Collapsibility of simplicial complexes of hypergraphs

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Abstract

Let $\mathcal{H}$ be an r-uniform hypergraph. We show that the simplicial complex whose simplices are the hypergraphs $\mathcal{F} \subset \mathcal{H}$ with covering number at most p is $\left(\binom{r+p}{r} - 1\right)$-collapsible. Similarly, the simplicial complex whose simplices are the pairwise intersecting hypergraphs $\mathcal{F} \subset \mathcal{H}$ is $\frac{1}{2}\binom{2r}{r}$-collapsible.

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1 Introduction

Let X be a finite simplicial complex. Let η be a simplex of X such that $|\eta| \leqslant d$ and η is contained in a unique maximal face $\tau \in X$. We say that the complex

$$X' = X \setminus \{\sigma \in X : \eta \subset \sigma \subset \tau\}$$

is obtained from X by an *elementary d-collapse*, and we write $X \xrightarrow{\eta} X'$.

The complex X is called *d-collapsible* if there exists a sequence of elementary d-collapses

$$X = X_1 \xrightarrow{\eta_1} X_2 \xrightarrow{\eta_2} \cdots \xrightarrow{\eta_{k-1}} X_k = \emptyset$$

from X to the void complex $\emptyset$. The *collapsibility* of X is the minimal d such that X is d-collapsible.

A simple consequence of d-collapsibility is the following:

Proposition 1 (Wegner [11, Lemma 1]). *If X is d-collapsible then it is homotopy equivalent to a simplicial complex of dimension smaller than d.*

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Let $\mathcal{H}$ be a finite hypergraph. We identify $\mathcal{H}$ with its edge set. The *rank* of $\mathcal{H}$ is the maximal size of an edge of $\mathcal{H}$.

A set C is a *cover* of $\mathcal{H}$ if $A \cap C \neq \emptyset$ for all $A \in \mathcal{H}$. The *covering number* of $\mathcal{H}$, denoted by $\tau(\mathcal{H})$, is the minimal size of a cover of $\mathcal{H}$.

For $p \in \mathbb{N}$, let

$$\mathrm{Cov}_{\mathcal{H},p} = \{\mathcal{F} \subset \mathcal{H} : \tau(\mathcal{F}) \leqslant p\}.$$

That is, $\mathrm{Cov}_{\mathcal{H},p}$ is a simplicial complex whose vertices are the edges of $\mathcal{H}$ and whose simplices are the hypergraphs $\mathcal{F} \subset \mathcal{H}$ that can be covered by a set of size at most p. Some topological properties of the complex $\mathrm{Cov}_{\binom{[n]}{r},p}$ were studied by Jonsson in [6].

The hypergraph $\mathcal{H}$ is called *pairwise intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{H}$. Let

$$\mathrm{Int}_{\mathcal{H}} = \{\mathcal{F} \subset \mathcal{H} : A \cap B \neq \emptyset \text{ for all } A, B \in \mathcal{F}\}.$$

So, $\mathrm{Int}_{\mathcal{H}}$ is a simplicial complex whose vertices are the edges of $\mathcal{H}$ and whose simplices are the hypergraphs $\mathcal{F} \subset \mathcal{H}$ that are pairwise intersecting.

Our main results are the following:

Theorem 2. *Let $\mathcal{H}$ be a hypergraph of rank r. Then $\mathrm{Cov}_{\mathcal{H},p}$ is $\left(\binom{r+p}{r} - 1\right)$-collapsible.*

Theorem 3. *Let $\mathcal{H}$ be a hypergraph of rank r. Then $\mathrm{Int}_{\mathcal{H}}$ is $\frac{1}{2}\binom{2r}{r}$-collapsible.*

The following examples show that these bounds are sharp:

- Let $\mathcal{H} = \binom{[r+p]}{r}$ be the complete r-uniform hypergraph on $r+p$ vertices. The covering number of $\mathcal{H}$ is $p+1$, but for any $A \in \mathcal{H}$ the hypergraph $\mathcal{H} \setminus \{A\}$ can be covered by a set of size p, namely by $[r+p] \setminus A$. Therefore the complex $\mathrm{Cov}_{\binom{[r+p]}{r},p}$ is the boundary of the $\left(\binom{r+p}{r} - 1\right)$-dimensional simplex, so it is homeomorphic to a $\left(\binom{r+p}{r} - 2\right)$-dimensional sphere. Hence, by Proposition 1, $\mathrm{Cov}_{\binom{[r+p]}{r},p}$ is not $\left(\binom{r+p}{r} - 2\right)$-collapsible.

