtt{SW}(D)$ of the sweep map image of $\overline{D}$. Our running example gives

$$\begin{pmatrix} \mathtt{SW}(D) \\ r(\overline{D})_{\mathtt{sorted}} \end{pmatrix} = \begin{pmatrix} S & W & W & S & S & W & W & W & S & W & W & W & W & W & W & W \\ 0 & 4 & 8 & 8 & 8 & 12 & 12 & 12 & 12 & 16 & 16 & 16 & 20 & 20 & 20 & 24 \end{pmatrix}. \tag{2}$$

Model 3: Visual path model. We can rotate and stretch the picture in model 1 so that the diagonal line becomes the horizontal axis. Then rational (m, n)-Dyck paths are paths from $(0,0)$ to $(m+n,0)$, with up steps $S=(1,m)$ and down steps $W=(1,-n)$, that never go below the horizontal axis. It is clear that the ranks are just the levels (or y-coordinates). The visualization of the ranks in this model allows us to have better understanding of many results. See [4, 5]. See [1, 3, 7, 9, 10] for further references and related results.

The invertibility problem is to reconstruct $\overline{D}$ from the sole knowledge of $\mathtt{SW}(D)$.

The sweep map is an active subject in recent decades. Variations and extensions have been found, and some classical bijections turn out to be the disguised version of the sweep map. See [2] for detailed information and references. One major problem in this subject is the invertibility of the sweep map. The bijectivity has been shown in a variety of special cases including the Fuss case when $m = kn \pm 1$ which is proved in [10],[8]and[6]. The invertibility of the sweep map, even for rational Dyck paths, remained open for over ten years. Surprisingly, a general result proving the invertibility of a class of sweep maps that were listed in [2], was recently given by Thomas-Williams in [11]. Based on the idea of Thomas and Williams, Garsia and Xin [5] gave a geometric construction for inverting the sweep map on (m,n)-rational Dyck paths. These algorithms are nice iteration algorithms, but are not linear: the number of iterations is measured by the sum of the ranks of $\overline{D}$.

By using a completely different approach, Garsia and Xin find a $O(m+n)$ algorithm for inverting sweep map on (m,n)-Dyck paths in the Fuss case $m = kn \pm 1$. This raises the following problem.

Problem: Is there a linear algorithm to invert the sweep map, at least for a more general class of paths?

We find positive answers for $\mathbf{k}^+$, $\mathbf{k}^-$, and $\mathbf{k}$-Dyck paths by extending Garsia-Xin's idea.

The paper is organized as follows. In this introduction, we have introduced the basic concepts. Section 2 extends the concept of rational Dyck paths to general Dyck paths, as well as the sweep map in this general setting. Then we introduce our main constructions, results, especially Theorems 2.14, which gives a linear inverting algorithm for $\mathbf{k}$-Dyck paths. Section 3 includes the basic facts of the sweep map and the proof of Theorem 2.14, but leave the proof of Lemma 3.3 in Section 4. Lemma 3.3 says that our walking algorithm (Algorithm 2.13) produces a permutation $\sigma(D)$ of the desired length. To prove this lemma, we give two different approaches in Section 4. In Section 5, we give the proof for $\mathbf{k}^+$-Dyck paths, which is very similar to that in [6]. We include the proof here for the sake of self completeness according to the referee's suggestions. Similarly, we give the proof for $\mathbf{k}^-$-Dyck paths in Section 6.

2 Notations and Main Results

2.1 The notation of general Dyck paths

We start by introducing general Dyck paths using model 3.

General Dyck paths are two dimensional lattice paths from $(0,0)$ to $(m+n,0)$ that never go below the horizonal axis. We use vectors $\mathbf{u} = (u_1,\dots,u_n)$ and $\mathbf{d} = (d_1,\dots,d_m)$

to specify the up steps and down steps, so that $\mathcal{D}_{\mathbf{u},-\mathbf{d}}$ is the set of general Dyck paths with up steps $(1, u_i)$, $1 \leqslant i \leqslant n$ from left to right, and down steps $(1, -d_j)$, $1 \leqslant j \leqslant m$ from left to right. Clearly the total length of up steps $|\mathbf{u}| = u_1 + u_2 + \cdots + u_n$ is equal to the total length of down steps $|\mathbf{d}| = d_1 + d_2 + \cdots + d_m$. It is convenient to use $\mathbf{1}_m$ to denote the m-dimensional vector with each entry 1. Thus $\mathbf{d} = d\mathbf{1}_m$ corresponds to the case $d_i = d$ for all i. We use the short hand notation $\mathcal{D}_{\mathbf{u}} = \mathcal{D}_{\mathbf{u},-\mathbf{1}_{|\mathbf{u}|}}$. Here we usually restrict u_i and d_j to be positive integers, but sometimes allowing rational numbers is convenient, because of the obvious isomorphism $\mathcal{D}_{\mathbf{u},-\mathbf{d}} \simeq \mathcal{D}_{k\mathbf{u},-k\mathbf{d}}$ for any positive integer k.

A general Dyck path $\overline{D}$ may be encoded as $\overline{D} = (a_1, a_2, \ldots, a_{m+n})$ with each entry either u_i or $-d_j$. The rank sequence $r(\overline{D}) = (0 = r_1, r_2, \ldots, r_{m+n})$ of $\overline{D}$ is defined as the partial sums $r_i = a_1 + a_2 + \cdots + a_{i-1} \geqslant 0$, called starting rank (or level) of the i-th step. Geometrically, r_i is just the level or y-coordinate of the starting point of the i-th step. The SW-word of $\overline{D}$ is $\mathtt{SW}(\overline{D}) = \sigma_1\sigma_2\cdots\sigma_{m+n}$ where $\sigma_i = S^{a_i}$ if $a_i > 0$ and $\sigma_i = W^{-a_i}$ if $a_i < 0$ (with $W^1 = W$). The sweep map D of $\overline{D}$ is obtained by reading its steps by their starting levels from bottom to top, and from right to left at the same level. This corresponds to sweeping the starting points of the steps from bottom to top using a line of slope ϵ for sufficiently small $\epsilon > 0$.

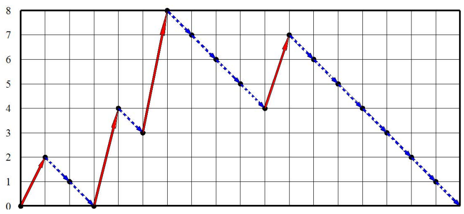
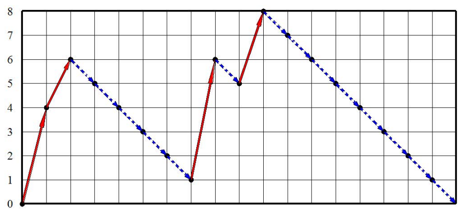
$\overline{D} =$  $\Longrightarrow D =$ 

Figure 3: An example of a $\mathbf{k}$-Dyck path and its sweep map image.

Figure 3 illustrates an example of $\mathbf{k}$-Dyck path $\overline{D}$ given by

$$\overline{D} = (2, -1, -1, 4, -1, 5, -1, -1, -1, -1, 3, -1, -1, -1, -1, -1, -1, -1),$$

where $\mathbf{k} = (2, 4, 5, 3)$. The SW-word of $\overline{D}$ and the rank sequence are given by

$$\begin{pmatrix} \mathtt{SW}(\overline{D}) \\ r(\overline{D}) \end{pmatrix} = \begin{pmatrix} S^2 & W & W & S^4 & W & S^5 & W & W & W & W & S^3 & W & W & W & W & W & W & W \\ 0 & 2 & 1 & 0 & 4 & 3 & 8 & 7 & 6 & 5 & 4 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

The sweep map image is

$$D = (4, 2, -1, -1, -1, -1, -1, 5, -1, 3, -1, -1, -1, -1, -1, -1, -1, -1),$$

with SW-word $\mathtt{SW}(D) = S^4S^2WWWWWS^5WS^3WWWWWWWW$.

Example 2.1.

1. *$\mathcal{D}_{\mathbf{1}_n}$ is the set of classical Dyck paths in the $n \times n$ square (rotated version).*

2. $\mathcal{D}_{m\mathbf{1}_n,-n\mathbf{1}_m}$ *is just* $\mathcal{D}_{m,n}$*, the set of rational* (m,n)*-Dyck paths.*

3. $\mathcal{D}_{k\mathbf{1}_n}$ *is just* $\mathcal{D}_{kn,n}$*, the set of* k*-Dyck paths of length* n*. (Their rank sequences differed by a factor* n*).*

In what follows, $\mathbf{k} = \vec{k} = (k_1, k_2, \dots, k_n)$ is always a vector of $n = \ell(\mathbf{k})$ positive integers. We will focus on $\mathbf{k}$-Dyck paths in $\mathcal{D}_{\mathbf{k}}$, i.e., general Dyck paths whose up steps are of lengths $k_1, \dots, k_n$ from left to right, and whose down steps are all of length 1. We will also consider $\mathbf{k}^+$-Dyck paths in $\mathcal{D}_{\mathbf{k}^+}$ and $\mathbf{k}^-$-Dyck paths in $\mathcal{D}_{\mathbf{k}^-}$, where $\mathbf{k}^\pm = \mathbf{k} \pm \frac{1}{n}\mathbf{1}_n$. Note that $\mathcal{D}_{\mathbf{k}^\pm} \simeq \mathcal{D}_{n\mathbf{k}\pm\mathbf{1}_n,-n\mathbf{1}_{|\mathbf{k}|\pm 1}}$. These are natural extensions of the Fuss case rational Dyck paths: k-Dyck paths are just $k\mathbf{1}_n$-Dyck paths; Fuss case $(kn \pm 1, n)$-Dyck paths are easily seen to be equivalent to $(k\mathbf{1}_n)^\pm$-Dyck paths.

