Choice functions in the intersection of matroids

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Abstract

We prove a common generalization of two results, one on rainbow fractional matchings [3] and one on rainbow sets in the intersection of two matroids [9]: Given $d = r\lceil k \rceil - r + 1$ functions of size (=sum of values) k that are all independent in each of r given matroids, there exists a rainbow set of $supp(f_i),\ i \leqslant d$, supporting a function with the same properties.

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1 Introduction

Let $\mathcal{F} = (F_1, \dots, F_m)$ be a family (namely, a multiset) of sets. A (partial) *rainbow set* for $\mathcal{F}$ is the image of a partial choice function. Namely, it is a set of the form $R = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$, where $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant k$, and $x_{i_j} \in F_{i_j}\ \ (j \leqslant k)$. Here it is assumed that R is a set, namely that the elements x_{i_j} are distinct. There are many theorems of the form "under some conditions there exists a rainbow set satisfying a prescribed condition". For example, the case where the condition is being full (representing all $F_i's$) is the subject of Hall's marriage theorem. The following theorem of Aharoni and Berger [1], which generalizes a result of Drisko [6], belongs to this family, and is a forefather of the results in the present paper:

Theorem 1. *Any family of $2k-1$ matchings of size k in a bipartite graph G have a rainbow matching of size k.*

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(Drisko's slightly narrower result was formulated in the language of Latin rectangles.) In [2] it was conjectured that almost the same is true in general graphs, namely that in any graph $2k$ matchings of size k have a rainbow matching of size k, and that for odd k the Drisko bound suffices - $2k-1$ matchings of size k have a rainbow matching of size k. This is far from being solved (in [2] the bound $3k-2$ was proved), but in [3] a fractional version of the conjecture was proved, in a more general setting. Recall that $\nu^*(F)$ denotes the largest total weight of a fractional matching in a hypergraph H.

Theorem 2 (Aharoni, Holzman and Jiang [3])**.** *Let m be a real number, let H be an r-uniform hypergraph and let $q \geqslant \lceil rk \rceil$ be an integer. Then any family $E_1, \ldots, E_q$ of sets of edges in H satisfying $\nu^*(E_j) \geqslant k$ for all $j \leqslant q$ has a rainbow set F of edges with $\nu^*(F) \geqslant k$. If H is r-partite then it suffices to assume that $q \geqslant r\lceil k \rceil - r + 1$ to obtain the same conclusion.*

Drisko's theorem is a special case, since in bipartite graphs $\nu^* = \nu$. The integral version of the theorem is false for $r > 2$. For $r = 3, k = 2$, for example, $rk - r + 1 = 4$, and the four matchings of size 2 in the complete $2 \times 2 \times 2$ 3-partite hypergraph do not have a rainbow matching of size 2, showing that 4 matchings of size 2 do not necessarily have a rainbow matching of size 2. In [4, 13] bounds are studied in the integral case, in particular showing a lower bound exponential in r.

Kotlar and Ziv proved a matroidal generalization of Theorem 1:

Theorem 3 (Kotlar and Ziv [9])**.** *Let $\mathcal{M}_1, \mathcal{M}_2$ be two matroids on the same vertex set V. Then any $2k-1$ sets $E_1, E_2, \ldots, E_{2k-1}$ of size k in $\mathcal{M}_1 \cap \mathcal{M}_2$ have a rainbow set of size k belonging to $\mathcal{M}_1 \cap \mathcal{M}_2$.*

Theorem 1 is obtained by taking $\mathcal{M}_1$ and $\mathcal{M}_2$ to be the two partition matroids whose parts are (respectively) the stars in the two sides of the bipartite graph.

The aim of this paper is to prove a matroidal generalization of the r-partite case of Theorem 2, along the lines of Theorem 3. By way of apology, most of the ideas are not new: the course of the proof follows closely that of Theorem 2. But there are points where the matroidal version poses its peculiar difficulties. In particular, in order for a perturbation argument used in the proof of Theorem 2 to be adapted to the matroidal case, we need to invoke some properties of matroids and of submodular functions. These appear in Lemmas 10, 12, 14, and in Theorem 13. These are possibly of some independent interest.

To formulate the main result, we need a matroidal generalization of the notion of fractional matchings. This involves the familiar notion of *matroid polytopes.* For a function f on a set V and a subset A of V, let $f[A] = \sum_{a \in A} f(a)$. We denote the total size of f, namely $f[V]$, by $|f|$.