- Let $\mathcal{H} = \binom{[2r]}{r}$ be the complete r-uniform hypergraph on $2r$ vertices. Any $A \in \mathcal{H}$ intersects all the edges of $\mathcal{H}$ except the edge $[2r] \setminus A$. Therefore the complex $\mathrm{Int}_{\binom{[2r]}{r}}$ is the boundary of the $\frac{1}{2}\binom{2r}{r}$-dimensional cross-polytope, so it is homeomorphic to a $\left(\frac{1}{2}\binom{2r}{r} - 1\right)$-dimensional sphere. Hence, by Proposition 1, $\mathrm{Int}_{\binom{[2r]}{r}}$ is not $\left(\frac{1}{2}\binom{2r}{r} - 1\right)$-collapsible.

A related problem was studied by Aharoni, Holzman and Jiang in [2], where they show that for any r-uniform hypergraph $\mathcal{H}$ and $p \in \mathbb{Q}$, the complex of hypergraphs $\mathcal{F} \subset \mathcal{H}$ with fractional matching number (or equivalently, fractional covering number) smaller than p is $(\lceil rp \rceil - 1)$-collapsible.

Our proofs rely on two main ingredients. The first one is the following theorem:

Theorem 4. *Let X be a simplicial complex on vertex set V. Let $S(X)$ be the collection of all sets $\{v_1, \ldots, v_k\} \subset V$ satisfying the following condition:*

There exist maximal faces $\sigma_1, \sigma_2, \dots, \sigma_{k+1}$ of X such that:

- $v_i \notin \sigma_i$ *for all* $i \in [k]$,
- $v_i \in \sigma_j$ *for all* $1 \leqslant i < j \leqslant k+1$.

Let $d'(X)$ be the maximum size of a set in $S(X)$. Then X is $d'(X)$-collapsible.

Theorem 4 is a special case of a more general result, due essentially to Matoušek and Tancer (who stated it in the special case where the complex is the nerve of a family of finite sets, and used it to prove the case $p = 1$ of Theorem 2; see [9]).

The second ingredient is the following combinatorial lemma, proved independently by Frankl and Kalai.

Lemma 5 (Frankl [4], Kalai [7]). *Let $\{A_1, \dots, A_k\}$ and $\{B_1, \dots, B_k\}$ be families of sets such that:*

- $|A_i| \leqslant r$, $|B_i| \leqslant p$ *for all* $i \in [k]$,
- $A_i \cap B_i = \emptyset$ *for all* $i \in [k]$,
- $A_i \cap B_j \neq \emptyset$ *for all* $1 \leqslant i < j \leqslant k$.

Then

$$k \leqslant \binom{r+p}{r}.$$

The paper is organized as follows. In Section 2 we present Matoušek and Tancer's bound on the collapsibility of a simplicial complex, and we prove Theorem 4. In Section 3 we present some results on the collapsibility of independence complexes of graphs. In Section 4 we prove our main results on the collapsibility of complexes of hypergraphs. Section 5 contains some generalizations of Theorems 2 and 3, which are obtained by applying different known variants of Lemma 5.

2 A bound on the collapsibility of a complex

Let X be a (non-void) simplicial complex on vertex set V. Fix a linear order $<$ on V. Let $\mathcal{A} = (\sigma_1, \dots, \sigma_m)$ be a sequence of faces of X such that, for any $\sigma \in X$, $\sigma \subset \sigma_i$ for some $i \in [m]$. For example, we may take $\sigma_1, \dots, \sigma_m$ to be the set of maximal faces of X (ordered in any way).

