Collapsibility of simplicial complexes of hypergraphs

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Abstract
Let $\mathcal{H}$ be an $r$-uniform hypergraph. We show that the simplicial complex whose simplices are the hypergraphs $\mathcal{F} \subset \mathcal{H}$ with covering number at most $p$ is $((r+p) - 1)$-collapsible. Similarly, the simplicial complex whose simplices are the pairwise intersecting hypergraphs $\mathcal{F} \subset \mathcal{H}$ is $\frac{1}{2}(2r)$-collapsible.

Mathematics Subject Classifications: 05E45, 05D05

1 Introduction
Let $X$ be a finite simplicial complex. Let $\eta$ be a simplex of $X$ such that $|\eta| \leq d$ and $\eta$ is contained in a unique maximal face $\tau \in X$. We say that the complex

$$X' = X \setminus \{\sigma \in X : \eta \subset \sigma \subset \tau\}$$

is obtained from $X$ by an elementary $d$-collapse, and we write $X \xrightarrow{\eta} X'$.

The complex $X$ is called $d$-collapsible if there exists a sequence of elementary $d$-collapses

$$X = X_1 \xrightarrow{m_1} X_2 \xrightarrow{m_2} \cdots \xrightarrow{m_{k-1}} X_k = \emptyset$$

from $X$ to the void complex $\emptyset$. The collapsibility of $X$ is the minimal $d$ such that $X$ is $d$-collapsible.

A simple consequence of $d$-collapsibility is the following:

**Proposition 1** (Wegner [11, Lemma 1]). If $X$ is $d$-collapsible then it is homotopy equivalent to a simplicial complex of dimension smaller than $d$.

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Let \( \mathcal{H} \) be a finite hypergraph. We identify \( \mathcal{H} \) with its edge set. The *rank* of \( \mathcal{H} \) is the maximal size of an edge of \( \mathcal{H} \).

A set \( C \) is a *cover* of \( \mathcal{H} \) if \( A \cap C \neq \emptyset \) for all \( A \in \mathcal{H} \). The *covering number* of \( \mathcal{H} \), denoted by \( \tau(\mathcal{H}) \), is the minimal size of a cover of \( \mathcal{H} \).

For \( p \in \mathbb{N} \), let

\[
\text{Cov}_{\mathcal{H},p} = \{ F \subset \mathcal{H} : \tau(F) \leq p \}.
\]

That is, \( \text{Cov}_{\mathcal{H},p} \) is a simplicial complex whose vertices are the edges of \( \mathcal{H} \) and whose simplices are the hypergraphs \( F \subset \mathcal{H} \) that can be covered by a set of size at most \( p \).

Some topological properties of the complex \( \text{Cov}(n,r)_p \) were studied by Jonsson in [6].

The hypergraph \( \mathcal{H} \) is called *pairwise intersecting* if \( A \cap B \neq \emptyset \) for all \( A, B \in \mathcal{H} \). Let

\[
\text{Int}_\mathcal{H} = \{ F \subset \mathcal{H} : A \cap B \neq \emptyset \text{ for all } A, B \in F \}.
\]

So, \( \text{Int}_\mathcal{H} \) is a simplicial complex whose vertices are the edges of \( \mathcal{H} \) and whose simplices are the hypergraphs \( F \subset \mathcal{H} \) that are pairwise intersecting.

Our main results are the following:

**Theorem 2.** Let \( \mathcal{H} \) be a hypergraph of rank \( r \). Then \( \text{Cov}_{\mathcal{H},p} \) is \( \left( \binom{r+p}{r} - 1 \right) \)-collapsible.

**Theorem 3.** Let \( \mathcal{H} \) be a hypergraph of rank \( r \). Then \( \text{Int}_\mathcal{H} \) is \( \frac{1}{2} \left( \binom{2r}{r} \right) \)-collapsible.

