On the Twelve-Point Theorem for ℓ-Reflexive Polygons

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Abstract

It is known that, adding the number of lattice points lying on the boundary of a reflexive polygon and the number of lattice points lying on the boundary of its polar, always yields 12. Generalising appropriately the notion of reflexivity, one shows that this remains true for "ℓ-reflexive polygons". In particular, there exist (for this reason) infinitely many (lattice inequivalent) lattice polygons with the same property. The first proof of this fact is due to Kasprzyk and Nill. The present paper contains a second proof (which uses tools only from toric geometry) as well as the description of complementary properties of these polygons and of the invariants of the corresponding toric log del Pezzo surfaces.

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1 Introduction

The purpose of this paper is to give a second proof of the so-called "Twelve-Point Theorem" for "ℓ-reflexive polygons" (see below Theorem 1.27), to explain where 12 comes from by taking a slightly different approach, and to provide some additional consequences of it from the point of view of toric geometry.

• **Polygons**. Let $P \subset \mathbb{R}^2$ be a (convex) *polygon*, i.e., the convex hull $\text{conv}(A)$ of a finite set $A \subset \mathbb{R}^2$ of at least 3 non-collinear points. We denote by $\text{Vert}(P)$ and $\text{Edg}(P)$ the set of its vertices and the set of its edges, respectively, and by ∂P and $\text{int}(P)$ its boundary and its interior, respectively. If the origin $\mathbf{0} \in \mathbb{R}^2$ belongs to $\text{int}(P)$, then its *polar polygon* is defined to be

$$P^\circ := \left\{ \mathbf{x} \in \mathbb{R}^2 \,\middle|\, \langle \mathbf{x}, \mathbf{y} \rangle \geqslant -1, \ \forall \mathbf{y} \in P \right\},$$

where $\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 + x_2 y_2$, for $\mathbf{x} = \binom{x_1}{x_2} \in \mathbb{R}^2$ and $\mathbf{y} = \binom{y_1}{y_2} \in \mathbb{R}^2$, is nothing but the usual inner product. Since $\mathbf{0} \in \mathrm{int}(P^\circ)$ and $(P^\circ)^\circ = P$, the polarity induces bijections

$$\mathrm{Vert}(P) \ni \mathbf{v} \longmapsto \{\mathbf{x} \in P^\circ \mid \langle \mathbf{x}, \mathbf{v} \rangle = -1\} \in \mathrm{Edg}(P^\circ), \tag{1.1}$$

and

$$\mathrm{Edg}(P) \ni F \longmapsto \{\mathbf{x} \in P^\circ \mid \langle \mathbf{x}, \mathbf{v} \rangle = \langle \mathbf{x}, \mathbf{v}' \rangle = -1\} \in \mathrm{Vert}(P^\circ), \tag{1.2}$$

with $\mathbf{v}, \mathbf{v}'$ denoting the vertices of F.

• **Lattices**. Since we shall deal with a special sort of lattice polygons, we first recall some basic properties of lattices (cf. [32, Ch. 1, §3]). Let $\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ denote the euclidean norm of any $\mathbf{x} \in \mathbb{R}^2$.

Proposition 1.1. *For any nonempty subset N of $\mathbb{R}^2$ the following conditions are equivalent*:

(i) *N is a discrete subgroup of the additive group $\mathbb{R}^2$* (i.e., *$\mathbf{n} - \mathbf{n}' \in N$ for all $\mathbf{n}, \mathbf{n}' \in N$, and for every $\mathbf{n} \in N$ there is a positive real number ε, s.t. $\mathbf{B}_\varepsilon(\mathbf{n}) \cap N = \{\mathbf{n}\}$, where $\mathbf{B}_\varepsilon(\mathbf{n}) := \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x} - \mathbf{n}\| \leqslant \varepsilon\}$), and N spans the entire $\mathbb{R}^2$ as $\mathbb{R}$-vector space.*

(ii) *There exists a set $\{\mathbf{b}_1, \mathbf{b}_2\}$ of two $\mathbb{R}$-linear independent vectors $\mathbf{b}_1, \mathbf{b}_2 \in \mathbb{R}^2$ such that*

$$N = \{k_1\, \mathbf{b}_1 + k_2 \mathbf{b}_2 \mid k_1, k_2 \in \mathbb{Z}\}.$$

Definition 1.2. A *lattice* in $\mathbb{R}^2$ is a nonempty subset N of $\mathbb{R}^2$ which satisfies the conditions of Proposition 1.1. A set $\{\mathbf{b}_1, \mathbf{b}_2\}$ as in 1.1 (ii) is said to be a ($\mathbb{Z}$-)*basis* of N. (N itself can be viewed as a free abelian group ($\mathbb{Z}$-module) of rank 2 generated by $\{\mathbf{b}_1, \mathbf{b}_2\}$.) If $\mathbf{b}_1 = \binom{\mathfrak{b}_{11}}{\mathfrak{b}_{21}}$ and $\mathbf{b}_2 = \binom{\mathfrak{b}_{12}}{\mathfrak{b}_{22}}$, we say that $\mathcal{B} = \begin{pmatrix} \mathfrak{b}_{11} & \mathfrak{b}_{12} \\ \mathfrak{b}_{21} & \mathfrak{b}_{22} \end{pmatrix}$ is the corresponding *basis matrix* of N.

Proposition 1.3. *If $N \subset \mathbb{R}^2$ is a lattice and $\mathbf{b}_1, \mathbf{b}_2 \in N$ are two $\mathbb{R}$-linear independent vectors, then $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis of N if and only if $\mathrm{conv}(\{\mathbf{0}, \mathbf{b}_1, \mathbf{b}_2\}) \cap N = \{\mathbf{0}, \mathbf{b}_1, \mathbf{b}_2\}$.*

Proposition 1.4. *Let $N \subset \mathbb{R}^2$ be a lattice with $\mathcal{B}$ as a basis matrix. Then a $\mathcal{B}' \in \mathrm{GL}_2(\mathbb{R})$ is a basis matrix of $N \Leftrightarrow \exists \mathcal{A} \in \mathrm{GL}_2(\mathbb{Z}) : \mathcal{B}' = \mathcal{B}\mathcal{A}$.*

Definition 1.5. Let $N \subset \mathbb{R}^2$ be a lattice with $\mathcal{B}$ as a basis matrix. The *determinant* of N is defined to be $\det(N) := |\det(\mathcal{B})|$. (By Proposition 1.4, $\det(N)$ does not depend on the particular choice of $\mathcal{B}$, because $\det(\mathcal{A}) \in \{\pm 1\}$ for all $\mathcal{A} \in \mathrm{GL}_2(\mathbb{Z})$.)

Proposition 1.6. *Let N' be a sublattice of a lattice $N \subset \mathbb{R}^2$. Suppose that $\{\mathbf{b}'_1, \mathbf{b}'_2\}$ and $\{\mathbf{b}_1, \mathbf{b}_2\}$ are bases of N' and N, respectively, and*

$$\mathbf{b}'_1 = u_{11}\mathbf{b}_1 + u_{12}\mathbf{b}_2, \quad \mathbf{b}'_2 = u_{21}\mathbf{b}_1 + u_{22}\mathbf{b}_2,$$

are the expressions of $\mathbf{b}'_1, \mathbf{b}'_2$ as integer linear combinations of $\mathbf{b}_1, \mathbf{b}_2$. Then the number of points of N which belong to the half-open parallelepiped $\Pi := \{\xi_1 \mathbf{b}'_1 + \xi_2 \mathbf{b}'_2 \mid \xi_1, \xi_2 \in [0,1)\}$ equals

$$\sharp(\Pi \cap N) = |\det(\mathcal{U})| = |N : N'| = \frac{\det(N')}{\det(N)},$$

where $\mathcal{U} := (u_{ij})_{1 \leqslant i,j \leqslant 2}$ and $|N : N'|$ the index of (the subgroup) N' in N.

Note 1.7. (i) $\mathbb{Z}^2 := \left\{ \left(\begin{smallmatrix}\lambda_1\\ \lambda_2\end{smallmatrix}\right) \middle| \, \lambda_1, \lambda_2 \in \mathbb{Z} \right\}$ is the *standard* (rectangular) *lattice* in $\mathbb{R}^2$ having $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ as basis matrix (and determinant $= 1$).
(ii) The automorphism group of the $\mathbb{R}$-vector space $\mathbb{R}^2$ is

$$\mathrm{Aut}(\mathbb{R}^2) := \mathrm{GL}(\mathbb{R}^2) = \{ \Phi_{\mathcal{A}} | \, \mathcal{A} \in \mathrm{GL}_2(\mathbb{R}) \} ,$$

where

$$\boxed{\mathbb{R}^2 \ni \left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right) \longmapsto \Phi_{\mathcal{A}}\left(\left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right) \right) := \mathcal{A}\left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right) \in \mathbb{R}^2,}$$

and

$$\mathrm{Aut}_{\mathbb{Z}^2}(\mathbb{R}^2) := \left\{ \Psi \in \mathrm{Aut}(\mathbb{R}^2) \, \middle| \Psi(\mathbb{Z}^2) = \mathbb{Z}^2 \right\} = \{ \Phi_{\mathcal{A}} \, | \mathcal{A} \in \mathrm{GL}_2(\mathbb{Z}) \} .$$

(iii) If $N \subset \mathbb{R}^2$ is an arbitrary lattice with $\mathcal{B}$ as a basis matrix, then $N = \Phi_{\mathcal{B}}(\mathbb{Z}^2)$, and the subgroup

$$\mathrm{Aut}_N(\mathbb{R}^2) := \left\{ \Psi \in \mathrm{Aut}(\mathbb{R}^2) \, | \Psi(N) = N \right\} = \{ \Phi_{\mathcal{B}\mathcal{A}\mathcal{B}^{-1}} \, | \mathcal{A} \in \mathrm{GL}_2(\mathbb{Z}) \}$$

of $\mathrm{Aut}(\mathbb{R}^2)$ consists of the so-called *unimodular N-transformations.*

Definition 1.8. Assume that $N \subset \mathbb{R}^2$ is a lattice with $\mathcal{B}$ as a basis matrix. Identifying the dual $\mathbb{Z}$-module $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ with $\{ \mathbf{x} \in \mathbb{R}^2 | \, \langle \mathbf{x}, \mathbf{n} \rangle \in \mathbb{Z}, \forall \mathbf{n} \in N \}$ we embed it as a lattice in $\mathbb{R}^2$ (and call it *dual lattice*) having $(\mathcal{B}^{\intercal})^{-1}$ as basis matrix, and determinant $\det(M) = (\det(N))^{-1}$. (The standard lattice $\mathbb{Z}^2$ is self-dual.)

• **Lattice polygons**. A *lattice polygon $P \subset \mathbb{R}^2$ w.r.t. a lattice $N \subset \mathbb{R}^2$* (or an *N-polygon*, for short) is a polygon with $\mathrm{Vert}(P) \subset N$. Let $\mathrm{POL}(N)$ be the set of all N-polygons. For $P \in \mathrm{POL}(N)$ the number $\sharp\,(P \cap N)$ is given by *Pick's formula* [57]:

$$\sharp\,(P \cap N) = \mathrm{area}_N(P) + \tfrac{1}{2}\sharp\,(\partial P \cap N) + 1. \tag{1.3}$$

Here, $\mathrm{area}_N(P)$ denotes the Lebesgue measure on $\mathbb{R}^2$ normalised w.r.t. N, so that half-open parallelepiped determined by the members of a basis of N has area 1. (In fact, in terms of the "usual" area, this equals $\frac{\mathrm{area}(P)}{\det(N)}$). If $kP := \{ k\,\mathbf{x} | \, \mathbf{x} \in P \}$ $(k \in \mathbb{Z}_{\geqslant 0})$ denotes the k-th dilation of P, it is known that the *Ehrhart polynomial*

$$\mathrm{Ehr}_N(P; k) := \sharp\,(kP \cap N) \in \mathbb{Q}\,[k]$$

of P (w.r.t. N) equals

$$\mathrm{Ehr}_N(P; k) = \mathrm{area}_N(P)k^2 + \tfrac{1}{2}\sharp\,(\partial P \cap N)\,k + 1 \tag{1.4}$$

and that $\mathrm{Ehr}_N(P; k) = \mathrm{Ehr}_N(\Upsilon(P); k)$ for all affine integral transformations Υ of $\mathbb{R}^2$ w.r.t. N (which are composed of unimodular N-transformations and N-translations). Furthermore, according to the reciprocity law [23, p. 50] for $\mathrm{Ehr}_N^{\mathrm{int}}(P; k) := \sharp\,(\mathrm{int}(kP) \cap N)$,

$$\mathrm{Ehr}_N^{\mathrm{int}}(P; k) = \mathrm{Ehr}_N(P; -k) = \mathrm{area}_N(P)k^2 - \tfrac{1}{2}\sharp\,(\partial P \cap N)\,k + 1. \tag{1.5}$$

• **The equivalence relation** "$\backsim_N$". On the set $\mathrm{POL}_{\mathbf{0}}(N) := \{ P \in \mathrm{POL}(N) | \, \mathbf{0} \in \mathrm{int}(P)\}$ we define the equivalence relation:

$$P_1 \backsim_N P_2 \underset{\text{def}}{\Longleftrightarrow} \exists \Psi \in \mathrm{Aut}_N(\mathbb{R}^2) : \Psi(P_1) = P_2.$$

If $P_1 \backsim_N P_2$, we say that P_1 and P_2 are *equivalent up to N-umimodular transformation.* If $P \in \mathrm{POL}_{\mathbf{0}}(N)$, we denote by $[P]_N := \{ R \in \mathrm{POL}_{\mathbf{0}}(N) | \, R \backsim_N P\}$ its equivalence class.

Definition 1.9. If $P \in \mathrm{POL}_{\mathbf{0}}(N)$, then for a fixed basis matrix $\mathcal{B} \in \mathrm{GL}_2(\mathbb{R})$ of N we have $N = \Phi_{\mathcal{B}}(\mathbb{Z}^2)$ with $\Phi_{\mathcal{B}} \in \mathrm{Aut}(\mathbb{R}^2)$. Thus, we may define the polygon

$$P^{\mathrm{st}} := \Phi_{\mathcal{B}^{-1}}(P) \in \mathrm{POL}_{\mathbf{0}}(\mathbb{Z}^2).$$

P^{st} will be called the *standard model* of P w.r.t. $\mathcal{B}$. By Proposition 1.4, $[P^{\mathrm{st}}]_{\mathbb{Z}^2}$ does not depend on the particular choice of $\mathcal{B}$.

If the induced bijection $\mathrm{POL}_{\mathbf{0}}(N)/\backsim_N \ni [P]_N \longmapsto [P^{\mathrm{st}}]_{\mathbb{Z}^2} \in \mathrm{POL}_{\mathbf{0}}(\mathbb{Z}^2)/\backsim_{\mathbb{Z}^2}$ is taken into account, it is sometimes convenient to work with the equivalence class of P^{st} instead of that of P (and with the standard lattice $\mathbb{Z}^2$ instead of N), e.g., when we draw figures, when we construct certain polygon classification lists etc. It is worth mentioning that

$$\mathrm{Ehr}_N(P;k) = \mathrm{Ehr}_{\mathbb{Z}^2}(P^{\mathrm{st}};k) \quad \text{for all } k \in \mathbb{Z}_{\geqslant 0},$$

because $\sharp\,(\partial P \cap N) = \sharp(\partial P^{\mathrm{st}} \cap \mathbb{Z}^2)$ and $\mathrm{area}_N(P) = \mathrm{area}_{\mathbb{Z}^2}(P^{\mathrm{st}})$.

• **LDP-polygons**. Let $N \subset \mathbb{R}^2$ be a lattice. A point $\mathbf{n} \in N \smallsetminus \{\mathbf{0}\}$ is said to be *primitive* (w.r.t. N) if

$$\mathrm{conv}(\{\mathbf{0}, \mathbf{n}\}) \cap N = \{\mathbf{0}, \mathbf{n}\}.$$

Definition 1.10. (i) A polygon $Q \in \mathrm{POL}_{\mathbf{0}}(N)$ is called *LDP-polygon* if it has primitive vertices.

(ii) Let $Q \subset \mathbb{R}^2$ be an LDP-polygon (w.r.t. N) and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual of our reference lattice N. For $F \in \mathrm{Edg}(Q)$ we denote by $\boldsymbol{\eta}_F \in M$ the unique primitive lattice point which defines an *inward-pointing* normal of F. The affine hull of F is of the form $\{ \mathbf{y} \in \mathbb{R}^2 | \, \langle -\boldsymbol{\eta}_F, \mathbf{y} \rangle = l_F \}$, for some positive integer l_F. This l_F is nothing but the integral distance between $\mathbf{0}$ and F, the so-called *local index* of F (w.r.t. Q). The *index* ℓ of Q is defined to be the positive integer $\ell := \mathrm{lcm}\{ l_F | \, F \in \mathrm{Edg}(Q)\}$. It is easy to prove that

$$\boxed{\ell = \min\{ k \in \mathbb{Z}_{>0} \, | \mathrm{Vert}(kQ^{\circ}) \subset M \}.} \tag{1.6}$$

(Note that the M-polygon ℓQ° is not necessarily an LDP-polygon w.r.t. M.)

For every positive integer ℓ we define

$$\mathrm{LDP}(\ell; N) := \{ [Q]_N \, | Q \in \mathrm{POL}_{\mathbf{0}}(N) \text{ is an LDP-polygon of index } \ell \}.$$

Theorem 1.11. $\sharp(\mathrm{LDP}(\ell; N)) < \infty$ *for all* $\ell \geqslant 1$.

This can be derived by using (more general) results of Hensley [35], and Lagarias & Ziegler [50]. LDP-polygons are of particular interest because their $\backsim_N$-classes parametrise the isomorphism classes of toric log del Pezzo surfaces. (See below §5.) LDP-triangles of index $\leqslant 3$ have been classified (up to unimodular transformation) in [18, §6] and [19]. More recently, this classification has been extended considerably in [45] via a certain algorithm, by means of which it is possible to produce the LDP-polygons of given index ℓ (up to unimodular transformation) by fixing a "special" edge and performing a prescribed successive addition of vertices. Of course, their cardinality grows rapidly as we increase indices! The classification is complete for $\ell \leqslant 17$.

Theorem 1.12 ([45]). *The values of the enumerating function $\ell \mapsto \sharp(\mathrm{LDP}(\ell; N))$ for $\ell \leqslant 17$ are those given in the following tables:*

ℓ	1	2	3	4	5	6	7	8	9
$\sharp(\mathrm{LDP}(\ell; N))$	16	30	99	91	250	379	429	307	690

ℓ	10	11	12	13	14	15	16	17
$\sharp(\mathrm{LDP}(\ell; N))$	916	939	1279	1142	1545	4312	1030	1892

Useful details for the structure of each of these $16+30+\cdots+1892 = 15346$ LDP-polygons (vertices of representatives of standard models w.r.t. a suitable coordinate system, interior and boundary lattice points, area, local indices, Ehrhart and Hilbert series etc.) are included in the database [9].

• **Reflexive polygons**. Firstly we focus on LDP-polygons of index 1.

Proposition 1.13. *Let $Q \subset \mathbb{R}^2$ be an LDP-polygon of index ℓ (w.r.t. N). Then the following conditions are equivalent*:

(i) *The polar polygon Q° of Q is an M-polygon, where $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.*

(ii) $\ell = 1$.

(iii) $l_F = 1$ *for all* $F \in \mathrm{Edg}(Q)$.

(iv) $\mathrm{int}(Q) \cap N = \{\mathbf{0}\}$.

(v) $\mathrm{Ehr}_N(Q; k) = \mathrm{Ehr}_N^{\mathrm{int}}(Q; k+1)$ *for all* $k \in \mathbb{Z}_{\geqslant 0}$.

Proof. By the definition of ℓ and by (1.6), the equivalences (i)⇔(ii)⇔(iii) are obvious. For (v)⇔(i) see Hibi [36, §35], [37]. The implication (v)⇒(iv) is apparently true (by applying (v) for $k = 0$). It suffices to show the validity of implication (iv)⇒(ii). Assuming that $\mathrm{int}(Q) \cap N = \{\mathbf{0}\}$, (1.5) gives

$$\begin{aligned} 1 &= \sharp\,(\mathrm{int}(Q) \cap N) = \mathrm{Ehr}_N^{\mathrm{int}}(Q; 1) = \mathrm{area}_N(Q) - \tfrac{1}{2}\sharp\,(\partial Q \cap N) + 1 \\ &\Rightarrow\ 2\mathrm{area}_N(Q) = \sharp\,(\partial Q \cap N)\,. \end{aligned} \tag{1.7}$$

Let F be an edge of Q having $\mathbf{n}, \mathbf{n}'$ as vertices. We observe that $l_F = \frac{1}{\sharp(F\cap N)-1} \cdot \frac{|\det(\mathbf{n},\mathbf{n}')|}{\det(N)}$. Next, we subdivide the triangle $T_F := \mathrm{cov}(\{\mathbf{0}, \mathbf{n}, \mathbf{n}'\})$ into $\sharp\,(F \cap N) - 1$ subtriangles, each of which having $\mathbf{0}$ and two consecutive lattice points of F as vertices. Obviously,

$$2\,\mathrm{area}_N(Q) = 2 \sum_{F \in\, \mathrm{Edg}(Q)} \mathrm{area}_N(T_F) = \sum_{F \in\, \mathrm{Edg}(Q)} (\sharp\,(F \cap N) - 1)\, l_F.$$

If there were an edge with local index > 1 (w.r.t. Q), we would have

$$2\,\mathrm{area}_N(Q) = \sum_{F \in \, \mathrm{Edg}(Q)} (\sharp\,(F \cap N) - 1)\, l_F > \sum_{F \in \, \mathrm{Edg}(Q)} (\sharp\,(F \cap N) - 1) = \sharp\,(\partial Q \cap N)\,,$$

contradicting (1.7). Thus, $l_F = 1$ for all $F \in \mathrm{Edg}(Q)$. □

Definition 1.14. An LDP-polygon $Q \subset \mathbb{R}^2$ (w.r.t. N) is called *reflexive polygon* (and (Q, N) *reflexive pair*) if it satisfies the conditions of Proposition 1.13.

Note 1.15. (i) If (Q, N) is a reflexive pair and M the dual of N, then (Q°, M) is again a reflexive pair. (See [3, Theorem 4.1.6, p. 510].)

(ii) The notion of reflexivity is extendable to lattice polytopes of *any* dimension $\geqslant 3$ via conditions 1.13 (i), (iii) and (v) (which remain equivalent). It was introduced by Batyrev in [3, §4]. Condition 1.13 (iv) is necessary (for a lattice polytope of dimension $\geqslant 3$ to be reflexive) but is not sufficient: There are several lattice polytopes of dimension $\geqslant 3$ which have the origin as the only interior lattice point without being reflexive. (Reflexive polytopes play a pivotal role in the so-called "combinatorial mirror symmetry"; cf. [16, Chapters 3 and 4] and [5]). On the other hand, in dimension 2 we meet nice lattice point enumerator identities like (1.8).

Theorem 1.16 (Twelve-Point Theorem)**.** *If (Q, N) is a reflexive pair and M the dual lattice of N, then*

$$\boxed{\sharp\,(\partial Q \cap N) + \sharp(\partial Q^\circ \cap M) = 12.} \tag{1.8}$$

One proof consists in case-by-case verification of (1.8) by passing through the explicit classification of reflexive polygons, i.e., by the so-called "exhaustion method".

