An explicit edge-coloring of K_n with six colors on every K_5

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Abstract

Let p and q be positive integers such that $1 \leqslant q \leqslant \binom{p}{2}$. A (p,q)-coloring of the complete graph on n vertices K_n is an edge coloring for which every p-clique contains edges of at least q distinct colors. We denote the minimum number of colors needed for such a (p,q)-coloring of K_n by $f(n,p,q)$. This is known as the Erdős-Gyárfás function. In this paper we give an explicit $(5,6)$-coloring with $n^{1/2+o(1)}$ colors. This improves the best known upper bound of $f(n,5,6) = O\left(n^{3/5}\right)$ given by Erdős and Gyárfás, and comes close to matching the order of the best known lower bound, $f(n,5,6) = \Omega\left(n^{1/2}\right)$.

Mathematics Subject Classifications: 05C55, 05D10

1 Introduction

Given two integers $s, t \geqslant 2$, the central question in classical Ramsey theory for graphs asks for the minimum number of vertices N for which any 2-coloring, say red and blue, of the edges of K_N must yield a red K_s or a blue K_t. We say that $N = R(s,t)$, the Ramsey number for s,t. This question generalizes to more than two colors in a natural way. That is, we let $R(s_1, \ldots, s_k)$ denote the minimum number of vertices N for which a coloring of the edges of K_N with k colors results in either an s_1-clique in the first color, or an s_2-clique in the second color, and so on. If all of the s_i are equal, then we call this the diagonal case for the multicolor Ramsey numbers and use the notation $R_k(s)$. A variation of the Ramsey problem is given by the following definition.

Definition 1. Let n, p, and q be positive integers such that $1 \leqslant q \leqslant \binom{p}{2}$. A **$(p,q)$-coloring** of the complete graph on n vertices K_n is an edge coloring, for some k,

$$c : E(K_n) \to \{1, 2, \ldots, k\},$$

for which every subset of p vertices of $V(K_n)$ spans at least q distinct edge colors. Let $f(n,p,q)$ denote the minimum number of colors k for which a (p,q)-coloring of K_n using k colors exists.

When $q=2$ in the above definition, then determining an upper bound for the function $f(n,p,2) \leqslant k$ is equivalent to giving a lower bound, $n+1 \leqslant R_k(p)$. Similarly, giving a lower bound $k \leqslant f(n,p,2)$ is equivalent to giving the upper bound $R_{k-1}(p)$.

1.1 Background

Erdős and Shelah [5, 6] introduced the function $f(n,p,q)$ in 1975, but it was not studied in depth until 1997 when Erdős and Gyárfás [7] looked at the growth rate of $f(n,p,q)$ as $n \to \infty$ for fixed values of p and q. They used the Lovász Local Lemma to give a general upper bound for the function,

$$f(n,p,q) \leqslant cn^{\frac{p-2}{\binom{p}{2}-q+1}}.$$

Other than this, they looked for threshold values for q in terms of p for which $f(n,p,q)$ "jumps" in order of magnitude. For instance, they showed that when

$$q = \binom{p}{2} - p + 3,$$

$f(n,p,q) = \Theta(n)$ and $f(n,p,q-1) = o(n)$. So the function becomes linear in n at this particular value of q. Similarly, they determined the exact values of q in terms of p at which $f(n,p,q)$ becomes quadratic in n and where $\binom{n}{2} - c \leqslant f(n,p,q)$ for some constant that depends only on p.

Left as an open question was determining the threshold value of q for which $f(n,p,q)$ first becomes $\Omega\left(n^{\epsilon}\right)$ for some fixed positive ϵ which depends only on p. They showed that

$$n^{\frac{1}{p-2}} - 1 \leqslant f(n,p,p).$$

So therefore, for any $q \geqslant p$ it follows that $f(n,p,q) = \Omega\left(n^{\frac{1}{p-2}}\right)$. However, it was unclear what the order of magnitude of $f(n,p,p-1)$ is in general. To this end they considered some small cases. When $p=3$, they pointed out that since determining $f(n,3,2)$ is equivalent to solving the multicolor Ramsey problem for 3-cliques, then

$$c_1 \frac{\log n}{\log \log n} \leqslant f(n,3,2) \leqslant c_2 \log n$$

for constants c_1, c_2. However, for $p=4$, they could not beat the probabilistic upper bound

$$f(n,4,3) = O(n^{1/2}).$$

For this reason, they called this the "most annoying" case.

In 1998, Mubayi [9] gave an explicit $(4,3)$-coloring using $n^{o(1)}$ colors. Specifically, he showed that

$$f(n,4,3) \leqslant e^{O(\sqrt{\log n})}.$$

In 2000, Mubayi and Eichhorn [4] demonstrated that for $p \geqslant 5$, this construction is in general a (p,q)-coloring for $q = 2\lceil \log_2 p \rceil - 2$. In 2015, Conlon, Fox, Lee, and Sudakov [3] finally proved that $f(n,p,p-1) = n^{o(1)}$ for all $p \geqslant 3$. We will discuss the construction they came up with to demonstrate this in Section 2.

In addition to their general results, Erdős and Gyárfás looked at several cases for small values of p. They found that

$$\frac{5}{6}(n-1) \leqslant f(n,4,5) \leqslant n$$

and that

$$f(n,9,34) = \binom{n}{2} - o(n^2).$$

Moreover, they singled out the cases of $(4,4)$ and $(5,9)$-colorings as being particularly interesting to look at. In 2000, Axenovich [1] gave a construction showing that $f(n,5,9) \leqslant n^{1+o(1)}$. Since Erdős and Gyárfás showed that $f(n,5,8) = \Theta(n)$, then this reduced the difference between the known upper and lower bounds for $f(n,5,9)$ to a factor of $n^{o(1)}$. In 2013, E. Krop and I. Krop [8] improved the lower bound to

$$\frac{7}{4}n - 3 \leqslant f(n,5,9).$$

They also improved the lower bound for the $(4,5)$ case to

$$\frac{5}{6}n + 1 \leqslant f(n,4,5).$$

In 2004, Mubayi [10] gave an explicit $(4,4)$-coloring which reduced the upper bound to

$$f(n,4,4) \leqslant n^{1/2+o(1)},$$

a "small" factor of $n^{o(1)}$ above the best known lower bound given by Erdős and Gyárfás of $n^{1/2} - 1$. We will discuss his construction in more detail in Section 2. Recently, Heath and I [2] gave an explicit $(5,5)$-coloring that uses only $n^{1/3+o(1)}$ colors. This is also a factor of $n^{o(1)}$ above the best known lower bound of $n^{1/3} - 1$.

1.2 Main Result

Here, we extend the ideas of the constructions from [10] and [2] to improve the probabilistic upper bound of $f(n,5,6)$ by giving an explicit $(5,6)$-coloring of K_n that uses few colors. The new upper bound comes close to matching the known lower bound in order of magnitude.

Theorem 2. *As $n \to \infty$,*

$$\left(\frac{5}{6}n - \frac{7}{12}\right)^{1/2} + \frac{1}{2} \leqslant f(n,5,6) \leqslant n^{1/2} 2^{O\left(\sqrt{\log_2 n}\log_2\log_2 n\right)}.$$

The lower bound comes from the following lemma, a generalization of an argument used by Erdős and Gyárfás [7] and stated explicitly in [3].