The following examples show that these bounds are sharp:

- Let \( \mathcal{H} = \binom{[r+p]}{r} \) be the complete \( r \)-uniform hypergraph on \( r+p \) vertices. The covering number of \( \mathcal{H} \) is \( p+1 \), but for any \( A \in \mathcal{H} \) the hypergraph \( \mathcal{H} \setminus \{ A \} \) can be covered by a set of size \( p \), namely by \( \binom{[r+p]}{r} \setminus A \). Therefore the complex \( \text{Cov}(\binom{[r+p]}{r})_p \) is the boundary of the \( \left( \binom{r+p}{r} - 1 \right) \)-dimensional simplex, so it is homeomorphic to a \( \left( \binom{r+p}{r} - 2 \right) \)-dimensional sphere. Hence, by Proposition 1, \( \text{Cov}(\binom{[r+p]}{r})_p \) is not \( \left( \binom{r+p}{r} - 2 \right) \)-collapsible.

- Let \( \mathcal{H} = \binom{[2r]}{r} \) be the complete \( r \)-uniform hypergraph on \( 2r \) vertices. Any \( A \in \mathcal{H} \) intersects all the edges of \( \mathcal{H} \) except the edge \( [2r] \setminus A \). Therefore the complex \( \text{Int}(\binom{[2r]}{r}) \) is the boundary of the \( \frac{1}{2} \left( \binom{2r}{r} \right) \)-dimensional cross-polytope, so it is homeomorphic to a \( \left( \frac{1}{2} \left( \binom{2r}{r} - 1 \right) \right) \)-dimensional sphere. Hence, by Proposition 1, \( \text{Int}(\binom{[2r]}{r}) \) is not \( \left( \frac{1}{2} \left( \binom{2r}{r} - 1 \right) \right) \)-collapsible.

A related problem was studied by Aharoni, Holzman and Jiang in [2], where they show that for any \( r \)-uniform hypergraph \( \mathcal{H} \) and \( p \in \mathbb{Q} \), the complex of hypergraphs \( F \subset \mathcal{H} \) with fractional matching number (or equivalently, fractional covering number) smaller than \( p \) is \( \left( \lceil rp \rceil - 1 \right) \)-collapsible.

Our proofs rely on two main ingredients. The first one is the following theorem:

**Theorem 4.** Let \( X \) be a simplicial complex on vertex set \( V \). Let \( S(X) \) be the collection of all sets \( \{v_1, \ldots, v_k\} \subset V \) satisfying the following condition:
There exist maximal faces $\sigma_1, \sigma_2, \ldots, \sigma_{k+1}$ of $X$ such that:

- $v_i \notin \sigma_i$ for all $i \in [k]$,
- $v_i \in \sigma_j$ for all $1 \leq i < j \leq k + 1$.

Let $d'(X)$ be the maximum size of a set in $S(X)$. Then $X$ is $d'(X)$-collapsible.

Theorem 4 is a special case of a more general result, due essentially to Matoušek and Tancer (who stated it in the special case where the complex is the nerve of a family of finite sets, and used it to prove the case $p = 1$ of Theorem 2; see [9]).

The second ingredient is the following combinatorial lemma, proved independently by Frankl and Kalai.

**Lemma 5** (Frankl [4], Kalai [7]). Let $\{A_1, \ldots, A_k\}$ and $\{B_1, \ldots, B_k\}$ be families of sets such that:

- $|A_i| \leq r$, $|B_i| \leq p$ for all $i \in [k]$,
- $A_i \cap B_i = \emptyset$ for all $i \in [k]$,
- $A_i \cap B_j \neq \emptyset$ for all $1 \leq i < j \leq k$.

Then

$$k \leq \binom{r + p}{r}.$$  

The paper is organized as follows. In Section 2 we present Matoušek and Tancer’s bound on the collapsibility of a simplicial complex, and we prove Theorem 4. In Section 3 we present some results on the collapsibility of independence complexes of graphs. In Section 4 we prove our main results on the collapsibility of complexes of hypergraphs. Section 5 contains some generalizations of Theorems 2 and 3, which are obtained by applying different known variants of Lemma 5.

## 2 A bound on the collapsibility of a complex

Let $X$ be a (non-void) simplicial complex on vertex set $V$. Fix a linear order $<$ on $V$. Let $A = (\sigma_1, \ldots, \sigma_m)$ be a sequence of faces of $X$ such that, for any $\sigma \in X$, $\sigma \subset \sigma_i$ for some $i \in [m]$. For example, we may take $\sigma_1, \ldots, \sigma_m$ to be the set of maximal faces of $X$ (ordered in any way).