For a simplex $\sigma \in X$, let $m_{X,\mathcal{A},<}(\sigma) = \min\{i \in [m] : \sigma \subset \sigma_i\}$. Let $i \in [m]$ and $\sigma \in X$ such that $m_{X,\mathcal{A},<}(\sigma) = i$. We define the *minimal exclusion sequence*

$$\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = (v_1, \dots, v_{i-1})$$

as follows: If $i = 1$ then $\mathrm{mes}_{X,\mathcal{A},<}(\sigma)$ is the empty sequence. If $i > 1$ we define the sequence recursively as follows:

Since $i > 1$, we must have $\sigma \not\subset \sigma_1$; hence, there is some $v \in \sigma$ such that $v \notin \sigma_1$. Let v_1 be the minimal such vertex (with respect to the order $<$).

Let $1 < j < i$ and assume that we already defined $v_1, \ldots, v_{j-1}$. Since $i > j$, we must have $\sigma \not\subset \sigma_j$; hence, there exists some $v \in \sigma$ such that $v \notin \sigma_j$.

- If there is a vertex $v_k \in \{v_1, \ldots, v_{j-1}\}$ such that $v_k \notin \sigma_j$, let v_j be such a vertex of minimal index k. In this case we call v_j *old at* j.
- If $v_k \in \sigma_j$ for all $k < j$, let v_j be the minimal vertex $v \in \sigma$ (with respect to the order $<$) such that $v \notin \sigma_j$. In this case we call v_j *new at* j.

Let $M_{X,\mathcal{A},<}(\sigma) \subset \sigma$ be the simplex consisting of all the vertices appearing in the sequence $\mathrm{mes}_{X,\mathcal{A},<}(\sigma)$. Let

$$d(X, \mathcal{A}, <) = \max\{|M_{X,\mathcal{A},<}(\sigma)| : \sigma \in X\}.$$

The following result was stated and proved in [9, Prop. 1.3] in the special case where X is the nerve of a finite family of sets (in our notation, $X = \mathrm{Cov}_{\mathcal{H},1}$ for some hypergraph $\mathcal{H}$).

Theorem 6. *The simplicial complex X is $d(X, \mathcal{A}, <)$-collapsible.*

The proof given in [9] can be easily modified to hold in this more general setting. Here we present a different proof.

Let X be a simplicial complex on vertex set V, and let $v \in V$. Let

$$X \setminus v = \{\sigma \in X : v \notin \sigma\}$$

and

$$\mathrm{lk}(X, v) = \{\sigma \in X : v \notin \sigma, \sigma \cup \{v\} \in X\}.$$

We will need the following lemma, proved by Tancer in [10]:

Lemma 7 (Tancer [10, Prop. 1.2])**.** *If $X \setminus v$ is d-collapsible and $\mathrm{lk}(X, v)$ is $(d-1)$-collapsible, then X is d-collapsible.*

Proof of Theorem 6. First, we deal with the case where X is a complete complex (i.e. a simplex). Then X is 0-collapsible; therefore, the claim holds.

For a general complex X, we argue by induction on the number of vertices of X. If $|V| = 0$, then $X = \{\emptyset\}$. In particular, it is a complete complex; hence, the claim holds.

Let $|V| > 0$, and assume that the claim holds for any complex with less than $|V|$ vertices. If $\sigma_1 = V$, then X is the complete complex on vertex set V, and the claim holds. Otherwise, let v be the minimal vertex (with respect to $<$) in $V \setminus \sigma_1$.

In order to apply Lemma 7, we will need the following two claims:

Claim 8. *The complex $X \setminus v$ is $d(X, \mathcal{A}, <)$-collapsible.*

Proof. For every $i \in [m]$, let $\sigma'_i = \sigma_i \setminus \{v\}$, and let $\mathcal{A}' = (\sigma'_1, \dots, \sigma'_m)$. Let $\sigma \in X \setminus v$. Since $v \notin \sigma$, then, for any $i \in [m]$, $\sigma \subset \sigma_i$ if and only if $\sigma \subset \sigma'_i$. Hence, every simplex $\sigma \in X \setminus v$ is contained in σ'_i for some $i \in [m]$ (since, by the definition of $\mathcal{A}$, $\sigma \subset \sigma_i$ for some $i \in [m]$). So, by the induction hypothesis, $X \setminus v$ is $d(X \setminus v, \mathcal{A}', <)$-collapsible.