Lemma 3 (Equation 11 in [3])**.** *Let $t = f(n,p,q)$, then*

$$f\left(\left\lceil\frac{n-1}{t}\right\rceil, p-1, q-1\right) \leqslant t.$$

Proof. Suppose we have a (p,q)-coloring of K_n with t colors. Fix some vertex x, then at least $\lceil\frac{n-1}{t}\rceil$ vertices must appear in a monochromatic neighborhood of x. The number of colors t must be enough to give a $(p-1,q-1)$-coloring on this set. □

This lemma gives the stated lower bound in Theorem 2 in the following way: Let $t = f(n,5,6)$, then by Lemma 3

$$f\left(\left\lceil\frac{n-1}{t}\right\rceil, 4, 5\right) \leqslant t.$$

So by the lower bound of $\frac{5}{6}n + 1 \leqslant f(n,4,5)$ given in [8] it follows that

$$\begin{aligned} \frac{5}{6}\left(\left\lceil\frac{n-1}{t}\right\rceil\right) + 1 &\leqslant t \\ \frac{5}{6}\left(\frac{n-1}{t}\right) + 1 &\leqslant t \\ 0 \leqslant t^2 - t - \frac{n}{6}(n-1) &= (t-\alpha_1)(t-\alpha_2) \end{aligned}$$

where $\alpha_1 = \left(\frac{5}{6}n - \frac{7}{12}\right)^{1/2} + \frac{1}{2}$ and $\alpha_2 = -\left(\frac{5}{6}n - \frac{7}{12}\right)^{1/2} + \frac{1}{2}$. Therefore, either $t \geqslant \left(\frac{5}{6}n - \frac{7}{12}\right)^{1/2} + \frac{1}{2}$ or $t \leqslant -\left(\frac{5}{6}n - \frac{7}{12}\right)^{1/2} + \frac{1}{2}$. The upper bound in the latter inequality is less than or equal to zero for all $n \geqslant 1$. Hence, the lower bound in the former inequality gives the only relevant condition for t. This gives us the lower bound stated in Theorem 2.

The construction providing the upper bound in Theorem 2 combines two existing constructions with some modification. The first was given recently by Conlon, Fox, Lee, and Sudakov [3] and was originally used to show

$$f(n,p,p-1) \leqslant 2^{16p(\log_2 n)^{1-1/(p-2)}\log_2\log_2 n}.$$

The second construction is a modified version of the "algebraic" part of the $(4,4)$-coloring given by Mubayi in [10].

This paper is structured as follows. Section 2 defines an explicit coloring of the edges of K_n which uses $n^{1/2}2^{O\left(\sqrt{\log_2 n}\log_2\log_2 n\right)}$ colors. Section 3 gives a series of lemmas which show that certain specified local configurations of colors do not occur in this coloring. Section 4 uses these lemmas to prove that this construction is a $(5,6)$-coloring by detailed case checking of every possible way that five vertices could span fewer than six colors.

2 The Construction

Given a positive integer n, let β be the positive integer for which

$$2^{(\beta-1)^2} < n \leqslant 2^{\beta^2},$$

and let q be the minimum odd prime power (that is, $q = p^k$ for some odd prime number p and some positive integer k) such that $n \leqslant (q-1)^2$. For each vertex $x \in V(K_n)$ we associate two objects - a unique binary string of length β^2 and a unique vector from $\mathbb{F}_q^2$, the two-dimensional vector space over the finite field with q elements, with no zero components. These objects are assigned to each vertex arbitrarily other than the condition that no two vertices can receive the same binary string and no two vertices can receive the same vector.

For a given vertex x we will denote the associated binary string by

$$\hat{x} = \left(x^{(1)}, x^{(2)}, \ldots, x^{(\beta)}\right)$$

where $x^{(i)}$ denotes the ith block of β bits of $\hat{x}$ for $i = 1, \ldots, \beta$. That is, we think of $\hat{x}$ as the concatenation of β binary strings each of length β. For a particular block, $x^{(i)}$, let $x_j^{(i)}$ denote its jth bit for $j = 1, \ldots, \beta$. That is,

$$x^{(i)} = \left(x_1^{(i)}, x_2^{(i)}, \ldots, x_\beta^{(i)}\right)$$

where $x_j^{(i)} \in \{0, 1\}$ for each j. Similarly, for a given vertex x we will denote its associated vector by

$$\vec{x} = (x_1, x_2)$$

where $x_1, x_2 \in \mathbb{F}_q$ and both x_1 and x_2 are nonzero.

In addition, we will sometimes wish to consider the vertices or their associated binary strings and vectors to be ordered. On the set of binary strings of length β, let $<_1$ denote the standard lexicographical order. Similarly, let $<_2$ denote the lexicographical order of binary strings of length β^2. So $\hat{x} <_2 \hat{y}$ if and only if there exists some i such that $1 \leqslant i \leqslant \beta$ for which $x^{(i)} <_1 y^{(i)}$ and $x^{(j)} = y^{(j)}$ for each $j = 1, \ldots, i-1$.

Now we extend $<_2$ to apply to the vertices of K_n and the associated vectors. For any two distinct vertices $x, y \in V(K_n)$,

$$x <_3 y \iff \hat{x} <_2 \hat{y}.$$

Similarly, define $<_4$ on the set of n vectors used in the construction by

$$\vec{x} <_4 \vec{y} \iff x <_3 y.$$

For any two distinct vertices $x, y \in V(K_n)$, we define the color of the edge xy in two parts, a part based on the binary strings $\hat{x}$ and $\hat{y}$ and a part based on the vectors $\vec{x}$ and $\vec{y}$. Formally, we will define the color of edge xy to be

$$\mathcal{C}(x, y) = \left(\varphi\left(\hat{x}, \hat{y}\right), \chi\left(\vec{x}, \vec{y}\right)\right).$$

The function $\mathcal{C}$ will be symmetric, $\mathcal{C}(x, y) = \mathcal{C}(y, x)$ for all distinct vertices x and y, so that it is well-defined for an edge.

2.1 The binary string coloring φ

The following coloring is the same that Heath and I used in [2] which is a modified version of the $(p, p-1)$-coloring defined by Conlon, Fox, Lee, and Sudakov [3]. Define

$$\varphi(\hat{x}, \hat{y}) = (\varphi_1(\hat{x}, \hat{y}), \varphi_2(\hat{x}, \hat{y}), \varphi_{3,1}(\hat{x}, \hat{y}), \ldots, \varphi_{3,\beta}(\hat{x}, \hat{y})).$$

We define

$$\varphi_1(\hat{x}, \hat{y}) = \min_{1 \leqslant i \leqslant \beta}\{i : x^{(i)} \neq y^{(i)}\},$$

and

$$\varphi_2(\hat{x}, \hat{y}) = \{x^{(i)}, y^{(i)}\}$$

where $i = \varphi_1(\hat{x}, \hat{y})$. Then for each $j = 1, \ldots, \beta$ we define

$$\varphi_{3,j}(\hat{x}, \hat{y}) = 0$$

if $x^{(j)} = y^{(j)}$, and otherwise

$$\varphi_{3,j}(\hat{x}, \hat{y}) = (\varphi_{3,j,1}(\hat{x}, \hat{y}), \varphi_{3,j,2}(\hat{x}, \hat{y})),$$

where

$$\varphi_{3,j,1}(\hat{x}, \hat{y}) = \min_{1 \leqslant i \leqslant \beta}\{i : x_i^{(j)} \neq y_i^{(j)}\},$$

and

$$\varphi_{3,j,2}(\hat{x}, \hat{y}) = \begin{cases} -1 & \left(\hat{x} <_2 \hat{y} \text{ and } y^{(j)} <_1 x^{(j)}\right) \text{ or } \left(\hat{y} <_2 \hat{x} \text{ and } x^{(j)} <_1 y^{(j)}\right) \\ 1 & \left(\hat{x} <_2 \hat{y} \text{ and } x^{(j)} <_1 y^{(j)}\right) \text{ or } \left(\hat{y} <_2 \hat{x} \text{ and } y^{(j)} <_1 x^{(j)}\right) \end{cases}.$$