For a simplex $\sigma \in X$, let $m_{X,A,<}(\sigma) = \min\{i \in [m] : \sigma \subset \sigma_i\}$. Let $i \in [m]$ and $\sigma \in X$ such that $m_{X,A,<}(\sigma) = i$. We define the minimal exclusion sequence

$$\text{mes}_{X,A,<}(\sigma) = (v_1, \ldots, v_{i-1})$$

as follows: If $i = 1$ then $\text{mes}_{X,A,<}(\sigma)$ is the empty sequence. If $i > 1$ we define the sequence recursively as follows:
Since \( i > 1 \), we must have \( \sigma \not\subset \sigma_1 \); hence, there is some \( v \in \sigma \) such that \( v \notin \sigma_1 \). Let \( v_1 \) be the minimal such vertex (with respect to the order \( \prec \)).

Let \( 1 < j < i \) and assume that we already defined \( v_1, \ldots, v_{j-1} \). Since \( i > j \), we must have \( \sigma \not\subset \sigma_j \); hence, there exists some \( v \in \sigma \) such that \( v \notin \sigma_j \).

- If there is a vertex \( v_k \in \{v_1, \ldots, v_{j-1}\} \) such that \( v_k \notin \sigma_j \), let \( v_j \) be such a vertex of minimal index \( k \). In this case we call \( v_j \) old at \( j \).

- If \( v_k \in \sigma_j \) for all \( k < j \), let \( v_j \) be the minimal vertex \( v \in \sigma \) (with respect to the order \( \prec \)) such that \( v \notin \sigma_j \). In this case we call \( v_j \) new at \( j \).

Let \( M_{X,A,\prec}(\sigma) \subset \sigma \) be the simplex consisting of all the vertices appearing in the sequence \( \text{mes}_{X,A,\prec}(\sigma) \). Let

\[
d(X, A, \prec) = \max\{|M_{X,A,\prec}(\sigma)| : \sigma \in X\}.
\]

The following result was stated and proved in [9, Prop. 1.3] in the special case where \( X \) is the nerve of a finite family of sets (in our notation, \( X = \text{Cov}_{H,1} \) for some hypergraph \( H \)).

**Theorem 6.** The simplicial complex \( X \) is \( d(X, A, \prec) \)-collapsible.

The proof given in [9] can be easily modified to hold in this more general setting. Here we present a different proof.

Let \( X \) be a simplicial complex on vertex set \( V \), and let \( v \in V \). Let

\[
X \setminus v = \{\sigma \in X : v \notin \sigma\}
\]

and

\[
\text{lk}(X, v) = \{\sigma \in X : v \notin \sigma, \sigma \cup \{v\} \in X\}.
\]

We will need the following lemma, proved by Tancer in [10]:

**Lemma 7** (Tancer [10, Prop. 1.2]). If \( X \setminus v \) is \( d \)-collapsible and \( \text{lk}(X, v) \) is \( (d-1) \)-collapsible, then \( X \) is \( d \)-collapsible.

**Proof of Theorem 6.** First, we deal with the case where \( X \) is a complete complex (i.e. a simplex). Then \( X \) is 0-collapsible; therefore, the claim holds.

For a general complex \( X \), we argue by induction on the number of vertices of \( X \). If \( |V| = 0 \), then \( X = \{\emptyset\} \). In particular, it is a complete complex; hence, the claim holds.

Let \( |V| > 0 \), and assume that the claim holds for any complex with less than \( |V| \) vertices. If \( \sigma_1 = V \), then \( X \) is the complete complex on vertex set \( V \), and the claim holds. Otherwise, let \( v \) be the minimal vertex (with respect to \( \prec \)) in \( V \setminus \sigma_1 \).

In order to apply Lemma 7, we will need the following two claims:

**Claim 8.** The complex \( X \setminus v \) is \( d(X, A, \prec) \)-collapsible.
Proof. For every \( i \in [m] \), let \( \sigma'_i = \sigma_i \setminus \{ v \} \), and let \( \mathcal{A}' = (\sigma'_1, \ldots, \sigma'_m) \). Let \( \sigma \in X \setminus v \). Since \( v \notin \sigma \), then, for any \( i \in [m] \), \( \sigma \subset \sigma_i \) if and only if \( \sigma \subset \sigma'_i \). Hence, every simplex \( \sigma \in X \setminus v \) is contained in \( \sigma'_i \) for some \( i \in [m] \) (since, by the definition of \( \mathcal{A} \), \( \sigma \subset \sigma_i \) for some \( i \in [m] \)). So, by the induction hypothesis, \( X \setminus v \) is \( d(X \setminus v, \mathcal{A}', <) \)-collapsible.

