Position sequences and a q-analogue for the modular hook length formula

Anthony Mendes
aamendes@calpoly.edu

Sam Lindbloom-Airey
slindblo@uw.edu

Department of Mathematics
California Polytechnic State University
San Luis Obispo, CA, 93407 U.S.A.

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Abstract

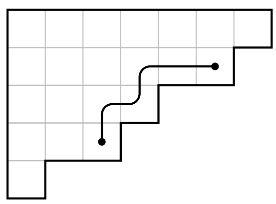
We prove a q-analogue of the modular hook length formula using position sequences. These position sequences, which correspond to moving the beads in a mathematical abacus, provide a new combinatorial interpretation for the characters of the irreducible representations of the symmetric group.

Mathematics Subject Classifications: 05E05, 05E10, 20C30

1 Introduction

Let n and k be positive integers and let λ and $\mu = (\mu_1, \mu_2, \dots)$ be integer partitions of n. A rim hook of length k is a sequence of k connected cells in the (English) Young diagram for λ that begins in a cell on the southeast boundary and travels up along the southeast edge such that its removal leaves the Young diagram of a smaller integer partition.

The sign of a rim hook ρ is $(-1)^{(\text{the number of rows spanned by } \rho)-1}$. For example, below is a rim hook of length 6 with sign $(-1)^{3-1} = +1$ inside of the Young diagram of the integer partition $(7, 6, 4, 3, 1)$:

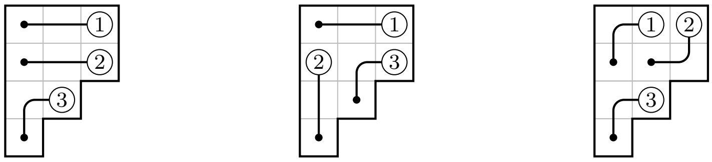


A rim hook tableau of shape λ and content μ is a filling of the cells of the Young diagram of λ with rim hooks of lengths $\mu_1, \mu_2, \dots$ labeled with $1, 2, \dots$ such that the removal of the last i rim hooks leaves the Young diagram of a smaller integer partition for all i. Let RHT^λ_μ be the set of all rim hook tableaux of shape λ and content μ.

The sign of a rim hook tableau T is the product of all of the signs of the rim hooks in T. We let

$$\chi^\lambda_\mu = \sum_{T \in RHT^\lambda_\mu} \operatorname{sign} T.$$

For example, all rim hook tableaux of shape $(3,3,2,1)$ and content $(3,3,3)$ are:



These three rim hook tableaux have sign -1 and so $\chi^{(3,3,2,1)}_{(3,3,3)} = -3$.

The numbers χ^λ_μ are of significant interest because they give

1. the value of the irreducible character of S_n indexed by λ on C_μ where C_μ denotes the conjugacy class containing the permutations with cycle type μ,

2. the coefficient of the Schur symmetric function s_λ in the power symmetric function p_μ, and

3. the coefficient of $|C_\mu| p_\mu$ in $n! s_\lambda$.

As such, rim hook tableaux have been extensively studied and can be found in most treatments of the representation theory of the symmetric group S_n and symmetric functions (see, for instance, [11, 14, 9]).

In the special case of $\mu = (1, \ldots, 1)$, rim hook tableaux of shape λ and type μ are standard tableaux and the number $\chi^\lambda_{(1,\ldots,1)}$ can be found using the hook length formula.

Theorem 1 (The hook length formula)**.** *If λ is an integer partition of n, then*

$$\chi^\lambda_{(1,\ldots,1)} = \frac{n!}{\prod_{c \in \lambda} h_c}$$

where the notation $c \in \lambda$ means that c is a cell in the Young diagram of λ and the hook length h_c is the length of the rim hook that begins in the same column as c and ends in the same row as c.

The hook length formula is a true crown jewel of enumerative combinatorics. Originally proved by Frame, Robinson, and Thrall [2], there are now a panoply of beautiful proofs (see final remark 10.3 and the references in [10]).

The hook length formula has been generalized in two different ways, the first of which involves the major index of a standard tableau. If $p = p_1 \cdots p_\ell$ is any sequence of integers, the major index of p, denoted $\operatorname{maj} p$, is equal to $\sum i$ where the sum runs over all indices

i such that $p_i > p_{i+1}$. Adapting this idea for standard tableaux, if $T \in RHT^{\lambda}_{(1,\ldots,1)}$ is a standard tableau, then the integer i is a descent in T if rim hook i appears in a row above that of rim hook $i+1$. The major index of T, denoted $\operatorname{maj} T$, is equal to $\sum i$ where the sum runs over all descents i in T.

For example, the descents in

1	3	4
2	6	8
5	7	
9		

are 1, 4, 6 and 8, and so the major index is 19.

The first generalization of the hook length formula is the q-hook length formula. It first appeared in [13] and was later proved using the elegant Hillman-Grassl algorithm [3].

Theorem 2 (The ***q***-hook length formula)**.** *If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is an integer partition of n, then*

$$\sum_{T \in RHT^{\lambda}_{(1,\ldots,1)}} q^{\operatorname{maj} T - (0\lambda_1 + 1\lambda_2 + \cdots)} = \frac{[n]_q!}{\prod_{c \in \lambda} [h_c]_q}$$

where $[n]_q = q^0 + \cdots + q^{n-1}$ and $[n]_q! = [n]_q \cdots [1]_q$ are the usual q-analogues of n and $n!$.

The second generalization of the hook length formula involves rim hook tableaux of shape λ and content $(k, \ldots, k)$. These rim hook tableaux are useful in the modular representation theory of the symmetric group and can be used to generalize the Robinson-Schensted-Knuth (RSK) algorithm [15].

All rim hook tableaux of shape λ with content $(k, \ldots, k)$ have the same sign (this is implied by (2.7.26) in [4]) and so $\left|\chi^{\lambda}_{(k,\ldots,k)}\right|$ is the number of rim hook tableaux of shape λ and type $(k, \ldots, k)$. The value of this quantity is given by the modular hook length formula, first proved in [1]. The modular hook length formula is less well known than Theorem 1 but is beginning to receive the attention it deserves [16].