We can now bound the number of colors used by φ total: φ_1 uses at most β colors, φ_2 uses at most $\binom{2^\beta}{2}$ colors, and $\varphi_{3,i}$ uses at most $1 + 2\beta$ colors for each $i = 1, \ldots, \beta$. Hence, overall, φ uses at most

$$\begin{aligned} \beta \cdot \binom{2^\beta}{2} \cdot (1 + 2\beta)^\beta &< (\beta + 1) \cdot \frac{1}{2}\left(2^\beta\right)^2 \cdot (2 + 2\beta)^\beta \\ &= (\beta + 1)^{\beta+1} \cdot 2^{3\beta-1} \\ &= 2^{(\beta+1)\log_2(\beta+1)+3\beta-1} \end{aligned}$$

colors. Since $2^{(\beta-1)^2} < n$, then it follows that $\beta < \sqrt{\log_2 n} + 1$. So the number of colors used by φ is at most

$$2^{(\sqrt{\log_2 n}+2)\log_2(\sqrt{\log_2 n}+2)+3\sqrt{\log_2 n}+2} = 2^{O(\sqrt{\log_2 n}\log_2\log_2 n)}.$$

2.2 The vector coloring χ

The following coloring is largely the same as the algebraic edge coloring constructed by Mubayi in [10] but with some modification. For vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, let

$$\chi(\vec{x}, \vec{y}) = (\chi_1(\vec{x}, \vec{y}), \chi_2(\vec{x}, \vec{y})))$$

where

$$\chi_1(\vec{x}, \vec{y}) = x_1 y_1 - x_2 - y_2,$$

and

$$\chi_2(\vec{x}, \vec{y}) = \begin{cases} 0 & x_1 = y_1 \\ (f(x_1, y_1), f(y_1, x_1)) & x_1 \neq y_1 \text{ and } \vec{x} <_4 \vec{y} \\ (f(y_1, x_1), f(x_1, y_1)) & x_1 \neq y_1 \text{ and } \vec{y} <_4 \vec{x} \end{cases}$$

where $f : \mathbb{F}_q \times \mathbb{F}_q \to \{S, T\}$ is a a function defined as follows.

For each $\alpha \in \mathbb{F}_q$ define the graph G_α by $V(G_\alpha) = \mathbb{F}_q \setminus \{\alpha\}$ and

$$E(G_\alpha) = \{xy : x + y = 2\alpha\}.$$

For each $x \in V(G_\alpha)$, there exists exactly one vertex $y = 2\alpha - x$ which it is adjacent to (since the field has odd characteristic, then $x \neq y$). Hence, the edges of G_α form a matching of the vertices, and so we can give a bipartition of the vertices,

$$\begin{aligned} V(G_\alpha) &= S_\alpha \cup T_\alpha \\ \emptyset &= S_\alpha \cap T_\alpha, \end{aligned}$$

such that every edge of G_α goes between a vertex of S_α and a vertex of T_α. For each $\alpha \in \mathbb{F}_q$ select such a bipartition S_α, T_α of the vertices of G_α arbitrarily.

Now, for any $\alpha, \beta \in \mathbb{F}_q$ define

$$f(\alpha, \beta) = \begin{cases} S & \beta \in S_\alpha \text{ or } \alpha = \beta \\ T & \beta \in T_\alpha \end{cases}$$

We can now give an upper bound on the number of colors produced by χ: χ_1 uses at most q colors while χ_2 uses five colors. Therefore, χ uses at most $5q$ colors. Since q was the minimum odd prime power such that $n \leqslant (q-1)^2$, then it follows that

$$\lfloor \sqrt{n} + 1 \rfloor \leqslant \sqrt{n} + 1 \leqslant q.$$

By Bertrand's Postulate, a prime number must exist between $\lfloor \sqrt{n} + 1 \rfloor$ and $2\left(\lfloor \sqrt{n} + 1 \rfloor\right)$. Since two is the only even prime, then this prime must be odd (since $n > 1$). And since any odd prime counts as an odd prime power itself, then it follow that

$$q < 2\left(\lfloor \sqrt{n} + 1 \rfloor\right) \leqslant 2\left(\sqrt{n} + 1\right).$$

Hence, χ gives at most $10\sqrt{n} + 5$ colors, and all together $\mathcal{C}$ uses at most

$$n^{1/2} 2^{O(\sqrt{\log_2 n} \log_2 \log_2 n)}$$

colors.

2.3 Example

For an example, suppose $n = 17$, then each vertex in K_{17} will be assigned a unique binary string from $\{0,1\}^9$ and a unique vector from $\mathbb{F}_7^2$ with no zero components. Let $x, y \in V(K_{17})$ be distinct vertices such that x is assigned $\hat{x} = (011, 101, 010)$ and $\vec{x} = (2,1)$ while y is assigned $\hat{y} = (111, 101, 001)$ and $\vec{y} = (3,2)$. Then $\hat{x} <_2 \hat{y}$ so $x <_3 y$ and $\vec{x} <_4 \vec{y}$. Moreover, suppose in the arbitrary bipartition of G_2 that $3 \in T_2$ and that in the bipartition of G_3, $2 \in S_3$. Then color of the edge between x and y is given by $\mathcal{C}(x,y) = (\varphi(\hat{x},\hat{y}), \chi(\vec{x},\vec{y}))$ where

$$\varphi(\hat{x},\hat{y}) = (1, \{011, 111\}, (1,1), 0, (2,-1)),$$

and $\chi(\vec{x},\vec{y}) = (3, (T,S))$.

3 Configurations Avoided by the Construction

We will now prove a series of lemmas which demonstrate that the construction $\mathcal{C}$ avoids certain "bad" configurations. This section is broken up into two parts. The first part contains lemmas which are primarily given to summarize arguments used multiple times in later lemmas. The second part contains lemmas which each show that a specific configuration is avoided by the construction.

3.1 Basic Lemmas

Lemma 4. *If $a, b, c \in V(K_n)$ are distinct vertices such that $a <_3 b <_3 c$, then $\mathcal{C}(a,b) \neq \mathcal{C}(b,c)$.*

Proof. Assume that $a, b, c \in V(K_n)$ are distinct vertices such that $a <_3 b <_3 c$. If $\varphi_1(\hat{a},\hat{b}) \neq \varphi_1(\hat{b},\hat{c})$, then $\mathcal{C}(a,b) \neq \mathcal{C}(b,c)$. If $\varphi_1(\hat{a},\hat{b}) = \varphi_1(\hat{b},\hat{c}) = i$, then $a <_3 b <_3 c$ implies that $a^{(i)} <_1 b^{(i)} <_1 c^{(i)}$. Hence,

$$\varphi_2(\hat{a},\hat{b}) = \{a^{(i)}, b^{(i)}\} \neq \{c^{(i)}, b^{(i)}\} = \varphi_2(\hat{b},\hat{c}).$$

So $\mathcal{C}(a,b) \neq \mathcal{C}(b,c)$. □

Lemma 5. *There are no monochromatic odd cycles under the edge coloring $\mathcal{C}$.*