Let \( \sigma \in X \setminus v \). We will show that \( \text{mes}_{X, \mathcal{A}, <}(\sigma) = \text{mes}_{X \setminus v, \mathcal{A}', <}(\sigma) \). Since for any \( i \in [m] \), \( \sigma \subset \sigma_i \) if and only if \( \sigma \subset \sigma'_i \), then the two sequences are of the same length. Let

\[
\text{mes}_{X, \mathcal{A}, <}(\sigma) = (v_1, \ldots, v_k)
\]

and

\[
\text{mes}_{X \setminus v, \mathcal{A}', <}(\sigma) = (v'_1, \ldots, v'_k).
\]

We will show that \( v_i = v'_i \) for all \( i \in [k] \). We argue by induction on \( i \). Let \( i \in [k] \), and assume that \( v_j = v'_j \) for all \( j < i \). Since \( v \notin \sigma \), then \( \sigma \setminus \sigma_i = \sigma \setminus \sigma'_i \). Therefore, for any \( j < i \), \( v_j \in \sigma \setminus \sigma_i \) if and only if \( v'_j = v_j \in \sigma \setminus \sigma'_i \). Hence, \( v_i \) is old at \( i \) if and only if \( v'_i \) is old at \( i \), and if \( v_i \) and \( v'_i \) are both old at \( i \), then \( v_i = v'_i \). Otherwise, both \( v_i \) and \( v'_i \) are new at \( i \). Then, \( v_i \) is the minimal vertex in \( \sigma \setminus \sigma_i \), and \( v'_i \) is the minimal vertex in \( \sigma \setminus \sigma'_i = \sigma \setminus \sigma_i \). Thus, \( v_i = v'_i \).

Therefore, \( |M_{X \setminus v, \mathcal{A}', <}(\sigma)| = |M_{X, \mathcal{A}, <}(\sigma)| \) for any \( \sigma \in X \setminus v \); hence,

\[
d(X \setminus v, \mathcal{A}', <) \leq d(X, \mathcal{A}, <).
\]

So, \( X \setminus v \) is \( d(X, \mathcal{A}, <) \)-collapsible.

\( \square \)

Claim 9. The complex \( \text{lk}(X, v) \) is \( (d(X, \mathcal{A}, <) - 1) \)-collapsible.

Proof. Let \( I = \{ i \in [m] : v \in \sigma_i \} \). For every \( i \in I \), let \( \sigma''_i \) \( = \sigma_i \setminus \{ v \} \). Write \( I = \{ i_1, \ldots, i_r \} \), where \( i_1 < \cdots < i_r \), and let \( \mathcal{A}'' = (\sigma''_1, \ldots, \sigma''_r) \).

For any \( \sigma \in \text{lk}(X, v) \), the simplex \( \sigma \cup \{ v \} \) belongs to \( X \); hence, there exists some \( i \in [m] \) such that \( \sigma \cup \{ v \} \subset \sigma_i \). Since \( v \in \sigma \cup \{ v \} \), we must have \( i \in I \), and therefore \( \sigma \subset \sigma''_i = \sigma_i \setminus \{ v \} \). So, by the induction hypothesis, \( \text{lk}(X, v) \) is \( d(\text{lk}(X, v), \mathcal{A}'', <) \)-collapsible.

Let \( \sigma \in \text{lk}(X, v) \). We will show that

\[
M_{X, \mathcal{A}, <}(\sigma \cup \{ v \}) = M_{\text{lk}(X, v), \mathcal{A}'', <}(\sigma) \cup \{ v \}.
\]

Let

\[
\text{mes}_{X, \mathcal{A}, <}(\sigma \cup \{ v \}) = (v_1, \ldots, v_n),
\]

and

\[
\text{mes}_{\text{lk}(X, v), \mathcal{A}'', <}(\sigma) = (u_1, \ldots, u_t).
\]

For any \( j \in [r] \), \( \sigma \subset \sigma''_{i_j} \) if and only if \( \sigma \cup \{ v \} \subset \sigma_{i_j} \). Also, for \( i \notin I \), \( \sigma \cup \{ v \} \notin \sigma_i \) (since \( v \notin \sigma_i \)). Therefore, \( n = i_{t+1} - 1 \).