Theorem 3 (The modular hook length formula)**.** *If λ is an integer partition of n such that $\chi^{\lambda}_{(k,\ldots,k)} \neq 0$, then*

$$\left|\chi^{\lambda}_{(k,\ldots,k)}\right| = \frac{(n/k)!}{\prod h_c/k}$$

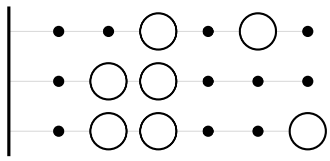
where the product is over all cells $c \in \lambda$ with h_c divisible by k.

The main result in this paper, Theorem 13, synthesizes the generalizations of the hook length formula in Theorems 2 and 3 to provide a q-analogue for the modular hook length formula. In order to prove Theorem 13 we introduce the concept of a position sequence. Position sequences are sequences of integers created from recording bead moves in a mathematical abacus. They provide a natural framework in which to understand rim hook tableaux, especially when interested in q-analogues.

The outline of the paper is as follows. In section 2 we introduce position sequences, the main tool needed to prove the q-modular hook length formula. Interesting connections with the RSK algorithm are made, a position sequence version of Theorem 2 is given, and we find a q-analogue for the entire character table for the symmetric group in section 2. Section 3 contains our proof of the q-modular hook length formula.

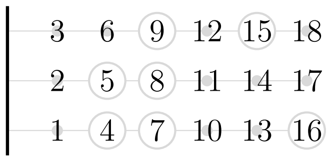
2 Position sequences

A k-abacus consists of k runners, each of which is a sequence of beads and empty places. For example, this is a 3-abacus with 7 beads:



(1)

Starting in the bottom left corner, label the beads and the empty places in the abacus with the integers $1, 2, \ldots$ by moving up each column, working left to right. For example, the above abacus is numbered



and there are beads at positions $4, 5, 7, 8, 9, 15$ and 16.

Each k-abacus A represents an integer partition. Let $b_1, \ldots, b_\ell$ be the labels of the beads on a k-abacus and let $\text{empty}(b_i)$ be the number of empty places with a label smaller than b_i. Then the integer partition corresponding to A is

$$\lambda_A = (\text{empty}(b_\ell), \ldots, \text{empty}(b_1)) .$$

For example, if A is the 3-abacus above, then $\lambda_A = (9, 9, 4, 4, 4, 3, 3)$.

Moving a bead in a k-abacus A from position i to an empty place in position $i - j$ removes a rim hook of length j from λ_A. When $j = k$ this move sends a bead one step left on its runner. The sign of the removed rim hook is $(-1)^b$ where b is the number of beads in positions between i and $i - j$ (see Section 2.7 in [4]). These facts have been used to great effect in proving classic results from symmetric function theory using abaci [6, 7].

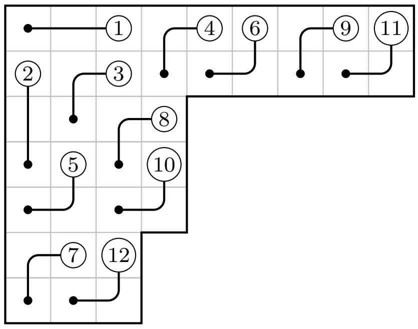
Therefore a rim hook tableau T of shape λ and content $\mu = (\mu_1, \ldots, \mu_\ell)$ can be interpreted as a sequence of bead moves in a k-abacus A such that the i^{th} bead move moves a bead in position j for some j to an empty place in position $j - \mu_{\ell+1-i}$ for $i = 1, \ldots, \ell$. The beads in A will be pushed as far as possible to the left after all of the moves. The sign of T is the product of the signs of the bead moves.

We record such a sequence of bead moves with a position sequence. A position sequence $p = p_1 \cdots p_\ell$ is the sequence of integers defined such that p_i is the empty position filled by a bead on bead move i. If A is a k-abacus with $\lambda_A = \lambda$, we let PS_μ^λ be the set of position sequences with bead moves of lengths given by μ. It follows that PS_μ^λ and RHT_μ^λ have the same number of elements.

For example, one position sequence when $\lambda = (9, 9, 4, 4, 4, 3, 3)$ and $\mu = (3, \ldots, 3)$ is

$$2 \;\; 13 \;\; 5 \;\; 12 \;\; 6 \;\; 1 \;\; 10 \;\; 3 \;\; 9 \;\; 6 \;\; 4 \;\; 7. \tag{2}$$

Starting with the 3-abacus displayed above, this position sequence says to move the bead in position 5 into position 2, then move the bead in position 16 to position 13, then move the bead in position 8 into position 5, and so on. The rim hook tableau of shape λ and content μ for this position sequence is:



This rim hook tableau was created by finding λ_A after each bead move and placing a rim hook in the removed cells.

The position sequence in (2) contains the subsequence 12 6 3 9 6. This subsequence comes from moving beads within the 3^{rd} runner in the 3-abacus (reading bottom to top), and so these numbers are congruent to 3 modulo 3. Furthermore, the position sequence contains the subsequences 6 9 and 3 6 because in order for the rightmost bead on the top runner to move into positions 9 and 6, the leftmost bead on the top runner must already have moved into positions 6 and 3. With this example as a guide we work towards characterizing position sequences in $PS^\lambda_{(k,\ldots,k)}$ by their subsequences.

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition. For each $j = 1, \ldots, \ell$, define I_j to be the sequence created by listing the integers in the interval $[j, \lambda_{\ell-j+1} + j - 1]$ in decreasing order. The sequence I_j gives the positions that the j^{th} bead in the 1-abacus will occupy when moved in a position sequence in $PS^\lambda_{(1,\ldots,1)}$. For example, the corresponding labeled 1-abacus for $\lambda = (4, 4, 3, 1)$ is

and the sequences $I_1, \ldots, I_4$ are

$$I_1 = 1, \qquad I_2 = 4 \;\; 3 \;\; 2, \qquad I_3 = 6 \;\; 5 \;\; 4 \;\; 3, \qquad \text{and} \qquad I_4 = 7 \;\; 6 \;\; 5 \;\; 4. \tag{3}$$

A shuffle of the sequences $I_1, \ldots, I_\ell$ is a sequence created by interleaving $I_1, \ldots, I_\ell$ such that each of $I_1, \ldots, I_\ell$ appears as a subsequence.