Proof. Let $x_1, \ldots, x_k \in V(K_n)$ such that

$$\mathcal{C}(x_i, x_{i+1}) = \mathcal{C}(x_k, x_1)$$

for each $i = 1, \ldots, k-1$. Then it follows that

$$\varphi_1(\hat{x}_i, \hat{x}_{i+1}) = \varphi_1(\hat{x}_k, \hat{x}_1),$$

and

$$\varphi_2(\hat{x}_i, \hat{x}_{i+1}) = \varphi_2(\hat{x}_k, \hat{x}_1),$$

for each $i = 1, \ldots, k-1$. Let $\varphi_1(\hat{x}_k, \hat{x}_1) = j$ and $\varphi_2(\hat{x}_k, \hat{x}_1) = \{a, b\}$. Without loss of generality we can assume that the jth block of $\hat{x}_1$ is a. Now, for each $i = 2, \ldots, k$, the jth block of $\hat{x}_i$ is a if the jth block of $\hat{x}_{i-1}$ is b, and the jth block of $\hat{x}_i$ is b if the jth block of $\hat{x}_{i-1}$ is a. If k is odd, then the jth block of $\hat{x}_k$ is a, a contradiction. Hence, k must be even. Therefore, no color class of $\mathcal{C}$ contains an odd cycle. □

Lemma 6. *Let $a, b, c, d \in V(K_n)$ such that $\hat{a} <_2 \hat{b} <_2 \hat{c} <_2 \hat{d}$. If $\mathcal{C}(a, b) = \mathcal{C}(c, d)$, then the clique formed by these four vertices spans at least four distinct edge colors. Moreover, if it spans exactly four colors, then $\mathcal{C}(a, c) = \mathcal{C}(b, d)$.*

Proof. Let $a, b, c, d \in V(K_n)$ such that $\hat{a} <_2 \hat{b} <_2 \hat{c} <_2 \hat{d}$ and $\mathcal{C}(a, b) = \mathcal{C}(c, d)$. Let

$$i = \varphi_1(\hat{a}, \hat{b}) = \varphi_1(\hat{c}, \hat{d}),$$

and let

$$\{x, y\} = \varphi_2(\hat{a}, \hat{b}) = \varphi_2(\hat{c}, \hat{d}).$$

Without loss of generality, we may assume that $a^{(i)} = x$ and $b^{(i)} = y$. Since $\hat{a} <_2 \hat{b}$, it follows that $x = a^{(i)} <_1 b^{(i)} = y$. Similarly, $c^{(i)} <_1 d^{(i)}$. Therefore, $c^{(i)} = x$ and $d^{(i)} = y$. Since $a^{(i)} = c^{(i)}$ and $b^{(i)} = d^{(i)}$, then

$$\varphi_{3,i}(\hat{a}, \hat{c}) = \varphi_{3,i}(\hat{b}, \hat{d}) = 0.$$

Also, $\hat{a} <_2 \hat{d}$ and $a^{(i)} <_1 d^{(i)}$ implies that $\varphi_{3,i}(\hat{a}, \hat{d}) \neq 0$ and that $\varphi_{3,i,2}(\hat{a}, \hat{d}) = 1$. Similarly, $\hat{b} <_2 \hat{c}$ and $c^{(i)} <_1 b^{(i)}$ implies that $\varphi_{3,i}(\hat{b}, \hat{c}) \neq 0$ and that $\varphi_{3,i,2}(\hat{b}, \hat{c}) = -1$.

Therefore, $\mathcal{C}$ gives at least three different edge colors between the sets $\{a, b\}$ and $\{c, d\}$, and if it yields exactly three, then it must be $\mathcal{C}(a, c) = \mathcal{C}(b, d)$.

All that remains to be shown is that the color $\mathcal{C}(a, b) = \mathcal{C}(c, d)$ is different than any color between the two sets. This is true since $\hat{b} <_2 \hat{c}$ implies that $b^{(j)} <_1 c^{(j)}$ where $j = \varphi_1(\hat{b}, \hat{c})$. Therefore, $j < i$ which implies that $a^{(j)} = b^{(j)}$ and $c^{(j)} = d^{(j)}$. Hence,

$$\varphi_1(\hat{a}, \hat{c}) = \varphi_1(\hat{a}, \hat{d}) = \varphi_1(\hat{b}, \hat{c}) = \varphi_1(\hat{b}, \hat{d}) = j$$

while

$$\varphi_1(\hat{a}, \hat{b}) = \varphi_1(\hat{c}, \hat{d}) = i.$$

□

Lemma 7. *Let $a, b, c \in V(K_n)$ be distinct vertices. If $\mathcal{C}(a, b) = \mathcal{C}(a, c)$, then $b_1 \neq c_1$ where $\vec{b} = (b_1, b_2)$ and $\vec{c} = (c_1, c_2)$.*

Proof. Let $a, b, c \in V(K_n)$ be distinct vertices such that $\mathcal{C}(a, b) = \mathcal{C}(a, c)$. Then

$$\begin{aligned} \chi_1(\vec{a}, \vec{b}) &= \chi_1(\vec{a}, \vec{c}) \\ a_1 b_1 - a_2 - b_2 &= a_1 c_1 - a_2 - c_2 \\ a_1(b_1 - c_1) &= b_2 - c_2 \end{aligned}$$

If $b_1 = c_1$, then $b_2 = c_2$ and so $\vec{b} = \vec{c}$, a contradiction of the assumption that b and c were distinct vertices. Hence, $b_1 \neq c_1$. □

Lemma 8. *Let $a, b, c, d \in V(K_n)$ be distinct vertices. If $\mathcal{C}(a,b) = \mathcal{C}(a,c)$ and $\mathcal{C}(d,b) = \mathcal{C}(d,c)$, then $a_1 = d_1$ where $\vec{a} = (a_1, a_2)$ and $\vec{d} = (d_1, d_2)$.*

Proof. Let $a, b, c, d \in V(K_n)$ be distinct vertices such that $\mathcal{C}(a,b) = \mathcal{C}(a,c)$ and $\mathcal{C}(d,b) = \mathcal{C}(d,c)$. Then

$$\chi_1(\vec{a}, \vec{b}) = \chi_1(\vec{a}, \vec{c})$$
$$\chi_1(\vec{d}, \vec{b}) = \chi_1(\vec{d}, \vec{c})$$

which implies that

$$a_1 b_1 - a_2 - b_2 = a_1 c_1 - a_2 - c_2$$
$$d_1 b_1 - d_2 - b_2 = d_1 c_1 - d_2 - c_2.$$

Therefore,

$$a_1(b_1 - c_1) = b_2 - c_2$$
$$d_1(b_1 - c_1) = b_2 - c_2.$$

By Lemma 7 we know that $b_1 - c_1 \neq 0$. Thus,

$$a_1 = (b_2 - c_2)(b_1 - c_1)^{-1} = d_1.$$ □

Lemma 9. *Let $a, b, c \in V(K_n)$ be distinct vertices such that $b_1 \neq a_1$, $c_1 \neq a_1$, and $b_1 + c_1 = 2a_1$ where $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, and $\vec{c} = (c_1, c_2)$. Then $\mathcal{C}(a,b) \neq \mathcal{C}(a,c)$.*

Proof. Let $a, b, c \in V(K_n)$ be distinct vertices such that $b_1 \neq a_1$, $c_1 \neq a_1$, and $b_1 + c_1 = 2a_1$ where $\vec{a} = (a_1, a_2)$, $\vec{b} = (b_1, b_2)$, and $\vec{c} = (c_1, c_2)$. Assume, towards a contradiction, that $\mathcal{C}(a,b) = \mathcal{C}(a,c)$.

Note that $b_1 + c_1 = 2a_1$ implies that b_1 and c_1 are adjacent in the graph G_{a_1}. So

$$f(a_1, b_1) \neq f(a_1, c_1).$$

Now, Lemma 4 implies that either $a <_3 b, c$ or $a >_3 b, c$. Thus, it is also the case that either $\vec{a} <_4 \vec{b}, \vec{c}$ or $\vec{a} >_4 \vec{b}, \vec{c}$.