The vertex \( v \) is the minimal vertex in \( V \setminus \sigma_1 \), therefore it is the minimal vertex in \( (\sigma \cup \{ v \}) \setminus \sigma_1 \). Hence, we have \( v_1 = v \). Now, let \( i > 1 \) such that \( i \notin I \). Then, \( v_i = v \) is the vertex of minimal index in the sequence \( (v_1, \ldots, v_{i-1}) \) that is contained in \( (\sigma \cup \{ v \}) \setminus \sigma_i \). Therefore, \( v_i = v \).
Finally, we will show that $v_{ij} = u_j$ for all $j \in [t]$. We argue by induction on $j$. Let $j \in [t]$, and assume that $v_{ij} = u_{i}$ for all $\ell < j$.

For any $k < i_j$, either $v_k = v$ (if $k \notin I$) or $v_k = u_{i}$ for some $\ell < j$ (if $k = i_{\ell} \in I$). Also, since $v \in \sigma_{i_j}$, we have $(\sigma \cup \{v\}) \setminus \sigma_{i_j} = \sigma \setminus \sigma''_{i_j}$. So, for any $k < i_j$, $v_k \in (\sigma \cup \{v\}) \setminus \sigma_{i_j}$ if and only if $k = i_{\ell}$ for some $\ell < j$ such that $u_{i} \in \sigma \setminus \sigma''_{i_j}$. Therefore, $v_{i_j}$ is old at $i_j$ if and only if $u_{i_j}$ is old at $j$, and if $v_{i_j}$ and $u_{i_j}$ are both old, then $v_{i_j} = u_{i_j}$. Otherwise, assume that $v_{i_j}$ is new at $i_j$ and $u_{i_j}$ is new at $j$. Then, $v_{i_j}$ is the minimal vertex in $(\sigma \cup \{v\}) \setminus \sigma_{i_j}$, and $u_{i_j}$ is the minimal vertex in $\sigma \setminus \sigma''_{i_j}$. Thus, $v_{i_j} = u_{i_j}$.

So, for any $\sigma \in \text{lk}(X, v)$ we obtain

$$|\text{M}_{\text{lk}(X, v), \mathcal{A}''}(\sigma)| = |\text{M}_{X, \mathcal{A}}(\sigma \cup \{v\})| - 1.$$ 

Hence,

$$d(\text{lk}(X, v), \mathcal{A}''_v, <) \leq d(X, \mathcal{A}, <) - 1.$$ 

So, $\text{lk}(X, v)$ is $(d(X, \mathcal{A}, <) - 1)$-collapsible. □

By Claim 8, Claim 9 and Lemma 7, $X$ is $d(X, \mathcal{A}, <)$-collapsible. □

Proof of Theorem 4. Let $\prec$ be some linear order on the vertex set $V$, and let $\mathcal{A} = (\sigma_1, \ldots, \sigma_m)$ be the sequence of maximal faces of $X$ (ordered in any way).

Let $i \in [m]$ and let $\sigma \in X$ with $m_{X, \mathcal{A}}(\sigma) = i$. Let $\text{mes}_{X, \mathcal{A}}(\sigma) = (v_1, \ldots, v_{i-1})$. Then $M_{X, \mathcal{A}}(\sigma) = \{v_1, \ldots, v_k\}$ for some $i_1 < \cdots < i_k \in [i-1]$ (these are exactly the indices $i_j$ such that $v_{i_j}$ is new at $i_j$). For each $j \in [k]$ we have $v_{i_j} \notin \sigma_{i_j}$. In addition, since $v_{i_j}$ is new at $i_j$, we have $v_{i_{\ell}} \in \sigma_{i_j}$ for all $\ell < j$. Let $i_{k+1} = i$. Since $m_{X, \mathcal{A}}(\sigma) = i = i_{k+1}$, we have $\sigma \subset \sigma_{i_{k+1}}$. In particular, $v_{i_{\ell}} \in \sigma_{i_{k+1}}$ for all $\ell < k + 1$.