Remark 2.2. *The sweep map of* $\mathbf{k}^+$*-Dyck paths can be fluctuated in the following sense: Let* $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ *with* $\epsilon_i > 0$ *for all* i *and* $|\epsilon| = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_n = 1$*. Then it is easy to see that the sweep map for* $(\mathbf{k}+\epsilon)$*-Dyck paths is the same as for* $\mathbf{k}^+$*-Dyck paths. Similarly,* $\mathbf{k}^-$*-Dyck paths can fluctuate as* $(\mathbf{k}-\epsilon)$*-Dyck paths.*

The sweep map of a $\mathbf{k}$-Dyck path is usually a $\mathbf{k}'$-Dyck path where $\mathbf{k}'$ is obtained from $\mathbf{k}$ by permuting its entries. Denote by $\mathcal{K}$ the set of all such $\mathbf{k}'$ and by $\mathcal{D}_{\mathcal{K}}$ the union of $\mathcal{D}_{\mathbf{k}'}$ for all such $\mathbf{k}'$. We also use the similar notation for $\mathcal{K}^\pm$ and $\mathcal{D}_{\mathcal{K}^\pm}$.

Theorem 2.3. *The sweep maps define bijections from* $\mathcal{D}_{\mathcal{K}^+}$*,* $\mathcal{D}_{\mathcal{K}^-}$*, and* $\mathcal{D}_{\mathcal{K}}$ *to themselves.*

It is known but nontrivial that the sweep map takes a Dyck path to another Dyck path (see, e.g., [2], [5], [7] for a proof), so the proof of the theorem boils down to construct the inverse image $\overline{D}$ from a given Dyck path D.

The three sets $\mathcal{D}_{\mathcal{K}}$, $\mathcal{D}_{\mathcal{K}^+}$, and $\mathcal{D}_{\mathcal{K}^-}$ are closely related as follows. For $D \in \mathcal{D}_{\mathcal{K}}$, let $\mathtt{SW}(D^+)$ be obtained from $\mathtt{SW}(D)$ by adding a W at the end and changing every S^a to $S^{a+\frac{1}{n}}$. Similarly let $\mathtt{SW}(D^-)$ be obtained from $\mathtt{SW}(D)$ by removing the final letter W and changing every S^a to $S^{a-\frac{1}{n}}$. It is clear that the map $D \mapsto D^+$ gives a bijection from $\mathcal{D}_{\mathcal{K}}$ to $\mathcal{D}_{\mathcal{K}^+}$. However, the map $D \mapsto D^-$ is a little different: it is a bijection from $\mathcal{D}^\circ_{\mathcal{K}}$ to $\mathcal{D}_{\mathcal{K}^-}$, where $\mathcal{D}^\circ_{\mathcal{K}}$ consists of paths $D \in \mathcal{D}_{\mathcal{K}}$ whose rank sequence $r(D)$ has only one 0 at r_1.

The bottom rectangle in Figure 4 illustrates the idea: Though the paths $D^+ \in \mathcal{D}_{\mathcal{K}^+}$ and $D \in \mathcal{D}_{\mathcal{K}}$ have almost identical pictures, their sweep map inverse images $\overline{D^+} \in \mathcal{D}_{\mathcal{K}^+}$ and $\overline{D} \in \mathcal{D}_{\mathcal{K}}$ may be very different, due to the different sweep order. Therefore, their inverting algorithms are also very different. For instance, below is an example when $\mathbf{k} = [2,1,3,1]$:

$$
\begin{array}{ll}
D & = (\ 1,\ \ 1,\ \ -1,\ \ 2,\ \ 3,\ \ -1, -1, -1,\ \ -1, -1,\ -1\ \ ) \\
D^+ & = (\ 1^+, 1^+,\ -1,\ 2^+,\ 3^+,\ -1, -1, -1,\ -1, -1, -1, -1\ ) \\
\overline{D} & = (\ 1,\ \ -1,\ 1,\ \ \ 3,\ \ -1, -1, -1,\ \ 2,\ \ -1, -1,\ -1\ \ ) \\
\overline{D^+} & = (\ 1^+, 2^+,\ -1,\ -1,\ 3^+,\ -1, -1, -1,\ -1,\ 1^+,\ -1, -1\ )
\end{array}
$$

2.2 The Filling algorithm and the $\vec{k}$ tableaux

In the Fuss case when $m = kn + 1$, a linear algorithm to invert the sweep map was discovered by Garsia and Xin in [6]. The algorithm relies on an intermediate family $\mathcal{T}_n^k$ of standard Young tableaux. The family $\mathcal{T}_n^k$ consists of $n \times (k+1)$ arrays with entries $1, 2, \ldots, kn+n$, column increasing from top to bottom and row increasing from left to right, with the additional property that for any pair of entries $a < d$ with d directly below a, the entries between a and d form a horizontal strip. That is, any pair of entries b, c with $a < b < c < d$ never appear in the same column. The standard Young tableau $T(D)$ constructed from the SW word of a path D encodes so much information about D. This allows us to invert the sweep map in the simplest possible way.

We find Garsia-Xin's construction naturally extends for $\mathbf{k}^+$-Dyck paths. The intermediate family $\mathcal{T}_n^k$ becomes the family $\mathcal{T}_{\mathbf{k}}$, where the only difference is that tableau $T \in \mathcal{T}_{\mathbf{k}}$ has $k_i + 1$ entries in the i-th column. Let $\mathbf{k}'$ be obtained from $\mathbf{k}$ by permuting its entries. Denote by $\mathcal{K}$ the set of all such $\mathbf{k}'$ and by $\mathcal{T}_{\mathcal{K}}$ the union of $\mathcal{T}_{\mathbf{k}'}$ for all such $\mathbf{k}'$.

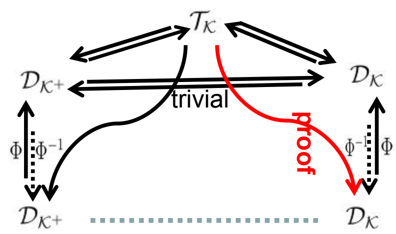


Figure 4: The idea for inverting the sweep map: Solid curve means easy, and dotted curve means difficult. So to obtain $\overline{D}$, we need the help of T.

The Filling Algorithm [6] is adapted in our case as follows, where the major change is the definition of *active*.

Algorithm 2.4 (Filling Algorithm). *Input: The SW-sequence* $\mathtt{SW}(D)$ *of a* $\mathbf{k}$*-Dyck path* $D \in \mathcal{D}_{\mathbf{k}}$.
Output: A tableau $T = T(D) \in \mathcal{T}_{\mathbf{k}}$.

1. *Start by placing a* 1 *in the top row and the first column.*
2. *If the second letter in* $\mathtt{SW}(D)$ *is an* S^* *we put a* 2 *on the top of the second column.*
3. *If the second letter in* $\mathtt{SW}(D)$ *is a* W *we place* 2 *below the* 1.
4. *At any stage the entry at the bottom of the* i*-th column but not in row* $k_i + 1$ *will be called* active.
5. *Having placed* $1, 2, \cdots, i-1$, *we place* i *immediately below the smallest active entry if the* i^{th} *letter in* $\mathtt{SW}(D)$ *is a* W, *otherwise we place* i *at the top of the first empty column.*

6. *We carry this out recursively until* $1, 2, \ldots, n + |\mathbf{k}|$ *have all been placed.*

We will denote by $t(T) = (t_1, \ldots, t_n)$ the top row entries of T from left to right, and similarly by $b(T) = (b_1, \ldots, b_n)$ the bottom entries. Note that the former is always increasing, but the latter is not. We denote by $b_\eta = \min(b(T))$, i.e., the smallest bottom entry appears in the η-th column. It will be convenient to denote by $c_1, c_2, \cdots, c_{k_\eta+1}$ the entries of the column η of T from top to bottom. See Figure 5 for an example.

$$T = \begin{array}{|c|c|c|c|}
\hline
1 & 2 & 8 & 10 \\
\hline
3 & 4 & 11 & 12 \\
\hline
5 & 6 & 13 & 14 \\
\hline
7 & & 15 & 16 \\
\hline
9 & & 17 & \\
\hline
 & & 18 & \\
\hline
\end{array}$$

$SW(D) = S^4S^2WWWWWS^5WS^3WWWWWWWW$
$\mathbf{k} = (4, 2, 5, 3)$
$t(T) = (1, 2, 8, 10)$
$b(T) = (9, 6, 18, 16)$
$b_2 = \min(b(T))$
$c_1 = 2, c_2 = 4, c_3 = 6$

Figure 5: The path D in Figure 3 and its filling tableau by the Filling Algorithm.

It is clear that the top row entries $t(T)$ uniquely determines T. Moreover, $\mathtt{SW}(D)$ can be recovered from $T(D)$ by placing letters S^{k_i} on the positions t_i of $T(D)$ and letters W in all the remaining $|\mathbf{k}|$ positions. Indeed, we have the following characterization.

Lemma 2.5. *An increasing sequence* $(t_1, \ldots, t_n)$ *is the top row entries* $t(T)$ *for some* $T \in \mathcal{T}_\mathbf{k}$ *if and only if* $t_i \leqslant k_1 + \cdots + k_{i-1} + i$ *holds for all* $i \geqslant 2$ *and* $t_1 = 1$.

Proof. By Filling Algorithm 2.4, we always insert 1 at row 1 column 1. For $i \geqslant 2$, we can insert t_i in row 1 column i if and only if the first $i-1$ columns have not been over filled, which is equivalent to $t_i \leqslant k_1 + \cdots + k_{i-1} + i$. ∎

Theorem 2.6. *The Filling Algorithm defines a bijection from* $\mathcal{D}_\mathbf{k}$ *to* $\mathcal{T}_\mathbf{k}$.

Proof. Suppose $D = (a_1, a_2, \ldots, a_{n+|\mathbf{k}|})$, where $a_j = k_i$ if $j = t_i$ and $a_j = -1$ if $j \neq t_i$. Then $D \in \mathcal{D}_\mathbf{k}$ if and only if all the partial sums $a_1 + \cdots + a_j \geqslant 0$. This is clearly equivalent to $t_1 = 1$ and $a_1 + \cdots + a_j \geqslant 0$ for $j = t_i - 1$, $i = 2, \ldots, n$. The theorem then follows by Lemma 2.5 and the fact that

$$0 \leqslant a_1 + \cdots + a_j = k_1 + \cdots + k_{i-1} - (t_i - 1 - (i-1)) = k_1 + \cdots + k_{i-1} + i - t_i$$

holds true for all $i \geqslant 2$. ∎

2.3 Walking algorithm for $\vec{\mathbf{k}}^{\pm}$-Dyck paths

The walking algorithm for inverting the sweep map on $\mathcal{D}_{kn\pm1,n}$ naturally extends to that of $\mathbf{k}^{\pm}$-Dyck paths. To state our results, we need to modify some notations. For $T \in \mathcal{T}_\mathbf{k}$, let T^+ be obtained from T by adding $n+|\mathbf{k}|+1$ below the entry $n+|\mathbf{k}|$. Let $\mathcal{T}_{\mathbf{k}}^+ = \{T^+ : T \in \mathcal{T}_{\mathbf{k}}\}$. The bottom entry of the i-th column of T^+ refers to the (k_i+1)-st entry for all i, i.e., $b(T^+) = b(T)$.