Let $\sigma \in X \setminus v$. We will show that $\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = \mathrm{mes}_{X\setminus v,\mathcal{A}',<}(\sigma)$. Since for any $i \in [m]$, $\sigma \subset \sigma_i$ if and only if $\sigma \subset \sigma'_i$, then the two sequences are of the same length. Let

$$\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = (v_1, \dots, v_k)$$

and

$$\mathrm{mes}_{X\setminus v,\mathcal{A}',<}(\sigma) = (v'_1, \dots, v'_k).$$

We will show that $v_i = v'_i$ for all $i \in [k]$. We argue by induction on i. Let $i \in [k]$, and assume that $v_j = v'_j$ for all $j < i$. Since $v \notin \sigma$, then $\sigma \setminus \sigma_i = \sigma \setminus \sigma'_i$. Therefore, for any $j < i$, $v_j \in \sigma \setminus \sigma_i$ if and only if $v'_j = v_j \in \sigma \setminus \sigma'_i$. Hence, v_i is old at i if and only if v'_i is old at i, and if v_i and v'_i are both old at i, then $v_i = v'_i$. Otherwise, both v_i and v'_i are new at i. Then, v_i is the minimal vertex in $\sigma \setminus \sigma_i$, and v'_i is the minimal vertex in $\sigma \setminus \sigma'_i = \sigma \setminus \sigma_i$. Thus, $v_i = v'_i$.

Therefore, $|M_{X\setminus v,\mathcal{A}',<}(\sigma)| = |M_{X,\mathcal{A},<}(\sigma)|$ for any $\sigma \in X \setminus v$; hence,

$$d(X \setminus v, \mathcal{A}', <) \leqslant d(X, \mathcal{A}, <).$$

So, $X \setminus v$ is $d(X, \mathcal{A}, <)$-collapsible. □

Claim 9. *The complex* $\mathrm{lk}(X, v)$ *is* $(d(X, \mathcal{A}, <) - 1)$*-collapsible.*

Proof. Let $I = \{i \in [m] : v \in \sigma_i\}$. For every $i \in I$, let $\sigma''_i = \sigma_i \setminus \{v\}$. Write $I = \{i_1, \dots, i_r\}$, where $i_1 < \cdots < i_r$, and let $\mathcal{A}'' = (\sigma''_{i_1}, \dots, \sigma''_{i_r})$.

For any $\sigma \in \mathrm{lk}(X, v)$, the simplex $\sigma \cup \{v\}$ belongs to X; hence, there exists some $i \in [m]$ such that $\sigma \cup \{v\} \subset \sigma_i$. Since $v \in \sigma \cup \{v\}$, we must have $i \in I$, and therefore $\sigma \subset \sigma''_i = \sigma_i \setminus \{v\}$. So, by the induction hypothesis, $\mathrm{lk}(X, v)$ is $d(\mathrm{lk}(X, v), \mathcal{A}'', <)$-collapsible.

Let $\sigma \in \mathrm{lk}(X, v)$. We will show that

$$M_{X,\mathcal{A},<}(\sigma \cup \{v\}) = M_{\mathrm{lk}(X,v),\mathcal{A}'',<}(\sigma) \cup \{v\}.$$

Let

$$\mathrm{mes}_{X,\mathcal{A},<}(\sigma \cup \{v\}) = (v_1, \dots, v_n),$$

and

$$\mathrm{mes}_{\mathrm{lk}(X,v),\mathcal{A}'',<}(\sigma) = (u_1, \dots, u_t).$$

For any $j \in [r]$, $\sigma \subset \sigma''_{i_j}$ if and only if $\sigma \cup \{v\} \subset \sigma_{i_j}$. Also, for $i \notin I$, $\sigma \cup \{v\} \not\subset \sigma_i$ (since $v \notin \sigma_i$). Therefore, $n = i_{t+1} - 1$.