Therefore, $M_{X, \mathcal{A}}(\sigma) \in S(X)$. Thus, $d(X, \mathcal{A}, <) \leq d'(X)$, and by Theorem 6, $X$ is $d'(X)$-collapsible. □

3 Collapsibility of independence complexes

Let $G = (V, E)$ be a graph. The independence complex $I(G)$ is the simplicial complex on vertex set $V$ whose simplices are the independent sets in $G$.

Definition 10. Let $k(G)$ be the maximal size of a set $\{v_1, \ldots, v_k\} \subset V$ that satisfies:

- $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$,
- There exist $u_1, \ldots, u_k \in V$ such that
  - $\{v_i, u_i\} \in E$ for all $i \in [k]$,
  - $\{v_i, u_j\} \notin E$ for all $1 \leq i < j \leq k$.

Proposition 11. $k(G) = d'(I(G))$.

Proof. Let $A = \{v_1, \ldots, v_k\} \in S(I(G))$. Then, there exist maximal faces $\sigma_1, \ldots, \sigma_{k+1}$ of $I(G)$ such that:
• $v_i \notin \sigma_i$ for all $i \in [k]$,
• $v_i \in \sigma_j$ for all $1 \leq i < j \leq k + 1$.

Let $i \in [k]$. Since $\sigma_i$ is a maximal independent set in $G$ and $v_i \notin \sigma_i$, there exists some $u_i \in \sigma_i$ such that $\{v_i, u_i\} \in E$.

Let $1 \leq i < j \leq k$. Since $v_i$ and $u_j$ are both contained in the independent set $\sigma_j$, we have $\{v_i, u_j\} \notin E$. Furthermore, since $A \subset \sigma_{k+1}$, $A$ is an independent set in $G$. That is, $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$. So, $A$ satisfies the conditions of Definition 10. Hence, $|A| \leq k(G)$; therefore, $d'(I(G)) \leq k(G)$.

Now, let $k = k(G)$, and let $v_1, \ldots, v_k, u_1, \ldots, u_k \in V$ such that

• $\{v_i, v_j\} \notin E$ for all $i \neq j \in [k]$,
• $\{v_i, u_i\} \in E$ for all $i \in [k]$,
• $\{v_i, u_j\} \notin E$ for all $1 \leq i < j \leq k$.

Let $i \in [k]$, and let $V_i = \{v_j : 1 \leq j < i\}$. Note that $V_i \cup \{u_i\}$ forms an independent set in $G$; therefore, it is a simplex in $I(G)$. Let $\sigma_i$ be a maximal face of $I(G)$ containing $V_i \cup \{u_i\}$. Since $\{v_i, u_i\} \in E$, we have $v_i \notin \sigma_i$.

The set $\{v_1, \ldots, v_k\}$ is also an independent set in $G$. Therefore, there is a maximal face $\sigma_{k+1} \in I(G)$ that contains it.

By the definition of $\sigma_1, \ldots, \sigma_{k+1}$, we have $v_i \in \sigma_j$ for $1 \leq i < j \leq k + 1$. Therefore, $\{v_1, \ldots, v_k\} \in S(I(G))$; so, $k(G) = k \leq d'(I(G))$. Hence, $k(G) = d'(I(G))$, as wanted. \qed

As an immediate consequence of Proposition 11 and Theorem 4, we obtain:

**Proposition 12.** The complex $I(G)$ is $k(G)$-collapsible.

Note that vertices $v_1, \ldots, v_k, u_1, \ldots, u_k \in V$ satisfying the conditions in Definition 10 must all be distinct. As a simple corollary, we obtain

**Corollary 13.** The independence complex of a graph $G = (V, E)$ on $n$ vertices is $\left\lfloor \frac{n}{2} \right\rfloor$-collapsible.

### 4 Complexes of hypergraphs

In this section we prove our main results, Theorems 2 and 3.

**Proof of Theorem 2.** Let $\mathcal{H}$ be a hypergraph of rank $r$ on vertex set $[n]$, and let

$\{A_1, \ldots, A_k\} \in S(\text{Cov}_{\mathcal{H}, p})$.