Theorem 1.17 (Classification of reflexive polygons)**.** *Let (Q, N) be a reflexive pair. Then Q has at most 6 vertices, and a representative of the equivalence class of its standard model (w.r.t. any basis matrix of N) is exactly one of the sixteen $\mathbb{Z}^2$-polygons $\mathcal{Q}_1, \ldots, \mathcal{Q}_{16}$ illustrated in Figure 1, whose vertices (in an anticlockwise order) are given in the second columns of the tables:*

i	vertices of $\mathcal{Q}_i$	$\sharp(\partial\mathcal{Q}_i \cap \mathbb{Z}^2)$
1	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{-1}$	3
2	$\binom{1}{0}, \binom{0}{1}, \binom{-2}{-1}$	4
3	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{0}, \binom{0}{-1}$	4
4	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}$	4
5	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{1}, \binom{-1}{-1}$	5
6	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}, \binom{0}{-1}$	5
7	$\binom{1}{-1}, \binom{-1}{2}, \binom{-1}{-1}$	6
8	$\binom{1}{0}, \binom{-1}{1}, \binom{-1}{-1}, \binom{1}{-1}$	6

i	vertices of $\mathcal{Q}_i$	$\sharp(\partial\mathcal{Q}_i \cap \mathbb{Z}^2)$
9	$\binom{1}{0}, \binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}, \binom{1}{-1}$	6
10	$\binom{1}{0}, \binom{1}{1}, \binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}, \binom{0}{-1}$	6
11	$\binom{-1}{-1}, \binom{1}{-1}, \binom{1}{0}, \binom{0}{1}, \binom{-1}{1}$	7
12	$\binom{-1}{-1}, \binom{0}{-1}, \binom{1}{0}, \binom{-1}{2}$	7
13	$\binom{-1}{-1}, \binom{1}{-1}, \binom{1}{0}, \binom{-1}{2}$	8
14	$\binom{-1}{-1}, \binom{1}{-1}, \binom{1}{1}, \binom{-1}{1}$	8
15	$\binom{-1}{-1}, \binom{1}{-1}, \binom{-1}{3}$	8
16	$\binom{-1}{-1}, \binom{2}{-1}, \binom{-1}{2}$	9

(1.9)

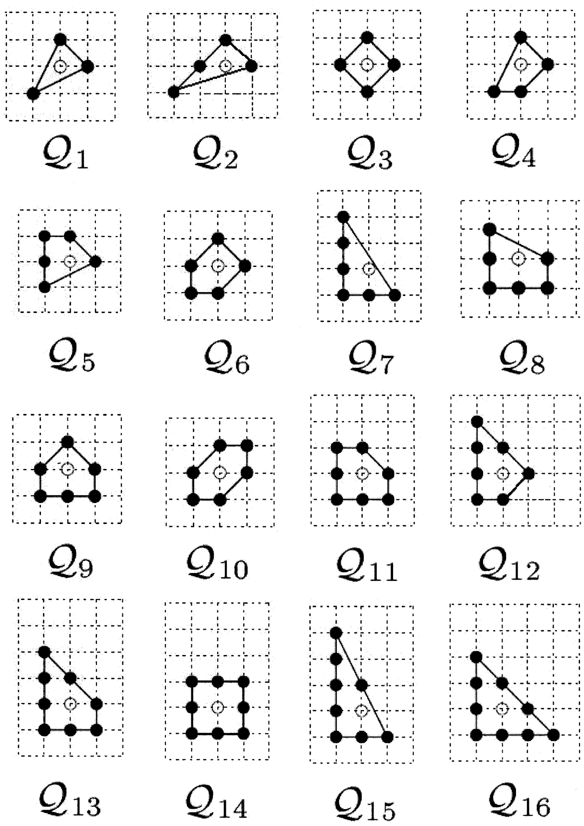


Figure 1: The sixteen $\mathbb{Z}^2$-polygons $\mathcal{Q}_1, \ldots, \mathcal{Q}_{16}$

Proof of Theorem 1.16 via Theorem 1.17. One checks directly that

$$\mathcal{Q}_{17-i} = \mathcal{Q}_i^\circ, \text{ for } i \in \{1,2,3,4,5,6\}, \tag{1.10}$$

and that the vertices of the polars of $\mathcal{Q}_7, \ldots, \mathcal{Q}_{10}$ are the following:

i	vertices of $\mathcal{Q}_i^\circ$
7	$\binom{-3}{-2}, \binom{1}{0}, \binom{0}{1}$
8	$\binom{-1}{-2}, \binom{1}{0}, \binom{0}{1}, \binom{-1}{0}$
9	$\binom{-1}{-1}, \binom{1}{-1}, \binom{1}{0}, \binom{0}{1}, \binom{-1}{0}$
10	$\binom{-1}{0}, \binom{0}{-1}, \binom{1}{-1}, \binom{1}{0}, \binom{0}{1}, \binom{-1}{1}$

Defining

$$\mathcal{A}_1 := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right), \ \mathcal{A}_2 := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \ \mathcal{A}_3 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \ \mathcal{A}_4 := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

we see that

$$\Phi_{\mathcal{A}_i}(\mathcal{Q}_{i+6}) = \mathcal{Q}_{i+6}^\circ \Longrightarrow [\mathcal{Q}_{i+6}]_{\mathbb{Z}^2} = [\mathcal{Q}_{i+6}^\circ]_{\mathbb{Z}^2}, \text{ for } i \in \{1,2,3,4\}. \tag{1.11}$$

The entries of the third columns of tables (1.9), combined with (1.10) and (1.11), give

$$\sharp(\partial\mathcal{Q}_i \cap \mathbb{Z}^2) + \sharp(\partial\mathcal{Q}_i^\circ \cap \mathbb{Z}^2) = 12, \text{ for all } i \in \{1, \ldots, 16\},$$

and therefore (1.8) is true. □

Note 1.18. (i) Although the above proof of (1.8) is elementary, it is not very enlightening because it does not explain where 12 comes from. Moreover, the *earliest* known proof of Theorem 1.17 (due to Batyrev [2] (reproduced in [4, Part A, Theorem 6.1.6])) makes use of formula (1.8)! (This happens implicitly also in [18, Theorem 6.10, pp. 108-109].)

(ii) The first purely combinatorial proof of Theorem 1.17 is due to Rabinowitz [60] who classified lattice polygons with exactly one interior lattice point. (In [60, Proposition 3, p. 89], the author failed to include $[\mathcal{Q}_{13}]_{\mathbb{Z}^2}$ but this inaccuracy can be removed easily by his own techniques.) For other proofs of Theorem 1.17 (which do not use (1.8)) the reader is referred (in chronological order) to Koelman [49, Theorem 4.2.4, p. 86], Sato [64, Theorem 6.22, p. 401], Nill [54, Proposition 3.4.1, pp. 55-57], and Kasprzyk [44, Proposition 5.2.4, pp. 59-60].

(iii) In fact, as it follows from the work of Batyrev mentioned in (i), the most natural interpretation for the presence of the number 12 in (1.8) arises from the application of the celebrated *Noether's formula* for the Euler–Poincaré characteristic of the structure sheaf of the minimally desingularized compact toric surface which is associated with a reflexive polygon. (See also [17, Theorem 10.5.10, pp. 510-511] and Note 7.5 below.) Max Noether [55] discovered this remarkable formula in 1870's in the framework of the theory of adjoints of algebraic surfaces. For comments on its early history and for a modern direct proof see Gray [30, §2], Hulek [43, §3] and Piene [58]. Besides, Noether's formula can be viewed, alternatively, as the Hirzebruch–Riemann–Roch formula [41, p. 154] in (complex) dimension 2. (The coefficient of t^2 in the expansion of $\frac{t}{\exp(t)-1}$ as a Taylor series equals 1/12.)

(iv) Poonen & Rodriguez-Villegas [59] added two new proofs of Theorem 1.16: (a) by stepping into the space of reflexive polygons, and (b) by exploiting basic properties of the universal cover of $\mathrm{SL}_2(\mathbb{R})$ and of the corresponding modular (cusp) forms of weight 12. (See also Castryck [11, §2].) Elementary geometric proofs (using reduction to the parallelogram case and Scott's inequality [66], respectively) are due to Cencelj, Repovš & Skopenkov [14], and Burns & O'Keeffe [10]. On the other hand, Hille & Skarke have shown in [39, Theorem 1.2] that there is a one-to-one correspondence between the set $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_{16}\}$ and certain relations of the generators $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Z})$ of length 12. Higashitani & Masuda [38] calculated the rotation number of *unimodular sequences* (see also Živaljević [70] for another approach), gave an alternative proof of the *generalised* Twelve-Point Theorem (for *legal loops*) of Poonen & Rodriguez-Villegas [59, §9.1], and studied the Ehrhart polynomials of *lattice multipolygons*. Finally, Batyrev & Schaller [6, Corollary 5.3] (and independently, Douai [21, §8]) proved that the so-called *stringy Libgober–Wood identity* for reflexive polygons is equivalent to (1.8).

Remark 1.19. For $i \in \{1, \ldots, 16\}$ let us define the $\mathbb{Z}^2$-polygons $\overline{\mathcal{Q}}_i$ as follows:

i	vertices of $\overline{\mathcal{Q}}_i$	i	vertices of $\overline{\mathcal{Q}}_i$	i	vertices of $\overline{\mathcal{Q}}_i$	i	vertices of $\overline{\mathcal{Q}}_i$
1	$\binom{0}{1}, \binom{-1}{-2}, \binom{1}{1}$	5	$\binom{0}{1}, \binom{-1}{0}, \binom{-1}{-2}, \binom{1}{1}$	9	$\binom{0}{1}, \binom{-1}{0}, \binom{0}{-1}, \binom{1}{-1}, \binom{1}{1}$	13	$\binom{0}{1}, \binom{-2}{-1}, \binom{1}{-1}, \binom{1}{1}$
2	$\binom{0}{1}, \binom{-1}{-1}, \binom{2}{1}$	6	$\binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}, \binom{0}{-1}, \binom{1}{1}$	10	$\binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}, \binom{0}{-1}, \binom{1}{0}, \binom{1}{1}$	14	$\binom{0}{1}, \binom{-2}{-1}, \binom{0}{-1}, \binom{2}{1}$
3	$\binom{0}{1}, \binom{-1}{-1}, \binom{0}{-1}, \binom{1}{1}$	7	$\binom{0}{1}, \binom{-3}{-2}, \binom{2}{1}$	11	$\binom{0}{1}, \binom{-1}{0}, \binom{-1}{-1}, \binom{1}{-1}, \binom{1}{1}$	15	$\binom{0}{1}, \binom{-2}{-1}, \binom{4}{1}$
4	$\binom{0}{1}, \binom{-1}{0}, \binom{0}{-1}, \binom{1}{1}$	8	$\binom{0}{1}, \binom{-2}{-1}, \binom{0}{-1}, \binom{1}{1}$	12	$\binom{0}{1}, \binom{-1}{0}, \binom{1}{-2}, \binom{1}{1}$	16	$\binom{0}{1}, \binom{-3}{-2}, \binom{3}{1}$

It is straightforward to see that $[\overline{\mathcal{Q}}_i]_{\mathbb{Z}^2} = [\mathcal{Q}_i]_{\mathbb{Z}^2}$ for all $i \in \{1, \dots, 16\}$,

$$[\overline{\mathcal{Q}}_{17-j}]_{\mathbb{Z}^2} = [\overline{\mathcal{Q}}_j^\circ]_{\mathbb{Z}^2} \text{ for } j \in \{1,2,3,4,5,6\}, \text{ and } [\overline{\mathcal{Q}}_{j+6}]_{\mathbb{Z}^2} = [\overline{\mathcal{Q}}_{j+6}^\circ]_{\mathbb{Z}^2} \text{ for } j \in \{1,2,3,4\}.$$

This *particular choice* of representatives from each of the 16 available equivalence classes is such that $\binom{0}{1} \in \mathrm{Vert}(\overline{\mathcal{Q}}_i)$ *for all* $i \in \{1, \dots, 16\}$, and will be convenient for the formulation of Theorem 6.9.

• **ℓ-Reflexive polygons**. Motivated by condition 1.13 (iii) one generalises Definition 1.14 as follows:

Definition 1.20. Let $Q \subset \mathbb{R}^2$ be an LDP-polygon of index ℓ (w.r.t. N). Q is called *ℓ-reflexive polygon* (and (Q, N) *ℓ-reflexive pair*) if $l_F = \ell$ *for all* $F \in \mathrm{Edg}(Q)$. (The terms *reflexive* and 1-*reflexive* polygon (or pair) coincide.)

Proposition 1.21. *If* (Q, N) *is an ℓ-reflexive pair, then*

$$\sharp\,(\partial Q \cap N) = \frac{2\,\mathrm{area}_N(Q)}{\ell}. \tag{1.12}$$

Proof. If $F \in \mathrm{Edg}(Q)$ with $\mathbf{n}, \mathbf{n}'$ as vertices, and $T_F := \mathrm{cov}(\{\mathbf{0}, \mathbf{n}, \mathbf{n}'\})$, then $l_F = \frac{1}{\sharp(F\cap N)-1} \cdot \frac{|\det(\mathbf{n},\mathbf{n}')|}{\det(N)} = \ell$ (by definition), and

$$\mathrm{area}_N(Q) = \sum_{F \in\, \mathrm{Edg}(Q)} \mathrm{area}_N(T_F) = \frac{\ell}{2}\left(\sum_{F \in\, \mathrm{Edg}(Q)} (\sharp\,(F \cap N) - 1)\right) = \frac{\ell}{2}\sharp\,(\partial Q \cap N)$$

because $\mathrm{area}_N(T_F) = \frac{1}{2}\ell\,(\sharp\,(F \cap N) - 1)$ for all $F \in \mathrm{Edg}(Q)$. □

Corollary 1.22. *If* (Q, N) *is an ℓ-reflexive pair and* $\sharp\,(\partial Q \cap N)$ *is odd, then* ℓ *is odd.*

Proof. By (1.3) and (1.12) we have

$$\sharp\,(Q \cap N) - 1 = \mathrm{area}_N(Q) + \tfrac{1}{2}\sharp\,(\partial Q \cap N) = (\tfrac{\ell+1}{2})\sharp\,(\partial Q \cap N)\,,$$

Hence, if $\sharp\,(\partial Q \cap N)$ is odd, then ℓ has to be odd. □

Next, we introduce the notation

$$\mathrm{RP}(\ell; N) := \{\,[Q]_N \in \mathrm{LDP}(\ell; N) \,|\, Q \text{ is an } \ell\text{-reflexive polygon}\}\,,$$

and for every integer $\nu \geqslant 3$ we set

$$\mathrm{RP}_{\nu}(\ell;N) := \{ [Q]_N \in \mathrm{RP}(\ell;N) \,|\, \sharp(\mathrm{Vert}(Q)) = \nu \} \,.$$

As will be seen in the sequel, there are no ℓ-reflexive polygons having more than 6 vertices and there are no ℓ-reflexive polygons with ℓ even (see Corollaries 7.6 and 7.8). For the time being, taking into account the precise polygon data from [9] we deduce the following:

Corollary 1.23. *The values of the enumerating functions*

$$\ell \mapsto \sharp(\mathrm{RP}_{\nu}(\ell;N)) \quad \text{and} \quad \ell \mapsto \sharp(\mathrm{RP}(\ell;N))$$

for $\nu \in \{3,4,5,6\}$ *and for odd* $\ell \leqslant 25$ *are those given in the table*:

ℓ	1	3	5	7	9	11	13	15	17	19	21	23	25
$\sharp(\mathrm{RP}_3(\ell;N))$	5	0	1	6	0	14	20	0	28	34	0	42	5
$\sharp(\mathrm{RP}_4(\ell;N))$	7	0	7	15	0	33	43	0	61	69	0	87	27
$\sharp(\mathrm{RP}_5(\ell;N))$	3	0	3	6	0	12	15	0	21	24	0	30	15
$\sharp(\mathrm{RP}_6(\ell;N))$	1	1	1	2	1	2	3	1	3	4	2	4	3
$\sharp(\mathrm{RP}(\ell;N))$	16	1	12	29	1	61	81	1	113	131	2	163	50

In addition, examples 1.28 show that $\bigcup_{\ell \text{ odd}} \mathrm{RP}_{\nu}(\ell;N)$ is an infinite set for all $\nu \in \{3,4,5,6\}$.

Definition 1.24. Let $Q \subset \mathbb{R}^2$ be an ℓ-reflexive polygon (w.r.t. N) *and* $M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$. The M-polygon

$$\boxed{Q^* := \ell Q^{\circ} \subset \mathbb{R}^2}$$

will be called the *dual* of Q. (The polar and the dual of Q coincide only if $\ell = 1$.)

Proposition 1.25. *If* (Q,N) *is an* ℓ*-reflexive pair and* M *the dual of* N*, then* (Q^*,M) *is again an* ℓ*-reflexive pair.*

Proof. Since the affine hull of every $F \in \mathrm{Edg}(Q)$ is of the form $\{\mathbf{y} \in \mathbb{R}^2 |\, \langle \boldsymbol{\eta}_F, \mathbf{y} \rangle = -\ell\}$, we have

$$Q = \bigcap_{F \in \mathrm{Edg}(Q)} \{\mathbf{y} \in \mathbb{R}^2 |\, \langle \boldsymbol{\eta}_F, \mathbf{y} \rangle \geqslant -\ell\} \Rightarrow \mathrm{Vert}(Q^*) = \{\boldsymbol{\eta}_F \,|F \in \mathrm{Edg}(Q)\} \subset M,$$

i.e., $\mathrm{Vert}(Q^*)$ consists of primitive lattice points, and $\mathbf{0} \in \mathrm{int}(Q^{\circ}) \subseteq \mathrm{int}(Q^*)$. Since the affine hull of any edge of Q^* is of the form $\{\mathbf{x} \in \mathbb{R}^2 |\, \langle \mathbf{x}, -\mathbf{n} \rangle = \ell\}$ for some $\mathbf{n} \in \mathrm{Vert}(Q)$, the integral distance between $\mathbf{0}$ and the edge equals ℓ. Hence, Q^* is an LDP-polygon of index ℓ w.r.t. M. □

Note 1.26. In analogy to (1.1), (1.2), we establish bijections:

$$\mathrm{Vert}(Q) \ni \mathbf{n} \longmapsto \{\mathbf{x} \in Q^* | \langle \mathbf{x}, \mathbf{n} \rangle = -\ell\} \in \mathrm{Edg}(Q^*), \tag{1.13}$$

and

$$\mathrm{Edg}(Q) \ni F \longmapsto \{\mathbf{x} \in Q^* | \langle \mathbf{x}, \mathbf{n} \rangle = \langle \mathbf{x}, \mathbf{n}' \rangle = -\ell\} = \boldsymbol{\eta}_F \in \mathrm{Vert}(Q^*), \tag{1.14}$$

where $\mathbf{n}, \mathbf{n}'$ denote the vertices of F.

Based on the involution

$$\{\ell\text{-reflexive pairs}\} \ni (Q, N) \longmapsto (Q^*, M) \in \{\ell\text{-reflexive pairs}\},$$

on (1.13) and (1.14), and on Theorems 6.8 and 6.9, we shall give a second proof of the following:

Theorem 1.27 (Twelve-Point Theorem for $\boldsymbol{\ell}$-Reflexive Pairs, [46])**.** *If (Q, N) is an ℓ-reflexive pair, then*

$$\boxed{\sharp\,(\partial Q \cap N) + \sharp(\partial Q^* \cap M) = 12.} \tag{1.15}$$

Examples 1.28. Let ℓ be a positive odd integer. We define ℓ-reflexive polygons w.r.t. the standard lattice $\mathbb{Z}^2$ as follows:

(i) If $3 \nmid \ell$ and $5 \nmid \ell$, then

$$Q := \mathrm{conv}\left(\left\{\binom{5}{-2\ell}, \binom{-1}{\ell}, \binom{-4}{\ell}\right\}\right) \tag{1.16}$$

is an ℓ-reflexive triangle having

$$Q^* := \ell Q^\circ = \mathrm{conv}\left(\left\{\binom{-\ell}{-2}, \binom{0}{-1}, \binom{\ell}{3}\right\}\right) \tag{1.17}$$

as its dual. Obviously,

$$\sharp\left(\partial Q \cap \mathbb{Z}^2\right) = \gcd(6, 3\ell) + \gcd(3, 0) + \gcd(9, 3\ell) = 3 \cdot 3 = 9, \quad \sharp\left(\partial Q^* \cap \mathbb{Z}^2\right) = 3.$$

(ii) If $3 \nmid \ell$, then

$$Q := \mathrm{conv}\left(\left\{\binom{3}{-\ell}, \binom{-1}{\ell}, \binom{-3}{\ell}, \binom{1}{-\ell}\right\}\right) \tag{1.18}$$

is an ℓ-reflexive quadrilateral with

$$Q^* = \mathrm{conv}\left(\left\{\binom{-\ell}{-2}, \binom{0}{-1}, \binom{\ell}{2}, \binom{0}{1}\right\}\right) \tag{1.19}$$

and

$$\sharp\left(\partial Q \cap \mathbb{Z}^2\right) = 8, \quad \sharp\left(\partial Q^* \cap \mathbb{Z}^2\right) = 4.$$

(iii) If $3 \nmid \ell$, then

$$Q := \mathrm{conv}\left(\left\{\binom{3}{-2\ell}, \binom{1}{0}, \binom{-1}{\ell}, \binom{-2}{\ell}, \binom{-1}{0}\right\}\right) \tag{1.20}$$

is an ℓ-reflexive pentagon with

$$Q^* = \operatorname{conv}\left(\left\{\binom{-\ell}{-1}, \binom{-\ell}{-2}, \binom{0}{-1}, \binom{\ell}{1}, \binom{\ell}{2}\right\}\right) \tag{1.21}$$

and

$$\sharp\left(\partial Q \cap \mathbb{Z}^2\right) = 7, \quad \sharp\left(\partial Q^* \cap \mathbb{Z}^2\right) = 5.$$

(iv) The hexagon

$$Q := \operatorname{conv}\left(\left\{\binom{1}{0}, \binom{-1}{\ell}, \binom{-2}{\ell}, \binom{-1}{0}, \binom{1}{-\ell}, \binom{2}{-\ell}\right\}\right) \tag{1.22}$$

is ℓ-reflexive having

$$Q^* = \operatorname{conv}\left(\left\{\binom{-\ell}{-2}, \binom{0}{-1}, \binom{\ell}{1}, \binom{\ell}{2}, \binom{0}{1}, \binom{-\ell}{-1}\right\}\right) \tag{1.23}$$

as its dual, with $[Q]_{\mathbb{Z}^2} = [Q^*]_{\mathbb{Z}^2}$ (because anticlockwise rotation through $\pi/2$ maps Q onto Q^*), and

$$\sharp\left(\partial Q \cap \mathbb{Z}^2\right) = \sharp\left(\partial Q^* \cap \mathbb{Z}^2\right) = 6.$$

For $\ell = 3$ this is illustrated in Figure 2. (Here, the $\ell = 3$ case gives an interesting example, because the associated toric log del Pezzo surface is the only log del Pezzo surface among those with Fano index 1, anticanonical degree $\geqslant 2$ and singularities of type $(2, 3)$ (in our notation), the regular locus of which has *non-trivial* fundamental group. See Corti & Heuberger [15, Proposition 1.8 (b), pp. 83-84].)

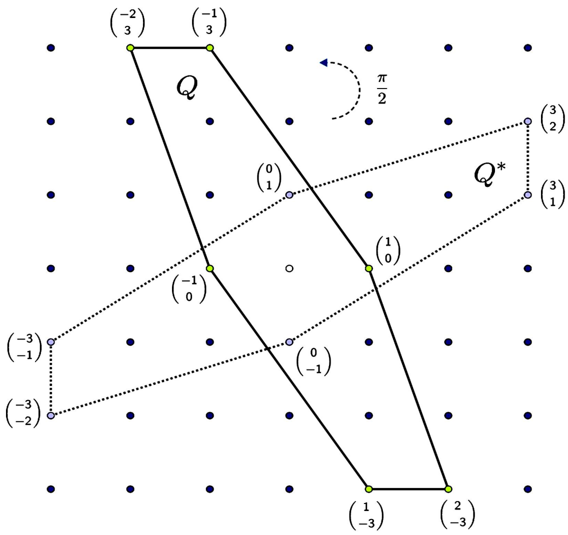


Figure 2: The $\ell = 3$ case

Since ℓ does not admit upper bound, the "exhaustion method" is apparently not the right method to verify formula (1.15). The first proof of Theorem 1.27 given by Kasprzyk and Nill in [46] is purely combinatorial, clear and short, and makes use of the so-called *ℓ-reflexive loops*. Nevertheless, it offers no essential information about the connection with the "classical" approach mentioned in 1.18 (iii). In [46, §1.6] the question was raised, whether there is also another *direct* argument arising from algebraic geometry in the case of ℓ-reflexive polygons. Here, maintaining the technique of *lattice change* from [46, §2.2] in our toolbox, we shall provide such an argument and a second proof of Theorem 1.27: Its disadvantage lies in that it is by no means short (as one has to translate everything into the language of toric varieties, and this requires several steps). On the other hand, among its main advantages are included: (a) Noether's formula remains *again* at least one assured reason for the appearance of 12 (in combination with other useful formulae in the $\ell > 1$ case), and (b) several other results are obtained by transferring the *duality concept* from the ℓ-reflexive polygons to the corresponding log del Pezzo surfaces.

More precisely, the paper is organised as follows: In Section 2 we focus on the two non-negative, relatively prime integers $p = p_\sigma$ and $q = q_\sigma$ parametrising the N-cones σ and characterising the two-dimensional toric singularities. Moreover, we describe briefly the minimal desingularization procedure by means of the negative-regular continued fraction expansion of $\frac{q}{q-p}$ and by determining the exceptional prime divisors after the Hilbert basis computation of the corresponding cone. In section 3 we recall the interrelation between lattice polygons and compact toric surfaces with a fixed ample divisor, and explain how one computes the area and the number of lattice points lying on the boundary of a lattice polygon via intersection numbers. (See Theorems 3.8 and 3.9.) In §4-§5 we indicate the manner in which we classify (up to isomorphism) compact toric surfaces via the WVE^{2}C-*graphs* and, in particular, toric log del Pezzo surfaces via LDP-*polygons*. Giving priority to those log del Pezzo surfaces which are associated with ℓ-*reflexive polygons* we present in §6 the geometric meaning of the *lattice change* from [46, §2.2] (which, in a sense, seems to be the standard method of reducing ℓ-reflexivity to 1-reflexivity): One may patch together canonical cyclic covers over the singularities in order to construct a finite holomorphic map of degree ℓ and to represent the surfaces under consideration as *global quotients* of *Gorenstein* del Pezzo surfaces by finite cyclic groups of order ℓ. Results of this geometric interpretation (e.g., Proposition 6.13, concerning the relation between the self-intersection numbers of the canonical divisors), combined with Noether's formula and other information derived from the desingularization, give rise to a new proof of Theorem 1.27 in §7 and to various consequences of it (upper bound for $\sharp(\mathrm{Vert}(Q))$, a proof of "oddness" of ℓ, a new approach of Suyama's formula, number-theoretic identities involving types of singularities, combinatorial triples, Dedekind sums etc.). In section 8 we discuss certain new phenomena which occur in the $\ell > 1$ case, and give typical examples. For instance, the *characteristic differences*

$$\sharp\,(\partial Q^* \cap M) - K^2_{X(N,\widetilde{\Delta}_Q)} = e(X(N,\widetilde{\Delta}_Q)) - \sharp\,(\partial Q \cap N)$$

$$\text{and}\quad \sharp\,(\partial Q \cap N) - K^2_{X(M,\widetilde{\Delta}_{Q^*})} = e(X(M,\widetilde{\Delta}_{Q^*})) - \sharp\,(\partial Q^* \cap M)$$

no longer vanish (as in the $\ell = 1$ case, where each 1-reflexive polygon has only the origin in its interior), but are equal to the number of lattice points lying on the boundary of $\mathbf{I}(Q^*)$ and $\mathbf{I}(Q)$, i.e., of the polygons defined as convex hulls of the (at least 4, non-collinear) interior lattice points of Q^* and Q, respectively. Finally, in §9 we verify (in the lowest dimension) the existence of a large number of families of *combinatorial mirror pairs* (of certain smooth curves of *high* genus, owing to this new wider framework of duality) and in §10 we state some open questions.