The vertex v is the minimal vertex in $V \setminus \sigma_1$, therefore it is the minimal vertex in $(\sigma \cup \{v\}) \setminus \sigma_1$. Hence, we have $v_1 = v$. Now, let $i > 1$ such that $i \notin I$. Then, $v_1 = v$ is the vertex of minimal index in the sequence $(v_1, \dots, v_{i-1})$ that is contained in $(\sigma \cup \{v\}) \setminus \sigma_i$. Therefore, $v_i = v$.

Finally, we will show that $v_{i_j} = u_j$ for all $j \in [t]$. We argue by induction on j. Let $j \in [t]$, and assume that $v_{i_\ell} = u_\ell$ for all $\ell < j$.

For any $k < i_j$, either $v_k = v$ (if $k \notin I$) or $v_k = u_\ell$ for some $\ell < j$ (if $k = i_\ell \in I$). Also, since $v \in \sigma_{i_j}$, we have $(\sigma \cup \{v\}) \setminus \sigma_{i_j} = \sigma \setminus \sigma''_{i_j}$. So, for any $k < i_j$, $v_k \in (\sigma \cup \{v\}) \setminus \sigma_{i_j}$ if and only if $k = i_\ell$ for some $\ell < j$ such that $u_\ell \in \sigma \setminus \sigma''_{i_j}$. Therefore, v_{i_j} is old at i_j if and only if u_j is old at j, and if v_{i_j} and u_j are both old, then $v_{i_j} = u_j$. Otherwise, assume that v_{i_j} is new at i_j and u_j is new at j. Then, v_{i_j} is the minimal vertex in $(\sigma \cup \{v\}) \setminus \sigma_{i_j}$, and u_j is the minimal vertex in $\sigma \setminus \sigma''_{i_j} = (\sigma \cup \{v\}) \setminus \sigma_{i_j}$. Thus, $v_{i_j} = u_j$.

So, for any $\sigma \in \mathrm{lk}(X, v)$ we obtain

$$|M_{\mathrm{lk}(X,v),\mathcal{A}'',<}(\sigma)| = |M_{X,\mathcal{A},<}(\sigma \cup \{v\})| - 1.$$

Hence,

$$d(\mathrm{lk}(X, v), \mathcal{A}'', <) \leqslant d(X, \mathcal{A}, <) - 1.$$

So, $\mathrm{lk}(X, v)$ is $(d(X, \mathcal{A}, <) - 1)$-collapsible. □

By Claim 8, Claim 9 and Lemma 7, X is $d(X, \mathcal{A}, <)$-collapsible. □

Proof of Theorem 4. Let $<$ be some linear order on the vertex set V, and let $\mathcal{A} = (\sigma_1, \ldots, \sigma_m)$ be the sequence of maximal faces of X (ordered in any way).

Let $i \in [m]$ and let $\sigma \in X$ with $m_{X,\mathcal{A},<}(\sigma) = i$. Let $\mathrm{mes}_{X,\mathcal{A},<}(\sigma) = (v_1, \ldots, v_{i-1})$. Then $M_{X,\mathcal{A},<}(\sigma) = \{v_{i_1}, \ldots, v_{i_k}\}$ for some $i_1 < \cdots < i_k \in [i-1]$ (these are exactly the indices i_j such that v_{i_j} is new at i_j). For each $j \in [k]$ we have $v_{i_j} \notin \sigma_{i_j}$. In addition, since v_{i_j} is new at i_j, we have $v_{i_\ell} \in \sigma_{i_j}$ for all $\ell < j$. Let $i_{k+1} = i$. Since $m_{X,\mathcal{A},<}(\sigma) = i = i_{k+1}$, we have $\sigma \subset \sigma_{i_{k+1}}$. In particular, $v_{i_\ell} \in \sigma_{i_{k+1}}$ for all $\ell < k + 1$.

Therefore, $M_{X,\mathcal{A},<}(\sigma) \in S(X)$. Thus, $d(X, \mathcal{A}, <) \leqslant d'(X)$, and by Theorem 6, X is $d'(X)$-collapsible. □

3 Collapsibility of independence complexes

Let $G = (V, E)$ be a graph. The independence complex $I(G)$ is the simplicial complex on vertex set V whose simplices are the independent sets in G.