Then, there exist maximal faces $\mathcal{F}_1, \ldots, \mathcal{F}_{k+1} \in \text{Cov}_{\mathcal{H}, p}$ such that

• $A_i \notin \mathcal{F}_i$ for all $i \in [k]$,
\begin{itemize}
  \item $A_i \in \mathcal{F}_j$ for all $1 \leq i < j \leq k + 1$.
\end{itemize}

For any $i \in [k + 1]$, there is some $C_i \subset [n]$ of size at most $p$ that covers $\mathcal{F}_i$. Since $\mathcal{F}_i$ is maximal, then, for any $A \in \mathcal{H}$, $A \in \mathcal{F}_i$ if and only if $A \cap C_i \neq \emptyset$. Therefore, we obtain
\begin{itemize}
  \item $A_i \cap C_i = \emptyset$ for all $i \in [k]$,
  \item $A_i \cap C_j \neq \emptyset$ for all $1 \leq i < j \leq k + 1$.
\end{itemize}

Hence, the pair of families
\[
\{A_1, \ldots, A_k, \emptyset\}
\]
and
\[
\{C_1, \ldots, C_k, C_{k+1}\}
\]
satisfies the conditions of Lemma 5; thus, $k + 1 \leq \binom{r + p}{r}$. Therefore,
\[
d'(\text{Cov}_{\mathcal{H},p}) \leq \binom{r + p}{r} - 1,
\]
and by Theorem 4, $\text{Cov}_{\mathcal{H},p}$ is $\left(\binom{r + p}{r} - 1\right)$-collapsible.

**Proof of Theorem 3.** Let $\mathcal{H}$ be a hypergraph of rank $r$ and let $G$ be the graph on vertex set $\mathcal{H}$ whose edges are the pairs $\{A, B\} \subset \mathcal{H}$ such that $A \cap B = \emptyset$. Then $\text{Int}_{\mathcal{H}} = I(G)$.

Let $k = k(G)$ and let $\{A_1, \ldots, A_k\} \subset \mathcal{H}$ that satisfies the conditions of Definition 10. That is,
\begin{itemize}
  \item $A_i \cap A_j \neq \emptyset$ for all $i \neq j \in [k]$,
  \item There exist $B_1, \ldots, B_k \in \mathcal{H}$ such that
    \begin{itemize}
      \item $A_i \cap B_i = \emptyset$ for all $i \in [k]$,
      \item $A_i \cap B_j \neq \emptyset$ for all $1 \leq i < j \leq k$.
    \end{itemize}
\end{itemize}

Then, the pair of families $\{A_1, \ldots, A_k, B_k, \ldots, B_1\}$ and $\{B_1, \ldots, B_k, A_k, \ldots, A_1\}$ satisfies the conditions of Lemma 5; therefore, $2k \leq \binom{2r}{r}$. Thus, by Proposition 12, $\text{Int}_{\mathcal{H}} = I(G)$ is $\frac{1}{2}(\binom{2r}{r})$-collapsible.

**5 More complexes of hypergraphs**

Let $\mathcal{H}$ be a hypergraph. A set $C$ is a $t$-transversal of $\mathcal{H}$ if $|A \cap C| \geq t$ for all $A \in \mathcal{H}$. Let $\tau_t(\mathcal{H})$ be the minimal size of a $t$-transversal of $\mathcal{H}$. The hypergraph $\mathcal{H}$ is pairwise $t$-intersecting if $|A \cap B| \geq t$ for all $A, B \in \mathcal{H}$. Let
\[
\text{Cov}^t_{\mathcal{H},p} = \{\mathcal{F} \subset \mathcal{H} : \tau_t(\mathcal{F}) \leq p\}
\]
and
\[
\text{Int}^t_{\mathcal{H}} = \{\mathcal{F} \subset \mathcal{H} : \mathcal{F} \text{ is pairwise } t\text{-intersecting}\}.
\]

The following generalization of Lemma 5 was proved by Füredi in [5].
Lemma 14 (Füredi [5]). Let \( \{A_1, \ldots, A_k\} \) and \( \{B_1, \ldots, B_k\} \) be families of sets such that:

- \( |A_i| \leq r, |B_i| \leq p \) for all \( i \in [k] \),
- \( |A_i \cap B_i| \leq t \) for all \( i \in [k] \),
- \( |A_i \cap B_j| > t \) for all \( 1 \leq i < j \leq k \).

Then

\[
    k \leq \left( \frac{r + p - 2t}{r - t} \right).
\]

We obtain the following:

Theorem 15. Let \( \mathcal{H} \) be a hypergraph of rank \( r \) and let \( t \leq \min\{r, p\} - 1 \). Then \( \text{Cov}_{\mathcal{H}, p}^{t+1} \) is \((\left( \frac{r+p-2t}{r-t} \right) - 1)\)-collapsible.