We use tools from discrete and classical toric geometry (adopting the standard terminology from [17], [24], [27], and [56]), and some basic facts and formulae from intersection theory (see [7, Ch. I], [18, §4], [28, §2.2-2.4, §7.1, and §15.2], and [53, §II(b)]), working over $\mathbb{C}$ and within the analytic category (with complex (analytic) spaces as objects, holomorphic maps as morphisms and biholomorphic maps as isomorphisms).

2 Two-dimensional lattice cones and toric surfaces

• **N-cones**. A 2-dimensional strongly convex polyhedral cone in $\mathbb{R}^2$ (with the origin $\mathbf{0} \in \mathbb{R}^2$ as its apex and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^2 \smallsetminus \{\mathbf{0}\}$ as generators) is a subset σ of $\mathbb{R}^2$ of the form

$$\sigma = \mathbb{R}_{\geqslant 0}\mathbf{x}_1 + \mathbb{R}_{\geqslant 0}\mathbf{x}_2 := \{\lambda_1\ \mathbf{x}_1 + \lambda_1 \mathbf{x}_2 |\ \lambda_1, \lambda_2 \in \mathbb{R}_{\geqslant 0}\},$$

where $\mathbf{x}_1, \mathbf{x}_2$ are $\mathbb{R}$-linearly independent, and $\sigma \cap (-\sigma) = \{\mathbf{0}\}$.

Definition 2.1. Let N be a lattice in $\mathbb{R}^2$ (as defined in 1.2). A 2-dimensional strongly convex polyhedral cone $\sigma = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2 \subset \mathbb{R}^2$, generated by $\mathbf{n}_1, \mathbf{n}_2 \in N$, will be referred to as *N-cone* having $\mathbf{0} \in \mathbb{R}^2$ as its apex (0-dimensional face) and $\mathbb{R}_{\geqslant 0}\mathbf{n}_1 := \{\lambda\ \mathbf{n}_1 |\ \lambda \in \mathbb{R}_{\geqslant 0}\}$ and $\mathbb{R}_{\geqslant 0}\mathbf{n}_2 := \{\lambda\ \mathbf{n}_2 |\ \lambda \in \mathbb{R}_{\geqslant 0}\}$ as its *rays* (1-dimensional faces). If for $j \in \{1, 2\}$, $\breve{\mathbf{n}}_j$ is the unique primitive point (w.r.t. N) belonging to the ray $\mathbb{R}_{\geqslant 0}\mathbf{n}_j$,then we shall say that $\breve{\mathbf{n}}_j$ is the *minimal generator* of $\mathbb{R}_{\geqslant 0}\mathbf{n}_j$ and that $\breve{\mathbf{n}}_1, \breve{\mathbf{n}}_2$ are the *minimal generators* of σ. (Since $\sigma = \mathbb{R}_{\geqslant 0}\breve{\mathbf{n}}_1 + \mathbb{R}_{\geqslant 0}\breve{\mathbf{n}}_2$, one can always replace arbitrary generators of σ by the minimal ones). On the set of N-cones we define (as we did on $\mathrm{POL}_{\mathbf{0}}(N)$ in §1) the equivalence relation:

$$\sigma_1 \backsim_N \sigma_2 \underset{\text{def}}{\Longleftrightarrow} \exists \Psi \in \mathrm{Aut}_N(\mathbb{R}^2) : \Psi(\sigma_1) = \sigma_2.$$

If $\sigma_1 \backsim_N \sigma_2$, we say that σ_1 and σ_2 are *equivalent up to umimodular transformation*. If σ is an N-cone, we denote by $[\sigma]_N := \{N\text{-cones } \tau |\ \tau \backsim_N \sigma\}$ its equivalence class.

Definition 2.2. If σ is an N-cone, then for a fixed basis matrix $\mathcal{B} \in \mathrm{GL}_2(\mathbb{R})$ of the lattice N we have $N = \Phi_{\mathcal{B}}(\mathbb{Z}^2)$ with $\Phi_{\mathcal{B}} \in \mathrm{Aut}(\mathbb{R}^2)$. Thus, we may define the *standard model* of σ w.r.t. $\mathcal{B}$, i.e., the $\mathbb{Z}^2$-cone $\sigma^{\mathrm{st}} := \Phi_{\mathcal{B}^{-1}}(\sigma)$. By Proposition 1.4, $[\sigma^{\mathrm{st}}]_{\mathbb{Z}^2}$ does not depend on the particular choice of $\mathcal{B}$.

If the induced bijection $[\sigma]_N \longmapsto [\sigma^{\mathrm{st}}]_{\mathbb{Z}^2}$ is taken into account, it is in many cases convenient to work with the equivalence class of σ^{st} instead of that of σ (and with the standard lattice $\mathbb{Z}^2$ instead of N).

Definition 2.3. Let $\sigma = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2$ be an N-cone with $\mathbf{n}_1, \mathbf{n}_2$ as minimal generators. The *multiplicity* of σ is the positive integer

$$\mathrm{mult}_N(\sigma) := \frac{|\det(\mathbf{n}_1, \mathbf{n}_2)|}{\det(N)} = \frac{\det(N')}{\det(N)} = \sharp(\Pi \cap N), \tag{2.1}$$

where $\Pi := \{\xi_1\mathbf{n}_1 + \xi_2\mathbf{n}_2\,|\xi_1, \xi_2 \in [0, 1)\}$ and N' the sublattice of N having $\{\mathbf{n}_1, \mathbf{n}_2\}$ as basis (see Proposition 1.6).

Proposition 2.4. *Let* $\sigma = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2$ *be an* N*-cone with* $\mathbf{n}_1 = \binom{n_{11}}{n_{21}}, \mathbf{n}_2 = \binom{n_{12}}{n_{22}} \in N$ *as minimal generators, and* $\mathcal{B} = \left(\begin{smallmatrix} \mathfrak{b}_{11} & \mathfrak{b}_{12} \\ \mathfrak{b}_{21} & \mathfrak{b}_{22} \end{smallmatrix}\right)$ *a basis matrix of* N*. Denote by*

$$\overline{\mathbf{n}}_1 = \binom{\overline{n}_{11}}{\overline{n}_{21}} := \Phi_{\mathcal{B}^{-1}}(\mathbf{n}_1) \in \mathbb{Z}^2 \quad \textit{and} \quad \overline{\mathbf{n}}_2 = \binom{\overline{n}_{12}}{\overline{n}_{22}} := \Phi_{\mathcal{B}^{-1}}(\mathbf{n}_2) \in \mathbb{Z}^2$$

the minimal generators of its standard model $\sigma^{\mathrm{st}} := \Phi_{\mathcal{B}^{-1}}(\sigma)$ *w.r.t.* $\mathcal{B}$*, and consider* $\kappa, \lambda \in \mathbb{Z}$*, such that*

$$\kappa\overline{n}_{11} - \lambda\overline{n}_{21} = 1. \tag{2.2}$$

(i) *If* $q := |\det(\overline{\mathbf{n}}_1, \overline{\mathbf{n}}_2)|$ *and if* p *denotes the unique integer with*

$$0 \leqslant p < q \quad \textit{and} \quad \kappa\overline{n}_{12} - \lambda\overline{n}_{22} \equiv p \,(\mathrm{mod}\, q)\,, \tag{2.3}$$

then $\gcd(p, q) = 1$*, and there exists a primitive point* $\overline{\mathbf{n}}_1' \in \mathbb{Z}^2$*, such that* $\overline{\mathbf{n}}_2 = p\overline{\mathbf{n}}_1 + q\overline{\mathbf{n}}_1'$*, where* $\{\overline{\mathbf{n}}_1, \overline{\mathbf{n}}_1'\}$ *is a basis of* $\mathbb{Z}^2$*.*

(ii) *If* ε *is the sign of* $\det(\overline{\mathbf{n}}_1, \overline{\mathbf{n}}_2)$ *and* $\mathcal{M}_\sigma := \begin{pmatrix} \frac{\varepsilon}{q}(\overline{n}_{22} - \overline{n}_{21}p) & \frac{\varepsilon}{q}(\overline{n}_{11}p - \overline{n}_{12}) \\ -\varepsilon\overline{n}_{21} & \varepsilon\overline{n}_{11} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$*, then*

$$\Phi_{\mathcal{M}_\sigma}(\sigma^{\mathrm{st}}) = \mathbb{R}_{\geqslant 0}\tbinom{1}{0} + \mathbb{R}_{\geqslant 0}\tbinom{p}{q}, \quad \textit{i.e.,} \quad \left[\sigma^{\mathrm{st}}\right]_{\mathbb{Z}^2} = \left[\mathbb{R}_{\geqslant 0}\tbinom{1}{0} + \mathbb{R}_{\geqslant 0}\tbinom{p}{q}\right]_{\mathbb{Z}^2}.$$

(iii) *If* $\mathbf{n}_1' := \Phi_{\mathcal{B}}(\overline{\mathbf{n}}_1')$*, then* $\{\mathbf{n}_1, \mathbf{n}_1'\}$ *is a basis of* N*, and* $\mathbf{n}_2 = p\mathbf{n}_1 + q\mathbf{n}_1'$ *with* $q = \mathrm{mult}_N(\sigma)$*.*
The above integers $p =: p_\sigma$ *and* $q =: q_\sigma$ *associated with* σ *do not depend on the particular choice of* $\mathcal{B}$*.*

Proof. For (i) see [19, Lemma 2.1, p. 222]. (ii) can be checked directly. (Note that $\det(\mathcal{M}_\sigma) = \varepsilon$.)
(iii) Obviously, $\Phi_{\mathcal{B}}(\{\overline{\mathbf{n}}_1, \overline{\mathbf{n}}_1'\}) = \{\mathbf{n}_1, \mathbf{n}_1'\}$ is a basis of N and

$$\mathbf{n}_2 = \Phi_{\mathcal{B}}(\overline{\mathbf{n}}_2) = p\Phi_{\mathcal{B}}(\overline{\mathbf{n}}_1) + q\Phi_{\mathcal{B}}(\overline{\mathbf{n}}_1') = p\mathbf{n}_1 + q\mathbf{n}_1'.$$

On the other hand, since

$$\overline{\mathbf{n}}_1 = \Phi_{\mathcal{B}^{-1}}(\mathbf{n}_1) = \tfrac{1}{\det(\mathcal{B})}\binom{n_{11}\mathfrak{b}_{22} - n_{21}\mathfrak{b}_{12}}{n_{21}\mathfrak{b}_{11} - n_{11}\mathfrak{b}_{21}} \text{ and } \overline{\mathbf{n}}_2 = \Phi_{\mathcal{B}^{-1}}(\mathbf{n}_2) = \tfrac{1}{\det(\mathcal{B})}\binom{n_{12}\mathfrak{b}_{22} - n_{22}\mathfrak{b}_{12}}{n_{22}\mathfrak{b}_{11} - n_{12}\mathfrak{b}_{21}},$$

we have

$$q = |\det(\overline{\mathbf{n}}_1, \overline{\mathbf{n}}_2)| = \tfrac{1}{|\det(\mathcal{B})|^2}\,|\det(\mathbf{n}_1, \mathbf{n}_2)|\,|\det(\mathcal{B})| = \frac{|\det(\mathbf{n}_1, \mathbf{n}_2)|}{\det(N)} = \mathrm{mult}_N(\sigma)\,.$$

Therefore, $q = q_\sigma$ does not depend on the choice of $\mathcal{B}$. In addition, if $\mathcal{B}'$ is another basis matrix of N, then $\mathcal{B}' = \mathcal{B}\mathcal{A}$ for some $\mathcal{A} = \left(\begin{smallmatrix} \mathfrak{a}_{11} & \mathfrak{a}_{12} \\ \mathfrak{a}_{21} & \mathfrak{a}_{22} \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$ (see Proposition 1.4). Let

$$\widetilde{\mathbf{n}}_1 := \Phi_{(\mathcal{B}\mathcal{A})^{-1}}(\mathbf{n}_1) = \Phi_{\mathcal{A}^{-1}}(\overline{\mathbf{n}}_1) = \tfrac{1}{\det(\mathcal{A})}\binom{\mathfrak{a}_{22}\overline{n}_{11} - \mathfrak{a}_{12}\overline{n}_{21}}{\mathfrak{a}_{11}\overline{n}_{21} - \mathfrak{a}_{21}\overline{n}_{11}}$$

and

$$\widetilde{\mathbf{n}}_2 := \Phi_{\mathcal{A}^{-1}}(\overline{\mathbf{n}}_2) = \tfrac{1}{\det(\mathcal{A})}\binom{\mathfrak{a}_{22}\overline{n}_{12} - \mathfrak{a}_{12}\overline{n}_{22}}{\mathfrak{a}_{11}\overline{n}_{22} - \mathfrak{a}_{21}\overline{n}_{12}}$$

be the minimal generators of the standard model of σ w.r.t. $\mathcal{B}'$. We consider $\widetilde{\kappa}, \widetilde{\lambda} \in \mathbb{Z}$, such that

$$\widetilde{\kappa}\tfrac{(\mathfrak{a}_{22}\overline{n}_{11}-\mathfrak{a}_{12}\overline{n}_{21})}{\det(\mathcal{A})} - \widetilde{\lambda}\tfrac{(\mathfrak{a}_{11}\overline{n}_{21}-\mathfrak{a}_{21}\overline{n}_{11})}{\det(\mathcal{A})} = \tfrac{\widetilde{\kappa}\mathfrak{a}_{22}+\widetilde{\lambda}\mathfrak{a}_{21}}{\det(\mathcal{A})}\overline{n}_{11} - \tfrac{\widetilde{\kappa}\mathfrak{a}_{12}+\widetilde{\lambda}\mathfrak{a}_{11}}{\det(\mathcal{A})}\overline{n}_{21} = 1.$$

The integers $\kappa := \frac{\widetilde{\kappa}\mathfrak{a}_{22}+\widetilde{\lambda}\mathfrak{a}_{21}}{\det(\mathcal{A})}$ and $\lambda := \frac{\widetilde{\kappa}\mathfrak{a}_{12}+\widetilde{\lambda}\mathfrak{a}_{11}}{\det(\mathcal{A})}$ satisfy (2.2). Hence,

$$\kappa\overline{n}_{12} - \lambda\overline{n}_{22} = \widetilde{\kappa}\tfrac{(\mathfrak{a}_{22}\overline{n}_{12}-\mathfrak{a}_{12}\overline{n}_{22})}{\det(\mathcal{A})} - \widetilde{\lambda}\tfrac{(\mathfrak{a}_{11}\overline{n}_{22}-\mathfrak{a}_{21}\overline{n}_{12})}{\det(\mathcal{A})} \equiv p \,(\mathrm{mod}\, q)\,,$$

and $p = p_\sigma$ is also independent of the choice of $\mathcal{B}$. □

Note 2.5. It should be stressed that $p = p_\sigma$ *does depend* on which minimal generators of σ is regarded as first and which as second (because of the defining conditions (2.2) and (2.3)). For this reason, by writting $\sigma = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2$, with $\mathbf{n}_1, \mathbf{n}_2$ as its minimal generators, their ordering will always be implicit (and p_σ well-defined). Proposition 2.7 gives the precise description of what happens by interchanging $\mathbf{n}_1$ and $\mathbf{n}_2$ or, more generally, by replacing σ with a $\tau \backsim_N \sigma$.

Definition 2.6. Let p, q two integers with $0 \leqslant p < q$ and $\gcd(p, q) = 1$. The *socius* $\widehat{p}$ of p (w.r.t. q) is defined to be the unique integer satisfying the conditions:

$$0 \leqslant \widehat{p} < q, \quad \gcd(\widehat{p}, q) = 1, \text{ and } p\widehat{p} \equiv 1 (\mathrm{mod}\, q).$$

Proposition 2.7. *If σ, τ are two N-cones, then the following conditions are equivalent:*
(i) $[\sigma]_N = [\tau]_N$ (i.e., *σ and τ are equivalent up to umimodular transformation*).
(ii) *For the integers $p_\sigma, p_\tau, q_\sigma, q_\tau$ associated with σ, τ (by Proposition 2.4) we have $q_\tau = q_\sigma$, and either $p_\tau = p_\sigma$ or $p_\tau = \widehat{p}_\sigma$.*

Proof. Let $\sigma^{\mathrm{st}}, \tau^{\mathrm{st}}$ be the standard models of σ, τ w.r.t. an arbitrary basis matrix $\mathcal{B}$ of the lattice N, and $\mathcal{M}_\sigma, \mathcal{M}_\tau \in \mathrm{GL}_2(\mathbb{Z})$ the corresponding matrices defined in 2.4 (iii), so that

$$\Phi_{\mathcal{M}_\sigma}(\sigma^{\mathrm{st}}) = \mathbb{R}_{\geqslant 0}\tbinom{1}{0} + \mathbb{R}_{\geqslant 0}\tbinom{p_\sigma}{q_\sigma} \text{ and } \Phi_{\mathcal{M}_\tau}(\tau^{\mathrm{st}}) = \mathbb{R}_{\geqslant 0}\tbinom{1}{0} + \mathbb{R}_{\geqslant 0}\tbinom{p_\tau}{q_\tau}.$$

(i)⇒(ii) If $[\sigma]_N = [\tau]_N$, then there is a matrix $\mathcal{A} \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\Phi_{\mathcal{B}\mathcal{A}\mathcal{B}^{-1}}(\sigma) = \tau \Rightarrow \Phi_{\mathcal{A}}(\sigma^{\mathrm{st}}) = \tau^{\mathrm{st}}.$$

Hence,

$$\Phi_{\mathcal{M}_\tau\mathcal{A}\mathcal{M}_\sigma^{-1}}\left(\mathbb{R}_{\geqslant 0}\tbinom{1}{0} + \mathbb{R}_{\geqslant 0}\tbinom{p_\sigma}{q_\sigma}\right) = \mathbb{R}_{\geqslant 0}\tbinom{1}{0} + \mathbb{R}_{\geqslant 0}\tbinom{p_\tau}{q_\tau},$$

and either

$$\Phi_{\mathcal{M}_\tau\mathcal{A}\mathcal{M}_\sigma^{-1}}\left(\tbinom{1}{0}\right) = \tbinom{1}{0} \text{ and } \Phi_{\mathcal{M}_\tau\mathcal{A}\mathcal{M}_\sigma^{-1}}\left(\tbinom{p_\sigma}{q_\sigma}\right) = \tbinom{p_\tau}{q_\tau}$$

or

$$\Phi_{\mathcal{M}_\tau\mathcal{A}\mathcal{M}_\sigma^{-1}}\left(\tbinom{1}{0}\right) = \tbinom{p_\tau}{q_\tau} \text{ and } \Phi_{\mathcal{M}_\tau\mathcal{A}\mathcal{M}_\sigma^{-1}}\left(\tbinom{p_\sigma}{q_\sigma}\right) = \tbinom{1}{0}.$$

Thus, either

$$\mathcal{M}_\tau \mathcal{A} \mathcal{M}_\sigma^{-1} = \begin{pmatrix} 1 & \frac{p_\tau - p_\sigma}{q_\sigma} \\ 0 & \frac{q_\tau}{q_\sigma} \end{pmatrix} \quad \text{or} \quad \mathcal{M}_\tau \mathcal{A} \mathcal{M}_\sigma^{-1} = \begin{pmatrix} p_\tau & \frac{1 - p_\sigma p_\tau}{q_\sigma} \\ q_\tau & -\frac{p_\sigma q_\tau}{q_\sigma} \end{pmatrix}.$$

In the first case $\det(\mathcal{M}_\tau \mathcal{A} \mathcal{M}_\sigma^{-1})$ has to be equal to 1, which means that $q_\tau = q_\sigma$ and $p_\tau - p_\sigma \equiv 0 (\operatorname{mod} q)$, i.e., $p_\tau = p_\sigma$ (because $0 \leqslant p_\sigma, p_\tau < q_\sigma = q_\tau$). In the second case, $\det(\mathcal{M}_\tau \mathcal{A} \mathcal{M}_\sigma^{-1}) = -1$, i.e., $q_\tau = q_\sigma$ and $1 - p_\sigma p_\tau \equiv 0 (\operatorname{mod} q) \Rightarrow p_\tau = \widehat{p}_\sigma$.
(ii)⇒(i) We set $\mathcal{A} := \mathcal{M}_\tau \mathcal{D} \mathcal{M}_\sigma^{-1}$ with $\mathcal{D} \in \mathrm{GL}_2(\mathbb{Z})$ being defined as follows:

$$\mathcal{D} := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \text{ if } p_\tau = p_\sigma, \quad \text{and} \quad \mathcal{D} := \begin{pmatrix} p_\tau & \frac{1 - p_\sigma p_\tau}{q_\sigma} \\ q_\sigma & -p_\sigma \end{pmatrix} \text{ if } p_\tau = \widehat{p}_\sigma.$$

Obviously, $\Phi_{\mathcal{B}\mathcal{A}\mathcal{B}^{-1}}(\sigma) = \tau$ with $\mathcal{A} \in \mathrm{GL}_2(\mathbb{Z})$, i.e., $[\sigma]_N = [\tau]_N$. □

Definition 2.8. Let σ be an N-cone. Since the two integers $p = p_\sigma$ and $q = q_\sigma$ associated with σ (by Proposition 2.4) parametrise uniquely the equivalence class $[\sigma]_N$ up to replacement of p by its socius $\widehat{p}$, we shall henceforth say that σ is *of type* (p, q) (or simply that σ is a (p, q)*-cone*).

Definition 2.9. If σ is an N-cone, then $\sigma^\vee := \{\mathbf{x} \in \mathbb{R}^2 | \langle \mathbf{x}, \mathbf{y} \rangle \geqslant 0, \forall \mathbf{y} \in \sigma\}$ is called the *dual cone* of σ.

Proposition 2.10. *If σ is an N-cone and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, then $\sigma^\vee$ is an M-cone, and $(\sigma^\vee)^\vee = \sigma$. In particular, if σ is a (p, q)-cone with $q > 1$, then $\sigma^\vee$ is a $(q - p, q)$-cone.*

Proof. For the first assertion see [27, §1.2]. If σ is a (p, q)-cone with $q > 1$, then by Propostion 2.4

$$\sigma = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2 = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}(p\mathbf{n}_1 + q\mathbf{n}_1'),$$

where $\{\mathbf{n}_1, \mathbf{n}_1'\}$ is a basis of the lattice N. Let $\{\mathbf{m}_1, \mathbf{m}_1'\}$ be the dual basis of M (i.e., $\langle \mathbf{m}_1, \mathbf{n}_1 \rangle = \langle \mathbf{m}_1', \mathbf{n}_1' \rangle = 1$ and $\langle \mathbf{m}_1, \mathbf{n}_1' \rangle = \langle \mathbf{m}_1, \mathbf{n}_1' \rangle = 0$). Then

$$\sigma^\vee = \mathbb{R}_{\geqslant 0}\mathbf{m}_1' + \mathbb{R}_{\geqslant 0}(q\mathbf{m}_1 - p\mathbf{m}_1') = \mathbb{R}_{\geqslant 0}\mathbf{m}_1' + \mathbb{R}_{\geqslant 0}((q - p)\mathbf{m}_1' + q(\mathbf{m}_1 - \mathbf{m}_1')) \tag{2.4}$$

where $\{\mathbf{m}_1', \mathbf{m}_1 - \mathbf{m}_1'\}$ is a basis of M. Thus the dual cone $\sigma^\vee$ of σ is a $(q - p, q)$-cone (because $0 < q - p < q$ and $\gcd(q - p, q) = 1$). □

• **Hibert basis**. $\sigma \cap N$ is an additive commutative monoid for any N-cone σ. It is known (by Gordan's Lemma [56, Proposition 1.1 (iii), p. 3]) that $\sigma \cap N$ is finitely generated (as a semigroup), and that among all generating systems there is a system $\mathrm{Hilb}_N(\sigma)$ of minimal cardinality, the so-called *Hilbert basis* of σ, which is uniquely determined (up to reordering of its elements) by the following characterisation:

$$\mathrm{Hilb}_N(\sigma) = \left\{ \mathbf{n} \in \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \;\middle|\; \begin{array}{l} \mathbf{n} \text{ cannot be expressed as sum of two} \\ \text{other vectors belonging to } \sigma \cap (N \smallsetminus \{\mathbf{0}\}) \end{array} \right\}.$$

• **Affine toric surfaces**. Let σ be an N-cone in $\mathbb{R}^2$ and M the dual lattice. We set $S_\sigma := \sigma^\vee \cap M$. Let $\mathbb{C}[S_\sigma] := \bigoplus_{\mathbf{m}\in S_\sigma} \mathbb{C}\mathbf{e}(\mathbf{m})$ be the $\mathbb{C}$-algebra with basis $\{\mathbf{e}(\mathbf{m})\,|\,\mathbf{m}\in S_\sigma\}$ consisting of formal elements which fulfill the exponential law:

$$\mathbf{e}(\mathbf{m})\mathbf{e}(\mathbf{m}') = \mathbf{e}(\mathbf{m}+\mathbf{m}'),\ \forall\,(\mathbf{m},\mathbf{m}')\in S_\sigma\times S_\sigma\ \ [\text{with } \mathbf{e}(\mathbf{0}) =: 1_{\mathbb{C}[S_\sigma]}].$$

Since S_σ is finitely generated (as a semigroup),

$$\exists \mathbf{m}_1,\ldots,\mathbf{m}_k \in S_\sigma :\ S_\sigma = \mathbb{Z}_{\geqslant 0}\mathbf{m}_1 + \cdots + \mathbb{Z}_{\geqslant 0}\mathbf{m}_k.$$

Thus $\mathbb{C}[S_\sigma]$ is generated by $\{\mathbf{e}(\mathbf{m}_1),\ldots,\mathbf{e}(\mathbf{m}_k)\}$ and $\mathbb{C}[S_\sigma]\cong\mathbb{C}[z_1,\ldots,z_k]/\mathcal{I}$, where $\mathcal{I}$ denotes the kernel of the $\mathbb{C}$-algebra epimorphism $\mathbb{C}[z_1,\ldots,z_k]\longrightarrow\mathbb{C}[S_\sigma]$ which maps z_j onto $\mathbf{e}(\mathbf{m}_j)$ for all $j\in\{1,\ldots,k\}$.