Lemma 4. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition. Then p is a position sequence in $PS^{\lambda}_{(1,\ldots,1)}$ if and only if p is a shuffle of $I_1, \ldots, I_\ell$ and each integer m in p that comes from I_j appears after every $m-1$ in p coming from $I_1, \ldots, I_{j-1}$ for all $j = 1, \ldots, \ell$.*

Proof. Suppose p is a position sequence in $PS^{\lambda}_{(1,\ldots,1)}$. Since I_j gives the positions the j^{th} bead moves into in a sequence of bead moves that correspond to a position sequence, each I_j appears as a subsequence in p with the order of the integers in I_j preserved. Therefore p is a shuffle of $I_1, \ldots, I_\ell$.

If a bead is moved into position m in a sequence of bead moves, then each of the beads to its left must already have been moved into in position $m-1$ or smaller. Therefore each integer m in p that comes from I_j must appear after every integer $m-1$ that appears in each of $I_1, \ldots, I_{j-1}$.

Now suppose p is a shuffle of $I_1, \ldots, I_\ell$ satisfying the condition in the statement of the theorem. The subsequence I_j in p represents moving the j^{th} bead from its starting position to its final position, and condition in the statement of the theorem guarantees that position m will be empty at the time when bead j is moved into position m. Therefore p represents a sequence of bead moves and so $p \in PS^{\lambda}_{(1,\ldots,1)}$. □

It will be convenient to break the k-abacus into k instances of 1-abaci. Let $\lambda^{(i)}$ be the integer partition found by considering the i^{th} runner reading bottom to top on the k-abacus as a 1-abacus. For example, $\lambda^{(1)} = (3, 1, 1)$, $\lambda^{(2)} = (1, 1)$, and $\lambda^{(3)} = (3, 2)$ for the abacus in (1).

Theorem 5. *Let λ be an integer partition such that $RHT^{\lambda}_{(k,\ldots,k)}$ is nonempty. Let $PS^{\lambda^{(i)}}_{k}$ be the set of position sequences $p \in PS^{\lambda^{(i)}}_{(1,\ldots,1)}$ with each integer j in p replaced with $kj + i$. Then p is a position sequence in $PS^{\lambda}_{(k,\ldots,k)}$ if and only if p is a shuffle of $p^{(1)}, \ldots, p^{(k)}$ for some $p^{(1)} \in PS^{\lambda^{(1)}}_{k}, \ldots, p^{(k)} \in PS^{\lambda^{(k)}}_{k}$.*

Proof. Suppose $p \in PS^{\lambda}_{(k,\ldots,k)}$ and let $p^{(i)}$ be the subsequence of p consisting of the values in p congruent to i modulo k. Each bead move on the k-abacus moves a bead on a runner one position to the left on the same runner. Therefore bead moves on a single runner must satisfy the conditions in Lemma 4, showing that $p^{(i)} \in PS^{\lambda^{(i)}}_{k}$ for $i = 1, \ldots, \ell$. This shows that p is a shuffle of $p^{(1)}, \ldots, p^{(k)}$ for some $p^{(1)} \in PS^{\lambda^{(1)}}_{k}, \ldots, p^{(k)} \in PS^{\lambda^{(k)}}_{k}$.

Now suppose p is a shuffle of $p^{(1)}, \ldots, p^{(k)}$ for some $p^{(1)} \in PS^{\lambda^{(1)}}_{k}, \ldots, p^{(k)} \in PS^{\lambda^{(k)}}_{k}$. Since bead moves on different runners do not influence each other, it follows from 4 that p corresponds to a sequence of bead moves on the k-abacus and so $p \in PS^{\lambda}_{(k,\ldots,k)}$. □

An increasing run in a sequence of integers is a maximal weakly increasing consecutive subsequence.

Lemma 6. *Let $A_1, \ldots, A_\ell$ be finite sequences of integers. We interpret each of $A_1, \ldots, A_\ell$ as having the same number r of increasing runs by possibly padding the beginning of each of $A_1, \ldots, A_\ell$ with empty increasing runs. Define $\hat{p}$ to be the shuffle of $A_1, \ldots, A_\ell$ with*

r increasing runs such that the i^{th} increasing run in $\hat{p}$ contains the integers in the i^{th} increasing runs in each of $A_1, \ldots, A_\ell$ sorted into increasing order for $i = 1, \ldots, r$. Then

$$\operatorname{maj}\hat{p} = \operatorname{maj}(A_1) + \cdots + \operatorname{maj}(A_\ell)$$

and this $\hat{p}$ is the unique shuffle of $A_1, \ldots, A_\ell$ with the minimum possible major index.

As an example, consider

$$A_1 = 1\ \ 2\ \ 4\ \ 3\ \ 6, \qquad A_2 = 3\ \ 2\ \ 3\ \ 3, \qquad \text{and} \qquad A_3 = 1\ \ 5\ \ 5.$$

Here A_1 and A_2 have 2 nonempty increasing runs and A_3 has 1 nonempty increasing run, so we interpret A_3 as having 2 increasing runs where the first increasing run is empty. Then we see $\hat{p} = 1\ 2\ 3\ 4\ 1\ 2\ 3\ 3\ 3\ 5\ 5\ 6$ and $\operatorname{maj}\hat{p} = 4 = 3{+}1{+}0 = \operatorname{maj} A_1{+}\operatorname{maj} A_2{+}\operatorname{maj} A_3$.

Proof. The assertion is trivially true when each of $A_1, \ldots, A_\ell$ is empty. We proceed by induction on $|A_1| + \cdots + |A_\ell|$.

Let $\widetilde{A_j}$ be A_j with its final increasing run removed. The only descent in A_j that does not appear in $\widetilde{A_j}$ is in position $|A_j| - |\widetilde{A_j}|$ and so

$$\operatorname{maj}\widetilde{A_j} + |A_j| - |\widetilde{A_j}| = \operatorname{maj} A_j \tag{4}$$

for $j = 1, \ldots, \ell$.

Let p be a shuffle of $A_1, \ldots, A_\ell$. There must be a descent in p at position

$$|A_1| + \cdots + |A_\ell| - |\widetilde{A_1}| - \cdots - |\widetilde{A_\ell}| \tag{5}$$

or greater because this is the position where the maximum possible final increasing run in any shuffle of $A_1, \ldots, A_\ell$ begins (this maximum possible final increasing run is created by combining the final increasing runs in $A_1, \ldots, A_\ell$ into one increasing sequence). If we define p' to be p with its final increasing run removed, then this implies

$$\operatorname{maj} p \geqslant |A_1| + \cdots + |A_\ell| - |\widetilde{A_1}| - \cdots - |\widetilde{A_\ell}| + \operatorname{maj} p'$$

where equality is achieved only when the final descent in p occurs in the position in (5).