Algorithm 2.7 (Walking Algorithm$^+$)**.** *Input: A tableau $T^+ = T(D^+) \in \mathcal{T}_{\mathbf{k}}^+$ with $b_\eta = \min(b(T^+))$.*
Output: A permutation $\sigma(D^+)$ through walk on T^+.

1. *Write in bold all the entries in T^+ that are by 1 more than a bottom row entry.*
2. *Go to row 1 column η and write the entry c_1.*
3. *If you are in row 1 go down the column to the bottom. If the entry there is r, then go to $r+1$ and write $r+1$.*
4. *If you are not in the first row go up the column one row. If the entry there is r and is not bold write r.*
5. *If the entry there is r and bold go to $r-1$ and continue until you run into a normal entry, then write it.*
6. *Suppose the closed walk is $\omega(T^+) = w_1 \to w_2 \to \cdots \to w_{|\mathbf{k}|+n+1}$ with $w_1 = c_1$, $w_{|\mathbf{k}|+n+1} = c_2$ and $w_s = 1$, then $\sigma(D^+) = w_s w_{s+1} \cdots w_{|\mathbf{k}|+n+1} w_1 \cdots w_{s-1}$.*

See Figure 6 for an example.

$$T^+ = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 8 & \mathbf{10} \\ \hline 3 & 4 & 11 & 12 \\ \hline 5 & 6 & 13 & 14 \\ \hline \mathbf{7} & & 15 & 16 \\ \hline 9 & & \mathbf{17} & \\ \hline & & 18 & \\ \hline & & \mathbf{19} & \end{array} \qquad T^{*+} = \begin{array}{|c|c|c|} \hline 1 & 8 & \mathbf{10} \\ \hline 3 & 11 & 12 \\ \hline 5 & 13 & 14 \\ \hline 7 & 15 & 16 \\ \hline 9 & \mathbf{17} & \\ \hline & 18 & \\ \hline & \mathbf{19} & \end{array}$$

$$\sigma(D^+) = 1\ 10\ 17\ 15\ 13\ 11\ 8\ 19\ 18\ 16\ 14\ 12\ 9\ \mathbf{6}\ \mathbf{4}\ \mathbf{2}\ 7\ 5\ 3$$

$$\sigma(D^{*+}) = 1\ 10\ 17\ 15\ 13\ 11\ 8\ 19\ 18\ 16\ 14\ 12\ 9\ \qquad 7\ 5\ 3$$

Figure 6: Walking Algorithm$^+$ applies to T^+ to give $\sigma(D^+)$, and applies to T^{*+} to give $\sigma(D^{*+})$, where T^{*+} is obtained from T^+ by removing column $\eta = 2$.

Once the permutation $\sigma(D^+)$ is obtained, the SW-word $SW\left(\overline{D^+}\right)$ of the inverse image $\overline{D^+}$ is easy to construct: we write one letter at a time by placing above each entry of $\sigma(D^+)$ an $S^{k_i^+}$ if that entry is t_i and a W if that entry is not in row 1. This done we can simply draw $\overline{D^+}$ by reading the sequence of letters of $SW\left(\overline{D^+}\right)$. Note that we will also use this construction for $\sigma(D^-)$ and $\sigma(D)$ in a similar way.

Theorem 2.8. *For a Dyck path $D^+ \in \mathcal{D}_{\mathbf{k}^+}$, the permutation $\sigma(D^+)$ that rearranges the letters of* `SW`(D^+) *in the order that gives the SW word of $\overline{D^+}$, is obtained by the Walking Algorithm$^+$ (Algorithm 2.7).*

The situation for $\mathcal{D}_{\mathbf{k}^-}$ is similar. Recall that $D^- \in \mathcal{D}_{\mathbf{k}^-}$ only when $D \in \mathcal{D}_{\mathbf{k}}^{\circ}$, i.e., the rank sequence $r(D)$ has only one 0 at $r_1 = 0$. It is not hard to see that the Filling Algorithm takes such D to

$$\mathcal{T}_{\mathbf{k}}^- = \{T \in \mathcal{T}_{\mathbf{k}} : t_i < k_1 + \cdots + k_{i-1} + i \text{ for } i \geqslant 2\},$$

where t_i denotes the top entry in the i-th column of T. The bottom entry of the i-th column of $T^- \in \mathcal{T}_{\mathbf{k}}^-$ refers to the $(k_i + 1)$-st entry for all i, i.e., $b(T^-) = b(T)$.

Theorem 2.9. *The Filling Algorithm defines a bijection from $\mathcal{D}_{\mathbf{k}^-}$ to $\mathcal{T}_{\mathbf{k}}^-$.*

Proof. Suppose $D^- = (a_1, a_2, \ldots, a_{n+|\mathbf{k}|-1})$, where $a_j = k_i^-$ if $j = t_i$ and $a_j = -1$ if $j \neq t_i$. Then $D^- \in \mathcal{D}_{\mathbf{k}^-}$ if and only if all the partial sums $a_1 + \cdots + a_j \geqslant 0$ for all j and the equality holds only when $j = n + |\mathbf{k}| - 1$. This is clearly equivalent to $t_1 = 1$ and $a_1 + \cdots + a_j > 0$ for $j = t_i - 1$, $i = 2, \ldots, n$. The theorem then follows by the following direct computation:

$$0 < a_1 + \cdots + a_j = k_1 + \cdots + k_{i-1} - \frac{i-1}{n} - (t_i - 1 - (i-1)) = k_1 + \cdots + k_{i-1} + i - \frac{i-1}{n} - t_i,$$

which is equivalent to

$$t_i < k_1 + \cdots + k_{i-1} + i, \text{ for all } i \geqslant 2.$$

∎

Algorithm 2.10 (Walking Algorithm$^-$)**.** *Input: A tableau $T^- = T(D^-) \in \mathcal{T}_{\mathbf{k}}^-$ with $b_\eta = \min(b(T^-))$.*
Output: A permutation $\sigma(D^-)$ through walking on T^-.

1. *Write in bold all the entries in T^- that are by 1 less than a bottom row entry.*
2. *Go to row 1 column η and write the entry c_1.*
3. *If you are in row 1 go down the column to the bottom. If the entry there is r go to $r - 1$ and write $r - 1$.*
4. *If you are not in the first row go up the column one row. If the entry there is r and is not bold write r.*
5. *If the entry there is r and bold go to $r + 1$ and continue until you run into a normal entry, then write it.*
6. *Suppose the closed walk is $\omega(T^-) = w_1 w_2 \cdots w_{|\mathbf{k}|+n-1}$ with $w_1 = c_1$, $w_{|\mathbf{k}|+n-1} = c_2$ and $w_s = 1$, then $\sigma(D^-) = w_s w_{s+1} \cdots w_{|\mathbf{k}|+n-1} w_1 \cdots w_{s-1}$.*

See Figure 7 for an example.

Theorem 2.11. *For a Dyck path $D^- \in \mathcal{D}_{\mathbf{k}^-}$, the permutation $\sigma(D^-)$ that rearranges the letters of* `SW`(D^-) *in the order that gives the SW word of $\overline{D^-}$, is obtained by the Walking Algorithm$^-$.*

$$T^- = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 6 & 13 \\ \hline 3 & 4 & \mathbf{9} & 15 \\ \hline 5 & 7 & 12 & \mathbf{17} \\ \hline 8 & \mathbf{10} & 14 & \\ \cline{1-3} 11 & & \mathbf{16} & \\ \cline{1-1}\cline{3-3} & & 18 & \\ \cline{3-3} \end{array} \qquad T^{*-} = \begin{array}{|c|c|c|}\hline 1 & 6 & 13 \\ \hline 3 & \mathbf{9} & 15 \\ \hline 5 & 12 & \mathbf{17} \\ \hline 8 & 14 & \\ \cline{1-2} 11 & \mathbf{16} & \\ \cline{1-2} & 18 & \\ \cline{2-2} \end{array}$$

$$\sigma(D^-) = 1\ 10\ 7\ 4\ 2\ 9\ 6\ 17\ 15\ 13\ 16\ 14\ 12\ 11\ 8\ 5\ 3$$
$$\sigma(D^{*-}) = 1\ 9\ 6\ 17\ 15\ 13\ 16\ 14\ 12\ 11\ 8\ 5\ 3$$

Figure 7: Walking Algorithm$^-$ applies to T^- to give $\sigma(D^-)$, and applies to T^{*-} to give $\sigma(D^{*-})$, where T^{*-} is obtained from T^- by removing column $\eta = 2$.

2.4 The walking algorithm for $\vec{\mathbf{k}}$-Dyck paths

After extending Garsia-Xin's idea for $\mathbf{k}^{\pm}$-Dyck paths, one might think the inverting algorithm will be similar for $\mathbf{k}$-Dyck paths. It turns out that the situation is quite different.

The new walking algorithm not only relies on the intermediate tableau $T = T(D) \in \mathcal{T}_{\mathbf{k}}$, but also on the *rank tableau* $R(D)$ constructed from T.

It is convenient to call numbers in T indices. We will use a ranking algorithm to construct the rank tableau $R(D)$ of T. For clarity, we start with the empty tableau of the shape of T, and successively *assign* each index a rank. By assigning index A a rank r, we mean to fill r into the box in $R(D)$ corresponding to index A in T.

Algorithm 2.12 (Ranking Algorithm)**.** *Input: A tableau $T = T(D) \in \mathcal{T}_{\mathbf{k}}$.*
Output: A rank tableau $R(D)$ of the same shape with T.