Definition 2.11. We denote by U_σ (or, more precisely, by $U_{\sigma,N}$, whenever it is necessary to stress that σ is an N-cone) *the affine toric surface* Spec($\mathbb{C}[S_\sigma]$) which is associated with σ and has $\mathbb{C}[S_\sigma]$ as coordinate ring. (U_σ is a 2-dimensional *normal* complex analytic variety embedded in $\mathbb{C}^k$ as vanishing locus of finitely many binomials which generate $\mathcal{I}$; see [56, Proposition 1.2, pp. 4-5]. To work with the embedding of U_σ into an affine complex space of minimal dimension it is enough to replace the arbitrary generating system $\{\mathbf{m}_1,\ldots,\mathbf{m}_k\}$ of S_σ by $\mathrm{Hilb}_M(\sigma^\vee)$.)

Next, we use the identifications

$$\mathrm{Hom}_{\text{semigr.}}(S_\sigma,\mathbb{C}) \overset{(a)}{\longleftrightarrow} \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathbb{C}[S_\sigma],\mathbb{C}) \overset{(b)}{\longleftrightarrow} \{\text{points of } U_\sigma\} \overset{(c)}{\longleftrightarrow} \text{Max-Spec}(\mathbb{C}[S_\sigma])$$

with (a) induced by

$$\mathrm{Hom}_{\text{semigr.}}(S_\sigma,\mathbb{C}) \ni u \longmapsto \theta_u \in \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathbb{C}[S_\sigma],\mathbb{C}),\quad \theta_u(\mathbf{e}(\mathbf{m})) := u(\mathbf{m}), \forall \mathbf{m}\in S_\sigma,$$

(b) induced by

$$\mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathbb{C}[S_\sigma],\mathbb{C}) \ni \theta_u \longmapsto (u(\mathbf{m}_1),\ldots,u(\mathbf{m}_k)) \in \{\text{points of } U_\sigma\},$$

and (c) by

$$\{\text{points of } U_\sigma\} \ni \mathfrak{p} \longmapsto \mathfrak{m}_{\mathfrak{p}} := \{\varphi\in\mathcal{O}_{U_\sigma,\mathfrak{p}}\,|\,\varphi(\mathfrak{p})=0\} \in \text{Max-Spec}(\mathbb{C}[S_\sigma]) := \left\{\begin{array}{c}\text{maximal ideals}\\ \text{of } \mathbb{C}[S_\sigma]\end{array}\right\},$$

where $\mathcal{O}_{U_\sigma,\mathfrak{p}}\cong\mathbb{C}[S_\sigma]_{\mathfrak{p}}$ is the local ring of U_σ at $\mathfrak{p}$ (i.e., the stalk of the structure sheaf $\mathcal{O}_{U_\sigma}$ of U_σ at $\mathfrak{p}$). The standard action of the algebraic torus

$$\mathbb{T} := \mathbb{T}_N := \mathrm{Hom}_{\mathbb{C}\text{-alg.}}(\mathbb{C}[M],\mathbb{C}) = \mathrm{Hom}_{\text{semigr.}}(M,\mathbb{C}) = \mathrm{Hom}_{\text{gr.}}(M,\mathbb{C}^\times) = N\otimes_{\mathbb{Z}}(\mathbb{C}^\times)\cong(\mathbb{C}^\times)^2 \tag{2.5}$$

on (the set of points $\mathrm{Hom}_{\text{semigr.}}(S_\sigma,\mathbb{C})$ of) U_σ can be conceived as multiplication of semigroup homomorphisms:

$$\mathbb{T}\times U_\sigma \ni (t,u) \longmapsto t\cdot u \in U_\sigma. \tag{2.6}$$

We denote by $\operatorname{orb}(\sigma) \in \operatorname{Hom}_{\text{semigr.}}(S_\sigma, \mathbb{C})$ (or, more precisely, by $\operatorname{orb}_N(\sigma)$, whenever it is necessary to stress that σ is an N-cone) the unique point of U_σ remaining fixed under (2.6), i.e., the semigroup homomorphism mapping $\mathbf{m} \in S_\sigma$ onto 1 whenever $\langle \mathbf{m}, \mathbf{y} \rangle = 0$ for all $\mathbf{y} \in \sigma$, and onto 0 otherwise. By Propositions 2.7 and 2.12 the type (p, q) of σ (up to replacement of p by its socius $\widehat{p}$) determines the isomorphism class of the germ $(U_\sigma, \operatorname{orb}(\sigma))$.

Proposition 2.12. *If σ, τ are two N-cones, then the following conditions are equivalent*:

(i) $[\sigma]_N = [\tau]_N$.

(ii) *There is a $\mathbb{T}_N$-equivariant analytic isomorphism $U_\sigma \xrightarrow{\cong} U_\tau$ mapping* $\operatorname{orb}(\sigma)$ *onto* $\operatorname{orb}(\tau)$.

Proof. It follows from [24, Ch. VI, Theorem 2.11, pp. 222-223]. □

Proposition 2.13. *Let σ be an N-cone of type (p, q). The following conditions are equivalent*:

(i) $q = 1$ (*and consequently,* $p = 0$).

(ii) *The minimal generators of σ constitute a basis of N.*

(iii) $U_\sigma = U_{\sigma,N} \cong \mathbb{C}^2$.

Proof. Since $q = \operatorname{mult}_N(\sigma)$ (by 2.4 (iii)), $q = 1$ if and only if the triangle having the origin and the two minimal generators of σ as vertices does not contain any additional lattice point (see (2.1)). Hence, the equivalence of (i) and (ii) follows from Proposition 1.3. For the proof of the equivalence of conditions (ii) and (iii) see [56, Theorem 1.10, p. 15]. □

If the conditions of Proposition 2.13 are satisfied, then σ is said to be a *basic N-cone.* The non-basic N-cones are characterised by the following:

Proposition 2.14. *Let σ be an N-cone of type (p, q). If $q > 1$, then $p \geqslant 1$ and* $\operatorname{orb}(\sigma) \in U_\sigma$ *is a cyclic quotient singularity.* (It is often called *cyclic quotient singularity of type*[1] (p, q).) *In particular,*

$$U_\sigma = U_{\sigma,N} \cong \mathbb{C}^2/G = \operatorname{Spec}(\mathbb{C}[z_1, z_2]^G), \textit{ where } G \subset \operatorname{GL}_2(\mathbb{C})$$

is the cyclic group of order q which is generated by $\operatorname{diag}(\zeta_q^{-p}, \zeta_q)$ (*with* $\zeta_q := \exp(2\pi\sqrt{-1}/q)$) *and acts on* $\mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[z_1, z_2])$ *linearly and faithfully.*

Proof. See Oda [56, Proposition 1.24, p.30]. □

Proposition 2.15. *Let σ be a non-basic N-cone of type (p, q). The following conditions are equivalent*:

(i) $\operatorname{orb}(\sigma) \in U_\sigma$ *is a Gorenstein singularity* (*i.e.,* $\mathcal{O}_{U_\sigma, \operatorname{orb}(\sigma)}$ *is a Gorenstein local ring*).

(ii) $p = 1$.

[1]In Reid's terminology [62, p. 370], it is called *cyclic quotient singularity of type* $\frac{1}{q}(q-p, 1)$.

Proof. Since the quotient space $U_\sigma \cong \mathbb{C}^2/G$, $G := \left\langle \mathrm{diag}(\zeta_q^{-p}, \zeta_q) \right\rangle$, is Gorenstein (as complex variety) if and only if $G \subset \mathrm{SL}_2(\mathbb{C})$ (see [69]), condition (i) is equivalent to $\zeta_q^{1-p} = 1$, i.e., to $p = 1$ (because $0 < p < q$). $\square$

• N-**fans**. A set Δ consisting of finitely many N-cones and their 0- and 1-dimensional faces (i.e., the origin and their rays) will be referred to as an N-*fan* if for any N-cones σ_1, σ_2 belonging to Δ with $\sigma_1 \neq \sigma_2$, the intersection $\sigma_1 \cap \sigma_2$ is either the singleton $\{\mathbf{0}\}$ or a common ray of σ_1 and σ_2. (We shall denote by $\Delta(1)$ and, respectively, by $\Delta(2)$ the set of rays and the set of N-cones of Δ,, and by $|\Delta|$ the *support* of Δ, i.e., the union of its elements.) If Δ is an N-fan, then using the so-called Glueing Lemma (for the affine toric surfaces $U_\sigma, \sigma \in \Delta(2)$) one defines the (normal) *toric surface* $X(N,\Delta)$ *associated with* Δ. The actions of the algebraic torus (2.5) on the affine toric surfaces $U_\sigma, \sigma \in \Delta(2)$, defined in (2.6) are compatible with the patching isomorphisms, giving the natural action of $\mathbb{T}$ on $X(N,\Delta)$ (which extends the multiplication in $\mathbb{T}$). All the orbits w.r.t. it are either of the form $\mathrm{orb}(\sigma)$, $\sigma \in \Delta(2)$, with $\mathrm{orb}(\sigma)$ the unique $\mathbb{T}$-fixed point of $U_\sigma \hookrightarrow X(N,\Delta)$ as defined before, or of the form $\mathrm{orb}(\varrho) := \mathrm{Hom}_{\mathrm{gr.}}(\varrho^\perp \cap M, \mathbb{C}\smallsetminus\{0\})$, $\varrho \in \Delta(1)$ (which are 1-dimensional subvarieties of $X(N,\Delta)$) with $M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ and $\varrho^\perp := \{\mathbf{x} \in \mathbb{R}^2 | \langle \mathbf{x}, \mathbf{y} \rangle = 0, \forall \mathbf{y} \in \varrho\}$, or, finally, $\mathrm{orb}(\{\mathbf{0}\}) = \mathbb{T}$. If D is a Weil divisor on $X(N,\Delta)$, then $D \sim D'$ for some $\mathbb{T}$-invariant Weil divisor D' (with "$\sim$" meaning linearly equivalent). It is known that every Weil divisor on $X(N,\Delta)$ is a $\mathbb{Q}$-Cartier divisor (see [27, p. 65]). We denote by $\mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X(N,\Delta))$ and $\mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X(N,\Delta))$ the groups of $\mathbb{T}$-invariant Weil and Cartier divisors on $X(N,\Delta)$, respectively. The first of them is described as follows:

$$\mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X(N,\Delta)) = \bigoplus_{\varrho \in \Delta(1)} \mathbb{Z}\,\mathbf{V}_\Delta(\varrho) \quad \begin{array}{l}\text{[where } \mathbf{V}_\Delta(\varrho) \text{ denotes} \\ \text{the topological closure of } \mathrm{orb}(\varrho)].\end{array} \tag{2.7}$$

Theorem 2.16 ($\mathbb{T}$-invariant Cartier divisors, [27, p. 62])**.** *If*

$$D = \sum_{\varrho \in \Delta(1)} \lambda_\varrho \mathbf{V}_\Delta(\varrho) \in \mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X(N,\Delta)) \quad (\lambda_\varrho \in \mathbb{Z}),$$

then

$$D \in \mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X(N,\Delta)) \Longleftrightarrow \left\{ \begin{array}{c} \text{for each } \sigma \in \Delta(2) \text{ there is an } \mathbf{l}_\sigma \in M \\ \text{such that } \langle \mathbf{l}_\sigma, \mathbf{n}_\varrho \rangle = -\lambda_\varrho, \ \forall \varrho \in \Delta(1) \cap \sigma \end{array} \right\}, \tag{2.8}$$

with $\mathbf{n}_\varrho$ *denoting the minimal generator of* $\varrho \in \Delta(1)$.

The group $\mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X(N,\Delta))$ can be also described in terms of Δ-support functions. A Δ-*support function* (*integral w.r.t.* N) is a function $h : |\Delta| \longrightarrow \mathbb{R}$ which is linear on each N-cone belonging to Δ, with $h(|\Delta| \cap N) \subset \mathbb{Z}$. Let $\mathrm{SF}(N,\Delta)$ denote the additive group of all Δ-support functions. Then

$$\mathrm{SF}(N,\Delta) \ni h \longmapsto D_h := -\sum_{\varrho \in \Delta(1)} h(\mathbf{n}_\varrho)\mathbf{V}_\Delta(\varrho) \in \mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X(N,\Delta)) \tag{2.9}$$

is a group isomorphism having

$$\mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X\,(N,\Delta)) \ni D = \sum_{\varrho\in\Delta(1)} \lambda_{\varrho}\mathbf{V}_{\Delta}(\varrho) \longmapsto h_D \in \mathrm{SF}\,(N,\Delta) \tag{2.10}$$

as its inverse (i.e., $D = D_{h_D}$ and $h = h_{D_h}$, cf. [17, Theorem 4.2.12, p. 184]), where

$$h_D|_{\sigma}(\mathbf{y}) := \langle \mathbf{l}_{\sigma}, \mathbf{y}\rangle\,,\ \forall\sigma\in\Delta\,(2)\,,\ \text{with } h_D(\mathbf{n}_{\varrho}) = -\lambda_{\varrho},\ \forall\varrho\in\Delta\,(1)\,. \tag{2.11}$$

Note 2.17 (Canonical divisor and the Euler number)**.** (i) The canonical (divisorial) sheaf $\omega_{X(N,\Delta)}$ on $X(N,\Delta)$ is isomorphic to $\mathcal{O}_{X(N,\Delta)}(-\sum_{\varrho\in\Delta(1)}\mathbf{V}_{\Delta}(\varrho))$ (see [27, §4.3, pp. 85-86] or [17, Theorem 8.2.3, pp. 366-367]). So we may define the Weil divisor (2.12) as *canonical divisor* of $X(N,\Delta)$:

$$K_{X(N,\Delta)} := -\sum_{\varrho\in\Delta(1)} \mathbf{V}_{\Delta}(\varrho). \tag{2.12}$$

(ii) The topological *Euler number*

$$e(X(N,\Delta)) := \sum_{j=0}^{4} (-1)^j \dim_{\mathbb{R}} H^j(X(N,\Delta),\mathbb{R})$$

of $X(N,\Delta)$ equals $\sharp\Delta\,(2)$ (see [27, p. 59]).

Let Δ be an N-fan. If Δ' is a *refinement* of Δ (i.e., if Δ' is an N-fan with $|\Delta'| = |\Delta|$ and each N-cone of Δ is a union of N-cones of Δ'), then the induced $\mathbb{T}$-equivariant holomorphic map

$$f : X(N,\Delta') \longrightarrow X(N,\Delta) \tag{2.13}$$

is proper and birational (see [56, Corollary 1.17, p. 23]). The singular locus of $X(N,\Delta)$ is

$$\mathrm{Sing}(X(N,\Delta)) = \{\mathrm{orb}(\sigma)\,|\sigma\in\Delta\,(2)\,,\ \sigma \text{ non-basic}\}\,.$$

In the case in which $\mathrm{Sing}(X(N,\Delta)) \neq \varnothing$, it is always possible to construct (by suitable successive N-cone subdivisions) a refinement Δ' of Δ such that $\mathrm{Sing}(X(N,\Delta')) = \varnothing$, i.e., such that (2.13) is a resolution of the singular points of $X(N,\Delta)$ (a *desingularization* of $X(N,\Delta)$). The so-called *minimal* desingularization $f : X(N,\widetilde{\Delta}) \longrightarrow X(N,\Delta)$ of a toric surface $X(N,\Delta)$ (which is unique, up to factorisation by an isomorphism) is that one arising from the coarsest refinement $\widetilde{\Delta}$ of Δ with $\mathrm{Sing}(X(N,\widetilde{\Delta})) = \varnothing$.

• **Intersection numbers.** If $X(N,\Delta)$ is smooth, the *intersection number* $D_1 \cdot D_2 \in \mathbb{Z}$ of two divisors D_1, D_2 on $X(N,\Delta)$ with compactly supported intersection is defined in the usual sense (see [28, 2.4.9, p. 40]). If $X(N,\Delta)$ is singular and compact, and D_1, D_2 two $\mathbb{Q}$-Weil divisors on $X(N,\Delta)$, we set

$$D_1 \cdot D_2 := f^{\star}(D_1)\cdot f^{\star}(D_2) \in \mathbb{Q}, \tag{2.14}$$

where $f : X(N,\Delta') \longrightarrow X(N,\Delta)$ is an arbitrary $\mathbb{T}$-equivariant desingularization of $X(N,\Delta)$ and $f^\star(D_j)$ the pullback of D_j, $j \in \{1,2\}$, via f in the sense of Mumford (see [53, pp. 17-18] and [18, §1]). It is easy to prove that (2.14) does not depend on the choice of the desingularization (cf. Fulton [28, 7.1.16, p. 125]).

• **Continued fractions and minimal desingularization of U_σ.** Let $\sigma = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2$ be an N-cone with $\mathbf{n}_1, \mathbf{n}_2$ as minimal generators. The affine toric surface U_σ can be viewed as $X(N,\Delta_\sigma)$ with $\Delta_\sigma := \{\{\mathbf{0}\}, \mathbb{R}_{\geqslant 0}\mathbf{n}_1, \mathbb{R}_{\geqslant 0}\mathbf{n}_2, \sigma\}$. If σ is *non-basic* of type (p,q) (as in Proposition 2.14), we consider the negative-regular continued fraction expansion

$$\frac{q}{q-p} = [\![b_1, b_2, \ldots, b_s]\!] := b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots \, b_{s-1} - \cfrac{1}{b_s}}} \tag{2.15}$$

of $\frac{q}{q-p}$ (with $b_s \geqslant 2$) and define recursively $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_s, \mathbf{u}_{s+1} \in N$ by setting

$$\mathbf{u}_0 := \mathbf{n}_1, \mathbf{u}_1 := \frac{1}{q}((q-p)\mathbf{n}_1 + \mathbf{n}_2), \text{ and } \mathbf{u}_{j+1} := b_j\mathbf{u}_j - \mathbf{u}_{j-1}, \quad \forall j \in \{1,\ldots,s\}. \tag{2.16}$$

It is easy to calculate $b_1, b_2, \ldots, b_s$ (see, e.g., [20, Lemma 3.4 (i)] and [8, §3]) and to verify that

$$\mathbf{u}_s = \frac{1}{q}(\mathbf{n}_1 + (q-\widehat{p})\mathbf{n}_2), \ \mathbf{u}_{s+1} = \mathbf{n}_2, \text{ and } b_j \geqslant 2 \text{ for all } j \in \{1,\ldots,s\}. \tag{2.17}$$

Note 2.18. (i) $p, \widehat{p}, q$ and the sum $b_1 + \cdots + b_s$ are related to each other via the formula

$$12\,\mathrm{DS}\,(p,q) = \sum_{j=1}^{s} (3 - b_j) + \frac{1}{q}\,(p + \widehat{p}) - 2, \tag{2.18}$$

where

$$\mathrm{DS}\,(p,q) := \sum_{\iota=1}^{q-1} \left(\!\!\left(\frac{\iota}{q}\right)\!\!\right) \left(\!\!\left(\frac{p\iota}{q}\right)\!\!\right)$$

is the *Dedekind sum* of the pair (p,q). (For an $x \in \mathbb{Q}$, $((x)) := x - \lfloor x \rfloor - \frac{1}{2}$ if $x \notin \mathbb{Z}$ and is 0 if $x \in \mathbb{Z}$.)

(ii) Since $\sigma^\vee = \mathbb{R}_{\geqslant 0}\mathbf{m}_1' + \mathbb{R}_{\geqslant 0}(q\mathbf{m}_1 - p\mathbf{m}_1')$ is a $(q-p,q)$-cone (see (2.4)), considering the negative-regular continued fraction expansion

$$\frac{q}{p} = \frac{q}{q-(q-p)} = [\![b_1^\vee, b_2^\vee, \ldots, b_t^\vee]\!]$$

of $\frac{q}{p}$ we may define analogously $\mathbf{u}_0^\vee, \mathbf{u}_1^\vee, \ldots, \mathbf{u}_t^\vee, \mathbf{u}_{t+1}^\vee \in M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$ by setting

$$\mathbf{u}_0^\vee := \mathbf{m}_1', \ \mathbf{u}_1^\vee := \mathbf{m}_1, \text{ and } \mathbf{u}_{j+1}^\vee := b_j^\vee\mathbf{u}_j^\vee - \mathbf{u}_{j-1}^\vee, \quad \forall j \in \{1,\ldots,t\}, \tag{2.19}$$

with

$$\mathbf{u}_t^\vee = \frac{1}{q}(\mathbf{m}_1' + \widehat{p}(q\mathbf{m}_1 - p\mathbf{m}_1')),\ \ \mathbf{u}_{t+1}^\vee = q\mathbf{m}_1 - p\mathbf{m}_1',\ \text{and}\ b_j^\vee \geqslant 2, \forall j \in \{1,\dots,t\}. \tag{2.20}$$

It is known (cf. [56, p. 29]) that

$$(b_1 + \cdots + b_s) - s = (b_1^\vee + \cdots + b_t^\vee) - t = s + t - 1. \tag{2.21}$$

Proposition 2.19. *If we define*

$$\Theta_\sigma := \ \text{conv}\,(\sigma \cap (N \smallsetminus \{\mathbf{0}\}))\,,\ \text{resp.},\ \ \Theta_{\sigma^\vee} := \ \text{conv}\,(\sigma^\vee \cap (M \smallsetminus \{\mathbf{0}\}))\,, \tag{2.22}$$

and denote by $\partial\Theta_\sigma^{\mathbf{cp}}$ (resp., by $\partial\Theta_{\sigma^\vee}^{\mathbf{cp}}$) the part of the boundary $\partial\Theta_\sigma$ (resp., of $\partial\Theta_{\sigma^\vee}$) containing only its compact edges, then

$$\text{Hilb}_N\,(\sigma) = \partial\Theta_\sigma^{\mathbf{cp}} \cap N = \{\mathbf{u}_j \ \mid 0 \leqslant j \leqslant s+1\}\,,$$

$$\text{Hilb}_M\,(\sigma^\vee) = \partial\Theta_{\sigma^\vee}^{\mathbf{cp}} \cap M = \left\{\mathbf{u}_j^\vee \ \mid 0 \leqslant j \leqslant t+1\right\}.$$

(See Figure 3.)

Proof. This follows from [56, pp. 26-29] and [20, Theorem 3.16, pp. 226-228]. □

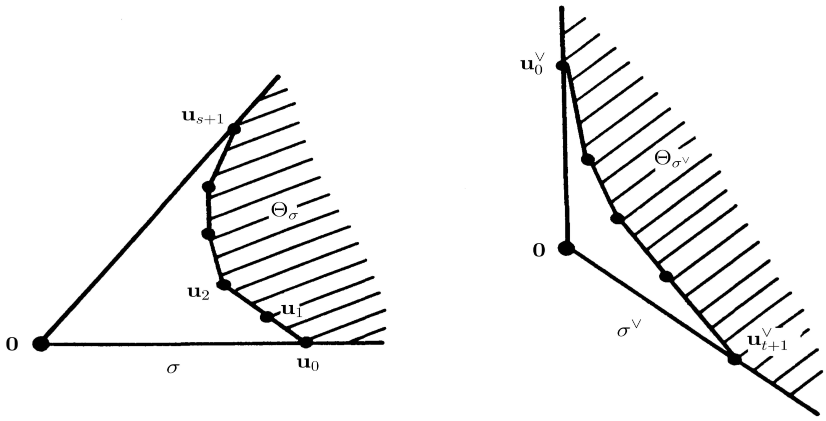


Figure 3: Θ_σ and $\Theta_{\sigma^\vee}$

Theorem 2.20 (Toric version of Hirzebruch's desingularization)**.** *The refinement*

$$\widetilde{\Delta}_\sigma := \{\{\mathbb{R}_{\geqslant 0}\,\mathbf{u}_j + \mathbb{R}_{\geqslant 0}\,\mathbf{u}_{j+1} \ \mid 1 \leqslant j \leqslant s+1\} \ \ \text{together with their faces}\}$$

of the N-fan Δ_σ (having the Hilbert basis elements of σ as minimal generators of its rays) contains only basic N-cones, and constitutes the coarsest refinement of Δ_σ with this property. Therefore, it gives rise to the construction of the minimal $\mathbb{T}$-equivariant resolution

$$f : X(N, \widetilde{\Delta}_\sigma) \longrightarrow X(N, \Delta_\sigma) = U_\sigma$$

of the singular point $\mathrm{orb}(\sigma) \in U_\sigma$*. The exceptional prime divisors w.r.t.* f *are*

$$E_j := \mathbf{V}_{\widetilde{\Delta}_\sigma}(\mathbb{R}_{\geqslant 0}\,\mathbf{u}_j) \cong \mathbb{P}^1_{\mathbb{C}}, \quad j \in \{1, \ldots, s\},$$

and have self-intersection number $E_j^2 := E_j \cdot E_j = -b_j$*.*

Proof. See Hirzebruch [40, pp. 15-20] who constructs $X(N, \widetilde{\Delta}_\sigma)$ by resolving the unique singularity lying over the origin of $\mathbb{C}^3$ in the normalisation of the hypersurface

$$\left\{ (z_1, z_2, z_3) \in \mathbb{C}^3 \middle|\ z_1^q - z_2 z_3^{q-p} = 0 \right\},$$

and Oda [56, pp. 24-30] for a proof which uses only tools from toric geometry. □

3 Compact toric surfaces and lattice polygons

Let N be a lattice in $\mathbb{R}^2$ and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. An N-fan Δ is said to be *complete* whenever $|\Delta| = \mathbb{R}^2$.