Let $A'_1, \ldots, A'_\ell$ be $A_1, \ldots, A_\ell$ but with possibly some of their tails trimmed so that p' is a shuffle of $A'_1, \ldots, A'_\ell$. Then $\widetilde{A_j}$ is equal to A'_j but maybe with some final integers deleted. Therefore we have $\operatorname{maj} A'_j \geqslant \operatorname{maj}\widetilde{A_j}$ for each $j = 1, \ldots, \ell$.

The induction hypothesis on p' gives that $\operatorname{maj} p' \geqslant \operatorname{maj} A'_1 + \cdots + \operatorname{maj} A'_\ell$ with equality holding if and only if p' is the unique shuffle of $A'_1, \ldots, A'_\ell$ with the minimum possible major index as described in the statement of the Lemma.

Putting these observations together gives

$$\begin{aligned}
\operatorname{maj} p &\geqslant |A_1| + \cdots + |A_\ell| - |\widetilde{A_1}| - \cdots - |\widetilde{A_\ell}| + \operatorname{maj} p' \\
&\geqslant |A_1| + \cdots + |A_\ell| - |\widetilde{A_1}| - \cdots - |\widetilde{A_\ell}| + \operatorname{maj} A'_1 + \cdots + \operatorname{maj} A'_\ell \\
&\geqslant |A_1| + \cdots + |A_\ell| - |\widetilde{A_1}| - \cdots - |\widetilde{A_\ell}| + \operatorname{maj}\widetilde{A_1} + \cdots + \operatorname{maj}\widetilde{A_\ell} \\
&= \operatorname{maj} A_1 + \cdots + \operatorname{maj} A_\ell
\end{aligned}$$

where the last line used (4).

Equality is achieved in this string of inequalities if and only if there is a descent exactly in the position in (5) (hence $A'_j = \widetilde{A_j}$ for all j) and p' is the unique shuffle of $A'_1, \ldots, A'_\ell$ with the minimum possible major index. In other words, equality is uniquely achieved when the shuffle p is the shuffle $\hat{p}$ as described in the statement of the Lemma. □

Corollary 7. *Let $\hat{p}$ be the position sequence in $PS^{\lambda}_{(1,\ldots,1)}$ with the minimum major index. Then $\hat{p}$ corresponds to finding the rightmost bead b in the 1-abacus that can be moved one position to the left, moving b one position to the left, and then iterating until no more moves can be made.*

Proof. Since I_j has length $\lambda_{\ell-j+1}$, the sequences $I_1, \ldots, I_\ell$ weakly increase in length. Let m be the minimum integer such that the sets $I_m, \ldots, I_\ell$ all have the same length. This means that λ has $\ell - m + 1$ copies of its largest part, and, on the 1-abacus, the sequence of beads and empty places ends with an empty place and then $\ell - m + 1$ consecutive beads.

Let z be the first integer in I_m. If b is the rightmost bead in the 1-abacus that can be moved one position to the left, then the appearance of z in a position sequence corresponds to moving b one position to the left. It remains to be shown that $\hat{p}$ begins with z.

Since each of $I_1, \ldots, I_{m-1}$ has a length less than that of I_m, we begin creating $\hat{p}$ by padding each of $I_1, \ldots, I_{m-1}$ with empty increasing runs. Then z appears first in $\hat{p}$ because z is the minimum integer appearing in the first increasing runs of $I_1, \ldots, I_\ell$. □

Theorem 8. *If $\hat{p}$ is the element in $PS^{\lambda}_{(k,\ldots,k)}$ with the minimum major index, then*

$$\operatorname{maj} \hat{p} = \sum \binom{x}{2}$$

where the sum runs over all parts x in the integer partitions $\lambda^{(1)}, \ldots, \lambda^{(k)}$.

Proof. Suppose $\lambda^{(i)} = (\lambda^{(i)}_1, \ldots, \lambda^{(i)}_{\ell_i})$ and let $p^{(i)}$ be an element of $PS^{\lambda^{(i)}}_k$. By Lemma 4, $p^{(i)}$ has $kI_j + i$ as subsequence for $j = 1, \ldots, \ell_i$. This subsequence has length $\lambda^{(i)}_{\ell_i - j+1}$, so it has major index $1 + 2 + \cdots + (\lambda^{(i)}_{\ell_i-j+1} - 1) = \binom{\lambda^{(i)}_{\ell_i-j+1}}{2}$. Using Lemma 6 on the sequences $kI_1 + i, \ldots, kI_{\ell_i} + i$ gives that the minimum major index over all elements in $PS^{\lambda^{(i)}}_k$ is

$$\binom{\lambda^{(i)}_1}{2} + \cdots + \binom{\lambda^{(i)}_{\ell_i}}{2}. \tag{6}$$

The second condition in Lemma 4 says that m must appear after a certain number of appearances of $m-1$. As can be seen using Corollary 7, this condition is preserved when using Lemma 6. Therefore this minimum major index is actually achieved by an element in $PS^{\lambda^{(i)}}_k$.

By Theorem 5, each element in $PS^{\lambda}_{(k,\ldots,k)}$ is a shuffle of sequences $p^{(1)}, \ldots, p^{(k)}$ with $p^{(i)} \in PS^{\lambda^{(i)}}_k$. Taking each $p^{(i)}$ to be the element with major index given in (6), another application of Lemma 6 completes the proof. □

As an example of Theorem 8, consider the k-abacus in (1). Since $\lambda^{(1)} = (3, 1, 1)$, $\lambda^{(2)} = (1, 1)$, and $\lambda^{(3)} = (3, 2)$, the minimum major index is $\binom{3}{2} + \binom{1}{2} + \cdots + \binom{3}{2} + \binom{2}{2} = 7$. Lemma 6 and the proof of Theorem 8 tell us that the unique position sequence $\hat{p}$ that achieves this minimum is $\hat{p} = 12\ \ 13\ \ 6\ \ 9\ \ 10\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ 6\ \ 7$.