Definition 10. Let $k(G)$ be the maximal size of a set $\{v_1, \ldots, v_k\} \subset V$ that satisfies:

- $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$,
- There exist $u_1, \ldots, u_k \in V$ such that
  - $\{v_i, u_i\} \in E$ for all $i \in [k]$,
  - $\{v_i, u_j\} \notin E$ for all $1 \leqslant i < j \leqslant k$.

Proposition 11. $k(G) = d'(I(G))$.

Proof. Let $A = \{v_1, \ldots, v_k\} \in S(I(G))$. Then, there exist maximal faces $\sigma_1, \ldots, \sigma_{k+1}$ of $I(G)$ such that:

- $v_i \notin \sigma_i$ for all $i \in [k]$,
- $v_i \in \sigma_j$ for all $1 \leqslant i < j \leqslant k+1$.

Let $i \in [k]$. Since σ_i is a maximal independent set in G and $v_i \notin \sigma_i$, there exists some $u_i \in \sigma_i$ such that $\{v_i, u_i\} \in E$.

Let $1 \leqslant i < j \leqslant k$. Since v_i and u_j are both contained in the independent set σ_j, we have $\{v_i, u_j\} \notin E$. Furthermore, since $A \subset \sigma_{k+1}$, A is an independent set in G. That is, $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$. So, A satisfies the conditions of Definition 10. Hence, $|A| \leqslant k(G)$; therefore, $d'(I(G)) \leqslant k(G)$.

Now, let $k = k(G)$, and let $v_1, \ldots, v_k, u_1, \ldots, u_k \in V$ such that

- $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$,
- $\{v_i, u_i\} \in E$ for all $i \in [k]$,
- $\{v_i, u_j\} \notin E$ for all $1 \leqslant i < j \leqslant k$.

Let $i \in [k]$, and let $V_i = \{v_j : 1 \leqslant j < i\}$. Note that $V_i \cup \{u_i\}$ forms an independent set in G; therefore, it is a simplex in $I(G)$. Let σ_i be a maximal face of $I(G)$ containing $V_i \cup \{u_i\}$. Since $\{v_i, u_i\} \in E$, we have $v_i \notin \sigma_i$.

The set $\{v_1, \ldots, v_k\}$ is also an independent set in G. Therefore, there is a maximal face $\sigma_{k+1} \in I(G)$ that contains it.

By the definition of $\sigma_1, \ldots, \sigma_{k+1}$, we have $v_i \in \sigma_j$ for $1 \leqslant i < j \leqslant k+1$. Therefore, $\{v_1, \ldots, v_k\} \in S(I(G))$; so, $k(G) = k \leqslant d'(I(G))$. Hence, $k(G) = d'(I(G))$, as wanted. ☐

As an immediate consequence of Proposition 11 and Theorem 4, we obtain:

Proposition 12. *The complex $I(G)$ is $k(G)$-collapsible.*

Note that vertices $v_1, \ldots, v_k, u_1, \ldots, u_k \in V$ satisfying the conditions in Definition 10 must all be distinct. As a simple corollary, we obtain

Corollary 13. *The independence complex of a graph $G = (V, E)$ on n vertices is $\lfloor \frac{n}{2} \rfloor$-collapsible.*

4 Complexes of hypergraphs

In this section we prove our main results, Theorems 2 and 3.

Proof of Theorem 2. Let $\mathcal{H}$ be a hypergraph of rank r on vertex set $[n]$, and let

$$\{A_1, \ldots, A_k\} \in S(\mathrm{Cov}_{\mathcal{H},p}).$$

Then, there exist maximal faces $\mathcal{F}_1, \ldots, \mathcal{F}_{k+1} \in \mathrm{Cov}_{\mathcal{H},p}$ such that

- $A_i \notin \mathcal{F}_i$ for all $i \in [k]$,

- $A_i \in \mathcal{F}_j$ for all $1 \leqslant i < j \leqslant k+1$.