1. *Successively assign $0, 1, 2, \ldots, k_1$ to the first column indices of T from top to bottom;*

2. *For i from 2 to n, if the top index of the i-th column is $A+1$, and the rank of index A is a, then assign the index $A+1$ rank a. Moreover, the ranks in the i-th column are successively $a, a+1, \ldots, a+k_i$ from top to bottom.*

See Figure 8 for an example of the Ranking Algorithm.

Observe that the indices are distinct, but the ranks are not. The largest (cell of) rank r (entry) is the rank r with the largest index. For instance in Figure 8, the rank 2 entries have indices $5, 6$, so the largest rank 2 entry is the rank 2 with index 6, whose box is located in row 3 column 2. Similarly, the smallest rank 2 entry is the rank 2 with index 5, whose box is located in row 3 column 1.

We find a way to obtain the permutation $\sigma(D)$ directly from $T(D)$ and $R(D)$ as follows.

$$T(D) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 8 & 10 \\ \hline 3 & 4 & 11 & 12 \\ \hline 5 & 6 & 13 & 14 \\ \hline 7 & & 15 & 16 \\ \hline 9 & & 17 & \\ \hline & & 18 & \\ \hline \end{array} \qquad \Longrightarrow \qquad R(D) = \begin{array}{|c|c|c|c|} \hline 0 & 0 & 3 & 4 \\ \hline 1 & 1 & 4 & 5 \\ \hline 2 & 2 & 5 & 6 \\ \hline 3 & & 6 & 7 \\ \hline 4 & & 7 & \\ \hline & & 8 & \\ \hline \end{array}$$

Figure 8: The tableau $T(D)$ in Figure 5 and its rank tableau.

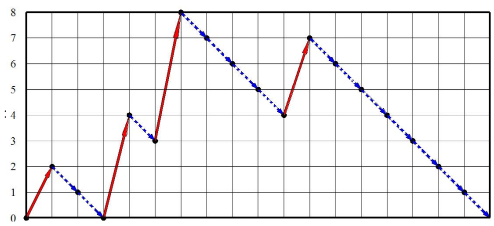
Algorithm 2.13 (Walking Algorithm)**.** *Input: The index-rank tableau* (T, R) *with* $T = T(D) \in \mathcal{T}_{\mathbf{k}}$ *and* $R = R(D)$.
Output: A permutation $\sigma(D)$ *through walking on* (T, R).

1. *In* $R(D)$*, go to the largest rank* 0 *entry. Mark this rank and write down its index;*
2. *Repeat the following steps until no unmarked rank can be selected.*
   (a) *If we are in row* 1*, then go to the bottom row in the same column. If the rank there is* r *then go to the largest unmarked rank* r*. Mark this rank and write down its index;*
   (b) *If we are not in row* 1*, then go up one box. If the rank there is* r *then go to the largest unmarked rank* r*. Mark this rank and write down its index.*

Theorem 2.14. *For a Dyck path* $D \in \mathcal{D}_{\mathbf{k}}$*, the permutation* $\sigma(D)$ *that rearranges the letters of* $\mathtt{SW}(D)$ *in the order that gives the SW word of* $\overline{D}$*, is obtained by the Walking Algorithm.*

Let us apply the Walking Algorithm (Algorithm 2.13) to the index-rank tableau (T, R) in Figure 8. The permutation $\sigma(D)$ is given on the top row. From it, one easily produces the middle $\mathtt{SW}(\overline{D})$, whose rank sequence is given in the third row for comparison. Now we clearly see that the ranks of $\overline{D}$ are exactly the ranks in $R(D)$.

$$\begin{pmatrix} \sigma(D) \\ \mathtt{SW}(\overline{D}) \\ r(\overline{D}) \end{pmatrix} = \begin{pmatrix} 2 & 6 & 4 & 1 & 11 & 8 & 18 & 17 & 15 & 13 & 10 & 16 & 14 & 12 & 9 & 7 & 5 & 3 \\ S^2 & W & W & S^4 & W & S^5 & W & W & W & W & S^3 & W & W & W & W & W & W & W \\ 0 & 2 & 1 & 0 & 4 & 3 & 8 & 7 & 6 & 5 & 4 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

$\overline{D}$ 

$$(T, R) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 8 & 10 \\ \hline 3 & 4 & 11 & 12 \\ \hline 5 & 6 & 13 & 14 \\ \hline 7 & & 15 & 16 \\ \hline 9 & & 17 & \\ \hline & & 18 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 3 & 4 \\ \hline 1 & 1 & 4 & 5 \\ \hline 2 & 2 & 5 & 6 \\ \hline 3 & & 6 & 7 \\ \hline 4 & & 7 & \\ \hline & & 8 & \\ \hline \end{array}$$

3 Some basic auxiliary facts about the sweep map

There are a number of auxiliary properties of the sweep map for $\mathbf{k}$-Dyck paths that need to be established to prove our basic results.

Lemma 3.1. *The Ranking Algorithm 2.12 assigns every index a rank.*

Proof. Assume to the contrary that i with $2 \leqslant i \leqslant n$ is the smallest such that the top index of the i-th column, say $A > 1$, can not be assigned a rank. This only happens when the index $A-1$ is not assigned a rank yet. But then $A-1$ must not belong to the first $i-1$ columns, contradicting the Filling Algorithm. ∎

Lemma 3.2. *Let $D \in \mathcal{D}_{\mathbf{k}}$ be a $\mathbf{k}$-Dyck path with rank tableau $R(D)$. Then the ranks are weakly increasing according to their indices in $T(D)$. In other words, if indices $1, 2, \ldots, n+|\mathbf{k}|$ are assigned ranks $r_1, r_2, r_3, \ldots, r_{n+|\mathbf{k}|}$, then $0 = r_1 \leqslant r_2 \leqslant r_3 \leqslant \ldots \leqslant r_{|\mathbf{k}|+n}$. More precisely, $r_i - r_{i-1}$ is either 0 or 1 for each $2 \leqslant i \leqslant |\mathbf{k}| + n$.*

Proof. We prove by induction on i.

For the base case $i = 2$, we need to consider the following two cases by using the Filling Algorithm 2.4.

1. Index 2 is in row 1 column 2. Then r_2 is assigned 0, so $r_2 - r_1 = 0 - 0 = 0$.

2. Index 2 is placed under index 1. Then r_2 is assigned 1, so $r_2 - r_1 = 1 - 0 = 1$.

Now assume by induction that $0 = r_1 \leqslant r_2 \leqslant r_3 \leqslant \ldots \leqslant r_i$ and that $0 \leqslant r_i - r_{i-1} \leqslant 1$ for $2 \leqslant i < |\mathbf{k}| + n$. We need to show that $0 \leqslant r_{i+1} - r_i \leqslant 1$.

There are three cases as follows.

Case 1: If index $i+1$ is in row 1, then $r_{i+1} = r_i$;

Case 2: If index $i+1$ is placed under index i, then $r_{i+1} = r_i + 1$;

Case 3: Otherwise, the index $i+1$ is not in row 1 and is not placed under index i. We use the fact that if the index j with $j \geqslant 2$ is in row 1, then $r_j = r_{j-1}$. Let i' be the smallest index with rank r_i. Then $r_{i'} = r_{i'+1} = \ldots = r_i$ and we need to consider the following two cases.

(i) If $i' \geqslant 2$ then it cannot be in the top row, for otherwise $r_{i'-1} = r_{i'}$ contradicting our choice of i'. Assume the index above i' is p, and the index above $i+1$ is q. Then $p < q < i' < i+1$ by Filling Algorithm, and we have $r_{i'} = r_p + 1$, $r_{i+1} = r_q + 1$ by Ranking Algorithm. Thus $r_q \leqslant r_{i'}$ by the induction hypothesis and we obtain $r_{i+1} - r_i \leqslant r_{i+1} - r_q = 1$. On the other hand, $r_{i+1} - r_i = r_q - r_p$, which is greater than or equal to 0 (again) by the induction hypothesis.

(ii) If $i' = 1$ then $0 = r_1 = r_2 = \cdots = r_i$. It follows that indices $1, 2, \ldots, i$ are all in row 1 and hence $i+1$ is placed under 1, which implies $0 = r_i < r_{i+1} = 1$. ∎

Lemma 3.3. *The permutation $\sigma(D)$ produced by Algorithm 2.13 has length $n + |\mathbf{k}|$.*

We will give two proofs of this lemma in the next section. The first one only considers the equal parameter case $T \in \mathcal{T}_n^k$. One will see that the idea extends but the notation becomes awkward for $T \in \mathcal{T}_{\mathbf{k}}$. The second one uses standard terminology from graph theory.

Proof of Theorem 2.14. By Lemma 3.3, we may write $\sigma(D) = a_1 a_2 \cdots a_{|\mathbf{k}|+n}$. Assume the corresponding ranks are $r_1, r_2, \ldots, r_{|\mathbf{k}|+n}$. Define $\overline{D}$ to be the SW-sequence obtained from $\sigma(D)$ by replacing each top row index t_i with an S^{k_i} and every other index by a W. We need to show that $\Phi(\overline{D}) = D$. This follows from the following two facts.

Fact 1: The rank sequence of $\overline{D}$ is exactly $(r_1, r_2, \ldots, r_{|\mathbf{k}|+n})$. This is consistent with the rule $r_{j+1} = r_j + k$ if a_j is equal to t_i and $r_{j+1} = r_j - 1$ if a_j is below row 1.

Fact 2: The sweep order is from right to left when two ranks are equal. This corresponds to that for equal rank entries, their corresponding indices are increasing from right to left in $\sigma(D)$. ∎

4 Proof of Lemma 3.3

4.1 First proof

We only consider the equal parameter case $T \in \mathcal{T}_n^k$. We need the following notation. One will see that the notation becomes awkward for $T \in \mathcal{T}_{\mathbf{k}}$.