Definition 4. [11] Let $\mathcal{M}$ be a matroid on a ground set V. The polytope of $\mathcal{M}$, denoted by $P(\mathcal{M})$, is

$$\{f \in \mathbb{R}_+^V \mid f[A] \leqslant \mathrm{rk}_{\mathcal{M}} A \ \text{ for } \ \text{every } A \subseteq V\}.$$

Edmonds [7] proved that all vertices of $P(\mathcal{M})$ are integral, and that this is true also for the intersection of two matroids.

Theorem 5 ([11])**.** *If $\mathcal{M}_1, \mathcal{M}_2$ are matroids on the same ground set, then the vertices of the polytope $P(\mathcal{M}_1) \cap P(\mathcal{M}_2)$ are integral.*

This is a corollary of another theorem of Edmonds, the classical matroid intersection theorem [7].

Our main result is:

Theorem 6. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_r$ be matroids on the same ground set V, and let k be a real number. Let $d = r\lceil k \rceil - r + 1$. Let $f_1, \ldots, f_d$ be non-negative real valued functions belonging to $\bigcap_{i \leqslant r} P(\mathcal{M}_i)$, satisfying $|f_j| \geqslant k$ for every $j \leqslant d$. Let $F_i = supp(f_i)$, $i \leqslant d$. Then there exists a function $f \in \bigcap_{i \leqslant r} P(\mathcal{M}_i)$ such that $supp(f)$ is a rainbow set of $(F_1, \ldots, F_d)$, and $|f| \geqslant k$.*

Theorem 3 follows. Let E_i, $i \leqslant 2k-1$ be sets as in that theorem. Applying Theorem 6 to the functions χ_{E_i}, $i \leqslant 2k-1$ (here and below χ_S is the characteristic function of the set S), yields a function $f \in P(\mathcal{M}_1) \cap P(\mathcal{M}_2)$ with $|f| \geqslant k$ whose support is a rainbow set for the E_i's. The function f is a convex combination of vertices of $P(\mathcal{M}_1) \cap P(\mathcal{M}_2)$, and since in this combination all coefficients are positive, the supports of these vertices are contained in $\mathrm{supp}(f)$. Among these there is at least one vertex g with $|g| \geqslant |f|$. By Theorem 5 g is integral, namely a $0,1$ function, meaning that it is a characteristic function of a set as in the conclusion of Theorem 3.

To obtain the r-partite case of Theorem 2 from Theorem 6, choose the matroids $\mathcal{M}_i$, $i \leqslant r$ to be the partition matroids on $\bigcup_{i \leqslant d} E_i$ defined by the stars in the i-th side V_i of the hypergraph. Namely, a set is independent in $\mathcal{M}_i$ if it does not contain two edges meeting in V_i. Then a function belongs to $\bigcap_i P(\mathcal{M}_i)$ if and only if it is a fractional matching. The condition $\nu^*(E_j) \geqslant k$ means that there exists a fractional matching $f_j \in \bigcap_i P(\mathcal{M}_i)$ with $\mathrm{supp}(f_j) \subseteq E_j$ and $|f_j| \geqslant k$ $(j \leqslant d)$. Applying Theorem 6 then yields a fractional matching f whose support is rainbow with respect to the sets E_j.

2 A Topological Tool

A *complex* is a downward-closed collection of sets, that in this context are called *faces*. Let $\mathcal{C}$ be a complex on a vertex set V. A face σ of $\mathcal{C}$ is called a *collapsor* if it is contained in a unique maximal face. The operation of removing from $\mathcal{C}$ all faces containing a collapsor σ is then called a *collapse*, and if $|\sigma| \leqslant d$ then it is called a *d-collapse*. We say that $\mathcal{C}$ is *d-collapsible* if it can be reduced to $\varnothing$ by a sequence of d-collapses. Wegner [14] observed that a d-collapsible complex is *d-Leray*, meaning that the homology groups of all induced complexes vanish in dimensions d and higher.

Our main tool will be a theorem of Kalai and Meshulam [8]. For a complex $\mathcal{C}$ let $\mathcal{C}^c$ be the collection of all non-$\mathcal{C}$-faces (namely, $\mathcal{C}^c := 2^V \setminus \mathcal{C}$).

Theorem 7 (Kalai-Meshulam [8])**.** *If $\mathcal{C}$ is d-collapsible, then every $d+1$ sets in $\mathcal{C}^c$ have a rainbow set belonging to $\mathcal{C}^c$.*

In fact, this is a special case of the main theorem in [8]. The way to derive it from the original theorem can be found in [3].