For any $i \in [k+1]$, there is some $C_i \subset [n]$ of size at most p that covers $\mathcal{F}_i$. Since $\mathcal{F}_i$ is maximal, then, for any $A \in \mathcal{H}$, $A \in \mathcal{F}_i$ if and only if $A \cap C_i \neq \emptyset$. Therefore, we obtain

- $A_i \cap C_i = \emptyset$ for all $i \in [k]$,
- $A_i \cap C_j \neq \emptyset$ for all $1 \leqslant i < j \leqslant k+1$.

Hence, the pair of families

$$\{A_1, \dots A_k, \emptyset\}$$

and

$$\{C_1, \dots, C_k, C_{k+1}\}$$

satisfies the conditions of Lemma 5; thus, $k+1 \leqslant \binom{r+p}{r}$. Therefore,

$$d'(\mathrm{Cov}_{\mathcal{H},p}) \leqslant \binom{r+p}{r} - 1,$$

and by Theorem 4, $\mathrm{Cov}_{\mathcal{H},p}$ is $\left(\binom{r+p}{r} - 1\right)$-collapsible. □

Proof of Theorem 3. Let $\mathcal{H}$ be a hypergraph of rank r and let G be the graph on vertex set $\mathcal{H}$ whose edges are the pairs $\{A, B\} \subset \mathcal{H}$ such that $A \cap B = \emptyset$. Then $\mathrm{Int}_{\mathcal{H}} = I(G)$.

Let $k = k(G)$ and let $\{A_1, \dots, A_k\} \subset \mathcal{H}$ that satisfies the conditions of Definition 10. That is,

- $A_i \cap A_j \neq \emptyset$ for all $i \neq j \in [k]$,
- There exist $B_1, \dots, B_k \in \mathcal{H}$ such that
  - $A_i \cap B_i = \emptyset$ for all $i \in [k]$,
  - $A_i \cap B_j \neq \emptyset$ for all $1 \leqslant i < j \leqslant k$.

Then, the pair of families $\{A_1, \dots, A_k, B_k, \dots, B_1\}$ and $\{B_1, \dots, B_k, A_k, \dots, A_1\}$ satisfies the conditions of Lemma 5; therefore, $2k \leqslant \binom{2r}{r}$. Thus, by Proposition 12, $\mathrm{Int}_{\mathcal{H}} = I(G)$ is $\frac{1}{2}\binom{2r}{r}$-collapsible. □

5 More complexes of hypergraphs

Let $\mathcal{H}$ be a hypergraph. A set C is a *t-transversal* of $\mathcal{H}$ if $|A \cap C| \geqslant t$ for all $A \in \mathcal{H}$. Let $\tau_t(\mathcal{H})$ be the minimal size of a t-transversal of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is *pairwise t-intersecting* if $|A \cap B| \geqslant t$ for all $A, B \in \mathcal{H}$. Let

$$\mathrm{Cov}^t_{\mathcal{H},p} = \{\mathcal{F} \subset \mathcal{H} : \tau_t(\mathcal{F}) \leqslant p\}$$

and

$$\mathrm{Int}^t_{\mathcal{H}} = \{\mathcal{F} \subset \mathcal{H} : \mathcal{F} \text{ is pairwise } t\text{-intersecting}\}.$$

The following generalization of Lemma 5 was proved by Füredi in [5].

Lemma 14 (Füredi [5]). *Let* $\{A_1, \dots, A_k\}$ *and* $\{B_1, \dots, B_k\}$ *be families of sets such that:*

- $|A_i| \leqslant r$, $|B_i| \leqslant p$ *for all* $i \in [k]$,
- $|A_i \cap B_i| \leqslant t$ *for all* $i \in [k]$,
- $|A_i \cap B_j| > t$ *for all* $1 \leqslant i < j \leqslant k$.