The RSK algorithm can be used to understand monotonic subsequences in words. Lemma 4 characterizes position sequences in terms of decreasing subsequences, and so it may not come as a surprise that there is a relationship between the RSK algorithm and position sequences.

Theorem 9. *The RSK algorithm produces the same insertion tableau P for every position sequence in $PS^{\lambda}_{(1,\ldots,1)}$.*

Proof. We will use Knuth equivalence, defined as follows. Let A and B be finite sequences with integer letters. An elementary Knuth operation on A is one of these two operations or their inverses:

1. If $x\ z\ y$ appears consecutively in A and $x \leqslant y < z$, then the order of these letters is changed to $z\ x\ y$ and the rest of A is left unchanged.

2. If $y\ z\ x$ appears consecutively in A and $x < y \leqslant z$, then the order of these letters is changed to $y\ x\ z$ and the rest of A is left unchanged.

Then A and B are defined to be Knuth equivalent if A can be transformed into B by a sequence of elementary Knuth operations. This is relevant because A and B are Knuth equivalent if and only if the RSK algorithm produces the same P tableau for A and B [5].

Let $\hat{p}$ be the position sequence in $PS^{\lambda}_{(1,\ldots,1)}$ with the minimum major index and let p be any other position sequence in $PS^{\lambda}_{(1,\ldots,1)}$. We will prove the theorem by showing that p and $\hat{p}$ are Knuth equivalent.

The theorem is clearly true when $|\lambda|$ is $0, 1$ or 2 because in these cases there is at most one position sequence in $PS^{\lambda}_{(1,\ldots,1)}$. We will prove the theorem true when the length of p is larger than 2 by induction on $|\lambda|$.

Let z be the first integer in $\hat{p}$. If p also begins with z, then we are done by induction on the remaining portion of p. If not, we will show that z can be moved into the first position of p using a sequence of elementary Knuth operations at which point the theorem again follows by induction on the remaining portion of p. For this it is enough to show that the leftmost appearance of z in p can be moved one position to the left in p by a sequence of elementary Knuth operations.

Since $\hat{p}$ begins with z, position z on the 1-abacus is initially empty and remains empty when performing moves in the order given by p until the move corresponding to the leftmost z. Furthermore, Corollary 7 implies that all integers larger than z in p appear to the right of the leftmost z. Therefore, if we define x to be the integer immediately to the left of the leftmost z in p, then $x + 1 < z$.

Let p' be the sequence of integers appearing to the right of the leftmost z in p. Then p' is a position sequence in $PS^{\widetilde{\lambda}}_{(1,\ldots,1)}$ for some integer partition $\widetilde{\lambda}$ such that $\widetilde{\lambda} \subset \lambda$. Let p'' be a position sequence in $PS^{\widetilde{\lambda}}_{(1,\ldots,1)}$ that begins with an integer y that satisfies $x \leqslant y < z$.

Such a p'' exists because the integer $x+1$ appears to the right of the leftmost z in p (because the bead moves corresponding to the x z in p leave position $x+1$ empty and that empty position must eventually be filled with a bead). Thus one possible bead move that would correspond to an acceptable value of y can be moving the leftmost possible bead to the right of x first.

By induction, RSK produces the same insertion tableau P for every position sequence in $PS^{\widetilde{\lambda}}_{(1,\ldots,1)}$ and therefore all position sequences in $PS^{\widetilde{\lambda}}_{(1,\ldots,1)}$ are Knuth equivalent. In particular, there is a sequence of elementary Knuth operations that turns p' into p''. Applying these same elementary Knuth operations to p yields a position sequence $p''' \in PS^{\lambda}_{(1,\ldots,1)}$ such that p and p''' are the same up until the leftmost z and such that the tail end of p''' is p''.

The sequence p''' now contains the consecutive sequence x z y with $x < y < z$. Using the first elementary Knuth operation, turn this into z x y. This still gives a valid position sequence because z x y corresponds to moving a bead into position z and then moving a bead into position x instead of vice versa. We have now successfully moved the leftmost z one position to the left using a sequence of Knuth operations, as needed. □

For example, the insertion tableau P found when applying the RSK algorithm to any $p \in PS^{\lambda}_{(1,\ldots,1)}$ when $\lambda = (4,4,3,1)$ is

1	2	3	4
3	4	5	
4	5	6	
6	7		

The reading word for a tableau is the word found by reading the rows left to right, bottom to top. The reading word for the above tableau is

$$6\ \ 7\ \ 4\ \ 5\ \ 6\ \ 3\ \ 4\ \ 5\ \ 1\ \ 2\ \ 3\ \ 4.$$

This word is also the position sequence $\hat{p} \in PS^{\lambda}_{(1,\ldots,1)}$ with the minimum major index.

Theorem 10. *Let λ be an integer partition and let P be the insertion tableau found when applying the RSK algorithm to any position sequence in $PS^{\lambda}_{(1,\ldots,1)}$. Then the shape of P is the conjugate partition λ' and the reading word for P is the position sequence $\hat{p} \in PS^{\lambda}_{(1,\ldots,1)}$ with the minimum major index.*

Proof. The position sequence $\hat{p}$ with the minimum major index is the reading word for some standard tableau, say $\hat{P}$. The construction of $\hat{p}$ given in Lemma 6 implies that $\hat{P}$ has shape λ' because the parts of λ' give the lengths of the increasing runs in $\hat{p}$.

Lemma 3.4.5 in [11] says that the insertion tableau P found when applying the RSK algorithm to $\hat{p}$ is also $\hat{P}$, and Theorem 9 says that all position sequences in PS^{λ} have the same insertion tableau P. □

Theorem 11 (The position sequence version of the $\boldsymbol{q}$-hook length formula)**.** *If* $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ *is an integer partition of* n *and* $\hat{p}$ *is the position sequence in* $PS^{\lambda}_{(1,\ldots,1)}$ *with the minimum major index, then*

$$\sum_{p \in PS^{\lambda}_{(1,\ldots,1)}} q^{\operatorname{maj} p - \operatorname{maj} \hat{p}} = \frac{[n]_q!}{\prod_{c\in\lambda}[h_c]_q}.$$

Proof. There are the same number of position sequences in $PS^{\lambda}_{(1,\ldots,1)}$ as there are rim hook tableaux of shape λ and type $(1,\ldots,1)$, which is also the number of rim hook tableaux of shape λ' and type $(1,\ldots,1)$. If P is the tableau of shape λ' with reading word $\hat{p}$ as in Theorem 10, then

$$\{(P,Q) : Q \text{ is a rim hook tableau of shape } \lambda' \text{ and type } (1,\ldots,1)\} \tag{7}$$

has the same size as $PS^{\lambda}_{(1,\ldots,1)}$. Since RSK is a bijection, applying RSK to $PS^{\lambda}_{(1,\ldots,1)}$ produces every element in (7) exactly once.