We will use Theorem 7 to reduce Theorem 6 to a topological statement. To state this, we first extend the definition of the fractional matching number ν^* to our matroidal setting. For each $W \subseteq V$, let

$$\nu^*(W) := \max\left\{|f| : f \in \bigcap_i P(\mathcal{M}_i),\ \mathrm{supp}(f) \subseteq W\right\}.$$

For a positive real k let $\mathcal{X}_k$ be the simplicial complex of all sets $W \subseteq V$ with $\nu^*(W) < k$.

Theorem 8. *$\mathcal{X}_k$ is $(r\lceil k\rceil - r)$-collapsible.*

Theorem 6 follows from Theorem 8. Indeed, as $\mathcal{X}_k$ is $(r\lceil k\rceil - r)$-collapsible, by Theorem 7 any $r\lceil k\rceil - r + 1$ sets not in $\mathcal{X}_k$ contain a rainbow set not in $\mathcal{X}_k$. Since $F \notin \mathcal{X}_k$ means that some $f \in \bigcap_{i \leqslant r} P(\mathcal{M}_i)$ supported on F satisfies $|f| \geqslant k$, Theorem 6 follows.

3 Proof of Theorem 8

A non-negative function $\mathbf{c} : 2^V \to \mathbb{R}_+$ is said to be *decreasing* if $c(A) \leqslant c(A')$ whenever $A \supseteq A'$. A non-negative function $\mathbf{c} : 2^V \to \mathbb{R}_+$ is said to be *submodular* if, whenever $A, B \subseteq V$, we have

$$c(A) + c(B) \geqslant c(A \cup B) + c(A \cap B).$$

Note that the rank function $\mathrm{rk}_{\mathcal{M}}$ of a matroid $\mathcal{M}$ is submodular [15].

Definition 9. If $\mathbf{c} : 2^V \to \mathbb{R}_+$ is decreasing and $\mathcal{M}$ is a matroid on V, let

$$P_{\mathbf{c}}(\mathcal{M}) := \{f \in \mathbb{R}_+^V \mid f[A] \leqslant c(A)\mathrm{rk}_{\mathcal{M}} A \text{ for every } \varnothing \neq A \subseteq V\}.$$

Note that excluding the $A = \varnothing$ inequality does not change the polytope.

We shall use the acronym PDS for "positive, decreasing and submodular". As in [3], we shall consider perturbations of $\mathcal{X}_k$. For this purpose, we shall need the following:

Lemma 10. *The polytope Q of PDS functions on 2^V has full dimension. Moreover, for any $b > 0$, the polytope $Q \cap \{c(V) = b\}$ has full dimension (namely $2^{|V|} - 1$) relative to the hyperplane $\{c(V) = b\}$, for any $b > 0$.*

Proof. To show the first claim, let $c(A) := 2|V|^2 - |A|^2$ for every $A \subseteq V$. We claim that $\mathbf{c} \in \mathit{interior}(Q)$. Clearly, $\mathbf{c}$ is strictly positive and strictly decreasing. To show strict submodularity, note that if $A \neq B \subseteq V$ then

$$\begin{aligned} &c(A) + c(B) - c(A \cup B) - c(A \cap B)\\ =&|A \cup B|^2 + |A \cap B|^2 - |A|^2 - |B|^2\\ =&\frac{1}{2}(|A \cup B| - |A \cap B|)^2 + \frac{1}{2}(|A| - |B|)^2 > 0, \end{aligned}$$

(To obtain the second equality we subtracted from both sides of the equation $\frac{1}{2}\big((|A \cup B| + |A \cap B|)^2 - (|A| + |B|)^2\big) = 0$). It is not necessary to check the case $A = B$, since in this case equality is true for any function.

To show the second claim, let $\mathbf{c}' := \frac{b}{|V|^2}\mathbf{c}$ for $\mathbf{c}$ as above. Then $\mathbf{c}'$ maintains the strictness of all inequalities defining Q, and satisfies $\mathbf{c}'(V) = b$. $\square$

Given an r-tuple $\mathbf{b} = (\mathbf{b}^1, \ldots, \mathbf{b}^r)$ of PDS functions on 2^V and a non-negative vector $\mathbf{a} = (a_v)_{v \in V}$, let $\nu^*_{\mathbf{a},\mathbf{b}}(W)$ be the largest possible value of $\mathbf{a} \cdot f$ among all $f \in \bigcap P_{\mathbf{b}^i}(\mathcal{M}_i)$ with $\mathrm{supp}(f) \subseteq W$. That is:

$$\begin{aligned} \nu^*_{\mathbf{a},\mathbf{b}}(W) := \max \quad & \sum_{v \in W} a_v f(v) \\ \text{s.t.} \quad & \sum_{v \in A} f(v) \leqslant b^i(A)\mathrm{rk}_{\mathcal{M}_i}(A) && \forall A \subseteq V, \forall i \in [r], \\ & \text{and } f(v) \geqslant 0 && \forall v \in W. \end{aligned}$$