Then

$$k \leqslant \binom{r+p-2t}{r-t}.$$

We obtain the following:

Theorem 15. *Let* $\mathcal{H}$ *be a hypergraph of rank* r *and let* $t \leqslant \min\{r,p\} - 1$. *Then* $\mathrm{Cov}_{\mathcal{H},p}^{t+1}$ *is* $\left(\binom{r+p-2t}{r-t} - 1\right)$*-collapsible.*

Theorem 16. *Let* $\mathcal{H}$ *be a hypergraph of rank* r *and let* $t \leqslant r-1$. *Then* $\mathrm{Int}_{\mathcal{H}}^{t+1}$ *is* $\frac{1}{2}\binom{2(r-t)}{r-t}$*-collapsible.*

Note that by setting $t = 0$ we recover Theorems 2 and 3. The proofs are essentially the same as the proofs of Theorems 2 and 3, except for the use of Lemma 14 instead of Lemma 5. The extremal examples are also similar: Let

$$\mathcal{H}_1 = \left\{ A \cup [t] : A \in \binom{[r+p-t] \setminus [t]}{r-t} \right\}$$

and

$$\mathcal{H}_2 = \left\{ A \cup [t] : A \in \binom{[2r-t] \setminus [t]}{r-t} \right\}.$$

The complex $\mathrm{Cov}_{\mathcal{H}_1}^{t+1}$ is the boundary of the $\left(\binom{r+p-2t}{r-t} - 1\right)$-dimensional simplex, hence it is not $\left(\binom{r+p-2t}{r-t} - 2\right)$-collapsible, and the complex $\mathrm{Int}_{\mathcal{H}_2}^{t+1}$ is the boundary of the $\frac{1}{2}\binom{2(r-t)}{r-t}$-dimensional cross-polytope, hence it is not $\left(\frac{1}{2}\binom{2(r-t)}{r-t} - 1\right)$-collapsible.

Restricting ourselves to special classes of hypergraphs we may obtain better bounds on the collapsibility of their associated complexes. For example, we may look at r-partite r-uniform hypergraphs (that is, hypergraphs $\mathcal{H}$ on vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_r$ such that $|A \cap V_i| = 1$ for all $A \in \mathcal{H}$ and $i \in [r]$). In this case we have the following result:

Theorem 17. *Let* $\mathcal{H}$ *be an* r*-partite* r*-uniform hypergraph. Then* $\mathrm{Int}_{\mathcal{H}}$ *is* 2^{r-1}*-collapsible.*

The next example shows that the bound on the collapsibility of $\mathrm{Int}_{\mathcal{H}}$ in Theorem 17 is tight: Let $\mathcal{H}$ be the complete r-partite r-uniform hypergraph with all sides of size 2. It has 2^r edges, and any edge $A \in \mathcal{H}$ intersects all the edges of $\mathcal{H}$ except its complement. Therefore the complex $\mathrm{Int}_{\mathcal{H}}$ is the boundary of the 2^{r-1}-dimensional cross-polytope, so it is homeomorphic to a $(2^{r-1}-1)$-dimensional sphere. Hence, by Proposition 1, $\mathrm{Int}_{\mathcal{H}}$ is not $(2^{r-1}-1)$-collapsible.

For the proof we need the following Lemma, due to Lovász, Nešetřil and Pultr.

Lemma 18 (Lovász, Nešetřil, Pultr [8, Prop. 5.3])**.** *Let* $\{A_1, \dots, A_k\}$ *and* $\{B_1, \dots, B_k\}$ *be families of subsets of* $V = V_1 \cup V_2 \cup \dots \cup V_r$ *such that:*

- $|A_i \cap V_j| = 1$, $|B_i \cap V_j| = 1$ *for all* $i \in [k]$ *and* $j \in [r]$,
- $A_i \cap B_i = \emptyset$ *for all* $i \in [k]$,
- $A_i \cap B_j \neq \emptyset$ *for all* $1 \leqslant i < j \leqslant k$.

Then $k \leqslant 2^r$.

A common generalization of Lemma 5 and Lemma 18 was proved by Alon in [3].

The proof of Theorem 17 is the same as the proof of Theorem 3, except that we replace Lemma 5 by Lemma 18. A similar argument was also used by Aharoni and Berger ([1, Theorem 5.1]) in order to prove a related result about rainbow matchings in r-partite r-uniform hypergraphs.

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