If $\vec{a} <_4 \vec{b}, \vec{c}$, then

$$\chi_2(\vec{a}, \vec{b}) = (f(a_1, b_1), f(b_1, a_1)),$$

and

$$\chi_2(\vec{a}, \vec{c}) = (f(a_1, c_1), f(c_1, a_1)).$$

Since we have established that $f(a_1, b_1) \neq f(a_1, c_1)$, then it follows that $\chi_2(\vec{a}, \vec{b}) \neq \chi_2(\vec{a}, \vec{c})$. Hence, $\mathcal{C}(a,b) \neq \mathcal{C}(a,c)$, a contradiction.

Similarly, if $\vec{a} >_4 \vec{b}, \vec{c}$, then

$$\chi_2(\vec{a}, \vec{b}) = (f(b_1, a_1), f(a_1, b_1)),$$

and

$$\chi_2(\vec{a}, \vec{c}) = (f(c_1, a_1), f(a_1, c_1)).$$

Again, since we have established that $f(a_1, b_1) \neq f(a_1, c_1)$, then it follows that $\chi_2(\vec{a}, \vec{b}) \neq \chi_2(\vec{a}, \vec{c})$. Hence, $\mathcal{C}(a,b) \neq \mathcal{C}(a,c)$, a contradiction. □

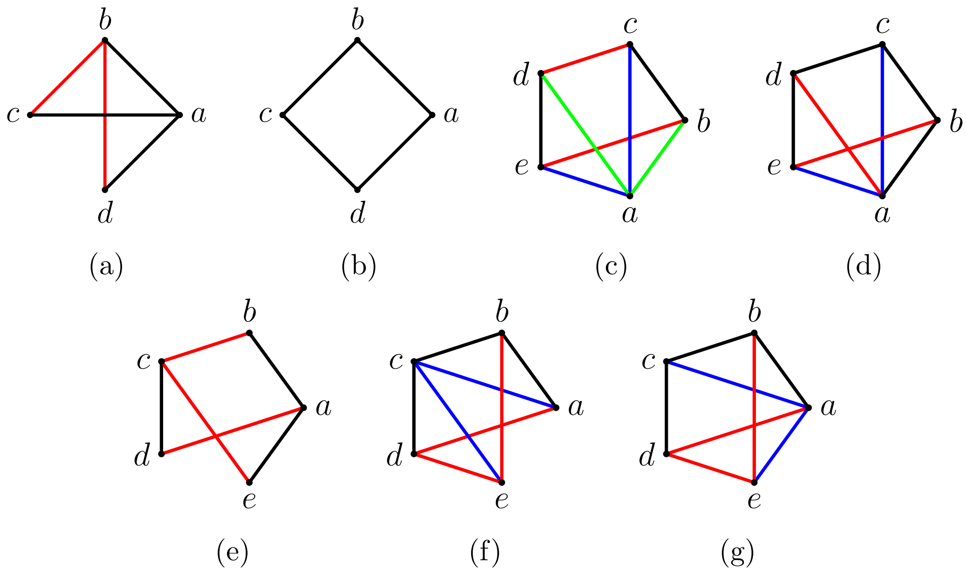


Figure 1: Configurations eliminated by $\mathcal{C}$.

3.2 Avoided Configurations

Lemma 10. *Given four distinct vertices $a, b, c, d \in V(K_n)$, it is not possible for both*

$$\mathcal{C}(a, b) = \mathcal{C}(a, c) = \mathcal{C}(a, d)$$

and $\mathcal{C}(b, c) = \mathcal{C}(b, d)$ as in Figure 1 (A).

Proof. Assume, towards a contradiction, that there exist $a, b, c, d \in V(K_n)$ such that $\mathcal{C}(a, b) = \mathcal{C}(a, c) = \mathcal{C}(a, d)$ and $\mathcal{C}(b, c) = \mathcal{C}(b, d)$. By Lemma 8 we know that $a_1 = b_1$. Hence, $\chi_2(\vec{a}, \vec{b}) = 0$ and so $\chi_2(\vec{a}, \vec{c}) = \chi_2(\vec{a}, \vec{d}) = 0$ as well. Hence, $c_1 = a_1 = d_1$. However, we know by Lemma 7 that $c_1 \neq d_1$, a contradiction. $\square$

Lemma 11. *Given four distinct vertices $a, b, c, d \in V(K_n)$, it is not possible for*

$$\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d) = \mathcal{C}(d, a)$$

as in Figure 1 (B).

Proof. Assume, towards a contradiction, that there exist $a, b, c, d \in V(K_n)$ such that

$$\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d) = \mathcal{C}(d, a).$$

By Lemma 7 we know that $a_1 \neq c_1$ but by Lemma 8 we get that $a_1 = c_1$, a contradiction. $\square$

Lemma 12. *Given five distinct vertices $a, b, c, d, e \in V(K_n)$, it is not possible for $\mathcal{C}(a, b) = \mathcal{C}(a, d)$, $\mathcal{C}(a, c) = \mathcal{C}(a, e)$, $\mathcal{C}(b, c) = \mathcal{C}(d, e)$, and $\mathcal{C}(c, d) = \mathcal{C}(e, b)$ as in Figure 1 (C).*

Proof. Let $a, b, c, d, e \in V(K_n)$ and assume towards a contradiction that $\mathcal{C}(a, b) = \mathcal{C}(a, d)$, $\mathcal{C}(a, c) = \mathcal{C}(a, e)$, $\mathcal{C}(b, c) = \mathcal{C}(d, e)$, and $\mathcal{C}(c, d) = \mathcal{C}(e, b)$. Then it follows that $\chi_1(\vec{a}, \vec{b}) = \chi_1(\vec{a}, \vec{d})$, $\chi_1(\vec{a}, \vec{c}) = \chi_1(\vec{a}, \vec{e})$, $\chi_1(\vec{b}, \vec{c}) = \chi_1(\vec{d}, \vec{e})$, and $\chi_1(\vec{c}, \vec{d}) = \chi_1(\vec{e}, \vec{b})$. Therefore,

$$\begin{aligned}
b_1c_1 - b_2 - c_2 &= d_1e_1 - d_2 - e_2 \\
e_1b_1 - e_2 - b_2 &= c_1d_1 - c_2 - d_2 \\
a_1c_1 - a_2 - c_2 &= a_1e_1 - a_2 - e_2.
\end{aligned}$$

By taking the difference of the first two equations and rearranging the third we get that

$$\begin{aligned}
(b_1 + d_1)(c_1 - e_1) &= 2(c_2 - e_2) \\
a_1(c_1 - e_1) &= c_2 - e_2.
\end{aligned}$$

We can then substitute $c_2 - e_2$ in the first equation with $a_1(c_1 - e_1)$ to get

$$(b_1 + d_1)(c_1 - e_1) = 2a_1(c_1 - e_1).$$

Now, Lemma 7 tells us that $c_1 \neq e_1$. Thus,

$$b_1 + d_1 = 2a_1.$$

Note that if either $b_1 = a_1$ or $d_1 = a_1$, then $\chi_2(\vec{a}, \vec{b}) = \chi_2(\vec{a}, \vec{d})$ implies that both must be true. But then $b_1 = d_1$, a contradiction of Lemma 7. Thus, we must assume that $b_1 \neq a_1$ and $d_1 \neq a_1$. Therefore, the fact that $b_1 + d_1 = 2a_1$ implies that $\mathcal{C}(a, b) \neq \mathcal{C}(a, d)$ by Lemma 9, a contradiction. □

Lemma 13. *Given five distinct vertices $a, b, c, d, e \in V(K_n)$, it is not possible for*

$$\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d) = \mathcal{C}(d, e),$$

$\mathcal{C}(a, c) = \mathcal{C}(a, e)$, and $\mathcal{C}(a, d) = \mathcal{C}(b, e)$ as in Figure 1 (D).