Theorem 3.1 (cf. [56, Theorem 1.11, p. 16])**.** *If* Δ *is an* N*-fan, then the toric surface* $X(N, \Delta)$ *is compact if and only if* Δ *is complete.*

• **Nef and ample Cartier divisors on compact toric surfaces**. From now on we shall work with a fixed complete N-fan Δ. For a given $D \in \mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X(N, \Delta))$, we set

$$\boxed{\begin{aligned} P_D := P_{D,\Delta} &:= \{\mathbf{x} \in \mathbb{R}^2 |\ \langle \mathbf{x}, \mathbf{y} \rangle \geqslant h_D(\mathbf{y}), \forall \mathbf{y} \in \mathbb{R}^2\} \\ &= \{\mathbf{x} \in \mathbb{R}^2 |\ \langle \mathbf{x}, \mathbf{n}_\varrho \rangle \geqslant h_D(\mathbf{n}_\varrho), \forall \varrho \in \Delta(1)\} \end{aligned}} \tag{3.1}$$

with $h_D \in \mathrm{SF}(N, \Delta)$ as defined in (2.11). (We write $P_{D,\Delta}$ instead of P_D if we wish to emphasise which our reference fan is.) P_D is bounded and its affine hull has dimension $\leqslant 2$. Moreover, there is a *unique* set $\{\mathbf{l}_\sigma |\ \sigma \in \Delta(2)\} \subset M$ of lattice points satisfying (2.8), and

$$H^0(X(N, \Delta), \mathcal{O}_{X(N,\Delta)}(D)) = \bigoplus_{\mathbf{m} \in P_D \cap M} \mathbb{C}\mathbf{e}(\mathbf{m}).$$

We denote by

$$\mathrm{UCSF}(N, \Delta) := \left\{ h \in \mathrm{SF}(N, \Delta) \,\middle|\, \begin{array}{c} h(t\mathbf{y} + (1-t)\,\mathbf{y}') \geqslant th(\mathbf{y}) + (1-t)h(\mathbf{y}'), \\ \text{for all } \mathbf{y}, \mathbf{y}' \in \mathbb{R}^2 \text{ and } t \in [0,1] \end{array} \right\}$$

and

$$\mathrm{SUCSF}(N, \Delta) := \left\{ h \in \mathrm{UCSF}(N, \Delta) \,\middle|\, \begin{array}{c} \text{for all } \sigma_1, \sigma_2 \in \Delta(2) \text{ with } \sigma_1 \neq \sigma_2, \\ h|_{\sigma_1}, h|_{\sigma_2} \text{ are different linear functions} \end{array} \right\}$$

the sets of upper convex and strictly upper convex Δ-support functions, respectively.

Theorem 3.2 (Neffity Criterion)**.** *If* $D = \sum_{\varrho \in \Delta(1)} \lambda_\varrho \mathbf{V}_\Delta(\varrho) \in \mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\Delta))$ $(\lambda_\varrho \in \mathbb{Z})$, *then the following conditions are equivalent*:

(i) *D is basepoint free, i.e., $\mathcal{O}_{X(N,\Delta)}(D)$ is generated by its global sections.*

(ii) $h_D \in \mathrm{UCSF}(N,\Delta)$.

(iii) $\mathbf{l}_\sigma \in P_D,\ \forall \sigma \in \Delta(2)$.

(iv) $P_D = \mathrm{conv}(\{\mathbf{l}_\sigma | \, \sigma \in \Delta(2)\})$.

(v) $\{\mathbf{l}_\sigma | \, \sigma \in \Delta(2)\}$ *is the vertex set of* P_D *(possibly with repetitions).*

(vi) $h_D(\mathbf{y}) = \min\{\langle \mathbf{m}, \mathbf{y}\rangle \, | \mathbf{m} \in P_D \cap M\} = \min\{\langle \mathbf{l}_\sigma, \mathbf{y}\rangle \, | \sigma \in \Delta(2)\},\ \forall \mathbf{y} \in \mathbb{R}^2$.

(vii) $D \cdot \mathbf{V}_\Delta(\varrho) \geqslant 0,\ \forall \varrho \in \Delta(1)$.

(viii) *D is "nef" (numerically effective), i.e., the intersection number of D with any (irreducible compact) curve on $X(N,\Delta)$ is non-negative.*

Proof. See [56, Theorem 2.7, pp. 76-77] and [17, Theorems 6.1.7, pp. 266-267, and 6.3.12, p. 291]. □

Theorem 3.3 (Ampleness Criterion)**.** *If* $D = \sum_{\varrho \in \Delta(1)} \lambda_\varrho \mathbf{V}_\Delta(\varrho) \in \mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\Delta))$ $(\lambda_\varrho \in \mathbb{Z})$, *then the following conditions are equivalent*:

(i) *D is ample.*

(ii) $h_D \in \mathrm{SUCSF}(N,\Delta)$.

(iii) $\mathbf{l}_\sigma \in P_D,\ \forall \sigma \in \Delta(2)$, *and* $\mathbf{l}_{\sigma_1} \neq \mathbf{l}_{\sigma_2}$ *for all* $\sigma_1, \sigma_2 \in \Delta(2)$ *with* $\sigma_1 \neq \sigma_2$.

(iv) *P_D is an M-polygon with* $\mathrm{Vert}(P_D) = \{\mathbf{l}_\sigma | \, \sigma \in \Delta(2)\}$ *(without repetitions).*

(v) $D \cdot \mathbf{V}_\Delta(\varrho) > 0,\ \forall \varrho \in \Delta(1)$.

Proof. The equivalence of the conditions (i), (ii), (iii) and (iv) follows from [56, Lemma 2.12, p. 82]. (i)⇔(v) is the so-called Toric Nakai Criterion (see [56, Theorem 2.18, p. 86, and Remark on p. 87]). □

Note 3.4. (i) If $D \in \mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\Delta))$ and $M \cap P_D = \{\mathbf{m}_0, \mathbf{m}_1, \ldots, \mathbf{m}_k\}$, then (according to a result of Ewald & Wessels [25]) D is ample if and only if D is very ample, i.e., if and only if D is nef and the map

$$X(N,\Delta) \ni \mathfrak{p} \longmapsto [\mathbf{e}(\mathbf{m}_0)(\mathfrak{p}) : \mathbf{e}(\mathbf{m}_1)(\mathfrak{p}) : \ldots : \mathbf{e}(\mathbf{m}_k)(\mathfrak{p})] \in \mathbb{P}_{\mathbb{C}}^k$$

is a closed embedding.

(ii) Compact toric surfaces are *projective* because there exist always ample $\mathbb{T}$-invariant Cartier divisors on them (see [17, Proposition 6.3.25, p. 297]).

Theorem 3.5. *The self-intersection number of a nef divisor* $D \in \mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\Delta))$ *is*

$$D^2 = 2\,\mathrm{area}_M(P_D). \tag{3.2}$$

Proof. This follows directly from the highest power term in the (generalised) Riemann-Roch formula:

$$\int_{X(N,\Delta)} [D]^2 = 2\,\mathrm{area}_M(P_D).$$

For details see [17, Theorems 13.4.1 (b), pp. 655-656, and 13.4.3, p. 657]. □

• **Lattice polygons and normal fans**. For given $D \in \mathrm{Div}_{\mathbb{C}}^{\mathbb{T}}(X(N,\Delta))$ we have defined $P_D = P_{D,\Delta}$ in (3.1) which is an M-polygon whenever D is ample. Starting with an M-polygon P one can, conversely, construct a compact toric surface $X(N,\Sigma_P)$ and a distinguished ample divisor D_P on it.

Definition 3.6. Let P be an M-polygon. For every $\mathbf{m} \in \mathrm{Vert}(P)$ we define the M-cone

$$\varpi_{\mathbf{m}} := \{\lambda(\mathbf{x}-\mathbf{m}) \mid \lambda \in \mathbb{R}_{\geqslant 0},\ \mathbf{x} \in P\}. \tag{3.3}$$

It is easy to see that

$$\boxed{\Sigma_P := \{ \text{ the } N\text{-cones } \{\varpi_{\mathbf{m}}^{\vee} \mid \mathbf{m} \in \mathrm{Vert}(P)\} \text{ together with their faces}\}}$$

is a complete N-fan. Σ_P is called the *normal fan* of P. Denoting by $\boldsymbol{\eta}_F \in N$ the (primitive) inward-pointing normal of an $F \in \mathrm{Edg}(P)$ (cf. 1.10 (ii)) we observe that

$$\varpi_{\mathbf{m}} = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \langle x, \boldsymbol{\eta}_F\rangle \geqslant 0 \text{ and } \langle x, \boldsymbol{\eta}_{F'}\rangle \geqslant 0\right\} \quad \text{and} \quad \varpi_{\mathbf{m}}^{\vee} = \mathbb{R}_{\geqslant 0}\boldsymbol{\eta}_F + \mathbb{R}_{\geqslant 0}\boldsymbol{\eta}_{F'},$$

where F, F' are the edges of P having $\mathbf{m}$ as their common vertex. Now writing P in the form

$$P = \bigcap_{F \in \mathrm{Edg}(P)} \left\{\mathbf{x} \in \mathbb{R}^2 \middle| \langle \mathbf{x}, \boldsymbol{\eta}_F\rangle \geqslant -\lambda_F\right\} \quad (\text{with } \lambda_F \in \mathbb{Z}, \forall F \in \mathrm{Edg}(P))$$

we set

$$\boxed{D_P := \sum_{F \in \mathrm{Edg}(P)} \lambda_F \mathbf{V}_{\Sigma_P}(\mathbb{R}_{\geqslant 0}\boldsymbol{\eta}_F) \in \mathrm{Div}_{\mathbb{C}}^{\mathbb{T}}(X(N,\Sigma_P)).}$$

Proposition 3.7. *D_P is ample and its support function $h_P := h_{D_P} : \mathbb{R}^2 \to \mathbb{R}$ (often called* the support function of P*) is defined as follows:*

$$h_P(\mathbf{y}) := \min\{\langle \mathbf{x}, \mathbf{y}\rangle \,|\mathbf{x} \in P\},\ \forall \mathbf{y} \in \mathbb{R}^2, \tag{3.4}$$

with $h_P(\boldsymbol{\eta}_F) = -\lambda_F, \forall F \in \mathrm{Edg}(P)$. Moreover, $P = P_{D_P}$, and

$$H^0(X(N,\Sigma_P), \mathcal{O}_{X(N,\Sigma_P)}(D_P)) = \bigoplus_{\mathbf{m} \in P \cap M} \mathbb{C}\mathbf{e}(\mathbf{m}). \tag{3.5}$$

Proof. See [17, Propositions 4.2.14, pp. 186-187, and 6.1.10 (a), pp. 269-270]. □

Next, we consider the set of pairs

$$\mathfrak{Y} := \left\{(X(N,\Delta), D) \,\middle|\, \Delta \text{ a complete } N\text{-fan and } D \in \mathrm{Div}_{\mathbb{C}}^{\mathbb{T}}(X(N,\Delta)) \text{ ample}\right\}.$$

Theorem 3.8. *If $(X(N,\Delta), D) \in \mathfrak{Y}$, then $\Delta = \Sigma_{P_D}$ and $D = D_{P_D}$. Thus,*

$$\boxed{\mathrm{POL}(M) \ni P \longmapsto (X(N,\Sigma_P), D_P) \in \mathfrak{Y} \quad \text{and} \quad \mathfrak{Y} \ni (X(N,\Delta), D) \longmapsto P_D \in \mathrm{POL}(M)}$$

are bijections which are inverses of each other.

Proof. It follows from Proposition 3.7, Theorem 3.3 and [17, Theorem 6.2.1, p. 277]. □

Theorem 3.9. *Let P be an M-polygon. If $f : X(N,\widetilde{\Sigma_P}) \longrightarrow X(N,\Sigma_P)$ is the minimal desingularization of $X(N,\Sigma_P)$, then the pullback $f^\star(D_P)$ of the ample divisor D_P is the unique nef divisor on $X(N,\widetilde{\Sigma_P})$ for which we have $P = P_{D_P} = P_{f^\star(D_P)}$ (or, more precisely, $P = P_{D_P,\Sigma_P} = P_{f^\star(D_P),\widetilde{\Sigma_P}}$), and for which*

$$\chi(\mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P))) = \mathrm{Ehr}_M(P;k), \quad \text{for all } k \in \mathbb{Z}_{\geqslant 0}. \tag{3.6}$$

(3.6) gives

$$D_P^2 = f^\star(D_P)^2 = 2\,\mathrm{area}_M(P) \tag{3.7}$$

and

$$-f^\star(D_P) \cdot K_{X(N,\widetilde{\Sigma_P})} = \sharp\,(\partial P \cap M). \tag{3.8}$$

Proof. For the first assertion see [17, Proposition 6.2.7, p. 281]. (D_P has strictly upper convex support function and therefore $f^\star(D_P)$ has upper convex support function, and $P = P_{D_P,\Sigma_P} = P_{f^\star(D_P),\widetilde{\Sigma_P}}$ because by Theorem 3.2 the M-polygon associated with a nef divisor is determined by its support function.) Now let k be an arbitrary non-negative integer. By the Demazure Vanishing Theorem (cf. [56, Theorem 2.7 (d), pp. 76-77] or [17, Theorem 9.2.3, p. 410]) we obtain

$$\dim_{\mathbb{C}} H^j(X(N,\widetilde{\Sigma_P}), \mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P)))) = 0, \text{ for } j = 1,2. \tag{3.9}$$

Thus, the Euler-Poincaré characteristic

$$\chi(X(N,\widetilde{\Sigma_P}), \mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P))) = \sum_{j=0}^{2} (-1)^j \dim_{\mathbb{C}} H^j(X(N,\widetilde{\Sigma_P}), \mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P))))$$

of the sheaf $\mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P))$ is computed via (3.9) and (3.5) as follows:

$$\begin{aligned} \chi(\mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P))) &= \dim_{\mathbb{C}} H^0(X(N,\widetilde{\Sigma_P}), \mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P)))) \\ &= \dim_{\mathbb{C}} H^0(X(N,\Sigma_P), \mathcal{O}_{X(N,\Sigma_P)}(kD_P)) \\ &= \dim_{\mathbb{C}} H^0(X(N,\Sigma_P), \mathcal{O}_{X(N,\Sigma_P)}(D_{kP})) \\ &= \sharp\,(kP \cap M), \end{aligned}$$

and (3.6) is therefore true. On the other hand, Riemann-Roch Theorem for the projective smooth toric surface $X(N,\widetilde{\Sigma_P})$ gives

$$\chi(\mathcal{O}_{X(N,\widetilde{\Sigma_P})}(kf^\star(D_P))) = \tfrac{1}{2} f^\star(D_P)^2 k^2 - \tfrac{1}{2}(f^\star(D_P) \cdot K_{X(N,\widetilde{\Sigma_P})})k + 1 \tag{3.10}$$

(cf. [28, Ex. 15.2.2, p. 289]). (3.7) and (3.8) follow from (3.6) and (1.4) after coefficient comparison. (To prove (3.7) one may alternatively use (3.2) for the divisor $f^\star(D_P)$.) □

4 WVE^{2}C-graphs and classification up to isomorphism

Given two complete N-fans Δ, Δ', under which conditions are the corresponding compact toric surfaces $X(N,\Delta)$ and $X(N,\Delta')$ biholomorphically equivalent, i.e., *isomorphic* in the analytic category? The answer to this question requires the use of the so-called "WVE^{2}C-graphs", the weights of which are the types of the N-cones of Δ and Δ', and some additional characteristic integers determined by the minimal desingularizations of $X(N,\Delta)$ and $X(N,\Delta')$ (see below Theorem 4.5). Let Δ be a complete N-fan, and let $\sigma_i = \mathbb{R}_{\geqslant 0}\mathbf{n}_i + \mathbb{R}_{\geqslant 0}\mathbf{n}_{i+1}$, $i \in \{1,\ldots,\nu\}$, be its N-cones (with $\nu \geqslant 3$ and $\mathbf{n}_i$ primitive for all $i \in \{1,\ldots,\nu\}$), enumerated in such a way that $\mathbf{n}_1,\ldots,\mathbf{n}_\nu$ go *anticlockwise* around the origin exactly once in this order. (*Convention.* We set $\mathbf{n}_{\nu+1} := \mathbf{n}_1$ and $\mathbf{n}_0 := \mathbf{n}_\nu$. In general, in definitions and formulae involving enumerated sets of numbers or vectors in which the index set $\{1,\ldots,\nu\}$ is meant as a cycle, we shall read the indices i "mod ν", even if it is not mentioned explicitly.) By (2.7) we have

$$\mathrm{Div}^{\mathbb{T}}_{\mathrm{W}}(X(N,\Delta)) = \bigoplus_{i=1}^{\nu} \mathbb{Z}\, C_i, \text{ where } C_i := \mathbf{V}_\Delta(\mathbb{R}_{\geqslant 0}\,\mathbf{n}_i), \ \forall i \in \{1,\ldots,\nu\}. \tag{4.1}$$

Suppose that σ_i is a (p_i,q_i)-cone for all $i \in \{1,\ldots,\nu\}$ and introduce the notation

$$I_\Delta := \{ i \in \{1,\ldots,\nu\} \mid q_i > 1 \}, \quad J_\Delta := \{ i \in \{1,\ldots,\nu\} \mid q_i = 1 \}, \tag{4.2}$$

to separate the indices corresponding to non-basic from those corresponding to basic N-cones. Obviously, $\mathrm{Sing}(X(N,\Delta)) = \{ \mathrm{orb}(\sigma_i) \mid i \in I_\Delta \}$. For all $i \in I_\Delta$ consider the negative-regular continued fraction expansions

$$\frac{q_i}{q_i - p_i} = \left[\!\left[b_1^{(i)}, b_2^{(i)}, \ldots, b_{s_i}^{(i)} \right]\!\right], \tag{4.3}$$

define recursively, in accordance with what is already mentioned in (2.16) and (2.17) for a single non-basic N-cone, lattice points $\mathbf{u}_0^{(i)}, \mathbf{u}_1^{(i)}, \ldots, \mathbf{u}_{s_i}^{(i)}, \mathbf{u}_{s_i+1}^{(i)} \in N$ by

$$\mathbf{u}_0^{(i)} := \mathbf{n}_i, \mathbf{u}_1^{(i)} := \frac{1}{q_i}((q_i - p_i)\mathbf{n}_i + \mathbf{n}_{i+1}), \text{ and } \ \mathbf{u}_{j+1}^{(i)} := b_j^{(i)}\mathbf{u}_j^{(i)} - \mathbf{u}_{j-1}^{(i)}, \quad \forall j \in \{1,\ldots,s_i\}, \tag{4.4}$$

with

$$\mathbf{u}_{s_i}^{(i)} = \frac{1}{q_i}(\mathbf{n}_i + (q_i - \widehat{p}_i)\mathbf{n}_{i+1}), \ \mathbf{u}_{s_i+1}^{(i)} = \mathbf{n}_{i+1}, \text{ and } b_j^{(i)} \geqslant 2, \quad \forall j \in \{1,\ldots,s_i\}, \tag{4.5}$$

and, finally, define the complete N-fan

$$\widetilde{\Delta} := \left\{ \begin{array}{c} \text{the } N\text{-cones } \{\sigma_i \mid i \in J_\Delta\} \text{ and} \\ \left\{ \mathbb{R}_{\geqslant 0}\,\mathbf{u}_j^{(i)} + \mathbb{R}_{\geqslant 0}\,\mathbf{u}_{j+1}^{(i)} \,\middle|\, i \in I_\Delta, \ j \in \{0,1,\ldots,s_i\} \right\}, \\ \text{together with their faces} \end{array} \right\}.$$

By construction, the induced $\mathbb{T}$-equivariant proper birational map

$$f : X(N,\widetilde{\Delta}) \longrightarrow X(N,\Delta)$$

is the minimal desingularization of $X(N,\Delta)$ (as we just patch together the minimal desingularizations of U_{σ_i}'s, $i \in I_\Delta$, established in Theorem 2.20). Setting

$$\begin{cases} E_j^{(i)} := \mathbf{V}_{\widetilde{\Delta}}(\mathbb{R}_{\geqslant 0}\, \mathbf{u}_j^{(i)}), & \forall i \in I_\Delta \text{ and } \forall j \in \{1,\ldots,s_i\}, \\ \overline{C}_i := \mathbf{V}_{\widetilde{\Delta}}(\mathbb{R}_{\geqslant 0}\, \mathbf{n}_i), & \forall i \in \{1,\ldots,\nu\}, \end{cases} \tag{4.6}$$

we observe that $\overline{C}_i$ is the strict transform of C_i w.r.t. f, $E^{(i)} := \sum_{j=1}^{s_i} E_j^{(i)}$ the exceptional divisor replacing the singular point $\mathrm{orb}(\sigma_i)$ via f, and

$$\mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\widetilde{\Delta})) \otimes_{\mathbb{Z}} \mathbb{Q} = \left(\bigoplus_{i=1}^{\nu} \mathbb{Q}\,\overline{C}_i\right) \oplus \left(\bigoplus_{i\in I_\Delta}\bigoplus_{j=1}^{s_i} \mathbb{Q}\, E_j^{(i)}\right).$$

The *discrepancy divisor* w.r.t. f is

$$K_{X(N,\widetilde{\Delta})} - f^\star(K_{X(N,\Delta)}) \sim \sum_{i\in I_\Delta} K(E^{(i)}), \tag{4.7}$$

where each of the $K(E^{(i)})$'s is a $\mathbb{Q}$-Cartier divisor (the *local canonical divisor* of $X(N,\widetilde{\Delta})$ at $\mathrm{orb}(\sigma_i)$ in the sense of [18, §1]) supported in the union $\bigcup_{j=1}^{s_i} E_j^{(i)}$. If $K_{X(N,\widetilde{\Delta})} \sim f^\star(K_{X(N,\Delta)})$, then f is said to be *crepant*.

Proposition 4.1. *f is crepant if and only if $X(N,\Delta)$ has at worst Gorenstein singularities.*

Proof. By Proposition 2.15, $X(N,\Delta)$ has at worst Gorenstein singularities if and only if $p_i = 1$ for all $i \in I_\Delta$. This can be shown to be equivalent to $K(E^{(i)}) \sim 0$, for all $i \in I_\Delta$, by using the explicit description of $K(E^{(i)})$'s given in [18, Proposition 4.4, pp. 94-95]. □

Definition 4.2 (The additional characteristic integers $\boldsymbol{r_i}$)**.** For every index $i \in \{1,\ldots,\nu\}$ we introduce integers r_i uniquely determined by the conditions:

$$r_i\mathbf{n}_i = \begin{cases} \mathbf{u}_{s_{i-1}}^{(i-1)} + \mathbf{u}_1^{(i)}, & \text{if } i \in I'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{u}_1^{(i)}, & \text{if } i \in I''_\Delta, \\ \mathbf{u}_{s_{i-1}}^{(i-1)} + \mathbf{n}_{i+1}, & \text{if } i \in J'_\Delta, \\ \mathbf{n}_{i-1} + \mathbf{n}_{i+1}, & \text{if } i \in J''_\Delta, \end{cases} \tag{4.8}$$

where

$$I'_\Delta := \{i \in I_\Delta \mid q_{i-1} > 1\}, \quad I''_\Delta := \{i \in I_\Delta \mid q_{i-1} = 1\},$$

and

$$J'_\Delta := \{i \in J_\Delta \mid q_{i-1} > 1\}, \quad J''_\Delta := \{i \in J_\Delta \mid q_{i-1} = 1\},$$

with I_Δ, J_Δ as in (4.2). The triples (p_i, q_i, r_i), $i \in \{1,\ldots,\nu\}$, will be referred to as the *combinatorial triples* of Δ.

Note 4.3. (i) The self-intersection number $(E_j^{(i)})^2$ of $E_j^{(i)}$ equals $-b_j^{(i)}$ for all $i \in I_\Delta$ and all $j \in \{1, \ldots, s_i\}$ (cf. Theorem 2.20). On the other hand, the opposite $-r_i$ of the integer r_i defined by (4.8) is nothing but the self-intersection number $\overline{C}_i^2$ of $\overline{C}_i$ for all $i \in \{1, \ldots, \nu\}$. For the proof of this fact, as well as for the computation of the intersection numbers of the rest of pairs of divisors (4.6) which constitute the given $\mathbb{Q}$-basis of $\mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N, \widetilde{\Delta})) \otimes_{\mathbb{Z}} \mathbb{Q}$, we refer to [18, Lemma 4.4, pp. 93-94].

(ii) The (fractional) intersection numbers $C_i \cdot C_{i'} \in \mathbb{Q}$ of any pair $C_i, C_{i'}$ of generators of $\mathrm{Div}_{\mathrm{W}}^{\mathbb{T}}(X(N, \Delta))$ (with $i, i' \in \{1, \ldots, \nu\}$) can be expressed in terms of the coordinates of the combinatorial triples of Δ and the socii of their first coordinates (see [18, Lemma 4.7, pp. 97-98]).

Definition 4.4. A *circular graph* is a plane graph whose vertices are points on a circle and whose edges are the corresponding arcs (on this circle, each of which connects two consecutive vertices). We say that a circular graph $\mathfrak{G}$ is *$\mathbb{Z}$-weighted at its vertices* and *double $\mathbb{Z}$-weighted at its edges* (and call it WVE^{2}C-*graph*, for short) if it is accompanied by two maps

$$\{\text{Vertices of } \mathfrak{G}\} \longmapsto \mathbb{Z}, \ \{\text{Edges of } \mathfrak{G}\} \longmapsto \mathbb{Z}^2,$$

assigning to each vertex an integer and to each edge a pair of integers, respectively. To the complete N-fan Δ (as described above) we associate an anticlockwise directed WVE^{2}C-graph $\mathfrak{G}_\Delta$ with

$$\{\text{Vertices of } \mathfrak{G}_\Delta\} = \{\mathbf{v}_1, \ldots, \mathbf{v}_\nu\} \ \text{ and } \ \{\text{Edges of } \mathfrak{G}_\Delta\} = \{\overline{\mathbf{v}_1\mathbf{v}_2}, \ldots, \overline{\mathbf{v}_\nu\mathbf{v}_1}\},$$

$(\mathbf{v}_{\nu+1} := \mathbf{v}_1)$, by defining its "weights" as follows:

$$\mathbf{v}_i \longmapsto -r_i, \qquad \overline{\mathbf{v}_i\mathbf{v}_{i+1}} \longmapsto (p_i, q_i), \ \forall i \in \{1, \ldots, \nu\}.$$

The *reverse graph* $\mathfrak{G}_\Delta^{\mathrm{rev}}$ of $\mathfrak{G}_\Delta$ is the directed WVE^{2}C-graph which is obtained by changing the double weight (p_i, q_i) of the edge $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$ into $(\widehat{p}_i, q_i)$ and reversing the initial anticlockwise direction of $\mathfrak{G}_\Delta$ into clockwise direction (see Figure 4).