If p is any word and RSK sends p to (P,Q), then $\operatorname{maj} p = \operatorname{maj} Q$ where $\operatorname{maj} p$ is the major index for sequences and $\operatorname{maj} Q$ is the major index for standard tableaux (see, for instance, [7]). This, combined with Theorem 8, gives

$$\sum_{p \in PS^{\lambda}_{(1,\ldots,1)}} q^{\operatorname{maj} p - \operatorname{maj} \hat{p}} = \sum_{Q \in RHT^{\lambda'}_{(1,\ldots,1)}} q^{\operatorname{maj} Q - \binom{\lambda_1}{2} - \cdots - \binom{\lambda_\ell}{2}}. \tag{8}$$

Let $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ be the conjugate partition to λ. Consider the tableau of shape λ' with $i-1$ occupying every entry in row i, like this example when $\lambda = (4,4,3,1)$:

0	0	0	0
1	1	1	
2	2	2	
3	3		

The sum of column j is $\binom{\lambda_j}{2}$. Summing the entries in this tableau column by column and then row by row, we find $\binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \cdots = 0\lambda'_1 + 1\lambda'_2 + \cdots$. Using this in (8) and then applying Theorem 2 gives

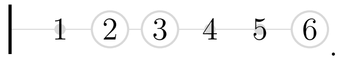
$$\sum_{Q \in RHT^{\lambda'}_{(1,\ldots,1)}} q^{\operatorname{maj} Q - \left(0\lambda'_1 + 1\lambda'_2 + \cdots\right)} = \frac{[n]_q!}{\prod_{c\in\lambda'}[h_c]_q} = \frac{[n]_q!}{\prod_{c\in\lambda}[h_c]_q},$$

as needed. □

Although this paper is most concerned with the situation where $\mu = (k,\ldots,k)$, we can use position sequences to give a q-analogue for χ^{λ}_{μ} when $\mu \neq (k,\ldots,k)$. The sign of a bead move from position i to position $i-j$ is $(-1)^b$ where b is the number of beads in positions between i and $i-j$. We define $\operatorname{sign} p$ to be the product of the signs of all of the bead moves given by the position sequence p. A q-analogue for χ^λ_μ can be defined as

$$\chi^\lambda_{\mu,q} = \sum_{p \in PS^\lambda_\mu} (\operatorname{sign} p)\, q^{\operatorname{maj} p}.$$

For example, consider $\lambda = (3, 1^2)$. The labeled 1-abacus for λ is



We have $\chi^{(3,1^2)}_{(2,1^3),q} = q + q^2 - q^4 - q^5$, found by moving beads of distances $1, 1, 1$ and 2. The relevant position sequences are below:

Position Sequence	Sign	Major Index
5 1 2 3	+1	1
1 5 2 3	+1	2
1 2 5 3	+1	3
5 4 1 2	−1	3
5 1 4 2	−1	4
1 5 4 2	−1	5

Doing this type of calculation as λ and μ range over all integer partitions of 5 gives a q-analogue for the character table for S_5:

	$C_{(1^5)}$	$C_{(2,1^3)}$	$C_{(2^2,1)}$	$C_{(3,1^2)}$	$C_{(3,2)}$	$C_{(4,1)}$	$C_{(5)}$
$\chi^{(5)}$	q^{10}	q^6	q^3	q^3	q	q	1
$\chi^{(4,1)}$	$q^6[4]_q$	$q^3+q^4+q^5-q^6$	q^2-q^3	$q+q^2-q^3$	$-q$	$1-q$	-1
$\chi^{(3,2)}$	$q^4[5]_q$	$q^2+q^3-q^5$	$q-q^2+q^3$	$q-q^2-q^3$	1	$-q$	0
$\chi^{(3,1^2)}$	$q^3\frac{[4]_q[3]_q}{[2]_q}$	$q+q^2-q^4-q^5$	$-q-q^2$	$1-q^2+q^3$	$q-1$	$q-1$	1
$\chi^{(2^2,1)}$	$q^2[5]_q$	$q-q^3-q^4$	$1-q+q^2$	$-q$	$-q$	q	0
$\chi^{(2,1^3)}$	$q[4]_q$	$1-q-q^2-q^3$	$q-1$	$-1+q+q^2$	1	$1-q$	-1
$\chi^{(1^5)}$	1	-1	1	1	-1	-1	1

The χ^λ row and C_μ column entry is $\chi^\lambda_{\mu,q}$. The first column was found using Theorem 11.

If ν is a rearrangement of the parts in μ, then $\chi^\lambda_\mu = \chi^\lambda_\nu$ (see [15, 8]). Unfortunately, the q-analogue $\chi^\lambda_{\mu,q}$ does not enjoy the same property. For example, we have $\chi^{(3,1^2)}_{(1,1,1,2),q} = q - q^5$ because the position sequence 4 1 2 3 with sign $+1$ has major index 1 and the position sequence 1 5 4 3 with sign -1 has major index 5.

3 The q-modular hook length formula

After this next Lemma we will be ready to prove our main result in Theorem 13.