Proof. Assume, towards a contradiction, that there exist five distinct vertices $a, b, c, d, e \in V(K_n)$ such that

$$\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d) = \mathcal{C}(d, e),$$

$\mathcal{C}(a, c) = \mathcal{C}(a, e)$, and $\mathcal{C}(a, d) = \mathcal{C}(b, e)$. We know that $\chi_1(\vec{a}, \vec{b}) = \chi_1(\vec{d}, \vec{e})$, $\chi_1(\vec{a}, \vec{d}) = \chi_1(\vec{b}, \vec{e})$, and $\chi_1(\vec{b}, \vec{c}) = \chi_1(\vec{c}, \vec{d})$ which implies that

$$\begin{aligned}
a_1b_1 - a_2 - b_2 &= d_1e_1 - d_2 - e_2 \\
a_1d_1 - a_2 - d_2 &= b_1e_1 - b_2 - e_2 \\
b_1c_1 - b_2 - c_2 &= c_1d_1 - c_2 - d_2
\end{aligned}$$

By taking the difference of the first two equations and rearranging the third we get that

$$\begin{aligned}
(a_1 + e_1)(b_1 - d_1) &= 2(b_2 - d_2) \\
c_1(b_1 - d_1) &= b_2 - d_2.
\end{aligned}$$

We can then substitute $b_2 - d_2$ in the first equation with $c_1(b_1 - d_1)$ to get

$$(a_1 + e_1)(b_1 - d_1) = 2c_1(b_1 - d_1).$$

Now, Lemma 7 tells us that $b_1 \neq d_1$. Thus, $a_1 + e_1 = 2c_1$.

By Lemma 8, we know that $a_1 = d_1$. Hence, $\mathcal{C}(a, d) = \mathcal{C}(b, e)$ implies that

$$\chi_2(\vec{b}, \vec{e}) = \chi_2(\vec{a}, \vec{d}) = 0.$$

Therefore, $b_1 = e_1$. Hence, $b_1 + d_1 = 2c_1$. Note that if either $b_1 = c_1$ or $d_1 = c_1$, then $\chi_2(\vec{b}, \vec{c}) = \chi_2(\vec{c}, \vec{d})$ implies that both must be true. But then $b_1 = d_1$, a contradiction of Lemma 7. Thus, we must assume that $b_1 \neq c_1$ and $d_1 \neq c_1$. Therefore, the fact that $b_1 + d_1 = 2c_1$ implies that $\mathcal{C}(b, c) \neq \mathcal{C}(c, d)$ by Lemma 9, a contradiction. ☐

Lemma 14. *Given five distinct vertices $a, b, c, d, e \in V(K_n)$, it is not possible for $\mathcal{C}(a, b) = \mathcal{C}(c, d) = \mathcal{C}(a, e)$ and $\mathcal{C}(b, c) = \mathcal{C}(d, a) = \mathcal{C}(c, e)$ as in Figure 1 (E).*

Proof. Assume, towards a contradiction, that $a, b, c, d, e \in V(K_n)$ are five distinct vertices such that $\mathcal{C}(a, b) = \mathcal{C}(c, d) = \mathcal{C}(a, e)$ and $\mathcal{C}(b, c) = \mathcal{C}(d, a) = \mathcal{C}(c, e)$. By Lemma 8 we know that $a_1 = c_1$. We allso know that since $\chi_1(\vec{a}, \vec{b}) = \chi_1(\vec{c}, \vec{d})$ and $\chi_1(\vec{b}, \vec{c}) = \chi_1(\vec{d}, \vec{a})$, then

$$\begin{aligned} a_1 b_1 - a_2 - b_2 &= c_1 d_1 - c_2 - d_2 \\ b_1 c_1 - b_2 - c_2 &= d_1 a_1 - d_2 - a_2. \end{aligned}$$

Using the fact that $a_1 = c_1$, the difference of these two equations gives us that $c_2 - a_2 = a_2 - c_2$. Hence, $2(c_2 - a_2) = 0$ which implies that $a_2 = c_2$ since we are working in a finite field of odd characteristic. Thus, $\vec{a} = \vec{c}$, a contradiction. ☐

Lemma 15. *Given five distinct vertices $a, b, c, d, e \in V(K_n)$, it is not possible for $\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d)$, $\mathcal{C}(a, d) = \mathcal{C}(d, e) = \mathcal{C}(e, b)$, and $\mathcal{C}(a, c) = \mathcal{C}(c, e)$ as in Figure 1 (F).*

Proof. Assume, towards a contradiction, that $a, b, c, d, e \in V(K_n)$ are five distinct vertices such that $\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d)$, $\mathcal{C}(a, d) = \mathcal{C}(d, e) = \mathcal{C}(e, b)$, and $\mathcal{C}(a, c) = \mathcal{C}(c, e)$. By Lemma 7 we know that $b_1 \neq d_1$. By Lemma 8 we know that $d_1 = c_1$. Hence,

$$\chi_2(\vec{b}, \vec{c}) = \chi_2(\vec{c}, \vec{d}) = 0.$$

Therefore, $b_1 = c_1 = d_1$, a contradiction. ☐

Lemma 16. *Given five distinct vertices $a, b, c, d, e \in V(K_n)$, it is not possible for $\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d)$, $\mathcal{C}(a, d) = \mathcal{C}(d, e) = \mathcal{C}(e, b)$, and $\mathcal{C}(a, c) = \mathcal{C}(a, e)$ as in Figure 1 (G).*

Proof. Assume, towards a contradiction, that $a, b, c, d, e \in V(K_n)$ are five distinct vertices such that $\mathcal{C}(a, b) = \mathcal{C}(b, c) = \mathcal{C}(c, d)$, $\mathcal{C}(a, d) = \mathcal{C}(d, e) = \mathcal{C}(e, b)$, and $\mathcal{C}(a, c) = \mathcal{C}(a, e)$. By Lemma 7 we know that $c_1 \neq e_1$. However, by Lemma 8 we know that $c_1 = e_1$, a contradiction. ☐

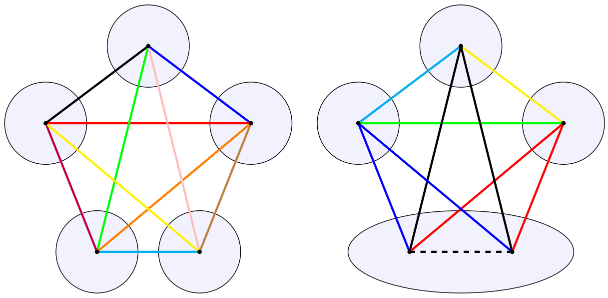


Figure 2: S with five or four equivalence classes under $\sim$. In (A), each edge gets its own distinct color. In (B), there are at least 6 "outside" colors between the equivalence classes plus one "inside" color denoted by the dashed line.

4 Proof of Theorem

Consider any set $S \subseteq V(K_n)$ of five distinct vertices. We will show that these vertices span at least six distinct edge colors under $\mathcal{C}$. Let

$$i = \min\{\varphi_1(\hat{x}, \hat{y}) : x, y \in S \text{ and } x \neq y\},$$

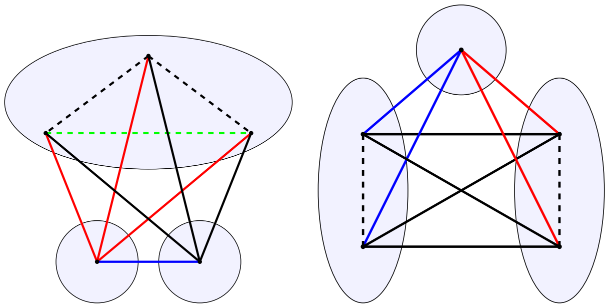
and define an equivalence relation on S by $x \sim y$ if and only if $x^{(i)} = y^{(i)}$. There are at least two and at most five equivalence classes under $\sim$. Note that the set of colors of edges which go in between two vertices in two different classes must be disjoint from the set of colors of edges which go in between vertices in the same class since $\varphi_1(\hat{x}, \hat{y}) = i$ if x and y are in different classes and $\varphi_1(\hat{x}, \hat{y}) > i$ if x and y belong to the same class. Moreover, the set of colors of the edges between any particular pair of classes must be disjoint from the set of colors of edges between a different pair of classes since $\varphi_2(\hat{x}, \hat{y}) = \{x^{(i)}, y^{(i)}\}$ for any pair of vertices x and y in two different classes of S.