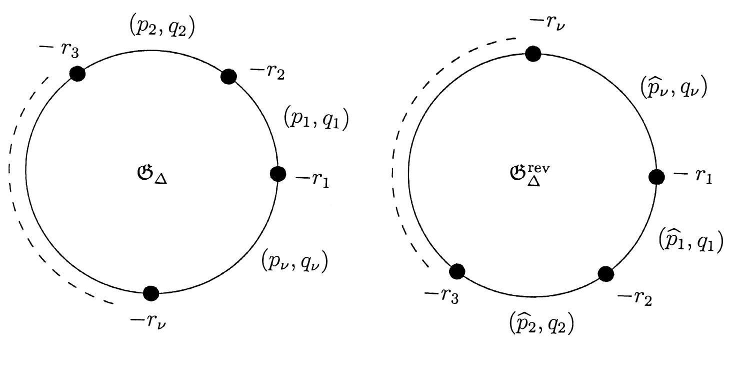


Figure 4: The directed WVE^{2}C-graphs $\mathfrak{G}_\Delta$ and $\mathfrak{G}_\Delta^{\mathrm{rev}}$

Theorem 4.5 (Classification up to isomorphism)**.** *Let Δ, Δ' be two complete N-fans. Then the following conditions are equivalent*:

(i) *The compact toric surfaces $X(N,\Delta)$ and $X(N,\Delta')$ are isomorphic.*

(ii) *Either $\mathfrak{G}_{\Delta'} \underset{\text{gr.}}{\cong} \mathfrak{G}_{\Delta}$ or $\mathfrak{G}_{\Delta'} \underset{\text{gr.}}{\cong} \mathfrak{G}_{\Delta}^{\text{rev}}$.*

Here "$\overset{\text{gr.}}{\cong}$" indicates graph-theoretic isomorphism (i.e., a bijection between the sets of vertices which preserves the corresponding weights). For further details and for the proof of Theorem 4.5 see [18, §5]. (*Conventions for the drawings.* When we draw concrete WVE^{2}C-graphs in the plane we attach, for simplification's sake, only the weight $-r_i$ at $\mathbf{v}_i$ without mentioning $\mathbf{v}_i$ itself, for $i \in \{1,\ldots,\nu\}$, and the double weight (p_i, q_i) at the edge $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, for $i \in I_\Delta$, while we leave edges $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, $i \in J_\Delta$, without any decoration in order to switch to the notation for the $\mathbb{Z}$-weighted circular graphs introduced by Oda in [56, pp. 42-46] which are used for the study of *smooth* compact toric surfaces.)

5 Toric log del Pezzo surfaces

A compact complex surface is called *log del Pezzo surface* if (a) it has at worst log-terminal singularities, i.e., quotient singularities, and (b) there is a positive integer multiple of its anticanonical divisor which is a Cartier ample divisor. The *index* of a log del Pezzo surface is defined to be the least positive integer having property (b). Every smooth compact toric surface possesses a unique *anticanonical model* (in the sense of Sakai [63]) which has to be a toric log del Pezzo surface; and conversely, every toric log del Pezzo surface is the anticanonical model of its minimal desingularization (see [18, Theorem 6.5, p. 106]).

Definition 5.1. Let $Q \in \mathrm{POL}_{\mathbf{0}}(N)$ be an LDP-polygon (see 1.10 (i)). For each edge $F \in \mathrm{Edg}(Q)$ we define the N-cone $\sigma_F := \{\lambda \mathbf{x} \,|\, \mathbf{x} \in F \text{ and } \lambda \in \mathbb{R}_{\geqslant 0}\}$ supporting F, and the complete N-fan

$$\boxed{\Delta_Q := \{\text{the } N\text{-cones } \sigma_F \text{ together with their faces} |\, F \in \mathrm{Edg}(Q)\}\,.}$$

$X(N,\Delta_Q)$ is said to be the compact toric surface *associated with* Q, and $\mathfrak{G}_{\Delta_Q}$ the WVE^{2}C-*graph of* Q.

Proposition 5.2. *Let Δ be a complete N-fan. Then the following conditions are equivalent*:

(i) *$X(N,\Delta)$ is a log del Pezzo surface of index ℓ.*

(ii) *There exists an LDP-polygon Q of index ℓ w.r.t. N (see 1.10 (ii)) such that $\Delta = \Delta_Q$.*

Proof. Suppose that $\ell := \min\left\{k \in \mathbb{Z}_{>0} \,\middle|\, -kK_{X(N,\Delta)} \in \mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\Delta)) \text{ and is ample}\right\}$. By Theorem 2.16 and (2.12),

$$-\ell K_{X(N,\Delta)} = \ell\Big(\sum_{\varrho \in \Delta(1)} \mathbf{V}_\Delta(\varrho)\Big) \in \mathrm{Div}_{\mathrm{C}}^{\mathbb{T}}(X(N,\Delta)),$$

which means that there is a unique set $\{\mathbf{l}_\sigma | \, \sigma \in \Delta(2)\} \subset M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, such that $\langle \mathbf{l}_\sigma, \mathbf{n}_\varrho \rangle = -\ell$ for $\varrho \in \Delta(1) \cap \sigma$. From the implication (i)⇒(iv) in Theorem 3.3 (applied for the divisor $D = -\ell K_{X(N,\Delta_Q)}$) we deduce that $P_{-\ell K_{X(N,\Delta)}}$ is an M-polygon with vertex set $\mathrm{Vert}(P_{-\ell K_{X(N,\Delta)}}) = \{\mathbf{l}_\sigma | \, \sigma \in \Delta(2)\}$ (without repetitions). We observe that the polygon $\frac{1}{\ell} P_{-\ell K_{X(N,\Delta)}} := \mathrm{conv}\big(\{\frac{1}{\ell}\, \mathbf{l}_\sigma | \, \sigma \in \Delta(2)\}\big)$ contains $\mathbf{0}$ in its interior. Since $\langle \frac{1}{\ell} \mathbf{l}_\sigma, \mathbf{n}_\varrho \rangle = -1$ for $\varrho \in \Delta(1) \cap \sigma$, its polar polygon is

$$(\frac{1}{\ell} P_{-\ell K_{X(N,\Delta)}})^\circ = \mathrm{conv}(\{\mathbf{n}_\varrho | \, \varrho \in \Delta(1)\}) \in \mathrm{POL}_{\mathbf{0}}(N)$$

(by (1.1) and (1.2)). Setting $Q := (\frac{1}{\ell} P_{-\ell K_{X(N,\Delta)}})^\circ$ we see that Q is an LDP-polygon because $\mathbf{n}_\varrho$ is primitive for all $\varrho \in \Delta(1)$. Moreover, by our hypothesis,

$$\ell = \min\{k \in \mathbb{Z}_{>0} \, | \mathrm{Vert}(kQ^\circ) \subset M\}.$$

Thus, the index of Q equals ℓ, $\Delta = \Delta_Q$, and (i)⇒(ii) is true. The proof of the reverse implication (ii)⇒(i) is similar. □

Proposition 5.3. *Let $Q, Q' \in \mathrm{POL}_{\mathbf{0}}(N)$ be two LDP-polygons. Then the following are equivalent*:

(i) $X(N, \Delta_Q)$ *and* $X(N, \Delta_{Q'})$ *are isomorphic.*

(ii) $[Q]_N = [Q']_N$.

Proof. We have $[Q]_N = [Q']_N$ if and only if there exist a basis matrix $\mathcal{B}$ of N and a matrix $\mathcal{A} \in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\Phi_{\mathcal{B}\mathcal{A}\mathcal{B}^{-1}}(Q) = Q' \Rightarrow \Phi_{\mathcal{A}\mathcal{B}^{-1}}(Q) = \Phi_{\mathcal{B}^{-1}}(Q') \Rightarrow \Phi_{\mathcal{A}}(Q^{\mathrm{st}}) = Q'^{\,\mathrm{st}} \Rightarrow [Q^{\mathrm{st}}]_{\mathbb{Z}^2} = [Q']_{\mathbb{Z}^2},$$

where $Q^{\mathrm{st}}, Q'^{\,\mathrm{st}}$ are the standard models of Q, and Q', respectively, w.r.t. $\mathcal{B}$. It is a easy to verify that this is equivalent to

$$\mathfrak{G}_{\Delta_{Q'^{\,\mathrm{st}}}} \underset{\mathrm{gr.}}{\cong} \begin{cases} \mathfrak{G}_{\Delta_{Q^{\mathrm{st}}}}, & \text{if } \det(\mathcal{A}) = 1, \\ \mathfrak{G}^{\mathrm{rev}}_{\Delta_{Q^{\mathrm{st}}}}, & \text{if } \det(\mathcal{A}) = -1, \end{cases} \Longleftrightarrow \mathfrak{G}_{\Delta_{Q'}} \underset{\mathrm{gr.}}{\cong} \begin{cases} \mathfrak{G}_{\Delta_Q}, & \text{if } \det(\mathcal{A}) = 1, \\ \mathfrak{G}^{\mathrm{rev}}_{\Delta_Q}, & \text{if } \det(\mathcal{A}) = -1. \end{cases}$$

Thus, (ii)⇔(i) can be seen to be true by making use of Theorem 4.5. □

Note 5.4. (i) By Propositions 5.2 and 5.3 the following map is a bijection:

$$\left\{ \begin{array}{c} \text{equivalence classes} \\ \text{of LDP-polygons} \\ \text{of index } \ell \text{ w.r.t. ``} \backsim_N \text{''} \end{array} \right\} \ni [Q]_N \longmapsto [X(N, \Delta_Q)] \in \left\{ \begin{array}{c} \text{isomorphism classes of toric} \\ \text{log Del Pezzo surfaces} \\ \text{of index } \ell \end{array} \right\}.$$

Thus, the classification of toric log del Pezzo surfaces of index ℓ *up to isomorphism* is equivalent to the classification of LDP-polygons of index ℓ *up to unimodular transformation.*

(ii) Let $Q \in \mathrm{POL}_{\mathbf{0}}(N)$ be an LDP-polygon. Enumerating the edges, say, $F_1, \ldots, F_\nu$ (and the vertices $\mathbf{n}_1, \dots, \mathbf{n}_\nu$) of Q anticlockwise (as in §4), with $F_i := \mathrm{conv}(\{\mathbf{n}_i, \mathbf{n}_{i+1}\})$ and $T_{F_i} := \mathrm{conv}(\{\mathbf{0}, \mathbf{n}_i, \mathbf{n}_{i+1}\})$, $i \in \{1, \dots, \nu\}$, and assuming that the N-cone

$$\sigma_i := \sigma_{F_i} = \mathbb{R}_{\geqslant 0}\mathbf{n}_i + \mathbb{R}_{\geqslant 0}\mathbf{n}_{i+1}$$

supporting F_i is of type (p_i, q_i), we obtain

$$q_i = \mathrm{mult}_N(\sigma_i) = \frac{\det(\mathbf{n}_i, \mathbf{n}_{i+1})}{\det(N)} = 2\,\mathrm{area}_N(T_{F_i}),\ \forall i \in \{1, \dots, \nu\}. \tag{5.1}$$

By Proposition 2.4 there exist a basis matrix $\mathcal{B}$ of N and a matrix $\mathcal{M}_{\sigma_i} \in \mathrm{GL}_2(\mathbb{Z})$ such that $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\sigma_i) = \Phi_{\mathcal{M}_{\sigma_i}}(\sigma_i^{\mathrm{st}}) = \mathbb{R}_{\geqslant 0}\binom{1}{0} + \mathbb{R}_{\geqslant 0}\binom{p_i}{q_i}$, where σ_i^{st} is the standard model of σ_i w.r.t. $\mathcal{B}$. Taking into account that $\mathrm{mult}_N(\sigma_i) = \mathrm{mult}_{\mathbb{Z}^2}(\sigma_i^{\mathrm{st}}) = \mathrm{mult}_{\mathbb{Z}^2}(\Phi_{\mathcal{M}_{\sigma_i}}(\sigma_i^{\mathrm{st}})) = q_i$, the local index l_{F_i} of F_i w.r.t. Q (as defined in 1.10 (ii)) is given by the formula

$$l_{F_i} = \frac{\mathrm{mult}_N(\sigma_i)}{\sharp\,(F_i \cap N) - 1} = \frac{\mathrm{mult}_{\mathbb{Z}^2}(\Phi_{\mathcal{M}_{\sigma_i}}(\sigma_i^{\mathrm{st}}))}{\sharp\left(\mathrm{conv}\left(\left\{\binom{1}{0}, \binom{p_i}{q_i}\right\}\right) \cap \mathbb{Z}^2\right) - 1} = \frac{q_i}{\gcd(q_i, p_i - 1)}. \tag{5.2}$$

Since we are mainly interested in the geometric properties of the toric log del Pezzo surfaces $X(N, \Delta_Q)$ and $X(M, \Delta_{Q^*})$ which are associated with *ℓ-reflexive* polygons Q and their duals $Q^* := \ell Q^\circ$, respectively, being defined in 1.20 and 1.24, let us first determine the WVE^{2}C-graphs $\mathfrak{G}_{\Delta_Q}$ and $\mathfrak{G}_{\Delta_{Q^*}}$ for the examples mentioned in 1.28. (For the WVE^{2}C-graphs of *all* 1-reflexive polygons cf. [18, Figures 8, 9 and 10, pp. 108-109].)

Examples 5.5. (i) The WVE^{2}C-graph $\mathfrak{G}_{\Delta_Q}$ of the ℓ-reflexive triangle (1.16) is shown in Figure 5. For $\ell \geqslant 7$ we set

$$\ell' := \begin{cases} \frac{1}{5}(6\ell - 1), & \text{if } \ell \equiv 1 (\mathrm{mod}\, 5) \\ \frac{1}{5}(3\ell - 1), & \text{if } \ell \equiv 2 (\mathrm{mod}\, 5) \\ \frac{1}{5}(8\ell + 1), & \text{if } \ell \equiv 3 (\mathrm{mod}\, 5) \\ \frac{1}{5}(9\ell - 1), & \text{if } \ell \equiv 4 (\mathrm{mod}\, 5) \end{cases} \quad \text{and} \quad \ell'' := \begin{cases} \frac{1}{4}(9\ell - 5), & \text{if } \ell \equiv 1 (\mathrm{mod}\, 4) \\ \frac{1}{4}(3\ell - 5), & \text{if } \ell \equiv 3 (\mathrm{mod}\, 4) \end{cases}$$

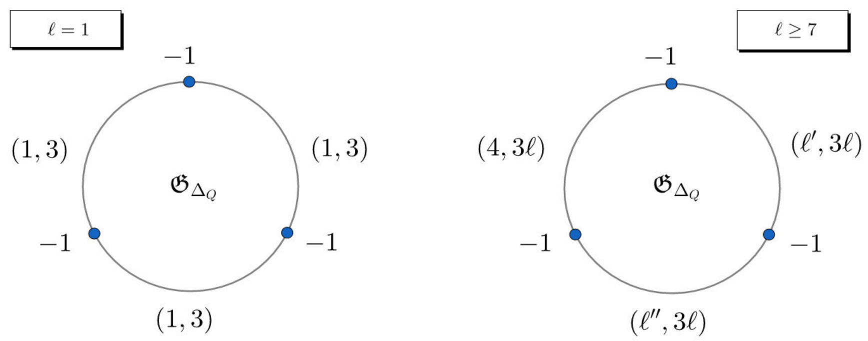


Figure 5: The WVE^{2}C-graph of the ℓ-reflexive triangle (1.16)

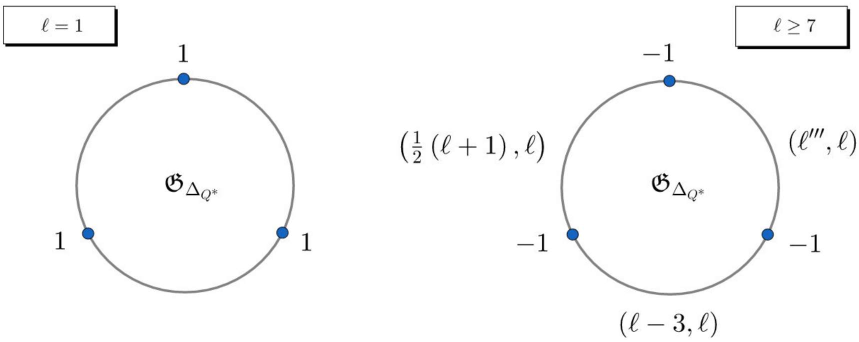


Figure 6: The WVE^{2}C-graph of the ℓ-reflexive triangle (1.17)

The WVE^{2}C-graph $\mathfrak{G}_{\Delta_{Q^*}}$ of its dual (1.17) is shown in Figure 6, where for $\ell \geqslant 7$ we set $\ell''' := \frac{2}{3}(\ell - 1)$ if $\ell \equiv 1(\mathrm{mod}\, 3)$ and $\ell''' := \frac{1}{3}(\ell - 2)$ if $\ell \equiv 2(\mathrm{mod}\, 3)$.
(ii) The WVE^{2}C-graph $\mathfrak{G}_{\Delta_Q}$ of the ℓ-reflexive quadrilateral (1.18) is illustrated in Figure 7, where for $\ell \geqslant 5$ we set $\ell' := \frac{4\ell-1}{3}$ whenever $\ell \equiv 1(\mathrm{mod}\, 3)$ and $\ell' := \frac{2\ell-1}{3}$ whenever $\ell \equiv 2(\mathrm{mod}\, 3)$.

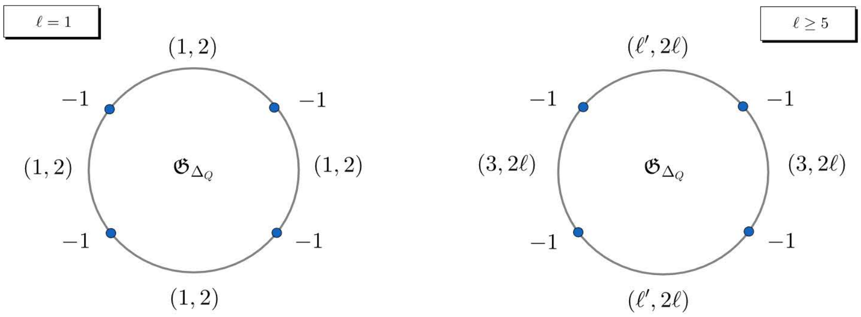


Figure 7: The WVE^{2}C-graph of the ℓ-reflexive quadrilateral (1.18)

For the WVE^{2}C-graph $\mathfrak{G}_{\Delta_{Q^*}}$ of its dual (1.19) see Figure 8.

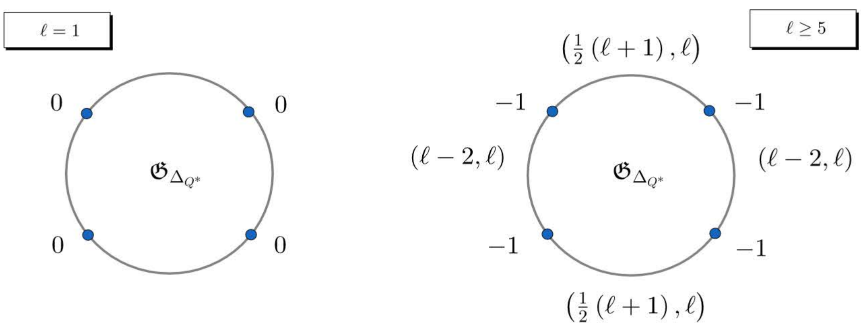


Figure 8: The WVE^{2}C-graph of the ℓ-reflexive quadrilateral (1.19)

(iii) The WVE^{2}C-graph $\mathfrak{G}_{\Delta_Q}$ of the ℓ-reflexive pentagon (1.20) is illustrated in Figure 9. For $\ell \geqslant 5$ we set $\ell' := \frac{1}{3}(2\ell + 1)$ if $\ell \equiv 1(\mathrm{mod}\, 3)$ and $\ell' := \frac{1}{3}(4\ell + 1)$ if $\ell \equiv 2(\mathrm{mod}\, 3)$.

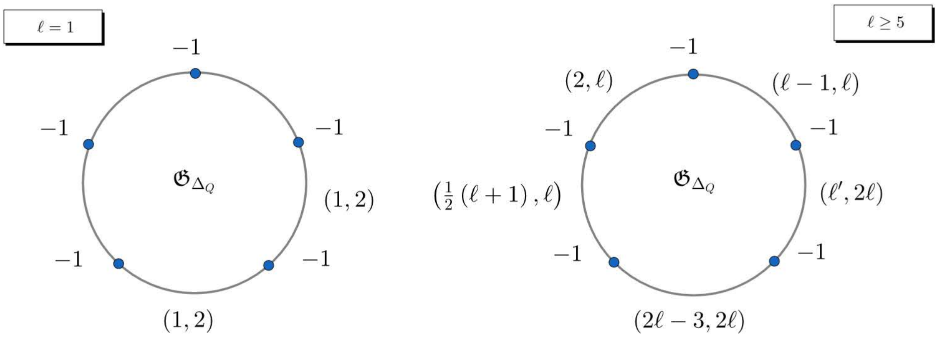


Figure 9: The WVE^{2}C-graph of the ℓ-reflexive pentagon (1.20)

The WVE^{2}C-graph $\mathfrak{G}_{\Delta_{Q^*}}$ of its dual (1.21) is given in Figure 10.

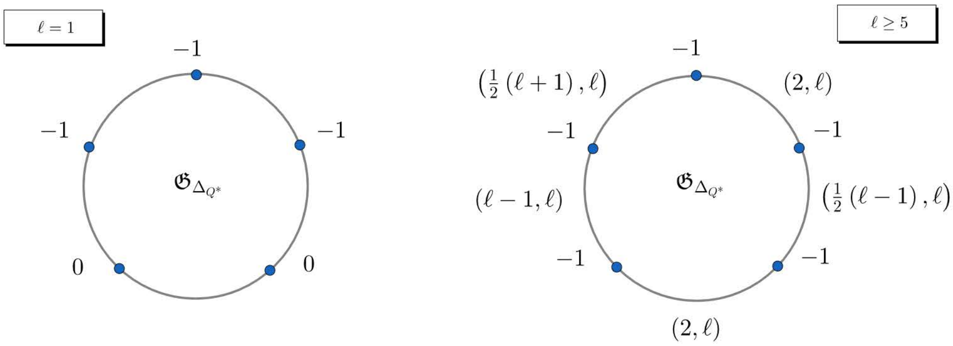


Figure 10: The WVE^{2}C-graph of the ℓ-reflexive pentagon (1.21)

(iv) The WVE^{2}C-graph $\mathfrak{G}_{\Delta_Q}$ of the ℓ-reflexive hexagon (1.22) is shown in Figure 11. Note that for its dual (1.23) we have $\mathfrak{G}_{\Delta_{Q^*}} \underset{\text{gr.}}{\cong} \mathfrak{G}^{\text{rev}}_{\Delta_Q}$ (If $\ell \geqslant 3$, the socii of $2, \frac{1}{2}(\ell+1), \ell-1$ w.r.t. ℓ are $\frac{1}{2}(\ell+1), 2$, and $\ell-1$, respectively.)

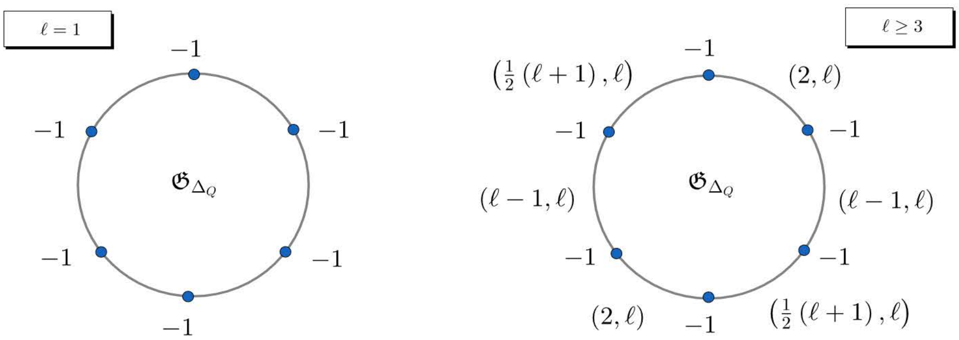


Figure 11: The WVE^{2}C-graph of the ℓ-reflexive hexagon (1.22)

If Q is an ℓ-reflexive polygon, examples 5.5 suggest that there should be a particular connection betweeen the combinatorial triples of Δ_Q and Δ_{Q^*} (and, consequently, between the WVE^{2}C-graphs $\mathfrak{G}_{\Delta_Q}$ and $\mathfrak{G}_{\Delta_{Q^*}}$) due to bijections (1.13) and (1.14). This will be clarified below in Propositions 7.10, 7.14 and 7.16.

6 Lattice change and cyclic covering trick whenever $\ell > 1$

• **Degree.** Let $f : X \longrightarrow Y$ be a proper holomorphic map between two complex (analytic) spaces. f is called *finite* if it is closed (as map) and for every $y \in Y$ the fibre $f^{-1}(\{y\})$ consists of finitely many points. f is called *generically finite* if there is a non-empty open subset $V \subset Y$ such that $f|_{f^{-1}(V)} : f^{-1}(V) \longrightarrow V$ is finite. If X and Y are complex *varieties*, f generically finite and $f(X)$ dense in Y, then the field extension defined by $f^\star : \mathbb{C}(Y) \hookrightarrow \mathbb{C}(X)$ is finite and $\deg(f) := [\mathbb{C}(X) : f^\star(\mathbb{C}(Y))]$ is said to be *the degree of* f. (Note that, in this case, the set $\{y \in V \,|\sharp(f^{-1}(\{y\})) = \deg(f)\}$ is dense in V.)