By linear programming duality, $\nu^*_{\mathbf{a},\mathbf{b}}(W)$ is equal to

$$\begin{aligned} \tau^*_{\mathbf{a},\mathbf{b}}(W) := \min \quad & \sum_{\substack{i \in [r] \\ A \subseteq V}} b^i(A)\mathrm{rk}_{\mathcal{M}_i}(A)h(i,A) \\ \text{s.t.} \quad & \sum_{\substack{i \in [r] \\ A \ni v}} h(i,A) \geqslant a_v && \forall v \in W \\ & h(i,A) \geqslant 0 && \forall A \subseteq V, i \in [r] \end{aligned}$$

Given a positive real number k, let $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ be the simplicial complex consisting of all sets $W \subseteq V$ for which $\nu^*_{\mathbf{a},\mathbf{b}}(W) < k$.

Theorem 11. *Let $\mathbf{a} \in \mathbb{R}^V_+$ and let $\mathbf{b}$ be an r-tuple of PDS functions on 2^V. Let $\underline{a} = \min_V\{a_v\}$, $\underline{b} = \min_{i \in [r]}\{b^i(V)\}$. Then $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ is $r\left\lfloor \frac{\overline{k}}{\underline{a}\underline{b}} \right\rfloor$-collapsible, where $\overline{k}$ is given by*

$$\overline{k} := \max\{\nu^*_{\mathbf{a},\mathbf{b}}(W) : W \in \mathcal{X}_{\mathbf{a},\mathbf{b},k}\} < k.$$

Theorem 8 is the special case of Theorem 11 obtained by fixing every $b^i(A) = 1$ and $\mathbf{a} = \mathbf{1}$. Theorem 11 applies since the constant-1 function is PDS. Here, $\mathcal{X}_k = \mathcal{X}_{\mathbf{a},\mathbf{b},k}$, $\underline{a} = \underline{b} = 1$, and $\lfloor \overline{k} \rfloor \leqslant \lceil k \rceil - 1$, yielding that $\mathcal{X}_k$ is $(r\lceil k \rceil - r)$-collapsible.

We prove Theorem 11 by induction on $|\mathcal{X}_{\mathbf{a},\mathbf{b},k}|$. Note that $|\mathcal{X}_{\mathbf{a},\mathbf{b},k}| > 1$, since $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ contains at least one nonempty set.

Following a crucial idea from [3], we may assume that generically, for every $W \subseteq V$ there is a unique function h on $[r] \times 2^V$ attaining the minimum in the program defining $\tau^*_{\mathbf{a},\mathbf{b}}(W)$. For, the set of all $\mathbf{b} = (\mathbf{b}^1, \ldots, \mathbf{b}^r)$ for which the optimum is not uniquely attained is the union of finitely many hyperplanes. By Lemma 10, it is possible to perturb the $\mathbf{b}^i$'s so as to avoid these hyperplanes, in a fashion sustaining the value of $\underline{b}$. If the perturbation is sufficiently small, $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ stays unaffected.

Now, we choose any $W \in \mathcal{X}_{\mathbf{a},\mathbf{b},k}$ such that:

$$\nu^*_{\mathbf{a},\mathbf{b}}(W) = \overline{k}, \text{ and } W \text{ is inclusion-minimal among all such sets.} \tag{†}$$

We prove that removing all supersets of W is an elementary $r\left\lfloor\frac{\bar{k}}{ab}\right\rfloor$-collapse in $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$. This requires the three claims ($\diamondsuit$), ($\clubsuit$), and ($\spadesuit$) as follows, which together will constitute the remainder of the proof of Theorem 11.