That is, if there are five different classes under the relation, then every pair of vertices x, y will yield a unique two-set $\{x^{(i)}, y^{(i)}\}$ and therefore, a unique $\varphi_2(\hat{x}, \hat{y})$. Hence, five equivalence classes of S means that this clique has ten different colors (Figure 2 (A)). Similarly, if there are four different equivalence classes of S, then there must be at least six different colors of edges that go in between classes plus the color of the edge contained within one of the classes giving at least seven edge colors overall (Figure 2 (B)).

We will now consider the cases where S has three or two equivalence classes. For convenience, we will refer to colors on edges that go in between two different classes as *outside* colors and those that are on edges inside a class as *inside* colors.

4.1 Three equivalence classes

If S has three classes, then either they were partitioned so that one class contained three vertices and the other two contained one vertex each or so that two of the classes each contained two vertices and the third class contained one. We will call these different



(a) (b)

Figure 3: S with three equivalence classes under $\sim$.

ways the five vertices could be partitioned into three classes as a $(3, 1, 1)$-partition and a $(2, 2, 1)$-partition respectively. In either case, there are at least three outside colors.

In a $(3, 1, 1)$-partition (Figure 3 (A)), there are at least two inside colors by Lemma 5. If there are three inside colors, the there are at least six colors overall. Otherwise, there are exactly two inside colors so by Lemma 10 there must be at least two distinct outside colors between the class with three vertices and either of the other classes. Therefore, there would be at least seven colors overall.

In a $(2, 2, 1)$-partition (Figure 3 (B)), there must be at least two distinct outside colors in between the two classes with two vertices by Lemma 11. Therefore, there are at least four outside colors and at least one inside color. If the two inside edges have distinct colors, then this gives at least six colors overall. By Lemma 6 if the two inside edges have the same color, then there are at least three distinct outside colors between them which again yields at least six distinct colors overall.

Finally, we consider the case when S has two equivalence classes. This can either split the vertices into one class of four and one part of one, a $(4, 1)$-partition, or into one part of three vertices and the other part of two, a $(3, 2)$-partition. We will consider these two cases in turn.

4.2 A $(4, 1)$-partition

First, note that there is at least one outside color. Let T be the class of four vertices. Let

$$j = \min\{\varphi_1(\hat{x}, \hat{y}) : x, y \in T \text{ and } x \neq y\},$$

and define an equivalence relation on T by $x \sim' y$ if and only if $x^{(j)} = y^{(j)}$. There are either four, three, or two equivalence classes in T under $\sim'$. If there are four such classes (Figure 4 (A)), then there are six inside colors and at least seven colors overall. If there are three classes (Figure 4 (B)), then there are at least four inside colors. If there are exactly four, then the outside edges must have at least two distinct colors by Lemma 10. Either way there must be at least six colors overall.

If T has two classes under $\sim'$, then it is either a $(3, 1)$-partition or a $(2, 2)$-partition. In either case, if there are at least five inside colors, then there are at least six colors overall

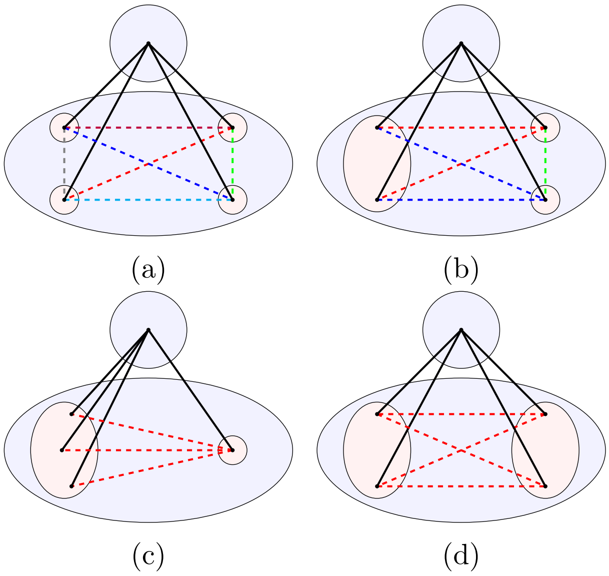


Figure 4: S with two equivalence classes under $\sim$ that give a $(4,1)$-partition of the vertices.

so we will assume there are at most four inside colors. In the case of a $(3,1)$-partition inside T (Figure 4 (C)), Lemma 5 implies that there are either three or two colors within the equivalence class of $\sim'$ that contains three vertices. If there are three, then to stay under four total inside colors, all of the edges between the two classes of $\sim'$ must all have the same color, but then the outside edges must have at least two different colors by Lemma 10 giving at least six colors overall. Otherwise, there are only two colors inside the class of three vertices. So by Lemma 10, the other inside edges must have at least two distinct colors and the outside edges must as well. This also gives at least six colors overall.

Next, consider a $(2,2)$-partition inside of T (Figure 4 (D)). Either the two edges contained completely inside of the equivalence classes of $\sim'$ are the same color or they are different. If they are the same, then by Lemma 6, there are at least four total inside colors. If there are five inside colors, then there are at least six overall. If there are exactly four inside colors, then again by Lemma 6, there is an alternating C_4 inside. Hence, by Lemma 12, there are at least two outside colors and therefore, at least six colors overall.

If the two inner-most edges are different colors, then there must be two additional inside edge colors by Lemma 11. So there are at least four inside colors. If there are at least five inside colors, then there are at least six colors overall. If there are exactly four inside colors, then either two edges with the same inside color are adjacent which would force at least two different outside colors by Lemma 10 or the inside colors between the two equivalence classes of $\sim'$ form an alternating C_4 which implies that there must be at least two outside colors by Lemma 12. In either case, there are six colors overall.

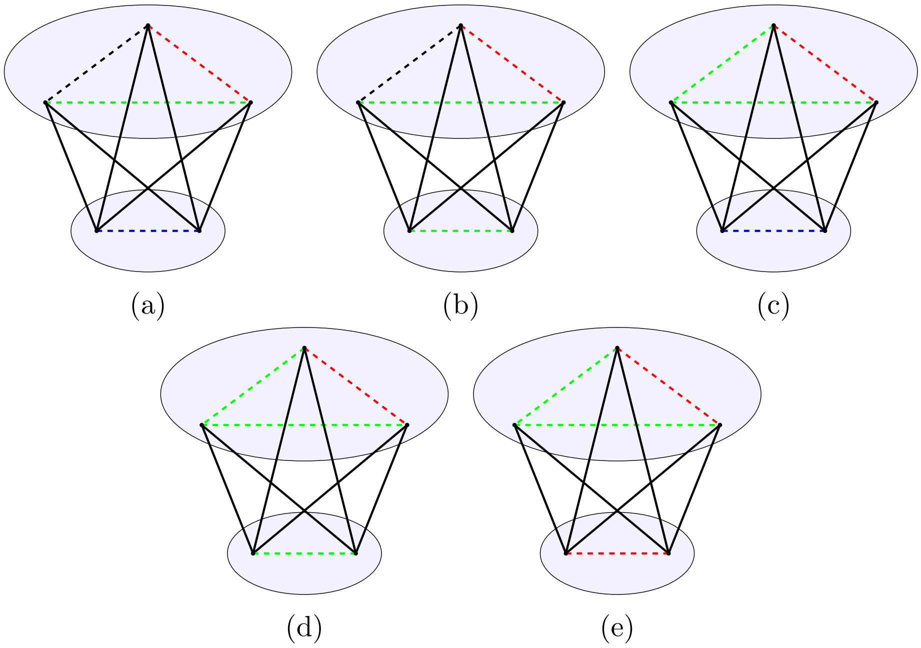


Figure 5: S with two equivalence classes under $\sim$ that give a $(3, 2)$-partition of the vertices.