Theorem 16. Let \( \mathcal{H} \) be a hypergraph of rank \( r \) and let \( t \leq r - 1 \). Then \( \text{Int}_{\mathcal{H}}^{t+1} \) is \( \frac{1}{2} \left( \frac{2(r-t)}{r-t} \right) \)-collapsible.

Note that by setting \( t = 0 \) we recover Theorems 2 and 3. The proofs are essentially the same as the proofs of Theorems 2 and 3, except for the use of Lemma 14 instead of Lemma 5. The extremal examples are also similar: Let

\[
    \mathcal{H}_1 = \left\{ A \cup [t] : A \in \left( \left[ \frac{r + p - t}{r - t} \right] \right) \right\}
\]

and

\[
    \mathcal{H}_2 = \left\{ A \cup [t] : A \in \left( \left[ \frac{2r - t}{r - t} \right] \right) \right\}.
\]

The complex \( \text{Cov}_{\mathcal{H}_1}^{t+1} \) is the boundary of the \((\left( \frac{r+p-2t}{r-t} \right) - 1)\)-dimensional simplex, hence it is not \((\left( \frac{r+p-2t}{r-t} \right) - 2)\)-collapsible, and the complex \( \text{Int}_{\mathcal{H}_2}^{t+1} \) is the boundary of the \( \frac{1}{2} \left( \frac{2(r-t)}{r-t} \right) \)-dimensional cross-polytope, hence it is not \((\frac{1}{2} \left( \frac{2(r-t)}{r-t} \right) - 1)\)-collapsible.

Restricting ourselves to special classes of hypergraphs we may obtain better bounds on the collapsibility of their associated complexes. For example, we may look at \( r \)-partite \( r \)-uniform hypergraphs (that is, hypergraphs \( \mathcal{H} \) on vertex set \( V = V_1 \cup V_2 \cup \cdots \cup V_r \) such that \( |A \cap V_i| = 1 \) for all \( A \in \mathcal{H} \) and \( i \in [r] \)). In this case we have the following result:

Theorem 17. Let \( \mathcal{H} \) be an \( r \)-partite \( r \)-uniform hypergraph. Then \( \text{Int}_{\mathcal{H}} \) is \( 2^{r-1} \)-collapsible.

The next example shows that the bound on the collapsibility of \( \text{Int}_{\mathcal{H}} \) in Theorem 17 is tight: Let \( \mathcal{H} \) be the complete \( r \)-partite \( r \)-uniform hypergraph with all sides of size 2. It has \( 2^r \) edges, and any edge \( A \in \mathcal{H} \) intersects all the edges of \( \mathcal{H} \) except its complement. Therefore the complex \( \text{Int}_{\mathcal{H}} \) is the boundary of the \( 2^{r-1} \)-dimensional cross-polytope, so it is homeomorphic to a \((2^{r-1} - 1)\)-dimensional sphere. Hence, by Proposition 1, \( \text{Int}_{\mathcal{H}} \) is not \((2^{r-1} - 1)\)-collapsible.

For the proof we need the following Lemma, due to Lovász, Nešetřil and Pultr.
Lemma 18 (Lovász, Nešetřil, Pultr [8, Prop. 5.3]). Let \( \{A_1, \ldots, A_k\} \) and \( \{B_1, \ldots, B_k\} \) be families of subsets of \( V = V_1 \cup V_2 \cup \cdots \cup V_r \) such that:

- \( |A_i \cap V_j| = 1, |B_i \cap V_j| = 1 \) for all \( i \in [k] \) and \( j \in [r] \),
- \( A_i \cap B_i = \emptyset \) for all \( i \in [k] \),
- \( A_i \cap B_j \neq \emptyset \) for all \( 1 \leq i < j \leq k \).

Then \( k \leq 2^r \).

A common generalization of Lemma 5 and Lemma 18 was proved by Alon in [3].

The proof of Theorem 17 is the same as the proof of Theorem 3, except that we replace Lemma 5 by Lemma 18. A similar argument was also used by Aharoni and Berger ([1, Theorem 5.1]) in order to prove a related result about rainbow matchings in \( r \)-partite \( r \)-uniform hypergraphs.

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References