$$W \text{ is contained in a unique facet.} \tag{$\diamondsuit$}$$

To prove ($\diamondsuit$), we follow [3], but reproduce the argument for completeness. Let $W^+ := \{v \in V : W \cup \{v\} \in \mathcal{X}_{\mathbf{a},\mathbf{b},k}\}$. Let $v \in W^+$ be arbitrary. By maximality of $\bar{k}$, we know $\nu^*_{\mathbf{a},\mathbf{b}}(W \cup \{v\}) = \nu^*_{\mathbf{a},\mathbf{b}}(W) = \bar{k}$, and hence $\tau^*_{\mathbf{a},\mathbf{b}}(W \cup \{v\}) = \tau^*_{\mathbf{a},\mathbf{b}}(W) = \bar{k}$. By our assumed perturbations, there exists a unique function h on $[r] \times 2^V$ attaining the minimum defining $\tau^*_{\mathbf{a},\mathbf{b}}(W)$. Since the function h' witnessing $\tau^*_{\mathbf{a},\mathbf{b}}(W \cup \{v\}) = \bar{k}$ is also feasible for $\tau^*_{\mathbf{a},\mathbf{b}}(W)$, it follows that $h' = h$, so h must satisfy the additional constraint $\sum_{i \in [r], A \ni v} h(i, A) \geqslant a_v$ for v. Since this is true for every $v \in W^+$, the function h satisfies the constraints for all vertices in $W \cup W^+$, witnessing $\tau^*_{\mathbf{a},\mathbf{b}}(W \cup W^+) = \bar{k}$. Thus $W \cup W^+ \in \mathcal{X}_{\mathbf{a},\mathbf{b},k}$ is the unique facet containing W, giving ($\diamondsuit$).

$$\text{If } W \text{ satisfies } (\dagger) \text{ then } |W| \leqslant r\left\lfloor\frac{\bar{k}}{ab}\right\rfloor. \tag{$\clubsuit$}$$

The proof of ($\clubsuit$) is the main place where new arguments are needed, beyond those appearing in [3]. These appear in Lemma 12, Theorem 13 and Lemma 14 below.

Let P_W be the polytope of functions f on $\mathbb{R}^W$ satisfying $f(v) \geqslant 0$ for all $v \in V$, and $\sum_{v \in A} f(v) \leqslant b^i(A)\mathrm{rk}_{\mathcal{M}_i}(A)$ for all $i \in [r]$ and $A \subseteq V$. Let f be a vertex of P_W at which the maximum value of $\sum_{v \in W} a_v f(v)$ is attained. This maximum is at least $\bar{k}$. Then f must satisfy $|W|$ linearly independent inequalities of the above kinds at equality. If $f(v) = 0$ were true for any v, then f would also witness $\nu^*_{\mathbf{a},\mathbf{b}}(W \setminus \{v\}) = \nu^*_{\mathbf{a},\mathbf{b}}(W) = \bar{k}$, contradicting minimality of W. So all $|W|$ equalities are of the form $\sum_{v \in A} f(v) = b^i(A)\mathrm{rk}_{\mathcal{M}_i}(A)$. For each $i \in [r]$ let w_i be the number of equalities of the form $\sum_A f(v) = b^i(A)\mathrm{rk}_{\mathcal{M}_i}(A)$ (so $\sum_i w_i = |W|$).

Let

$$\mathcal{F}_i^f := \Big\{A \subseteq V : \sum_{v \in A} f(v) = b^i(A)\mathrm{rk}_{\mathcal{M}_i}(A)\Big\},$$

so the set $\{\chi_A \mid A \in \mathcal{F}_i^f\}$ consists of w_i linearly independent vectors.

We can take advantage of these w_i sets as follows. Recall that χ_S denotes the indicator vector of S. We use the term "*chain* of length r of sets" for a collection of r distinct non-empty sets, totally ordered by inclusion.

Lemma 12. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a family of sets, closed under intersections and unions. If $\{\chi_S : S \in \mathcal{F}\}$ linearly spans (over the reals or rationals) a space of dimension t, then $\mathcal{F}$ contains a chain of length t.*

Proof. We proceed by induction on t. It is obvious when $t = 1$. For $t \geqslant 2$, we may assume that there exists a chain $\varnothing \neq A_1 \subsetneq \cdots \subsetneq A_{t-1}$ of length $t-1$. Since $\{\chi_S : S \in \mathcal{F}\}$ spans a t-dimensional space, there exists a non-empty set $A \in \mathcal{F}$ such that $\chi_A \notin U := \mathrm{span}(\{\chi_{A_i} :$

$i < t\}$). If $A \not\subseteq A_{t-1}$, then letting $A_t = A_{t-1} \cup A$ yields the desired chain of length t. Thus we may assume $A \subseteq A_{t-1}$.