Lemma 12. *If λ is an integer partition and $PS_k^{\lambda^{(i)}}$ is as in Theorem 5, then*

$$\frac{\sum_{p\in PS^{\lambda}_{(k,\dots,k)}} q^{\operatorname{maj} p}}{(1-q)\cdots(1-q^{|\lambda|/k})} = \prod_{i=1}^{k} \frac{\sum_{p^{(i)}\in PS_k^{\lambda^{(i)}}} q^{\operatorname{maj} p^{(i)}}}{(1-q)\cdots\left(1-q^{|\lambda^{(i)}|}\right)}. \tag{9}$$

Proof. The $1/((1-q)\cdots(1-q^{|\lambda|/k}))$ term on the left side of (9) is the generating function for integer partitions with no part larger than $|\lambda|/k$. Conjugating such an integer partition gives an integer partition that has exactly $|\lambda|/k$ parts with parts of size 0 allowed. The length of $p \in PS^{\lambda}_{(k,\dots,k)}$ is $|\lambda|/k$ and so the left side of (9) is equal to

$$\sum q^{\operatorname{maj} p + |\pi|}$$

where the sum runs over all possible $p \in PS^{\lambda}_{(k,\dots,k)}$ and all possible integer partitions π with parts of size 0 allowed such that the lengths of π and p are the same. Similarly, the right side of (9) is equal to

$$\sum q^{\operatorname{maj} p^{(1)} + \cdots + \operatorname{maj} p^{(k)} + |\pi^{(1)}| + \cdots + |\pi^{(k)}|}$$

where the sum runs over all possible $p^{(i)} \in PS_k^{\lambda^{(i)}}$ and all possible integer partitions $\pi^{(1)}, \dots, \pi^{(k)}$ with parts of size 0 allowed such that the lengths of $\pi^{(i)}$ and $p^{(i)}$ are the same for each i.

We will prove the lemma by exhibiting a bijection φ which sends pairs of the form (p, π) where p is a position sequence in $PS^{\lambda}_{(k,\dots,k)}$ and π is an integer partition with 0 parts allowed such that p and π have the same length to tuples of the form $\left(p^{(1)}, \pi^{(1)}, \dots, p^{(k)}, \pi^{(k)}\right)$ where $p^{(i)} \in PS_k^{\lambda^{(i)}}$ and $\pi^{(i)}$ have the same length for each i. The bijection φ will have the weight preserving property that

$$\operatorname{maj} p + |\pi| = \operatorname{maj} p^{(1)} + \cdots + \operatorname{maj} p^{(k)} + |\pi^{(1)}| + \cdots + |\pi^{(k)}|.$$

Let $p = p_1 p_2 \cdots$ be a position sequence in $PS^{\lambda}_{(k,\dots,k)}$ and $\pi = (\pi_1, \pi_2, \dots)$ be an integer partition with 0 parts allowed such that p and π have the same length. Let $c(i,j)$ be the position of the j^{th} integer in p that is congruent to i modulo k. Define $p^{(i)} = p_1^{(i)} p_2^{(i)} \cdots$ where $p_j^{(i)} = p_{c(i,j)}$ and define $\pi^{(i)} = (\pi_1^{(i)}, \pi_2^{(i)}, \dots)$ where

$$\begin{aligned} \pi_j^{(i)} = \pi_{c(i,j)} &+ (\text{the number of descents in } p \text{ at position } c(i,j) \text{ or greater}) \\ &- (\text{the number of descents in } p^{(i)} \text{ at position } j \text{ or greater}). \end{aligned} \tag{10}$$

The bijection φ is defined to send (p, π) to $\left(p^{(1)}, \pi^{(1)}, \dots, p^{(k)}, \pi^{(k)}\right)$.

For example, suppose $\lambda = (9, 9, 4, 4, 4, 3, 3)$, $k = 3$,

$$\begin{array}{lcccccccccccccl} p = & 2 & 13 & 5 & 12 & 6 & 1 & 10 & 3 & 9 & 6 & 4 & 7 & \text{and} \\ \pi = & (\,2, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 1, & 0, & 0, & 0). \end{array}$$

Then it can be found that

$$\begin{array}{lll} p^{(1)} = 13 \quad 1 \quad 10 \quad 4 \quad 7\ , & p^{(2)} = 2 \quad 5\ , & p^{(3)} = 12 \quad 6 \quad 3 \quad 9 \quad 6\ , \\ \pi^{(1)} = (5, \quad 3, \quad 3, \quad 0, \quad 0), & \pi^{(2)} = (8, \quad 6), & \pi^{(3)} = (3, \quad 3, \quad 2, \quad 2, \quad 1), \end{array}$$

and we see that $\operatorname{maj} p + |\pi| = 47 = \operatorname{maj} p^{(1)} + \operatorname{maj} p^{(2)} + \operatorname{maj} p^{(3)} + |\pi^{(1)}| + |\pi^{(2)}| + |\pi^{(3)}|$.

The sequences $p^{(1)}, \dots, p^{(k)}$ are indeed elements in $PS_k^{\lambda^{(i)}}$ because of Theorem 5.

Suppose $p^{(i)}$ has a descent in position $m \geqslant j$. Since p is a shuffle of $p^{(1)}, \dots, p^{(k)}$, the integer $p_m^{(i)}$ appears to the left of $p_{m+1}^{(i)}$ in p, and so there must be at least one descent between $p_m^{(i)}$ and $p_{m+1}^{(i)}$ in p. Thus every descent in $p^{(i)}$ in position j or greater has at least one corresponding descent in p in position $c(i,j)$ or greater. Therefore

$$(\text{the number of descents in } p \text{ at position } c(i,j) \text{ or greater})$$

is at least as large as

$$(\text{the number of descents in } p^{(i)} \text{ at position } j \text{ or greater})$$

and the difference of these quantities weakly decreases as j increases. We can now conclude that $\pi^{(1)}, \dots, \pi^{(k)}$ are indeed integer partitions.

The function φ is a bijection because we can describe its inverse. Suppose we are given the tuple

$$\left(p^{(1)}, \pi^{(1)}, \dots, p^{(k)}, \pi^{(k)}\right) \tag{11}$$

where $p^{(i)} \in PS_k^{\lambda^{(i)}}$ and $\pi^{(i)}$ have the same length for each i. Define $\widehat{\pi}^{(i)} = (\widehat{\pi}_1^{(i)}, \widehat{\pi}_2^{(i)}, \dots)$ to be the integer partition such that

$$\widehat{\pi}_j^{(i)} = \pi_j^{(i)} + (\text{the number of descents in } p^{(i)} \text{ at position } j \text{ or greater}).$$

This definition of $\widehat{\pi}^{(i)}$ increments each of the j parts $\pi_1^{(i)}, \dots, \pi_j^{(i)}$ by 1 if there is a descent in $p^{(i)}$ at position j, and so $|\widehat{\pi}^{(i)}| = \operatorname{maj} p^{(i)} + |\pi^{(i)}|$. Furthermore, this definition implies that $p_j^{(i)} < p_{j+1}^{(i)}$ for every value of j that satisfies $\widehat{\pi}_j^{(i)} = \widehat{\pi}_{j+1}^{(i)}$.