Let D be a Dyck path in $\mathcal{D}_{kn,n}$ with index-rank tableau $(T(D), R(D))$. For any integer r, we denote by $n(r)$ the number of r's appearing in $R(D)$, so $n(r) = 0$ if $r < 0$. We also denote by $n^-(r)$ and $n^\wedge(r)$ the number of r's in the top row and the number of r's below the top row respectively. Similarly, we denote by $n_-(r)$ and $n_\vee(r)$ the number of r's in the bottom row and the number of r's above the bottom row respectively. Then the following three equalities are clear.

$$n(r) = n_-(r) + n_\vee(r), \qquad n_-(r) = n^-(r-k), \qquad n_\vee(r) = n^\wedge(r+1). \tag{3}$$

Lemma 4.1. *Let $D \in \mathcal{D}_{m,n}$ be a Dyck path, where $m = kn$. Then we have the following basic properties.*

1. *The ranks of D have the common divisor n.*

2. *For a word $\omega \in S^n W^m$ and $1 \leqslant i \leqslant m+n$ denote by $a_i(\omega)$ and $b_i(\omega)$, the numbers of "W" and "S" respectively that occur in the first i letters of ω. It is important to notice that we will have $\omega = \mathtt{SW}(D)$ for some $D \in \mathcal{D}_{m,n}$ if and only if*

$$b_i(\omega)m - a_i(\omega)n \geqslant 0 \qquad \text{for all } 1 \leqslant i \leqslant m+n.$$

3. *If a rank r appears in $R(D)$, it will appear at most n times and*

$$n(r) = n^-(r-k) + n^\wedge(r+1).$$

In particular, we have

$$n(0) = n^\wedge(1).$$

Proof. 1. Because we start with assigning 0 to the south end of the first North step, this done we add an $m = kn$ as we go North and subtract an n as we go East, all the ranks are divisible by n.

2. In fact after $a_i(\omega)$ letters W and $b_i(\omega)$ letters S, the corresponding path has reached a lattice point of coordinates $\big(a_i(\omega), b_i(\omega)\big)$, this point is above the diagonal $(0,0) \to (m,n)$ if and only if

$$\frac{b_i(\omega)}{a_i(\omega)} \geqslant \frac{n}{m}.$$

3. Since every column of $R(D)$ is strictly increasing, any rank r may appear at most n times. The equality $n(r) = n^-(r-k) + n^\wedge(r+1)$ follows from equation (3).

∎

Lemma 4.2. *In the equal parameter case, Algorithm 2.13 terminates only after we mark the smallest rank* 1 *entry and write down its index.*

Proof. Suppose the algorithm terminates after we mark a rank r entry and write down its index a.

If a is in row 1, then all rank $r+k$ entries have been marked. But to mark a rank $r+k$ entry, we can either go from a rank r whose index is in row 1, or go from a rank $r+k+1$ entry whose index is not in row 1. Thus the number of marked rank $r+k$ entry is at most $n^-(r) + n^\wedge(r+k+1) - 1$, which is equal to $n(r+k) - 1$ by the equality in Lemma 4.1(3). This is a contradiction.

If a is not in row 1, then $r \geqslant 1$ and there is no unmarked rank $r-1$ for the Walking Algorithm to terminate. We have the following two cases.

1. If $r > 1$, then we show the contradiction that there is at least one unmarked rank $r-1$ entry so that the algorithm will not terminate. Or equivalently the number of marked $r-1$ is at most $n(r-1) - 1$. According to the Walking Algorithm, to mark a rank $r-1$ entry, we can either go from a rank $r-1-k$ whose index is in the top row or go from a rank r entry whose index is not in the top row. Thus the number of marked rank $r-1$ entry is at most $n^-(r-1-k) + n^\wedge(r) - 1$, which is equal to $n(r-1) - 1$ by the equality in Lemma 4.1(3).

2. If $r = 1$ and we stopped at the non-smallest rank 1 entry, then we show the contradiction that there is at least one unmarked rank 0 entry so that the algorithm will not terminate. Or equivalently the number of marked ranks 0 entries is at most $n(0) - 1$. The reason is similar. According to the Walking Algorithm, to mark a rank 0 entry, we can either mark the largest rank 0 entry at the first step or go from a rank 1 entry whose index is not in the top row. Since we stopped at the non-smallest rank 1 entry, the number of marked ranks 0 entry is at most $1 + n^\wedge(1) - 2 = n^\wedge(1) - 1$, which is equal to $n(0) - 1$ by the equality in Lemma 4.1(3).

∎

First Proof of Lemma 3.3. Now we are ready to show that Algorithm 2.13 terminates after we write down all of the $m+n$ indices. Assume to the contrary that the algorithm terminates after we write down only $p < m+n$ indices. Denote the resulting $\sigma(D)$ by $a_{i_1}, a_{i_2}, a_{i_3}, \dots, a_{i_p}$, with corresponding ranks $r_{i_1}, r_{i_2}, r_{i_3}, \dots, r_{i_p}$. Then $r_{i_1} = 0, r_{i_2} = k$, and $r_{i_p} = 1$ is the smallest rank 1 by Lemma 4.2. Note that all rank 0 indices have been written. In particularly, index 1 is written. Moreover, for each j with $1 \leqslant j < p$ either $r_{i_j} + k = r_{i_{j+1}}$ or $r_{i_j} - 1 = r_{i_{j+1}}$.

Arrange the unwritten indices in increasing order as $1 < a_{i_{p+1}} < a_{i_{p+2}} < \cdots < a_{i_{m+n}}$, and denote their corresponding ranks by $r_{i_{p+1}} \leqslant r_{i_{p+2}} \leqslant r_{i_{p+3}} \leqslant \ldots \leqslant r_{i_{m+n}}$. By our choice $a_{i_{p+1}} - 1$ is a written index, so assume $a_{i_j} = a_{i_{p+1}} - 1$ for some $j \leqslant p$. Now we have two cases, both leading to contradictions.

1. If $r_{i_{p+1}} = r_{i_j}$, then the Walking Algorithm would have preferred to writing index $a_{i_{p+1}}$ than a_{i_j} when visiting rank r_{i_j}.

2. If $r_{i_j} < r_{i_{p+1}}$, then $a_{i_{p+1}}$ is not in row 1. By definition of $a_{i_{p+1}}$ and the increasing ranks of the indices in Lemma 3.2, all rank $\alpha = r_{i_j}$ entries have been marked. But to mark a rank α entry, we can either go from a rank $\alpha - k$ in top row, or from a rank $\alpha + 1$ below top row (including the index $a_{i_{p+1}}$ with $r_{i_{p+1}}$). These implies that there are at most $n(\alpha - k) + n(\alpha + 1) - 1 = n(\alpha) - 1$ (by 4.1(3)) rank α entries have been marked, a contradiction. ∎

4.2 Second proof

Our second proof assumes basic knowledge of graph theory.

Firstly we construct a digraph G_R from the rank tableau $R = R(D)$ as follows. The vertices of G_R are the ranks appearing in R. Each directed edge of G is associated with an index of T: i) if the index is t_i and has rank a, then the directed edge is $a \to a + k_i$; ii) if the box is not in row 1 and has rank b, then the directed edge is $b \to b - 1$.

Each rank a of G_R is associated with a set $S(a)$ consisting of all indices with rank a. Denote by $F(T) = \{t_1, \dots, t_n\}$ the set of first row indices of T arranged increasingly.

Lemma 4.3. *The digraph G_R of a rank tableau $R = R(D)$ is balanced. That is, each rank a of G has in-degree equal to out-degree, and equal to $|S(a)|$.*

Proof. Let C_i be the digraph obtained by restricting G to the i-th column of R. Then C_i is clearly a directed cycle $r \to r + k_i \to \cdots \to r + 2 \to r + 1 \to r$, where r is the rank in row 1. The in-degree and out-degree of a rank a are both 1 if a appears in C_i and are 0 if otherwise. The lemma then follows since G is the union of C_i for $i = 1, 2, \dots, n$. ∎

The Walking Algorithm can be restated as a modified Eulerian tour as follows.

Algorithm 4.4 (Walking Algorithm G)**.** *Input: The digraph G_R together with $S(a)$ associated to every vertex a, and $F(T)$ as above.*
Output: A permutation $\sigma(D)$ through walking on G_R.

1. *In G_R, go to rank* 0*, mark the largest index in* $S(0)$ *and write down it;*

2. *Repeat the following steps.*

   (a) *Suppose we are at rank* r *and has just marked* t_i *in* $F(T)$*. If there is no unmarked index in* $S(r+k_i)$*, then terminates. Otherwise, go to rank* $r+k_i$*, mark the largest unmarked index in* $S(r+k_i)$ *and write down it;*

   (b) *Suppose we are at rank* r *and has just marked an index not in* $F(T)$*. If there is no unmarked index in* $S(r-1)$*, then terminates. Otherwise, go to rank* $r-1$*, mark the largest unmarked index in* $S(r-1)$ *and write down it.*

Second Proof of Lemma 3.3. Assume to the contrary that the algorithm terminates after we write down $\sigma(D) = a_1a_2\cdots a_p$ for $p < |\mathbf{k}| + n$. Then $r(a_1) = 0$.

Firstly we claim that $r(a_p) = 1$, and the algorithm terminates when we are trying to go to rank 0. This is simply due to the following observations: i) the in-degree and out-degree of rank r is $|S(r)|$; ii) every time when marking an index in $S(r)$, we used one in-degree and one out-degree (including the attempt to go out); iii) the assumption that every index of $S(r)$ has been marked implies that all the in-degree and out-degree has been used, and hence there are no more edges in G_R directed to rank r. The only exceptional case is when $r = 0$, because we starting by marking the largest rank 0 index without using its in-degree.

Now $C = r(a_1) \to r(a_2) \to \cdots r(a_p) \to r(a_1)$ is a directed cycle contained in G_R as a subgraph. Let b be the smallest unwritten index. Then $b - 1 = a_i \geqslant 1$ for some $i \leqslant p$, since all rank 0 indices have been written. By Lemma 3.2 $r(b)-r(a_i)$ is either 0 or 1. Both cases lead to contradictions. i) If $r(b) = r(a_i)$, then the Walking Algorithm would have preferred to writing index b than a_i when visiting rank $r(b)$. ii) If $r(b) = r(a_i) + 1$, then b is not in row 1 by Ranking Algorithm, and hence will be associated to a directed edge $e = r(b) \to r(b) - 1$ in G_R. Now observe that $S(r(b) - 1)$ are all contained $\sigma(D)$. This implies that the directed cycle C contains all directed edges into $r(b) - 1$, in particular the edge e. Then the Walking Algorithm must have written b already, which contradicts the choice of b. ∎

5 Proof of Theorem 2.8

For a vector $\mathbf{k} = (k_1, k_2, \cdots, k_n)$ ($\ell(\mathbf{k}) = n$) and $T^+ = T(D^+) \in \mathcal{T}^+_{\mathbf{k}}$, let $\omega(T^+)$ be the closed walk in the entries of T^+ yielded by Algorithm 2.7.