• **Étale holomorphic maps**. For any complex space X let us denote by $\mathcal{O}_X$ its *structure sheaf* and by Ω^1_X *the sheaf of germs of holomorhic* 1-*forms* on X (or *the cotangent sheaf on* X, cf. [26, §2.9 and §2.21]). If X and Y are two complex spaces, $f : X \longrightarrow Y$ a holomorphic map and $Df : f^\star\Omega^1_Y \longrightarrow \Omega^1_X$ the associated homomorphism (which is determined by means of the Jacobian), then one defines *the sheaf* $\Omega^1_{X|Y} := \mathrm{Coker}(Df)$ *of germs of relative* 1-*forms w.r.t.* f. The holomorphic map f is called *flat at* $x \in X$ if the stalk $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module. ($\mathcal{O}_{X,x}$ becomes an $\mathcal{O}_{Y,f(x)}$-module via the natural homomorphism $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$.) f is called *flat* if it is flat at all points of X. f is said to be *étale at a point* $x \in X$ if it is flat at x and simultaneously *unramified at* x, i.e., $\mathfrak{m}_{Y,f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ (where $\mathfrak{m}_{X,x}$ and $\mathfrak{m}_{Y,f(x)}$ denote the maximal ideals of the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,f(x)}$, respectively). f is called *étale* if it is étale at all points of X. f is, in particular, étale if and only if it is flat and $\Omega^1_{X|Y} = 0$.

• **Analytic spectrum.** Let X be a complex space and $\mathcal{G}$ be an arbitrary sheaf of $\mathcal{O}_X$-modules (*an* $\mathcal{O}_X$*-module*, for short). $\mathcal{G}$ is said to be *of finite type* at $x \in X$ if there is an open neighborhood $\mathcal{U}_x$ of x and a $\mathcal{G}|_{\mathcal{U}_x}$-epimorhism $\mathcal{O}_X^{\kappa_x} \to \mathcal{G}|_{\mathcal{U}_x}$ for a positive integer κ_x. $\mathcal{G}$ is called *of finite type* on X if it is of finite type at all points $x \in X$. $\mathcal{G}$ is *coherent* if $\mathcal{G}$ is of finite type on X and, in addition, for every $x \in X$ and every finite system $\mathfrak{s}_1, \ldots, \mathfrak{s}_\kappa \in \mathcal{G}(\mathfrak{U}_x)$ of sections over an open neighborhood $\mathfrak{U}_x$ of x the sheaf of relations $\mathrm{Rel}_x(\mathfrak{s}_1, \ldots, \mathfrak{s}_\kappa)$ (which is the kernel of the $\mathcal{G}|_{\mathfrak{U}_x}$-homomorhism $\mathcal{O}^\kappa_{\mathfrak{U}_x} \to \mathcal{G}|_{\mathfrak{U}_x}$ determined by $\mathfrak{s}_1, \ldots, \mathfrak{s}_\kappa$) is of finite type at x. If $\mathcal{G}$ happens to be a sheaf of $\mathcal{O}_X$-algebras (*an* $\mathcal{O}_X$*-algebra*, for short), i.e., if $\mathcal{G}_x$ is an $\mathcal{O}_{X,x}$-module and at the same time endowed with a *ring structure* for all $x \in X$, then the following is of particular importance.

Theorem 6.1. *Let X be a complex space and $\mathcal{G}$ be a coherent $\mathcal{O}_X$-algebra. Then there exists a unique (up to analytic isomorphism) complex space* $\mathrm{Specan}(\mathcal{G})$, *the so-called* analytic spectrum *of* $\mathcal{G}$, *as well as a finite holomorphic map* $\pi : \mathrm{Specan}(\mathcal{G}) \longrightarrow X$, *such that*

(i) *there is an isomorphism* $\pi_*(\mathcal{O}_{\mathrm{Specan}(\mathcal{G})}) \cong \mathcal{G}$, *and*

(ii) *there is a bijection* $\pi^{-1}(x) \leftrightarrow \mathrm{Max\text{-}Spec}(\mathcal{G}_x)$ *between the set of points of the fibre of* π *over* x *and the set of maximal ideals of the stalk of* $\mathcal{G}$ *at* x, *for all* $x \in X$.

For a rough local description of this "spectrum" in the analytic category we refer to [26, pp. 59-62] and [48, 45.B.1, p. 172], and for more details on the construction and the main properties of π to Houzel [42].

• **Normal complex varieties which are $\mathbb{Q}$-Gorenstein.** If X is a normal complex variety, then its Weil divisors can be described by means of "divisorial" sheaves.

Lemma 6.2. ([34, 1.6]). *For a coherent $\mathcal{O}_X$-module $\mathcal{F}$ the following conditions are equivalent*:
(i) $\mathcal{F}$ *is* reflexive (*i.e.*, $\mathcal{F} \cong \mathcal{F}^{\vee\vee}$, *with* $\mathcal{F}^{\vee} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ *denoting the dual of* $\mathcal{F}$) *and has rank one.*
(ii) *If* X^0 *is a non-singular open subvariety of* X *with* $\mathrm{codim}_X(X \smallsetminus X^0) \geqslant 2$, *then* $\mathcal{F}|_{X^0}$ *is invertible and*

$$\mathcal{F} \cong \iota_\star(\mathcal{F}|_{X^0}) \cong \iota_\star \iota^\star(\mathcal{F}),$$

where $\iota : X^0 \hookrightarrow X$ *denotes the inclusion map.*

The *divisorial sheaves* are exactly those satisfying the above conditions. Since a divisorial sheaf is torsion free, there is a non-zero section $\mathfrak{s} \in H^0(X, \mathcal{M}_X \otimes_{\mathcal{O}_X} \mathcal{F})$, with $H^0(X, \mathcal{M}_X \otimes_{\mathcal{O}_X} \mathcal{F}) \cong \mathbb{C}(X) \cdot \mathfrak{s}$, and $\mathcal{F}$ can be considered as a subsheaf of the constant sheaf $\mathcal{M}_X$ of meromorphic functions of X, i.e., as a special *fractional ideal sheaf.* Let $\mathcal{M}_X^*$ and $\mathcal{O}_X^*$ be the sheaves of germs of not identically vanishing meromorphic functions and of nowhere vanishing holomorphic functions on X, respectively.

Proposition 6.3. ([61, Appendix of §1]) *The correspondence*

$$\left\{ \begin{array}{c} \textit{classes of Weil} \\ \textit{divisors on } X \\ \textit{(w.r.t. rational equivalence)} \end{array} \right\} \ni \{D\} \overset{\mathfrak{d}}{\longmapsto} \mathcal{O}_X(D) \in \left\{ \begin{array}{c} \textit{divisorial} \\ \textit{coherent} \\ \textit{subsheaves of } \mathcal{M}_X \end{array} \right\} \Big/ \; H^0(X, \mathcal{O}_X^*)$$

with $\mathcal{O}_X(D)$ *defined by sending every non-empty open set* $\mathcal{U}$ *of* X *onto*

$$\Gamma(\mathcal{U}, \mathcal{O}_X(D)) := \mathcal{O}_X(D)(\mathcal{U}) := \{\varphi \in \mathcal{M}_X^*(\mathcal{U}) \mid (\mathrm{div}(\varphi) + D)|_{\mathcal{U}} \geqslant 0\} \cup \{\mathbf{0}\},$$

is a bijection, and induces a $\mathbb{Z}$-module isomorphism. In fact, to avoid torsion, one defines this $\mathbb{Z}$-module structure by setting

$$\mathfrak{d}(\{D_1 + D_2\}) := (\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{\vee\vee} \text{ and } \; \mathfrak{d}(\{jD\}) := \mathcal{O}_X(jD)^{\vee\vee},$$

for any Weil divisors D, D_1, D_2 and $j \in \mathbb{Z}$.

Let now $\Omega^1_{\mathrm{Reg}(X)}$ be the cotangent sheaf on $\mathrm{Reg}(X) := X \smallsetminus \mathrm{Sing}(X) \overset{\iota}{\hookrightarrow} X$, and for $j \geqslant 2$ let us set $\Omega^j_{\mathrm{Reg}(X)} := \bigwedge^{j} \Omega^1_{\mathrm{Reg}(X)}$. The canonical divisor K_X of X is that one, the class of which is mapped by $\mathfrak{d}$ onto the *canonical divisorial sheaf* $\omega_X := \iota_*\left(\Omega^{\dim_{\mathbb{C}}(X)}_{\mathrm{Reg}(X)}\right)$. Note that $\omega_X = \omega_X^{[1]} := \mathcal{O}_X(K_X)$ and that $\omega_X^{[j]} := \mathcal{O}_X(jK_X) = (\omega_X^{\otimes j})^{\vee\vee} = \iota_*((\Omega^{\dim_{\mathbb{C}}(X)}_{\mathrm{Reg}(X)})^{\otimes j})$ for all $j \in \mathbb{Z}$.

Definition 6.4. X is called $\mathbb{Q}$-*Gorenstein* if its canonical divisorial sheaf $\omega_X = \mathcal{O}_X(K_X)$ is such that K_X is $\mathbb{Q}$-Cartier divisor. If X is $\mathbb{Q}$-Gorenstein, then we set

$$\mathrm{index}(X) := \min\{j \in \mathbb{Z}_{\geqslant 1} \,|\, jK_X \text{ is Cartier}\}.$$

• **Canonical cyclic coverings**. Given a point x_0 of a normal $\mathbb{Q}$-Gorenstein complex variety X, we consider an affine neighborhood U of x_0 representing the set germ at x_0, and a nowhere vanishing section $\mathfrak{s} \in H^0(U, \mathcal{O}_U(-\mathrm{index}(U)K_U)$ such that

$$H^0(U, \mathcal{O}_U(-\mathrm{index}(U)K_U) = H^0(U, \omega_U^{[\text{-index}(U)]}) \cong \mathcal{O}_U \cdot \mathfrak{s} \cong \mathcal{O}_U.$$

If $\alpha \in \omega_U^{[i]}, \beta \in \omega_U^{[j]}$ and $\mathfrak{v}_U : \omega_U^{[i]} \otimes \omega_U^{[j]} \longrightarrow \omega_U^{[i+j]}$ is the natural map, then the coherent $\mathcal{O}_U$-module

$$\mathcal{R}_U := \mathcal{O}_U \oplus \omega_U \oplus \omega_U^{[2]} \oplus \cdots \oplus \omega_U^{[\mathrm{index}(U)-1]},$$

equipped with the multiplication "$\odot$" being induced by setting

$$\alpha \odot \beta := \begin{cases} \mathfrak{v}_U(\alpha \otimes \beta) \in \omega_U^{[i+j]}, & \text{if } i+j \leqslant \mathrm{index}(U) - 1, \\ \mathfrak{v}_U(\alpha \otimes \beta) \cdot \mathfrak{s} \in \omega_U^{[i+j-\mathrm{index}(U)]}, & \text{if } i+j \geqslant \mathrm{index}(U), \end{cases}$$

becomes an $\mathcal{O}_U$-algebra. Let $\pi_U : \mathrm{Specan}(\mathcal{R}_U) \longrightarrow U$ be the corresponding finite holomorphic map constructed by Theorem 6.1. Wahl (in the algebraic category, cf. [68, Appendix, pp. 260-262]) and Reid [61, Appendix of §1, pp. 281-285] were the first who initiated the use of π_U in order to replace U by $U^{\mathrm{can}} := \mathrm{Specan}(\mathcal{R}_U)$ *of index* 1 in the case in which x_0 is *singular*.

Theorem 6.5. ([61, 1.9], [62, §3.6] and [51, 4-5-1 & 4-5-2, pp. 183-186]) *The pair* $(U^{\mathrm{can}}, \pi_U)$ *has (and is up to an analytic isomorphism uniquely determined by) the following properties*:

(i) U^{can} *is a normal complex variety and the fiber* $\pi_U^{-1}(\{x_0\})$ *is a singleton (say* $\{y_0\}$*).*

(ii) *The field extension* $\mathbb{C}(U^{\mathrm{can}})$ *of* $\mathbb{C}(U)$ *is Galois with Galois group* $G_U \cong \mathbb{Z}/(\mathrm{index}(U))\mathbb{Z}$ *and with a generator* $\mathfrak{g}$ *of* G_U *acting on* $\mathcal{R}_U$ *as follows:*

$$(\mathfrak{g}, \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_{\mathrm{index}(U)-1}) \longmapsto \alpha_0 + \alpha_1\zeta_{\mathrm{index}(U)} + \alpha_2\zeta_{\mathrm{index}(U)}^2 \cdots + \alpha_{\mathrm{index}(U)-1}\zeta_{\mathrm{index}(U)}^{\mathrm{index}(U)-1}$$

(*with* $\zeta_{\mathrm{index}(U)} := \exp(2\pi\sqrt{-1}/\mathrm{index}(U)), \alpha_0 \in \mathcal{O}_U$ *and* $\alpha_i \in \omega_U^{[i]}$ *for* $i \in \{1, \ldots, \mathrm{index}(U) - 1\}$).

(iii) π_U *is étale in codimension*[2] 1.

(iv) $\mathcal{O}_{U^{\mathrm{can}}}(K_{U^{\mathrm{can}}}) \cong \mathcal{O}_{U^{\mathrm{can}}}$, *i.e.,* $K_{U^{\mathrm{can}}}$ *is a Cartier divisor, and* U^{can} *is a* $\mathbb{Q}$*-Gorenstein affine complex variety of index* 1.

(v) *There is a non-vanishing section* $\mathfrak{s}' \in H^0(U^{\mathrm{can}}, \mathcal{O}_{U^{\mathrm{can}}}(K_{U^{\mathrm{can}}}))$ *around the point* y_0 *which is semi-invariant w.r.t. the action of* G_U *and on which* G_U *acts faithfully.*

($\pi_U : U^{\mathrm{can}} \longrightarrow U$ is said to be the *canonical cyclic cover of* U of degree $\deg(\pi_U) = \mathrm{index}(U)$.)

[2]This means étale outside a subvariety of codimension $\geqslant 2$.

Remark 6.6. (i) In particular, $\pi_U : U^{\mathrm{can}} \longrightarrow U$ is surjective[3] and can be viewed as the quotient map by an appropriate identification $U \cong U^{\mathrm{can}}/G_U$.

(ii) If $\varphi \in \mathbb{C}(U)$ is such that $\mathrm{div}(\varphi) + K_U = 0$, then the polynomial $\mathsf{T}^{\mathrm{index}(U)} - \varphi$ is irreducible in $\mathbb{C}(U)[\mathsf{T}]$, the Galois extension

$$\mathbb{C}(U)[\sqrt[\mathrm{index}(U)]{\varphi}] = \mathbb{C}(U)[\mathsf{T}]/(\mathsf{T}^{\mathrm{index}(U)} - \varphi)$$

of $\mathbb{C}(U)$ has G_U as Galois group, and

$$U^{\mathrm{can}} \cong \mathrm{Spec}\Big(\bigoplus_{j=0}^{\mathrm{index}(U)-1} \Gamma(U, \omega_U^{[j]}) \cdot (\sqrt[\mathrm{index}(U)]{\varphi})^j\Big), \quad \forall j \in \{0, 1, \ldots, \mathrm{index}(U) - 1\}.$$

• **Back to our specific toric log del Pezzo surfaces.** Let ℓ be an integer > 1, and (Q, N), (Q^*, M) two ℓ-reflexive pairs, where $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, with $X(N, \Delta_Q)$ and $X(M, \Delta_{Q^*})$ the corresponding toric log del Pezzo surfaces. Assume that $\mathbf{n}_1, \ldots, \mathbf{n}_\nu$ are the vertices of Q ordered anticlockwise, and for $i \in \{1, \ldots, \nu\}$ define $F_i := \mathrm{conv}(\{\mathbf{n}_i, \mathbf{n}_{i+1}\})$ to be the i-th edge of Q (as in 5.4 (ii)) and $\sigma_i := \sigma_{F_i} = \mathbb{R}_{\geqslant 0}\mathbf{n}_i + \mathbb{R}_{\geqslant 0}\mathbf{n}_{i+1}$ the N-cone of type (p_i, q_i) supporting F_i. It is easy to verify that

$$U_{\sigma_i,N} \cong \mathrm{Spec}(\mathbb{C}[z_1, z_2]^{G_i}), \text{ where } G_i := \langle \mathrm{diag}(\zeta_{q_i}^{-p_i}, \zeta_{q_i}) \rangle \subset \mathrm{GL}_2(\mathbb{C}) \text{ (as in Proposition 2.14)},$$

is $\mathbb{Q}$-Gorenstein and that $\mathrm{index}(U_{\sigma_i,N}) = l_{F_i}$, where $l_{F_i} = q_i/\gcd(q_i, p_i - 1)$ is the *local index* of F_i (w.r.t. Q) as defined in 1.10 (ii). (See [18, Notes 3.19, p. 89, and 4.5 (b), p.96].)

Lemma 6.7. *For every $i \in \{1, \ldots, \nu\}$ let $\Lambda_{F_i} \subseteq N$ be the sublattice generated by the lattice points of F_i. The canonical cyclic cover*

$$\pi_{U_{\sigma_i,N}} : \mathrm{Specan}(\mathcal{R}_{U_{\sigma_i,N}}) \longrightarrow U_{\sigma_i,N} \cong \mathbb{C}^2/G_i = \mathrm{Spec}(\mathbb{C}[z_1, z_2]^{G_i}) \tag{6.1}$$

has degree l_{F_i}, with

$$\mathrm{Specan}(\mathcal{R}_{U_{\sigma_i,N}}) \cong U_{\sigma_i,\Lambda_{F_i}} \cong \mathbb{C}^2/G_i' = \mathrm{Spec}(\mathbb{C}[z_1, z_2]^{G_i'}), \text{ where } G_i' := G_i \cap \mathrm{SL}_2(\mathbb{C}),$$

and can be viewed as the quotient map by the identification

$$U_{\sigma_i,N} \cong U_{\sigma_i,\Lambda_{F_i}}/(N/\Lambda_{F_i}) \quad \text{with} \quad N/\Lambda_{F_i} \cong G_i/G_i' \cong \mathbb{Z}/l_{F_i}\mathbb{Z}.$$

Moreover, $\pi^{-1}_{U_{\sigma_i,N}}(\{\mathrm{orb}_N(\sigma_i)\}) = \{\mathrm{orb}_{\Lambda_{F_i}}(\sigma_i)\}$, where the point $\mathrm{orb}_{\Lambda_{F_i}}(\sigma_i) \in U_{\sigma_i,\Lambda_{F_i}}$ is either nonsingular or a Gorenstein cyclic quotient singularity of type $(1, \frac{q_i}{l_{F_i}})$.

Proof. Consider an arbitrary $i \in \{1, \ldots, \nu\}$. Firstly, $\mathrm{index}(U_{\sigma_i,N}) = l_{F_i} = |N : \Lambda_{F_i}|$. Secondly,

$$\mathrm{diag}(\zeta_{q_i}^{-p_i\ell}, \zeta_{q_i}^{\ell}) = \mathrm{diag}(\zeta_{\frac{q_i}{l_{F_i}}}^{-p_i}, \zeta_{\frac{q_i}{l_{F_i}}}) = \mathrm{diag}(\zeta_{\frac{q_i}{l_{F_i}}}^{-1}, \zeta_{\frac{q_i}{l_{F_i}}}),$$

[3]Since π_U is finite and surjective, we have, in particular, $\dim_{\mathbb{C}}(U^{\mathrm{can}}) = \dim_{\mathbb{C}}(U)$.

and therefore

$$G_i' := G_i \cap \mathrm{SL}_2(\mathbb{Z}) = \mathrm{Ker}(G_i \xrightarrow{\det} \mathbb{C}^\times) = \left\langle \mathrm{diag}(\zeta_{\frac{q_i}{l_{F_i}}}^{-1}, \zeta_{\frac{q_i}{l_{F_i}}}) \right\rangle, \quad \text{with} \quad |G_i'| = \frac{q_i}{l_{F_i}}.$$

Since $G_i \cong \mathbb{Z}/q_i\mathbb{Z}$, $G_i' \cong \mathbb{Z}/\frac{q_i}{l_{F_i}}\mathbb{Z}$ and $G_i/G_i' \cong \mathbb{Z}/l_{F_i}\mathbb{Z}$, the diagram

$$\begin{array}{ccccccccc}
 & & & & & & \{0\} & & \\
 & & & & & & \downarrow & & \\
\{0\} & \longrightarrow & \mathbb{Z}\mathbf{n}_i \oplus \mathbb{Z}\mathbf{n}_{i+1} & \hookrightarrow & \Lambda_{F_i} & \twoheadrightarrow & \mathbb{Z}/\frac{q_i}{l_{F_i}}\mathbb{Z} & \longrightarrow & \{0\} \\
 & & \| & & \downarrow & & \downarrow & & \\
\{0\} & \longrightarrow & \mathbb{Z}\mathbf{n}_i \oplus \mathbb{Z}\mathbf{n}_{i+1} & \hookrightarrow & N & \twoheadrightarrow & \mathbb{Z}/q_i\mathbb{Z} & \longrightarrow & \{0\} \\
 & & \downarrow & & \| & & \downarrow & & \\
\{0\} & \longrightarrow & \Lambda_{F_i} & \hookrightarrow & N & \twoheadrightarrow & \mathbb{Z}/l_{F_i}\mathbb{Z} & \longrightarrow & \{0\} \\
 & & & & & & \downarrow & & \\
 & & & & & & \{0\} & &
\end{array}$$

(the three rows and the last column of which are short exact sequences of additive groups) combined with [17, Proposition 1.13.18 and Ex. 1.3.20, pp. 44-46] gives $U_{\sigma_i,\Lambda_{F_i}} \cong \mathbb{C}^2/G_i'$, $U_{\sigma_i,N} \cong \mathbb{C}^2/G_i$, and

$$U_{\sigma_i,N} \cong U_{\sigma_i,\Lambda_{F_i}}/(G_i/G_i') \cong \mathbb{C}^2/G_i'/(G_i/G_i').$$

Now we apply Theorem 6.5 for $U_{\sigma_i,N}$. For every $j \in \{0, 1, \ldots, l_{F_i} - 1\}$ the divisor $-jK_{U_{\sigma_i,N}}$ is $\mathbb{T}_N$-invariant and $\Gamma(U_{\sigma_i,N}, \omega_{U_{\sigma_i,N}}^{[j]})$ is a reflexive $\mathbb{C}[\sigma_i^\vee \cap M]$-module of rank 1. Therefore $\mathcal{R}_{U_{\sigma_i,N}} := \bigoplus_{j=0}^{l_{F_i}-1} \omega_{U_{\sigma_i,N}}^{[j]}$ is a $\mathbb{T}_N$-invariant $\mathcal{O}_{U_{\sigma_i,N}}$-algebra, its analytic spectrum $\mathrm{Specan}(\mathcal{R}_{U_{\sigma_i,N}})$ is an affine *toric* surface which is $\mathbb{Q}$-Gorenstein and of index 1 (which means that it is a two-dimensional Gorenstein variety[4]), and the canonical cover map (6.1) is equivariant. Setting $\varphi_i := -\frac{\mathbf{de}(\mathbf{m}_i)}{\mathbf{e}(\mathbf{m}_i)} \wedge \frac{\mathbf{de}(\mathbf{m}_i')}{\mathbf{e}(\mathbf{m}_i')}$, with $\{\mathbf{m}_i, \mathbf{m}_i'\}$ a basis of M, we have $\mathrm{div}(\varphi_i) = -K_{U_{\sigma_i,N}}$ (cf. Oda [56, p. 71]), $\mathsf{T}^{l_{F_i}} - \varphi_i$ is irreducible in $\mathbb{C}(U_{\sigma_i,N})[\mathsf{T}]$ and the Galois extension $\mathbb{C}(U_{\sigma_i,N})[\sqrt[l_{F_i}]{\varphi_i}]$ of $\mathbb{C}(U_{\sigma_i,N})$ has a cyclic Galois group, say $G_i'' \cong \mathbb{Z}/l_{F_i}\mathbb{Z}$ (because $\deg(\pi_{U_{\sigma_i,N}}) = l_{F_i}$). Since

$$\mathrm{Specan}(\mathcal{R}_{U_{\sigma_i,N}}) \cong \mathrm{Spec}\left(\bigoplus_{j=0}^{l_{F_i}-1} \Gamma(U_{\sigma_i,N}, \omega_{U_{\sigma_i,N}}^{[j]}) \cdot (\sqrt[l_{F_i}]{\varphi_i})^j \right)$$

is a *Gorenstein* toric affine surface, it suffices for our purposes to recall that it has to appear as the quotient of $\mathbb{C}^2$ by a finite cyclic subgroup H_i of $\mathrm{SL}_2(\mathbb{C})$ acting diagonally. W.l.o.g. we may assume that $\mathrm{Specan}(\mathcal{R}_{U_{\sigma_i,N}}) \cong \mathbb{C}^2/H_i \cong U_{\sigma_i,L_i}$ (i.e., the toric affine

[4] If a $\mathbb{Q}$-Gorenstein variety of index 1 is Cohen-Macauley, then it is a *Gorenstein variety*, i.e., the local ring at each of its points is a Gorenstein ring.