For $i = 2, \ldots, t$ let $B_i = A_i \setminus A_{i-1}$ and let $B_1 = A_1$. Note that $\chi_{B_i} = \chi_{A_i} - \chi_{A_{i-1}} \in \mathrm{span}(\{\chi_{A_i} : i < t\})$. If for some $i \leqslant t$ neither $B_i \cap A = \varnothing$ nor $B_i \subseteq A$, then $(A \cup A_{i-1}) \cap A_i \in \mathcal{F}$ lies strictly between A_{i-1} and A_i, so its addition forms the desired chain. We may thus assume that there is no such B_i.

Let $S = \{i \leqslant t : B_i \subseteq A\}$. By the above assumption $A = \bigsqcup_{i \in S} B_i$. Hence $\chi_A = \sum_{i \in S} \chi_{B_i} \in U$, a contradiction. $\square$

We wish to show that each $\mathcal{F}_i^f$ satisfies the condition of Lemma 12, namely it is closed under intersections and unions. Indeed, for the usual matroid polytopes, it is a well-known fact (see Lemma 14 below). Extending this to skew polytopes first requires the following result.

Theorem 13. *If* $\mathbf{c}, \mathbf{r}$ *are nonnegative submodular functions on a lattice of sets,* $\mathbf{c}$ *is decreasing and* $\mathbf{r}$ *is increasing, then* $\mathbf{c} \cdot \mathbf{r}$ *is submodular.*

This may be folklore, and it closely resembles a standard fact on the product of convex functions (see e.g. [5, 3.32]), but Lovász's celebrated method [10] for linearly extending submodularity to convexity does not behave well under taking products, and so we could not establish a direct implication. The only explicit reference we found is a question answered at [12]. For completeness we provide a proof here.

Proof. We wish to show that, for any $A, B \subseteq V$,

$$c(A \cup B) r(A \cup B) + c(A \cap B) r(A \cap B) - c(A) r(A) - c(B) r(B) \leqslant 0.$$

For a real-valued function h on a lattice, let $D_T h(S) := h(S \cup T) - h(S)$ be the "difference" operator applied to h. In this terminology, a function h is submodular if and only if

$$D_{B \setminus A} D_{A \setminus B}(h)(A \cap B) \leqslant 0.$$

We shall show that $D_S D_R(cr)$ is non-positive for any sets S, R. To see this, write:

$$\begin{aligned} c(S \cup R) r(S \cup R) - c(S) r(S) = {} & c(S \cup R)(r(S \cup R) - r(S)) \\ & + (c(S \cup R) - c(S)) r(S) \end{aligned}$$

gives us the product rule $D_R(cr)(S) = c(S \cup R) D_R r(S) + (D_R c(S))(r(S))$. Letting $T_R h(X)$ denote $h(X \cup R)$ for any h, this says

$$D_R(cr) = (T_R c)(D_R r) + (D_R c) r.$$

Applying this twice gives

$$\begin{aligned} D_S D_R(cr) = {} & D_S\big((T_R c)(D_R r)\big) + D_S\big((D_R c)(r)\big) \\ = {} & T_S T_R c \cdot D_S D_R r + \big(D_S T_R c\big) \cdot D_R r \\ & + \big(D_R T_S c\big) \cdot D_S r + \big(D_S D_R c\big) \cdot r. \end{aligned}$$

All four products above are non-positive, as can be seen from the following:

- $c, r \geqslant 0$ by nonnegativity,
- $D_R D_T r, D_R D_T c \leqslant 0$ by submodularity,
- $D_R c, D_T c \leqslant 0$ as $\mathbf{c}$ decreasing,
- $D_T r, D_R r \geqslant 0$ as $\mathbf{r}$ increasing. □

Lemma 14. *Let $\mathcal{M}$ be a matroid on V, $\mathbf{c}$ a PDS function on 2^V, f a point in $P_{\mathbf{c}}(\mathcal{M})$, and W a subset of V. Let $\mathcal{F}$ be the family of all subsets A of W satisfying*

$$\sum_{v \in A} f(v) = c(A)\mathrm{rk}(A). \tag{1}$$

Then $\mathcal{F}$ is closed under intersections and unions.