Define $\widehat{\pi} = (\widehat{\pi}_1, \widehat{\pi}_2, \dots)$ to be the integer partition found by sorting the parts of $\widehat{\pi}^{(1)}, \dots, \widehat{\pi}^{(k)}$ into weakly decreasing order. Define $p = p_1 p_2 \cdots$ to be the shuffle of $p^{(1)}, \dots, p^{(k)}$ such that $p_j^{(i)}$ appears in the same position in p as $\widehat{\pi}_j^{(i)}$ appears in $\widehat{\pi}$ and such that $p_j < p_{j+1}$ for every value of j that satisfies $\widehat{\pi}_j = \widehat{\pi}_{j+1}$.

If $p_j > p_{j+1}$, then $\widehat{\pi}_j > \widehat{\pi}_{j+1}$, and so

$$\widehat{\pi}_j \geqslant (\text{the number of descents in } p \text{ at position } j \text{ or greater})$$

for all j. Define the integer partition $\pi = (\pi_1, \pi_2, \dots)$ such that

$$\pi_j = \widehat{\pi}_j - (\text{the number of descents in } p \text{ at position } j \text{ or greater}).$$

The function φ sends the pair (p, π) to the tuple in (11) because this pair can easily be used to find the above $\widehat{\pi}$ and $p^{(i)}$, which in turn can be used to show that the above $\pi_j^{(i)}$ matches that given in (10). Therefore φ is a bijection.

Since $\widehat{\pi}$ is decremented by 1 for each position j of a descent in p, we have $|\pi| = |\widehat{\pi}| - \operatorname{maj} p$ and so

$$\begin{aligned}\operatorname{maj} p + |\pi| &= |\widehat{\pi}| \\ &= |\widehat{\pi}^{(1)}| + \cdots |\widehat{\pi}^{(k)}| \\ &= \operatorname{maj} p^{(1)} + \cdots + \operatorname{maj} p^{(k)} + |\pi^{(1)}| + \cdots + |\pi^{(k)}|.\end{aligned}$$

The bijection φ is weight preserving, as needed. □

Theorem 13 (The ***q***-modular hook length formula). *If λ is an integer partition of n such that $PS^{\lambda}_{(k,\ldots,k)}$ is nonempty and $\hat{p}$ is the position sequence in $PS^{\lambda}_{(k,\ldots,k)}$ with the minimum major index, then*

$$\sum_{p \in PS^{\lambda}_{(k,\ldots,k)}} q^{\operatorname{maj} p - \operatorname{maj} \hat{p}} = \frac{[n/k]_q!}{\prod [h_c/k]_q}$$

where the product is over all cells $c \in \lambda$ with h_c divisible by k.

Proof. After dividing both sides of Theorem 11 by $(1-q)\cdots(1-q^{|\lambda^{(i)}|})$ and moving terms around, we have

$$\frac{\sum_{p^{(i)} \in PS^{\lambda^{(i)}}_{(1,\ldots,1)}} q^{\operatorname{maj} p^{(i)}}}{(1-q)\cdots(1-q^{|\lambda^{(i)}|})} = q^{\operatorname{maj} \hat{p}^{(i)}} \prod_{c \in \lambda^{(i)}} \frac{1}{1-q^{h_c}}.$$

Since the major index of a position sequence $p^{(i)} \in PS^{\lambda^{(i)}}_k$ is unchanged if each integer $j \in p^{(i)}$ is replaced with the integer $kj + i$, we can set $PS^{\lambda}_{(1,\ldots,1)}$ in the above equation to $PS^{\lambda^{(i)}}_{(1,\ldots,1)}$. Multiply the expressions together when each of $PS^{\lambda^{(1)}}_k, \ldots, PS^{\lambda^{(k)}}_k$ replaces $PS^{\lambda}_{(1,\ldots,1)}$ in the above equation and then use Lemma 12 to arrive at

$$\frac{\sum_{p \in PS^{\lambda}_{(k,\ldots,k)}} q^{\operatorname{maj} p}}{(1-q)\cdots(1-q^{n/k})} = \prod_{i=1}^{k} q^{\operatorname{maj} \hat{p}^{(i)}} \prod_{c^{(i)} \in \lambda^{(i)}} \frac{1}{1-q^{h_{c^{(i)}}}} \tag{12}$$

where $\hat{p}^{(i)}$ is the position sequence in $PS^{\lambda^{(i)}}_k$ with the minimum major index.

Each cell $c \in \lambda$ corresponds to a pair (e, b) such that e is an empty position on the k-abacus, b is a position of a bead on the k-abacus, and $e < b$. The hook length of c is $b - e$. This is divisible by k if and only if both positions e and b appear on the same runner of the k-abacus. Thus each cell $c \in \lambda$ with hook length kj corresponds to a cell $c^{(i)} \in \lambda^{(i)}$ with hook length j. Therefore the right side of (12) is equal to

$$q^{\operatorname{maj} \hat{p}^{(1)} + \cdots + \operatorname{maj} \hat{p}^{(k)}} \prod \frac{1}{1-q^{h_c/k}}$$

where the product is over all cells $c \in \lambda$ with h_c divisible by k.

There is a sequence of n/k bead moves of length k in the k-abacus that leave the empty partition (pushing all beads flush to the left) because $PS^{\lambda}_{(k,\ldots,k)}$ is nonempty, and so there are exactly n/k such pairs (e, b) where e and b are on the same runner. Any pairs (e, b) where e and b are not on the same runner do not correspond to cells $c \in \lambda$ with h_c divisible by k, and therefore there are exactly n/k cells $c \in \lambda$ with h_c divisible by k.

Theorem 8 implies $\operatorname{maj} \hat{p} = \operatorname{maj} \hat{p}^{(1)} + \cdots + \operatorname{maj} \hat{p}^{(k)}$, and so we have now shown that

$$\frac{\sum_{p \in PS^{\lambda}_{(k,\ldots,k)}} q^{\operatorname{maj} p}}{(1-q)\cdots(1-q^{n/k})} = q^{\operatorname{maj} \hat{p}} \prod \frac{1}{1-q^{h_c/k}}.$$

where the product is over all cells $c \in \lambda$ with h_c divisible by k. The result follows after multiplying by $(1-q)\cdots\left(1-q^{n/k}\right)$ and simplifying. □

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