It is convenient to make the following convention in this section. We fix the positive integer η with b_η=min$(b(T^+))$. Let $c_1, c_2, \cdots, c_{k_\eta+1}$ be the entries of the column η of T^+. Denote by $\mathbf{k}^*$ the vector obtained by removing the k_η from $\mathbf{k}$ and denote by T^{*+} the tableau obtained by removing from T^+ the η-th column. By the induction hypothesis, if we apply Algorithm 2.7 with respect to T^{*+}, as if its letters are contiguous, then we obtain a closed walk $\omega(T^{*+})$ on the entries of T^{*+}.

The closed walk $\omega(T^+)$ and $\omega(T^{*+})$ are closely related.

Lemma 5.1. *Let $D^+ \in \mathcal{D}_{\mathbf{k}^+}$ be a Dyck path, $T^+ = T(D^+)$ be its tableau, b_η=min$(b(T^+))$ and let T^{*+} be obtained from T by removing the η-th column. Then the following properties hold true.*

1. *As a closed walk $\omega(T^+)$ contains the column η segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1$.*

2. *Omitting the column η segment from $\omega(T^+)$ gives $\omega(T^{*+})$. More precisely, if $\omega(T^+)$ contains the segment $c' \to c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1 \to c_{k_\eta+1}+1$, then replacing this segment by $c' \to c_{k_\eta+1}+1$ gives $\omega(T^{*+})$.*

3. *$\ell(\omega(T^+)) = \ell(\omega(T^{*+})) + k_\eta + 1$*

4. *$\omega(T^+)$ is a closed walk of length $|\mathbf{k}| + n + 1$.*

Proof.

1. Since $\omega(T^+)$ is a closed walk containing c_1, it must return to c_1. Now c_j has indegree 1 from c_{j+1} for $j = 1, 2, \cdots, k_\eta$. It follows that $\omega(T^+)$ must contain the segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1$.

2. The directed edges of $\omega(T^+)$ and $\omega(T^{*+})$ are the same if both ends do not involve column η entries. The directed edges in $\omega(T^+)$ that involve column η entries are $c_{j+1} \to c_j$ for $1 \leqslant j \leqslant k_\eta$, and $c_1 \to c_{k_\eta+1}+1$, together with $c' \to c_{k_\eta+1}$ for some entry $c' \in T^{*+}$. We claim that the only directed edge in $\omega(T^{*+})$ that involves an entry of column η is $c' \to c_{k_\eta+1}+1$. This is because in T^+ we will go from c' to $c_{k_\eta+1}+1$, a bold faced letter, and then to $c_{k_\eta+1}$. While in T^{*+}, $c_{k_\eta+1}+1$ is not bold faced, so in $\omega(T^{*+})$ we have the directed edge $c' \to c_{k_\eta+1}+1$. This is equivalent to replacing the segment $c' \to c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1 \to c_{k_\eta+1}+1$ in $\omega(T^+)$ by $c' \to c_{k_\eta+1}+1$ to obtain $\omega(T^{*+})$.

3. This is a direct consequence of (2).

4. Follows by induction on n and part (3). ∎

Now we are ready to prove Theorem 2.8, which is restated as follows.

Theorem 5.2. *On the side of each edge $p \to q$ of $\omega(T^+)$ let us place an $S^{k_j^+}$ if p is in row 1 column j of T^+ and a W otherwise. This done, the SW sequence of the path $\Phi^{-1}(D^+)$ is simply obtained by reading all these edge labels starting from $p = 1$ and following the directed edges of $\omega(T^+)$.*

Proof. Let $\overline{D^+}$ be the path which results from this SW sequence. The path $\overline{D^+}$ will go from $(0, 0)$ to $(|\mathbf{k}| + n + 1, 0)$. Let us compute the sequence of ranks starting by assigning 0 to $p = 1$ then inductively (following $\omega(T^+)$) for each edge $p \to q$ set $\texttt{rank}(q)$=$\texttt{rank}(p) + k_j + \frac{1}{n}$ or $\texttt{rank}(q)$=$\texttt{rank}(p) - 1$ according as the label of $p \to q$ is an $S^{k_j^+}$ (when $p = t_j$) or a W (when $p \neq t_j$ for all j). To show that $\overline{D^+}$ is a Dyck path we must prove that all these ranks are non-negative. We will do this by showing that

$$\texttt{rank}(j+1) - \texttt{rank}(j) > 0 \ (\mathit{for\ all}\ 1 \leqslant j \leqslant |\mathbf{k}| + n). \tag{4}$$

In fact, this not only yields that $\overline{D^+}$ is a Dyck path but we will also obtain that $SW(D^+)$ is a rearrangement of the steps of $\overline{D^+}$ by increasing ranks of their starting entries, proving that $\overline{D^+} = \Phi^{-1}(D^+)$.

We need the following fact: Suppose the path $\prod_{p \dashrightarrow q}$ from p to q contains edge labels $S^{k^+_{j_m}}(m = 1, 2, \cdots, b_{p,q})$ and $a_{p,q}$ edge labels W. Then we have the identity

$$\texttt{rank}(q) - \texttt{rank}(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b_{p,q}}} + \frac{b_{p,q}}{n} - a_{p,q}.$$

Thus to prove (4) we need only to show that $k_{j_1} + k_{j_2} + \cdots + k_{j_{b_{p,p+1}}} + \frac{b_{p,p+1}}{n} - a_{p,p+1} > 0$ for all $1 \leqslant p \leqslant |\mathbf{k}| + n$.

We will prove the theorem by induction on $n = \ell(\mathbf{k})$. The case $n = 1$ is trivial, so we assume the theorem holds for $n - 1$, which is the length of $\mathbf{k}^*$. To prove the theorem for n, we will reduce paths in $\omega(T^+)$ to that in $\omega(T^{*+})$ by omitting column η segment. The induction hypothesis implies that if $p < q$ in T^{*+}, then

$$\texttt{rank}^*(q) - \texttt{rank}^*(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p,q}}} + \frac{b^*_{p,q}}{n-1} - a^*_{p,q} > 0,$$

where the notation should be adapted to T^{*+}.

Claim: if $p < q$ in T^+, and $p, q \neq c_u$ for $u \leqslant k_\eta$, then $\texttt{rank}(q) - \texttt{rank}(p) > 0$.

This is because in the assumed cases, the path $\prod_{p \dashrightarrow q}$ in $\omega(T^+)$ is reduced to a path $\prod^*_{p' \dashrightarrow q'}$ in $\omega(T^{*+})$, where $p' = p + \chi(p = c_{k_\eta+1})$ and $q' = q + \chi(q = c_{k_\eta+1})$. Clearly $p' \leqslant q'$. Now the path $\prod_{p \dashrightarrow q}$ contains either all or none of the column η segment:

i) if $\prod_{p \dashrightarrow q}$ does not contain column η segment, then it is also in $\omega(T^{*+})$. We have $b^*_{p',q'} = b_{p,q}$, $a^*_{p',q'} = a_{p,q}$, and

$$\texttt{rank}^*(q') - \texttt{rank}^*(p') = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p',q'}}} + \frac{b^*_{p',q'}}{n-1} - a^*_{p',q'} > 0$$

$$\Rightarrow \texttt{rank}(q) - \texttt{rank}(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b_{p,q}}} + \frac{b_{p,q}}{n} - a_{p,q} > 0,$$

because all k_{j_m}, $a_{p,q}$, $b_{p,q}$ are integers and $b^*_{p',q'} = b_{p,q} \leqslant n - 1$;

ii) if $\prod_{p \dashrightarrow q}$ contains the whole column η segment, then the path reduces to $\prod^*_{p' \dashrightarrow q'}$. We have

$$\texttt{rank}^*(q') - \texttt{rank}^*(p') = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p',q'}}} + \frac{b^*_{p',q'}}{n-1} - a^*_{p',q'} \geqslant 0$$

for $b^*_{p',q'} \leqslant n - 1$ and the equality holds only when $p = c_{k_\eta+1}, q = c_{k_\eta+1} + 1$. This situation leads to $b^*_{p',q'} = a^*_{p',q'} = 0$. Then

$$\texttt{rank}(q) - \texttt{rank}(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p',q'}}} + k_\eta + \frac{b^*_{p',q'} + 1}{n} - (a^*_{p',q'} + k_\eta) > 0,$$

for the same reason as in case i). This completes the proof of the claim.

Now we prove (4) by dealing with three cases (not necessarily mutually exclusive):

- Case 1: $p, p+1 \neq c_u$ for $u \leqslant k_\eta$. By the claim, $\texttt{rank}(p+1) > \texttt{rank}(p)$.
- Case 2: Both p and $p+1$ are in column η. In this case we will have $p = c_u$ and $p+1 = c_{u+1}$ for some $1 \leqslant u \leqslant k_\eta$. Since in $\omega(T^+)$ we have the directed edge $c_{u+1} \to c_u$, it follows that $\texttt{rank}(p+1) - \texttt{rank}(p) = 1$.
- Case 3: If exactly one of p and $p+1$ equals c_u for $u \leqslant k_\eta$, then we will transform the path from p to $p+1$ to another path q to $q' > q$ by adding and removing a same number of down steps (i.e., $(1, -1)$), so that the Claim applies and we deduce that

$$\texttt{rank}(p+1) - \texttt{rank}(p) = \texttt{rank}(q') - \texttt{rank}(q) > 0.$$

We divide into two subcases as follows.