4.3 A $(3, 2)$-partition

Now assume that $\sim$ yields a $(3, 2)$-partition of S. Lemma 11 implies that there are at least two outside colors. The number of inside edges and Lemma 5 tell us that there are between two and four inside colors. If there are four inside colors (Figure 5 (A)), then we immediately get at least six colors overall. If there are three inside colors and the color of the edge inside the equivalence class with two vertices is repeated on some edge in the equivalence class with three vertices (Figure 5 (B)), then by Lemma 6 we know that there are at least three outside colors and therefore, at least six overall.

Otherwise, three inside colors implies that the equivalence class with three vertices contains two edges with the same color. If the outside edges only have two distinct colors (Figure 5 (C)), then by Lemma 10, each vertex in the class with two vertices must be adjacent to both outside colors to the other three vertices. Therefore, each of these vertices has two outside edges of one color and one of the other color. If both vertices have the same color for their same-colored outside edges, then Lemma 11 implies that these edges cannot all go to the same pair of vertices in the class with three vertices. Hence, the five vertices either create the configuration eliminated by Lemma 13 (one pair goes to the vertex on either side of the non-repeated color in the class with three vertices and the other does not) or the one eliminated by Lemma 12 (neither pair goes to the non-repeated color in the class with three vertices).

Therefore, we must assume that the outside color on two edges for one of these two vertices must only be used once for the other and vice versa. If the two edges of the repeated colors from each vertex go to the same pair of vertices in the class with three vertices (either to an edge in the repeated color or to the edge with the non-repeated color), then the five vertices have created the configuration eliminated by Lemma 14.

So it must be the case that the two edges in the same color from the first vertex go to a different pair in the class with three vertices than the two edges in the same color from the second vertex go to. Either each pair of edges goes to a pair of vertices whose edge is the repeated inside color or only of them does and the other pair goes to vertices in the class with three vertices whose edge has the non-repeated color. In the first case, the vertices create the configuration eliminated by Lemma 16. In the other case, the vertices create the configuration eliminated by Lemma 15. Thus, we have exhausted all possible ways to have three inside colors and only two outside colors. So three inside colors implies that there are at least three outside colors, giving six colors overall.

Finally, if there are only two inside colors, then the edge in the class with two vertices must share a color with an edge in the other class so by Lemma 6, there must be at least three outside colors. If there are four or more outside colors, then there are at least six colors overall. So let's assume that there are exactly three outside colors. If the edge in the class with two vertices repeats the color that is repeated inside the other class (Figure 5 (D)), then by Lemma 6, the five vertices yield the configuration eliminated by Lemma 14. So this cannot happen.

Otherwise, the edge in the class with two vertices repeats the color that appears only once inside the other class (Figure 5 (E)). That is, let the vertices involved be a and b in the class with two vertices and c, d, and e in the class of three vertices such that $\mathcal{C}(a,b) = \mathcal{C}(d,e)$ and $\mathcal{C}(c,d) = \mathcal{C}(c,e)$. By Lemma 6 we know that there are at least three outside colors between the edge ab and the edge de. If there are four such outside colors, then there are six overall.

Let $\varphi_1(\hat{a},\hat{b}) = \varphi_1(\hat{d},\hat{e}) = k$. Moreover, we may assume that $a^{(k)} = d^{(k)}$ and $b^{(k)} = e^{(k)}$ since $\varphi_2(\hat{a},\hat{b}) = \varphi_2(\hat{d},\hat{e}) = \{d^{(k)}, e^{(k)}\}$. This implies that $\varphi_{3,k,1}(\hat{a},\hat{e}) = \varphi_{3,k,1}(\hat{b},\hat{d})$ since in both cases this gives the index of the first bit where $d^{(k)}$ differs from $e^{(k)}$.

Now, note that if $c^{(k)} = d^{(k)}$, then $\varphi_{3,k}(\hat{c},\hat{d}) = 0$ and $\varphi_{3,k}(\hat{c},\hat{e}) \neq 0$, a contradiction since we assume that $\mathcal{C}(c,d) = \mathcal{C}(c,e)$. So $c^{(k)} \neq d^{(k)}$, and, for the same reason, $c^{(k)} \neq e^{(k)}$. Moreover, $\mathcal{C}(c,d) = \mathcal{C}(c,e)$ implies that $\varphi_{3,k,1}(\hat{c},\hat{d}) = \varphi_{3,k,1}(\hat{c},\hat{e})$. This, in turn, implies that the first bit where $c^{(k)}$ differs from $d^{(k)}$ is the same as the first bit where $c^{(k)}$ differs from $e^{(k)}$. Since a bit is either zero or one, then $d^{(k)}$ must agree with $e^{(k)}$ at the bit given by $\varphi_{3,k,1}(\hat{c},\hat{d}) = \varphi_{3,k,1}(\hat{c},\hat{e})$.

Thus,

$$\varphi_{3,k,1}(\hat{c},\hat{a}) \neq \varphi_{3,k,1}(\hat{a},\hat{e}), \varphi_{3,k,1}(\hat{b},\hat{d}).$$

So there must be at least four outside colors, and therefore, at least six colors overall.

5 Conclusion

The result of this paper, when combined with the results of [1, 2, 4, 7], leaves $q = 7$ as the only remaining case for which the order of magnitude gap between the upper and lower bounds of $f(n, 5, q)$ is some power of n. In this case, Lemma 3 and the fact that

$f(n,4,6) = \binom{n}{2}$ gives a lower bound of

$$f(n,5,7) = \Omega(n^{2/3}),$$

and the general probabilistic upper bound from [7] gives

$$f(n,5,7) = O(n^{3/4}).$$

It would be interesting to see if, as here and in [2, 10], some kind of combination of a coloring using vectors over a finite field with the coloring developed in [3] could yield a better upper bound of $n^{2/3+o(1)}$ for that case. The following list gives a *rough* picture of the best known upper and lower bounds on the Erdös-Gyárfás function for the non-trivial cases when $p = 5$.

$$\begin{aligned}
\log n/\log\log n \leqslant & f(n,5,2) \leqslant \log n \\
\log n \leqslant & f(n,5,3) \leqslant n^{o(1)} \\
\log n \leqslant & f(n,5,4) \leqslant n^{o(1)} \\
n^{1/3} \leqslant & f(n,5,5) \leqslant n^{1/3+o(1)} \\
n^{1/2} \leqslant & f(n,5,6) \leqslant n^{1/2+o(1)} \\
n^{2/3} \leqslant & f(n,5,7) \leqslant n^{3/4} \\
& f(n,5,8) = \Theta(n) \\
n \leqslant & f(n,5,9) \leqslant n^{1+o(1)}
\end{aligned}$$

Also of interest would be to extend the construction presented here to larger values of p. For instance, using Lemma 3 we get a lower bound for $f(n,6,9)$ on the order of $n^{1/2}$. Currently, $\mathcal{C}$ is not a $(6,9)$-coloring as far as I can tell, but perhaps it could be after some modifications are made.

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