Proof. Let $A, B \in \mathcal{F}$, so $\sum_A f(v) = c(A)\mathrm{rk}(A)$ and $\sum_B f(v) = c(B)\mathrm{rk}(B)$. Then

$$\begin{aligned}
\sum_{v \in A \cup B} f(v) &\leqslant c(A \cup B)\mathrm{rk}(A \cup B) \\
&\leqslant c(A)\mathrm{rk}(A) + c(B)\mathrm{rk}(B) - c(A \cap B)\mathrm{rk}(A \cap B) \\
&= \sum_{v \in A} f(v) + \sum_{v \in B} f(v) - c(A \cap B)\mathrm{rk}(A \cap B) \\
&\leqslant \sum_{v \in A} f(v) + \sum_{v \in B} f(v) - \sum_{v \in A \cap B} f(v) \\
&= \sum_{v \in A \cup B} f(v).
\end{aligned}$$

The second inequality is the submodularity of $\mathbf{c} \cdot \mathrm{rk}$. The first and last inequalities follow from the fact that $f \in P_{\mathbf{c}}(\mathcal{M})$. Since equality should hold throughout, it follows that $A \cup B, A \cap B \in \mathcal{F}$. □

Lemma 14 enables application of Lemma 12 to $\mathcal{F} := \mathcal{F}_i^f$. We obtain a chain $\varnothing \neq A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{w_i}$ in $\mathcal{F}_i^f$. Thus

$$0 < \sum_{v \in A_1} f(v) < \sum_{v \in A_2} f(v) < \cdots < \sum_{v \in A_{w_i}} f(v)$$

as $f(v) > 0$ for each $v \in W$. We may rewrite this as

$$0 < b^i(A_1)\mathrm{rk}_{\mathcal{M}_i}(A_1) < \cdots < b^i(A_{w_i})\mathrm{rk}_{\mathcal{M}_i}(A_{w_i}).$$

But $\mathbf{b}^i$ is decreasing, so $b^i(A_1) \geqslant b^i(A_2) \geqslant \ldots \geqslant b^i(A_{w_i})$. Thus

$$0 < \mathrm{rk}_{\mathcal{M}_i}(A_1) < \mathrm{rk}_{\mathcal{M}_i}(A_2) < \cdots < \mathrm{rk}_{\mathcal{M}_i}(A_{w_i}),$$

Since ranks are integers, it follows that $\mathrm{rk}_{\mathcal{M}_i}(A_{w_i}) \geqslant w_i$.

Thus in fact, for each $i \in [r]$:

$$\underline{ab} w_i \leqslant \underline{a} b^i(A_{w_i}) \mathrm{rk}_{\mathcal{M}_i}(A_{w_i}) = \underline{a} \sum_{v \in A_{w_i}} f(v) \leqslant \sum_{v \in W} a_v f(v) = \overline{k},$$

and by integrality $w_i \leqslant \left\lfloor \frac{\overline{k}}{\underline{ab}} \right\rfloor$. So we conclude

$$|W| = \sum_{i \in [r]} w_i \leqslant \sum_{i \in [r]} \left\lfloor \frac{\overline{k}}{\underline{ab}} \right\rfloor = r \left\lfloor \frac{\overline{k}}{\underline{ab}} \right\rfloor,$$

which proves (♣).

(♠) Suppose W satisfies (†) and let $\mathcal{X}'$ be the complex obtained by removing from $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ all faces containing W. Then there exists $\mathbf{a}' \in \mathbb{R}_+^V$, satisfying $r \left\lfloor \frac{\overline{k}}{\underline{a'b}} \right\rfloor \leqslant r \left\lfloor \frac{k}{\underline{ab}} \right\rfloor$, for which

$$\mathcal{X}' = \{W' \subseteq V : \nu^*_{\mathbf{a}',\mathbf{b}}(W') < \overline{k}\} = \mathcal{X}_{\mathbf{a}',\mathbf{b},\overline{k}}.$$

The proof of (♠) follows a parallel argument in [3]. We claim that there is some $\epsilon > 0$ for which $\mathcal{X}' = \mathcal{X}_{\mathbf{a}',\mathbf{b},\overline{k}}$ is satisfied by the objective coefficients $\mathbf{a}'$ defined coordinate-wise by:

$$a'_v := \begin{cases} a_v - \epsilon & \text{if } v \notin W, \\ a_v & \text{if } v \in W. \end{cases}$$

First consider any $W' \subseteq V$ that wasn't even in $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ to begin with, so that $\nu^*_{\mathbf{a},\mathbf{b}}(W') \geqslant k$. The feasibility regions for $\nu^*_{\mathbf{a},\mathbf{b}}(W')$ and $\nu^*_{\mathbf{a}',\mathbf{b}}(W')$ are the same, so if ϵ is sufficiently small relative to $k - \overline{k}$, it follows $\nu^*_{\mathbf{a}',\mathbf{b}}(W') \geqslant \overline{k}$, so that $W' \notin \mathcal{X}_{\mathbf{a}',\mathbf{b},\overline{k}}$ either.