1. if $p = c_u$ and $p+1$ is not in column η. Assume $p+1$ is in column j with entries $d_1, d_2, \cdots, d_{k_j+1}$. Since $d_{k_j+1} > c_{k_\eta+1} = \min(b(T^+))$, we may assume $p+1 = d_v$ for some $v < k_j+1$. Now after $c_u = p$ and $d_v = p+1$ are inserted into T^+, c_u and d_v are both active, the next entries inserted into the two columns must be subsequently c_{u+1} and then d_{v+1}, c_{u+2}, and so on. It follows that $c_u < d_v < c_{u+1} < d_{v+1} < \cdots < d_{v+k_\eta-u} < c_{k_\eta+1} < d_{v+k_\eta+1-u}$. Now we transform $\prod_{p \dashrightarrow p+1}$ to $\prod_{q \dashrightarrow q'}$, where $q = c_{k_\eta+1}$ and $q' = d_{v+k_\eta+1-u} > q$ so that the Claim applies. It remains to show that this transform does not change the total weight of the path. Observe that $\omega(T^+)$ contains the segment $d_{v+k_\eta+1-u} \to d_{v+k_\eta-u} \to \cdots \to d_v$. This is due to the fact that $c_{k_\eta+1}+1$ is the smallest bold faced number and that $c_{k_\eta+1} > d_{v+k_\eta-u}$. By Lemma 5.1 $\omega(T^+)$ is a full cycle containing the segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_u$. Thus the path $\prod_{p \dashrightarrow p+1}$ looks like $p = c_u \to c_{u-1} \to \cdots \to d_{v+k_\eta+1-u} \to d_{v+k_\eta-u} \to \cdots \to d_v = p+1$. By adding $k_\eta+1-u$ down steps at the beginning and removing $k_\eta+1-u$ down steps at the end, we do not change the total weight of the path and obtain $q = c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to d_{v+k_\eta+1-u} = q'$, the path $\prod_{q \dashrightarrow q'}$ in $\omega(T^+)$, as desired.
2. if $p+1 = c_u$ and p is not in column η. The situation is similar to (1). We will include the details here for convenience. Assume p is in column j with entries $d_1, d_2, \cdots, d_{k_j+1}$. Since $d_{k_j+1} > c_{k_\eta+1} = \min(b(T^+))$, we may assume $p+1 = d_v$ for some $v < k_j+1$. Now after $d_v = p$ and $c_u = p+1$ are inserted into T^+, d_v and c_u are both active, the next entries inserted into the two columns must be subsequently d_{v+1} and then c_{u+1}, d_{v+2}, and so on. It follows that $d_v < c_u < d_{v+1} < c_{u+1} < \cdots < d_{v+k_\eta+1-u} < c_{k_\eta+1}$. Now we transform $\prod_{p \dashrightarrow p+1}$ to $\prod_{q \dashrightarrow q'}$, where $q = d_{v+k_\eta+1-u}$ and $q' = c_{k_\eta+1} > q$ so that the Claim applies. It remains to show that this transform does not change the total weight of the path. Observe that $\omega(T^+)$ contains the segment $d_{v+k_\eta+1-u} \to d_{v+k_\eta-u} \to \cdots \to d_v$. This is due to the fact that $c_{k_\eta+1}+1$ is the smallest bold faced number and that $c_{k_\eta+1} > d_{v+k_\eta-u}$. By Lemma 5.1 $\omega(T^+)$ is a full cycle containing the segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_u$. Thus

the path $\prod_{p \dashrightarrow p+1}$ looks like $p = d_v \to \cdots \to c_{k_\eta+1} \to \cdots \to c_u = p+1$. By adding $k_\eta + 1 - u$ down steps at the beginning and removing $k_\eta + 1 - u$ down steps at the end, we do not change the total weight of the path and obtain $q = d_{v+k_\eta+1-u} \to \cdots \to d_v \to \cdots \to c_{k_\eta+1} = q'$, the path $\prod_{q \dashrightarrow q'}$ in $\omega(T^+)$, as desired.

∎

6 Proof of Theorem 2.11

For a vector $\mathbf{k} = (k_1, k_2, \cdots, k_n)$ of length $\ell(\mathbf{k}) = n$ and $T^- = T(D^-) \in \mathcal{T}_{\mathbf{k}}^-$, let $\omega(T^-)$ be the closed walk in the entries of T^- yielded by Algorithm 2.10.

It is convenient to make the following convention in this section. We fix the positive integer η with b_η=min$(b(T^-))$. Let $c_1, c_2, \cdots, c_{k_\eta+1}$ be the entries of the column η of T^-. Denote by $\mathbf{k}^*$ the vector obtained by removing the k_η from $\mathbf{k}$ and denote by T^{*-} the tableau obtained by removing from T^- the η-th column. By the induction hypothesis, if we apply Algorithm 2.10 with respect to T^{*-}, as if its letters are contiguous, then we obtain a closed walk $\omega(T^{*-})$ on the entries of T^{*-}.

Unlike the case for $\mathcal{T}_{\mathbf{k}}^+$, we need the following lemma.

Lemma 6.1. *Let $T^- = T(D^-) \in \mathcal{T}_{\mathbf{k}}^-$ be a given tableau. Suppose $b_\eta = \min(b(T^-))$ (note that in $\mathcal{T}_n^k$, this η is always 1). Then there is at most one bold faced entry in column η of T^-. Moreover, if $\ell(\mathbf{k}) \geqslant 2$, then column η can only have the bottom entry b_η as the possible bold faced entry.*

Proof. i): The case $\ell(\mathbf{k}) = 1$ is trivial.

ii): By definition, $b_\eta - 1$ is the smallest bold faced entry and b_η can only be bold faced when $b_\eta = b_j - 1$ for some j. Thus it is sufficient to show that $b_\eta - 1$ is not in column η of T^-. Suppose to the contrary that $b_\eta - 1$ is in this column. We may assume that column η has $s, s+1, \ldots, b_\eta$ for $s \leqslant b_\eta - 1$ but not has $s-1$. Now $s-1$ is not a bottom entry by the fact $b_\eta = \min(b(T^-))$. i) if $s > 1$ then $s-1$ is in another column, and by Filling Algorithm 2.4, $s+1$ would have been placed under $s-1$, a contradiction; ii) if $s = 1$ then column η has entries $1, 2, \ldots, b_\eta$, then $\eta = 1$, $b_1 = k_1 + 1$, which forces $t_2 = k_1 + 2$. This contradicts the definition of T^-. ∎

The closed walk $\omega(T^-)$ and $\omega(T^{*-})$ are closely related.

Lemma 6.2. *Let $D^- \in \mathcal{D}_{\mathbf{k}^-}$ be a Dyck path ($\ell(\mathbf{k}) = n \geqslant 2$), $T^- = T(D^-)$ be its tableau, b_η=min$(b(T^-))$ and let T^{*-} be obtained from T by removing the η-th column. Then the following properties hold true.*

1. *As a closed walk $\omega(T^-)$ contains the column η segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1$.*
2. *Omitting the column η segment from $\omega(T^-)$ gives $\omega(T^{*-})$. More precisely, if $\omega(T^-)$ contains the segment $c' \to c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1 \to c_{k_\eta+1} - 1$, then replacing this segment by $c' \to c_{k_\eta+1} - 1$ gives $\omega(T^{*-})$.*

3. $\ell(\omega(T^-)) = \ell(\omega(T^{*-})) + k_\eta + 1$.

4. $\omega(T^-)$ *is a closed walk of length* $|\mathbf{k}| + n - 1$.

Proof.

1. Since $\omega(T^-)$ is a closed walk containing c_1, it must return to c_1. Now c_j has indegree 1 from c_{j+1} for $j = 1, 2, \cdots, k_\eta$ by Algorithm 2.10 and Lemma 6.1. It follows that $\omega(T^-)$ must contain the segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1$. (When $n = 1$, $\omega(T^-)$ only contains the segment $c_{k_1} \to c_{k_1-1} \to \cdots \to c_1$.)

2. The directed edges of $\omega(T^-)$ and $\omega(T^{*-})$ are the same if both ends do not involve column η entries. The directed edges in $\omega(T^-)$ that involve column η entries are $c_{j+1} \to c_j$ for $1 \leqslant j \leqslant k_\eta$, and $c_1 \to c_{k_\eta+1} - 1$, together with $c' \to c_{k_\eta+1}$ for some entry $c' \in T^{*-}$. We claim that the only directed edge in $\omega(T^{*-})$ that involves an entry of column η is $c' \to c_{k_\eta+1} - 1$. This is because in T^- we will go from c' to $c_{k_\eta+1} - 1$, a bold faced letter, and then to $c_{k_\eta+1}$. While in T^{*-}, $c_{k_\eta+1} - 1$ is not bold faced, so in $\omega(T^{*-})$ we have the directed edge $c' \to c_{k_\eta+1} - 1$. This is equivalent to replacing the segment $c' \to c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_1 \to c_{k_\eta+1} - 1$ in $\omega(T^-)$ by $c' \to c_{k_\eta+1} - 1$ to obtain $\omega(T^{*-})$.

3. This is a direct consequence of (2).

4. Follows by induction on n and part (3). ∎

Now we are ready to prove Theorem 2.11, which is restated as follows.

Theorem 6.3. *On the side of each edge* $p \to q$ *of* $\omega(T^-)$ *let us place an* $S^{k_j^-}$ *if* p *is in row* 1 *column* j *of* T^- *and a* W *otherwise. This done, the SW sequence of the path* $\Phi^{-1}(D^-)$ *is simply obtained by reading all these edge labels starting from* $p = 1$ *and following the directed edges of* $\omega(T^-)$.

Proof. Let $\overline{D^-}$ be the path which results from this SW sequence. The path $\overline{D^-}$ will go from $(0, 0)$ to $(|\mathbf{k}| + n - 1, 0)$. Let us compute the sequence of ranks starting by assigning 0 to $p = 1$ then inductively (following $\omega(T^-)$) for each edge $p \to q$ set $\texttt{rank}(q)$=$\texttt{rank}(p) + k_j - \frac{1}{n}$ or $\texttt{rank}(q)$=$\texttt{rank}(p) - 1$ according as the label of $p \to q$ is an $S^{k_j^-}$ or a W. To show that $\overline{D^-}$ is a Dyck path we must prove that all these ranks are non-negative. We will do this by showing that

$$\texttt{rank}(j + 1) - \texttt{rank}(j) > 0 \ (\textit{for all } 1 \leqslant j \leqslant |\mathbf{k}| + n - 2). \tag{5}$$

In fact, this not only yields that $\overline{D^-}$ is a Dyck path but we will also obtain that $SW(D^-)$ is a rearrangement of the steps of $\overline{D^-}$ by increasing ranks of their starting entries, proving that $\overline{D^-} = \Phi^{-1}(D^-)$.

We need the following fact: Suppose the path $\prod_{p \dashrightarrow q}$ from p to q contains edge labels $S^{k_{j_m}^-} (m = 1, 2, \cdots, b_{p,q})$ and $a_{p,q}$ edge labels W. Then we have the identity

$$\texttt{rank}(q) - \texttt{rank}(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b_{p,q}}} - \frac{b_{p,q}}{n} - a_{p,q}.$$

Thus to prove (5) we need only to show that $k_{j_1} + k_{j_2} + \cdots + k_{j_{b_{p,p+1}}} - \frac{b_{p,p+1}}{n} - a_{p,p+1} > 0$ for all $1 \leqslant p \leqslant |\mathbf{k}| + n - 2$.

We will prove the theorem by induction on $n = \ell(\mathbf{k})$. The case $n = 1$ is trivial, so we assume the theorem holds for $n - 1$, which is the length of $\mathbf{k}^*$. To prove the theorem for n, we will reduce paths in $\omega(T^-)$ to that in $\omega(T^{*-})$ by omitting column η segment. The induction hypothesis implies that if $p < q$ in T^{*-}, then

$$\mathtt{rank}^*(q) - \mathtt{rank}^*(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p,q}}} - \frac{b^*_{p,q}}{n-1} - a^*_{p,q} > 0,$$

where the notation should be adapted to T^{*-}.

Claim: if $p < q$ in T^-, and $p, q \neq c_u$ for $u \leqslant k_\eta$, then $\mathtt{rank}(q) - \mathtt{rank}(p) > 0$.

This is because in the assumed cases, the path $\prod_{p \dashrightarrow q}$ in $\omega(T^-)$ is reduced to a path $\prod^*_{p' \dashrightarrow q'}$ in $\omega(T^{*-})$, where $p' = p - \chi(p = c_{k_\eta+1})$ and $q' = q - \chi(q = c_{k_\eta+1})$. Clearly $p' \leqslant q'$. Now the path $\prod_{p \dashrightarrow q}$ contains either all or none of the column η segment:

i) if $\prod_{p \dashrightarrow q}$ does not contain column η segment, then we need to consider the following two sub-cases:

Case a): If $p' < q'$, then the path reduces to $\prod^*_{p' \dashrightarrow q'}$ with $b^*_{p',q'} = b_{p,q}$ and $a^*_{p',q'} = a_{p,q}$, we have

$$\mathtt{rank}^*(q') - \mathtt{rank}^*(p') = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p',q'}}} - \frac{b^*_{p',q'}}{n-1} - a^*_{p',q'} > 0$$

$$\Rightarrow \mathtt{rank}(q) - \mathtt{rank}(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b_{p,q}}} - \frac{b_{p,q}}{n} - a_{p,q} > 0;$$

Case b): If $p' = q'$, then this situation happens when $p = c_{k_\eta+1} - 1,\ q = c_{k_\eta+1}$

$$\Rightarrow \mathtt{rank}(q) - \mathtt{rank}(p) = -(k_\eta - \frac{1}{n}) + k_\eta = \frac{1}{n} > 0.$$

ii) if $\prod_{p \dashrightarrow q}$ contains the whole column η segment, then the path reduces to $\prod^*_{p' \dashrightarrow q'}$, we have

$$\mathtt{rank}^*(q') - \mathtt{rank}^*(p') = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p',q'}}} - \frac{b^*_{p',q'}}{n-1} - a^*_{p',q'} > 0$$

$$\Rightarrow \mathtt{rank}(q) - \mathtt{rank}(p) = k_{j_1} + k_{j_2} + \cdots + k_{j_{b^*_{p',q'}}} + k_\eta - \frac{b^*_{p',q'} + 1}{n} - (a^*_{p',q'} + k_\eta) > 0,$$

because all k_{j_m}, $a^*_{p',q'}$, $b^*_{p',q'}$ are integers and $b^*_{p',q'} \leqslant n - 1$. This completes the proof of the claim.

Now we prove (5) by dealing with three cases (not necessarily mutually exclusive):

- Case 1: $p, p+1 \neq c_u$ for $u \leqslant k_\eta$. By the claim, $\mathtt{rank}(p+1) > \mathtt{rank}(p)$.
- Case 2: Both p and $p+1$ are in column η. In this case we will have $p = c_u$ and $p + 1 = c_{u+1}$ for some $1 \leqslant u \leqslant k_\eta$. Since in $\omega(T^-)$ we have the directed edge $c_{u+1} \to c_u$, it follows that $\mathtt{rank}(p+1) - \mathtt{rank}(p) = 1$.

- Case 3: If exactly one of p and $p+1$ equals c_u for $u \leqslant k_\eta$, then we will transform the path from p to $p+1$ to another path q to $q' > q$ by adding and removing a same number of down steps, so that the Claim applies and we deduce that

$$\texttt{rank}(p+1) - \texttt{rank}(p) = \texttt{rank}(q') - \texttt{rank}(q) > 0.$$

We divide into two subcases as follows.

1. if $p = c_u$ and $p+1$ is not in column η. Unlike the case for $\mathbf{k}^+$-Dyck paths, we need to consider one extra situation. Assume $p+1$ is in column j with entries $d_1, d_2, \cdots, d_{k_j}, d_{k_j+1}$. Since $d_{k_j+1} > c_{k_\eta+1} = \min(b(T^-))$, we may assume $p+1 = d_v$ for some $v < k_j+1$. Now after $c_u = p$ and $d_v = p+1$ are inserted into T^-, c_u and d_v are both active, the next entries inserted into the two columns must be subsequently c_{u+1} and then d_{v+1}, c_{u+2}, and so on. It follows that $c_u < d_v < c_{u+1} < d_{v+1} < \cdots < c_{k_\eta} < d_{v+k_\eta-u} < c_{k_\eta+1} < d_{v+k_\eta+1-u}$. Now recall that $c_{k_\eta+1} - 1$ is the smallest bold faced number. This forces us to consider the extra situation that if $d_{v+k_\eta-u}$ is bold faced or not: i) if $c_{k_\eta+1} - 1 > d_{v+k_\eta-u}$, then we transform $\prod_{p\dashrightarrow p+1}$ to $\prod_{q\dashrightarrow q'}$, where $q = c_{k_\eta+1}$ and $q' = d_{v+k_\eta+1-u} > q$ so that the Claim applies. It remains to show that this transform does not change the total weight of the path. Observe that $\omega(T^-)$ contains the segment $d_{v+k_\eta+1-u} \to d_{v+k_\eta-u} \to \cdots \to d_v$. By Lemma 6.2 $\omega(T^-)$ is a full cycle containing the segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_u$. Thus the path $\prod_{p\dashrightarrow p+1}$ looks like $p = c_u \to c_{u-1} \to \cdots \to d_{v+k_\eta+1-u} \to d_{v+k_\eta-u} \to \cdots \to d_v = p+1$. By adding $k_\eta + 1 - u$ down steps at the beginning and removing $k_\eta + 1 - u$ down steps at the end, we do not change the total weight of the path and obtain $q = c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to d_{v+k_\eta+1-u} = q'$, the path $\prod_{q\dashrightarrow q'}$ in $\omega(T^-)$, as desired; ii) if $c_{k_\eta+1} - 1 = d_{v+k_\eta-u}$, i.e, $d_{v+k_\eta-u}$ is the smallest bold faced number, then observe that $\omega(T^-)$ contains the segment $d_{v+k_\eta+1-u} \to c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to \underbrace{c_u \to \cdots \to c_1} \to \underbrace{d_{v+k_\eta-u} \to \cdots \to d_v}$. We have

$$\texttt{rank}(p+1) - \texttt{rank}(p) = -(u-1) + k_\eta - \frac{1}{n} - (k_\eta - u) = 1 - \frac{1}{n} > 0.$$

2. if $p+1 = c_u$ and p is not in column η. The situation is similar to (1). We will include the details here for convenience. Assume p is in column j with entries $d_1, d_2, \cdots, d_{k_j+1}$. Since $d_{k_j+1} > c_{k_\eta+1} = \min(b(T^-))$, we may assume $p+1 = d_v$ for some $v < k_j+1$. Now after $d_v = p$ and $c_u = p+1$ are inserted into T^-, d_v and c_u are both active, the next entries inserted into the two columns must be subsequently d_{v+1} and then c_{u+1}, d_{v+2}, and so on. It follows that $d_v < c_u < d_{v+1} < c_{u+1} < \cdots < d_{v+k_\eta+1-u} < c_{k_\eta+1}$. Now we transform $\prod_{p\dashrightarrow p+1}$ to $\prod_{q\dashrightarrow q'}$, where $q = d_{v+k_\eta+1-u}$ and $q' = c_{k_\eta+1} > q$ so that the Claim applies. It remains to show that this transform does not change the total weight of the path. Observe that $\omega(T^-)$ contains the segment $d_{v+k_\eta+1-u} \to d_{v+k_\eta-u} \to \cdots \to d_v$. This is due to the fact that $c_{k_\eta+1} - 1$ is the

smallest bold faced number and that $c_{k_\eta+1} > d_{v+k_\eta+1-u} > d_{v+k_\eta-u}$. By Lemma 6.2 $\omega(T^-)$ is a full cycle containing the segment $c_{k_\eta+1} \to c_{k_\eta} \to \cdots \to c_u$. Thus the path $\prod_{p\dashrightarrow p+1}$ looks like $p = d_v \to \cdots \to c_{k_\eta+1} \to \cdots \to c_u = p+1$. By adding $k_\eta + 1 - u$ down steps at the beginning and removing $k_\eta + 1 - u$ down steps at the end, we do not change the total weight of the path and obtain $q = d_{v+k_\eta+1-u} \to d_v \to \cdots \to c_{k_\eta+1} = q'$, the path $\prod_{q\dashrightarrow q'}$ in $\omega(T^-)$, as desired. ∎

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