Next, pick any $W' \subseteq V$ previously in $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$, but which contained W so was removed in the collapse. As before, let f be an optimiser for the LP defining $\nu^*_{\mathbf{a},\mathbf{b}}(W)$, so $\mathbf{a} \cdot f = \overline{k}$ but also $\mathrm{supp}(f) = W \subseteq W'$. This way, f is also feasible for the linear program defining $\nu^*_{\mathbf{a},\mathbf{b}}(W')$. But whenever $a'_v < a_v$, $e \notin W$ and hence $f(v) = 0$ by minimality of W. Hence $\nu^*_{\mathbf{a}',\mathbf{b}}(W') \geqslant \mathbf{a}' \cdot f = \mathbf{a} \cdot f = \nu^*_{\mathbf{a},\mathbf{b}}(W') = \overline{k}$. Thus $W' \notin \mathcal{X}_{\mathbf{a}',\mathbf{b},\overline{k}}$.

Finally, take some $W' \subseteq V$ previously in $\mathcal{X}_{\mathbf{a},\mathbf{b},k}$ and not fully containing W. Note $W \cap W' \subsetneq W$. We wish to show $\nu^*_{\mathbf{a}',\mathbf{b}}(W') < \overline{k}$ for deducing $W' \in \mathcal{X}_{\mathbf{a}',\mathbf{b},\overline{k}}$, so assume for contradiction $\nu^*_{\mathbf{a}',\mathbf{b}}(W') \geqslant \overline{k}$, as witnessed by some $g \in \bigcap P_{\mathbf{b}^i}(\mathcal{M}_i), \mathrm{supp}(g) \subseteq W'$ with $\mathbf{a}' \cdot g \geqslant \overline{k}$. We cannot have $\mathrm{supp}(g) \subseteq W \cap W'$. For otherwise g would also witness $\nu^*_{\mathbf{a},\mathbf{b}}(W \cap W') \geqslant \mathbf{a}' \cdot g = \mathbf{a} \cdot g \geqslant \overline{k}$, hence $\nu^*_{\mathbf{a},\mathbf{b}}(W \cap W') = \overline{k}$ by maximality of $\overline{k}$, and this would contradict inclusion-minimality of W. So there is at least one $e_0 \in \mathrm{supp}(g) \setminus W$. So $g(v_0) > 0$ and $a'_{v_0} < a_{v_0}$ means $\sum_{v \in W'} a_v g(v) > \sum_{v \in W'} a'_v g(v) \geqslant \overline{k}$, still contradicting maximality of $\overline{k}$.

So, by inductive hypothesis, $\mathcal{X}_{\mathbf{a}',\mathbf{b},\overline{k}}$ is indeed $r \left\lfloor \frac{\overline{k}}{\underline{a'b}} \right\rfloor$-collapsible, and since $\overline{k} < k$, we can make ϵ small enough to guarantee $r \left\lfloor \frac{\overline{k}}{\underline{a'b}} \right\rfloor \leqslant r \left\lfloor \frac{k}{\underline{ab}} \right\rfloor$.

4 Closing Remarks

Theorem 6 provides a matroidal generalisation of Theorem 2. While the proof method goes via a weighted version, Theorem 11, this does not seem to also generalise the corresponding theorem for weighted fractional matchings (see [3, Theorem 3.2] for the non-r-partite version).

Indeed, suppose we are given an r-partite hypergraph H with parts $V = V_1 \cup \cdots \cup V_r$, along with vertex weights $\{b_v : v \in V\}$. We wish to define a collection of polytopes $\{P_i : i \in [r]\}$ with the following property. A weighted collection of edges $\{x_e : e \in E(H)\}$ is a fractional matching with respect to the $\{b_v\}$'s if and only if $x \in \bigcap_{i\in[r]} P_i$. To do so in a way that would generalise the usual (non-weighted) case would suggest we let

$$P_i := \{x \in \mathbb{R}_+^{E(H)} : \forall A \subset E(H), x[A] \leqslant b^i(A)|(\cup A) \cap V_i|\},$$

for some $b^i : 2^{E(H)} \to \mathbb{R}_+$ satisfying $b^i(N(v)) = b_v$ for every $v \in V_i$ (where $N(v)$ denotes all edges of H incident to v). This can be done by letting $b^i(A) := \max\{b_v : v \in (\cup A) \cap V_i\}$- this way, all inequalities not of the form $A = N(v)$ for some $v \in V_i$ are redundant. But while these b^i's are submodular, they are not decreasing, and hence Theorem 11 does not apply.

It is therefore natural to ask what is the largest family of functions for which Theorem 11 holds.

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