surface associated with the *same* cone σ_i but with respect to *another* lattice $L_i \subset \mathbb{R}^2$, such that $|L_i : \mathbb{Z}\mathbf{n}_i \oplus \mathbb{Z}\mathbf{n}_{i+1}| = |H_i|$), that $H_i \subseteq G'_i$ and that $\pi^{-1}_{U_{\sigma_i,N}}(\{\mathrm{orb}_N(\sigma_i)\}) = \{\mathrm{orb}_{L_i}(\sigma_i)\}$. Using the equivariant holomorphic map determined by the dotted arrow in the diagram:

$$
\begin{array}{ccccccc}
\mathbb{C}^2 & \overset{\cong}{\longleftrightarrow} & U_{\sigma_i,\mathbb{Z}\mathbf{n}_i\oplus\mathbb{Z}\mathbf{n}_{i+1}} & = & U_{\sigma_i,\mathbb{Z}\mathbf{n}_i\oplus\mathbb{Z}\mathbf{n}_{i+1}} & \overset{\cong}{\longleftrightarrow} & \mathbb{C}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C}^2/G'_i & \overset{\cong}{\longleftrightarrow} & U_{\sigma_i,\Lambda_{F_i}} & \leftarrow\,-\,-\,-\,- & U_{\sigma_i,L_i} & \overset{\cong}{\longleftrightarrow} & \mathbb{C}^2/H_i \\
\downarrow & & \downarrow & \swarrow{\scriptstyle \pi_{U_{\sigma_i,N}}} & \downarrow & & \\
\mathbb{C}^2/G_i & \overset{\cong}{\longleftrightarrow} & U_{\sigma_i,N} & \overset{\cong}{\longleftrightarrow} & U_{\sigma_i,L_i}/G''_i & &
\end{array}
$$

we verify that $\Lambda_{F_i}/L_i \cong G'_i/H_i$. On the other hand, the restriction

$$\xi_i := \pi_{U_{\sigma_i,N}}\Big|_{U_{\sigma_i,L_i}\smallsetminus\{\mathrm{orb}_{L_i}(\sigma_i)\}} : U_{\sigma_i,L_i}\smallsetminus\{\mathrm{orb}_{L_i}(\sigma_i)\} \twoheadrightarrow U_{\sigma_i,N}\smallsetminus\{\mathrm{orb}_N(\sigma_i)\}$$

is an étale holomorphic map (and, in particular, a topological, i.e., an unramified covering map), and

$$G''_i \cong \pi_1(U_{\sigma_i,N}\smallsetminus\{\mathrm{orb}_N(\sigma_i)\})/\xi_{i*}(\pi_1(U_{\sigma_i,L_i}\smallsetminus\{\mathrm{orb}_{L_i}(\sigma_i)\}))$$

(where $\pi_1(\dots)$ denotes the fundamental group of these pathwise connected spaces, cf. [52, Theorem 2.8, p. 18]). Furthermore, the composite of the étale holomorphic maps

$$\mathbb{C}^2\smallsetminus\{\mathbf{0}\} \twoheadrightarrow U_{\sigma_i,L_i}\smallsetminus\{\mathrm{orb}_{L_i}(\sigma_i)\} \twoheadrightarrow U_{\sigma_i,N}\smallsetminus\{\mathrm{orb}_N(\sigma_i)\}$$

(where $\mathbb{C}^2\smallsetminus\{\mathbf{0}\}$ is the universal cover of $U_{\sigma_i,L_i}\smallsetminus\{\mathrm{orb}_{L_i}(\sigma_i)\}$ which is simply connected) gives the following short exact sequence of fundamental groups:

$$
\begin{array}{ccccccccc}
\pi_1(\mathbb{C}^2\smallsetminus\{\mathbf{0}\}) & \longrightarrow & \pi_1(U_{\sigma_i,L_i}\smallsetminus\{\mathrm{orb}_{L_i}(\sigma_i)\}) & \overset{\xi_{i*}}{\hookrightarrow} & \pi_1(U_{\sigma_i,N}\smallsetminus\{\mathrm{orb}_N(\sigma_i)\}) & \twoheadrightarrow & G''_i & \longrightarrow & \{1\} \\
\| & & \| & & \| & & \| & & \| \\
\{1\} & \longrightarrow & H_i & \hookrightarrow & G_i & \twoheadrightarrow & G''_i & \longrightarrow & \{1\}
\end{array}
$$

Since H_i, G_i and G''_i are *finite* groups, we have

$$l_{F_i} = |G''_i| = |G_i| \,/\, |H_i| = \frac{q_i}{|H_i|} \Rightarrow |H_i| = \frac{q_i}{l_{F_i}} = |G'_i|\,,$$

and we conclude that $H_i = G'_i$ and $L_i = \Lambda_{F_i}$. Finally, it is by construction obvious that the orbit $\mathrm{orb}_{\Lambda_{F_i}}(\sigma_i) \in U_{\sigma_i,\Lambda_{F_i}} \cong \mathbb{C}^2/G'_i$ is either a smooth point (whenever G'_i is trivial) or a cyclic quotient singularity of type $(1, \frac{q_i}{l_{F_i}})$ (whenever $|G'_i| > 1$). $\square$

Now let $\Lambda_Q \subseteq N$ be the sublattice generated by the boundary lattice points of Q and $\Lambda_{Q^*} \subseteq M$ be the sublattice generated by the boundary lattice points of Q^*.

Theorem 6.8. (Kasprzyk & Nill [46, §2]) *We have* $\mathrm{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z}) = \frac{1}{\ell}\Lambda_{Q^*}$ *and*

$$|N : \Lambda_Q| = \ell = |M : \Lambda_{Q^*}| .$$

In addition, (Q, Λ_Q) *and* (Q^*, Λ_{Q^*}) *(with* $Q^* = \ell Q^\circ$*) are* 1*-reflexive pairs, where* (Q^*, Λ_{Q^*}) *is to be identified with* $(Q^\circ, \mathrm{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z}))$*.*

The "beauty" of being ℓ-reflexive is mainly embodied in the following property: All *local indices* of the edges F_i of Q coincide with the *index* ℓ of the toric log del Pezzo surface $X(N, \Delta_Q)$, and this allows us to patch together the canonical cyclic covers over the affine neighborhoods of its singularities in order to create a single *global* finite holomorphic map π_Q of *degree* ℓ and represent $X(N, \Delta_Q)$ as a *global* quotient space.

Theorem 6.9. *There is an equivariant (w.r.t. the actions of the algebraic tori* $\mathbb{T}_{\Lambda_Q}$ *and* $\mathbb{T}_N$*) finite holomorphic map*

$$\pi_Q : X(\Lambda_Q, \Delta_Q) \longrightarrow X(N, \Delta_Q) \tag{6.2}$$

which has degree ℓ *and coincides with the quotient map by the identification*

$$X(N, \Delta_Q) \cong X(\Lambda_Q, \Delta_Q)/(N/\Lambda_Q) \quad \textit{with} \quad \mathrm{Ker}[\mathbb{T}_{\Lambda_Q} \to \mathbb{T}_N] \cong N/\Lambda_Q \cong \mathbb{Z}/\ell\,\mathbb{Z}.$$

Moreover, there exist bases $\mathcal{B}$ *and* $\mathcal{B}^\diamondsuit$ *of the lattices* Λ_Q *and* N*, respectively, as well as a* $k \in \{1, \ldots, \ell - 1\}$ *with* $\gcd(k, \ell) = 1$ *and exactly one* $j \in \{1, \ldots, 16\}$*, such that* $\Phi_{\mathcal{A}_{\ell,k}}(\overline{\mathcal{Q}}_j) = Q^\diamondsuit$*, where* $\overline{\mathcal{Q}}_1, \ldots, \overline{\mathcal{Q}}_{16}$ *are the representatives of the* 16 *equivalent classes of the* 1*-reflexive* $\mathbb{Z}^2$*-polygons given in the table of* 1.19,

$$\Phi_{\mathcal{A}_{\ell,k}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2,\ \binom{x_1}{x_2} \longmapsto \Phi_{\mathcal{A}_{\ell,k}}\left(\binom{x_1}{x_2}\right) := \mathcal{A}_{\ell,k}\binom{x_1}{x_2}, \quad \textit{with}^{5} \quad \mathcal{A}_{\ell,k} := \left(\begin{smallmatrix} \ell & 0 \\ k & 1 \end{smallmatrix}\right),$$

$Q = \Phi_{\mathcal{B}}(\overline{\mathcal{Q}}_j)$*, and* $Q^\diamondsuit := \Phi_{\mathcal{B}^{\diamondsuit\,-1}}(Q)$*. Hence, the dotted arrow (which denotes the* $\mathbb{T}_{\mathbb{Z}^2}$*-equivariant holomorphic map induced by* $\Phi_{\mathcal{A}_{\ell,k}}$*) in the following diagram*

$$\begin{array}{ccc} X(\mathbb{Z}^2, \Delta_{\overline{\mathcal{Q}}_j}) & \dashrightarrow & X(\mathbb{Z}^2, \Delta_{Q^\diamondsuit}) \\ \cong \updownarrow & & \updownarrow \cong \\ X(\Lambda_Q, \Delta_Q) & \xrightarrow{\ \pi_Q\ } & X(N, \Delta_Q) \end{array}$$

can be viewed again as a quotient map.

Proof. Since Q is ℓ-reflexive, we have $l_{F_i} = \ell$ and $U_{\sigma_i, \Lambda_{F_i}} = U_{\sigma_i, \Lambda_Q}$, and for the canonical cyclic covers $\pi_{U_{\sigma_i,N}}$ which are constructed by Lemma 6.7 we obtain

$$\pi_{U_{\sigma_i,N}}\Big|_{U_{\sigma_i,\Lambda_Q} \cap U_{\sigma_{i+1},\Lambda_Q}} = \pi_{U_{\sigma_{i+1},N}}\Big|_{U_{\sigma_i,\Lambda_Q} \cap U_{\sigma_{i+1},\Lambda_Q}},$$

[5] Note that $\det(\mathcal{A}_{\ell,k}) = \ell$.

for all $i \in \{1, \ldots, \nu\}$. Since $\left\{ U_{\sigma_i, \Lambda_Q} \,\middle|\, i \in \{1, \ldots, \nu\} \right\}$ is an open covering of $X(\Lambda_Q, \Delta_Q)$, we may patch them together by setting

$$\pi_Q(\mathfrak{x}) := \pi_{U_{\sigma_i,N}}(\mathfrak{x}), \quad \forall \mathfrak{x} \in U_{\sigma_i, \Lambda_Q}.$$

π_Q is by definition a finite holomorphic map of degree $\ell = |N : \Lambda_Q|$, with

$$\mathrm{Ker}[\mathbb{T}_{\Lambda_Q} \longrightarrow \mathbb{T}_N] = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z})/M, \mathbb{C}^\times) \cong N/\Lambda_Q \cong \mathbb{Z}/\ell\,\mathbb{Z},$$

and it suffices to apply [56, Corollary 1.16, pp. 22-23] or [17, Proposition 3.3.7, pp. 127-128]. On the other hand, (Q, Λ_Q) is an 1-reflexive pair, and utilising suitable bases $\mathcal{B}$ and $\mathcal{B}^\diamondsuit$ of the lattices Λ_Q and N, respectively, we may transfer Q to $\mathbb{Z}^2$-polygons. To define carefully the matrix $\mathcal{A}_{\ell,k}$, so that $\Phi_{\mathcal{A}_{\ell,k}}$ maps $\overline{\mathcal{Q}}_j$ onto $Q^\diamondsuit$, one has to make use of the Hermite normal form. (For details see [46, Corollary 13].) □

Example 6.10. The $\mathbb{Z}^2$-triangle $Q := \mathrm{conv}\left(\left\{ \binom{0}{1}, \binom{14}{3}, \binom{-21}{-5} \right\}\right)$ is 7-reflexive, and via $\mathcal{A}_{7,1}$ we get $\Phi_{\mathcal{A}_{7,1}}(\overline{\mathcal{Q}}_7) = Q$. The toric del Pezzo surface $X(\mathbb{Z}^2, \Delta_Q)$ has three cyclic quotient singularities: One of type $(5, 14)$, one of type $(16, 21)$, and one of type $(5, 7)$. $X(\mathbb{Z}^2, \Delta_{\overline{\mathcal{Q}}_7})$ inherits a Gorenstein cyclic quotient singularity of type $(1, 2)$ over the first, a Gorenstein cyclic quotient singularity of type $(1, 3)$ over the second, and a smooth point over the third.

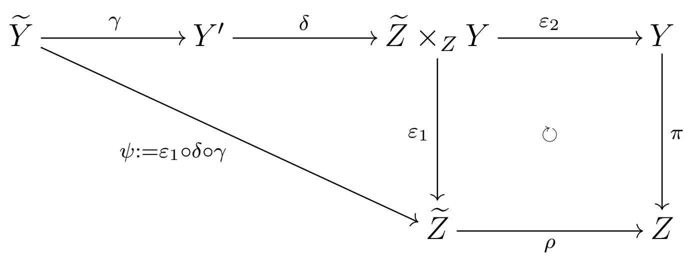
Remark 6.11. Clearly, Theorem 6.9 gives $\sharp(\mathrm{RP}(\ell; N)) \leqslant 16\,\phi(\ell)$, where ϕ is Euler's totient function, but this is only a rough upper bound. In fact, $\sharp(\mathrm{RP}(\ell; N))$ depends essentially on number-theoretic restrictions on the weights of the possible WVE^{2}C-graphs. In practice, for the classification of ℓ-reflexive polygons and for the construction of precise tables like those in [9], one has to perform ad-hoc tests to distinguish lattice-inequivalent polygons. (Cf. Grinis & Kasprzyk [31] for a more general discussion on the normal forms of lattice polytopes.)

Lemma 6.12. *Let Y and Z be two normal projective surfaces and $\pi : Y \longrightarrow Z$ be a generically finite and surjective holomorphic map of degree d. If D_1, D_2 are two $\mathbb{Q}$-Weil divisors on Z, then*

$$D_1 \cdot D_2 = \frac{1}{d}(\pi^\star(D_1) \cdot \pi^\star(D_2)), \tag{6.3}$$

where $\pi^\star(D_j)$ is the pullback of D_j, $j \in \{1, 2\}$, via π (in the sense of [28, p. 32]*).*

Proof. Denoting by $\rho : \widetilde{Z} \longrightarrow Z$ the minimal desingularization of the surface Z, by $\delta : Y' \longrightarrow \widetilde{Z} \times_Z Y$ the normalisation of the fiber product $\widetilde{Z} \times_Z Y$, and by $\gamma : \widetilde{Y} \longrightarrow Y'$ the minimal desingularization of Y', we obtain a commutative diagram of the form:



Since both $\widetilde{Z}$ and $\widetilde{Y}$ are *smooth*, and $\psi : \widetilde{Y} \longrightarrow \widetilde{Z}$ is generically finite and surjective (of degree d), we have

$$\begin{aligned} D_1 \cdot D_2 &:= \rho^\star(D_1) \cdot \rho^\star(D_2) && \text{(by [53, pp. 17-18])} \\ &= \tfrac{1}{d}(\psi^\star(\rho^\star(D_1)) \cdot \psi^\star(\rho^\star(D_2))) && \text{(by [7, Proposition I.8 (ii), pp. 4-5]).} \end{aligned}$$

On the other hand,

$$\begin{aligned} &\psi^\star(\rho^\star(D_1)) \cdot \psi^\star(\rho^\star(D_2)) = (\rho \circ \psi)^\star(D_1) \cdot (\rho \circ \psi)^\star(D_2) \\ &= (\pi \circ \varepsilon_2 \circ \delta \circ \gamma)^\star(D_1) \cdot (\pi \circ \varepsilon_2 \circ \delta \circ \gamma)^\star(D_2) \\ &= (\varepsilon_2 \circ \delta \circ \gamma)^\star(\pi^\star(D_1)) \cdot (\varepsilon_2 \circ \delta \circ \gamma)^\star(\pi^\star(D_2)) \\ &= \pi^\star(D_1) \cdot \pi^\star(D_2) \quad \text{(by [53, pp. 17-18] and [28, 7.1.16, p. 125])} \end{aligned}$$

and therefore (6.3) is true. □

Proposition 6.13. *The self-intersection number of the canonical divisor of* $X(N, \Delta_Q)$ *is*

$$K^2_{X(N,\Delta_Q)} = \frac{1}{\ell} K^2_{X(\Lambda_Q,\Delta_Q)}. \tag{6.4}$$

Correspondingly, the self-intersection number of the canonical divisor of $X(M, \Delta_{Q^*})$ *is*

$$K^2_{X(M,\Delta_{Q^*})} = \frac{1}{\ell} K^2_{X(\Lambda_{Q^*},\Delta_{Q^*})}. \tag{6.5}$$

Proof. Let $\iota : \mathrm{Reg}(X(\Lambda_Q, \Delta_Q)) \hookrightarrow X(\Lambda_Q, \Delta_Q)$ and $\iota' : \mathrm{Reg}(X(N, \Delta_Q)) \hookrightarrow X(N, \Delta_Q)$ be the natural inclusions of the regular loci of $X(\Lambda_Q, \Delta_Q)$ and $X(N, \Delta_Q)$ into themselves. Obviously,

$$\pi_Q^{-1}(\mathrm{Reg}(X(N, \Delta_Q))) \subseteq \mathrm{Reg}(X(\Lambda_Q, \Delta_Q))$$

and

$$\mathrm{codim}_{X(\Lambda_Q,\Delta_Q)}(X(\Lambda_Q, \Delta_Q) \smallsetminus \pi_Q^{-1}(\mathrm{Reg}(X(N, \Delta_Q)))) = 2.$$

Since

$$\pi_Q|_{\pi_Q^{-1}(\mathrm{Reg}(X(N,\Delta_Q)))} : \pi_Q^{-1}(\mathrm{Reg}(X(N, \Delta_Q))) \longrightarrow \mathrm{Reg}(X(N, \Delta_Q))$$

is an étale holomorphic map, we have

$$\Omega^1_{\pi_Q^{-1}(\mathrm{Reg}(X(N,\Delta_Q)))|\mathrm{Reg}(X(N,\Delta_Q))} = 0 \Longrightarrow \Omega^1_{\pi_Q^{-1}(\mathrm{Reg}(X(N,\Delta_Q)))} \cong \pi_Q^\star(\Omega^1_{\mathrm{Reg}(X(N,\Delta_Q))}).$$

Passing to $\bigwedge^2 \ldots$ and taking into account 6.2 (ii) this implies

$$\mathcal{O}_{X(\Lambda_Q,\Delta_Q)}(K_{X(\Lambda_Q,\Delta_Q)}) = \omega_{X(\Lambda_Q,\Delta_Q)} = \iota_\star(\Omega^2_{\mathrm{Reg}(X(\Lambda_Q,\Delta_Q))}) = \iota_\star(\Omega^2_{\pi_Q^{-1}(\mathrm{Reg}(X(N,\Delta_Q)))})$$

$$\iota_\star(\pi_Q^\star(\Omega^2_{\mathrm{Reg}(X(N,\Delta_Q))})) = \pi_Q^\star(\iota'_\star(\Omega^2_{\mathrm{Reg}(X(N,\Delta_Q))})) = \pi_Q^\star(\omega_{X(N,\Delta_Q)}) = \mathcal{O}_{X(\Lambda_Q,\Delta_Q)}(\pi_Q^\star(K_{X(N,\Delta_Q)})),$$

i.e., $K_{X(\Lambda_Q,\Delta_Q)} \sim \pi_Q^\star(K_{X(N,\Delta_Q)})$. Furthermore, both $X(N, \Delta_Q)$ and $X(\Lambda_Q, \Delta_Q)$ are projective. (See 3.4 (ii).) Thus (6.3) can be applied for the finite holomorphic map (6.2) of degree ℓ and for the $\mathbb{Q}$-Weil divisor $D_1 = D_2 = K_{X(N,\Delta_Q)}$ giving

$$K^2_{X(N,\Delta_Q)} = \frac{1}{\ell} \pi_Q^\star(K_{X(N,\Delta_Q)})^2 = \frac{1}{\ell} K^2_{X(\Lambda_Q,\Delta_Q)},$$

i.e., (6.4). The proof of the equality (6.5) is similar. □

7 Second proof and consequences of Theorem 1.27

• **Notation and basic facts**. Let ℓ be a positive integer. Throughout this section we shall work with fixed ℓ-reflexive pairs (Q,N) and (Q^*,M), where $M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$, and with the corresponding toric log del Pezzo surfaces $X(N,\Delta_Q)$ and $X(M,\Delta_{Q^*})$. Let $\mathbf{n}_1 = \binom{n_{1,1}}{n_{2,1}}, \ldots, \mathbf{n}_\nu = \binom{n_{1,\nu}}{n_{2,\nu}}$ be the vertices of Q ordered anticlockwise, and

$$F_i := \mathrm{conv}(\{\mathbf{n}_i, \mathbf{n}_{i+1}\}),\ i \in \{1,\ldots,\nu\},$$

be the edges of Q (as in §4 and in 5.4 (ii)). In these terms, the bijections (1.13) and (1.14) become

$$\mathrm{Vert}(Q) \ni \binom{n_{1,i}}{n_{2,i}} = \mathbf{n}_i \longmapsto F_i^* := \ \mathrm{conv}(\{\mathbf{m}_{i-1}, \mathbf{m}_i\}) \in \ \mathrm{Edg}(Q^*),$$

and

$$\mathrm{Edg}(Q) \ni F_i \longmapsto \mathbf{m}_i := \boldsymbol{\eta}_{F_i} = \begin{pmatrix} n_{1,i} & n_{2,i} \\ n_{1,i+1} & n_{2,i+1} \end{pmatrix}^{-1} \binom{-\ell}{-\ell} = \tfrac{\ell}{\det(\mathbf{n}_i,\mathbf{n}_{i+1})} \binom{n_{2,i}-n_{2,i+1}}{n_{1,i+1}-n_{1,i}} \in \ \mathrm{Vert}(Q^*),$$

respectively. (By definition, F_i, F_i^* preserve the involution, i.e., we have $\boldsymbol{\eta}_{F_i^*} = \mathbf{n}_i$, for all indices $i \in \{1,\ldots,\nu\}$. Note that the vertices $\mathbf{m}_1,\ldots,\mathbf{m}_\nu$ of Q^* are also equipped with anticlockwise order.) Next, for $i \in \{1,\ldots,\nu\}$ denote by $\sigma_i := \sigma_{F_i} = \mathbb{R}_{\geqslant 0}\mathbf{n}_i + \mathbb{R}_{\geqslant 0}\mathbf{n}_{i+1}$ the N-cone supporting F_i, by $\sigma_i^* := \sigma_{F_i^*} = \mathbb{R}_{\geqslant 0}\mathbf{m}_{i-1} + \mathbb{R}_{\geqslant 0}\mathbf{m}_i$ the M-cone supporting F_i^*, and assume that σ_i is a (p_i,q_i)-cone with $q_i = 2\,\mathrm{area}_N(T_{F_i})$ (see (5.1)), and that σ_i^* is a (p_i^*,q_i^*)-cone with $q_i^* = 2\,\mathrm{area}_M(T_{F_i^*})$.

Definition 7.1 (Auxiliary cones)**.** For $i \in \{1,\ldots,\nu\}$ the N-cone

$$\tau_i := \mathbb{R}_{\geqslant 0}(\frac{\ell}{q_{i-1}}(\mathbf{n}_{i-1}-\mathbf{n}_i)) + \mathbb{R}_{\geqslant 0}(\frac{\ell}{q_i}(\mathbf{n}_{i+1}-\mathbf{n}_i))$$

will be called *the auxiliary cone* associated with the vertex $\mathbf{n}_i$ of Q. Analogously, the M-cone

$$\tau_i^* := \mathbb{R}_{\geqslant 0}(\frac{\ell}{q_i^*}(\mathbf{m}_{i-1}-\mathbf{m}_i)) + \mathbb{R}_{\geqslant 0}(\frac{\ell}{q_{i+1}^*}(\mathbf{m}_{i+1}-\mathbf{m}_i))$$

will be the auxiliary cone associated with $\mathbf{m}_i \in \mathrm{Vert}(Q^*)$. (Their generators given here are the minimal ones.)

Lemma 7.2. *$\sigma_i^* = \tau_i^\vee$ and $\sigma_i = (\tau_i^*)^\vee$ for all $i \in \{1,\ldots,\nu\}$.*

Proof. For each $i \in \{1,\ldots,\nu\}$ the minimal generators of τ_i are

$$\tfrac{\ell}{q_{i-1}}\binom{n_{1,i-1}-n_{1,i}}{n_{2,i-1}-n_{2,i}} \ \text{ and } \ \tfrac{\ell}{q_i}\binom{n_{1,i+1}-n_{1,i}}{n_{2,i+1}-n_{2,i}}.$$

Since σ_i^* is (p_i^*,q_i^*)-cone, we have $\mathbf{m}_i = p_i^*\mathbf{m}_{i-1} + q_i^*\mathbf{m}_{i-1}'$, where $\{\mathbf{m}_{i-1},\mathbf{m}_{i-1}'\}$ is a basis of M. The corresponding basis matrix is

$$\mathcal{B} := \tfrac{\ell}{\det(\mathbf{n}_{i-1},\mathbf{n}_i)} \begin{pmatrix} n_{2,i-1}-n_{2,i} & \frac{1}{q_i^*}\left(\frac{\det(\mathbf{n}_{i-1},\mathbf{n}_i)}{\det(\mathbf{n}_i,\mathbf{n}_{i+1})}(n_{2,i}-n_{2,i+1}) - p_i^*(n_{2,i-1}-n_{2,i})\right) \\ n_{1,i}-n_{1,i-1} & \frac{1}{q_i^*}\left(\frac{\det(\mathbf{n}_{i-1},\mathbf{n}_i)}{\det(\mathbf{n}_i,\mathbf{n}_{i+1})}(n_{1,i+1}-n_{1,i}) - p_i^*(n_{1,i}-n_{1,i-1})\right) \end{pmatrix}.$$

Thus, the members of the dual basis of $\{\mathbf{m}_i, \mathbf{m}'_i\}$ are

$$(\mathcal{B}^{\intercal})^{-1}\binom{1}{0} = \frac{\det(\mathbf{n}_i,\mathbf{n}_{i+1})}{\ell(\det(\mathbf{n}_{i-1},\mathbf{n}_i)+\det(\mathbf{n}_i,\mathbf{n}_{i+1})-\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1}))}\begin{pmatrix} \frac{\det(\mathbf{n}_{i-1},\mathbf{n}_i)(n_{1,i+1}-n_{1,i})}{\det(\mathbf{n}_i,\mathbf{n}_{i+1})} - p_i^*\left(n_{1,i}-n_{1,i-1}\right) \\ -\frac{\det(\mathbf{n}_{i-1},\mathbf{n}_i)(n_{2,i}-n_{2,i+1})}{\det(\mathbf{n}_i,\mathbf{n}_{i+1})} + p_i^*\left(n_{2,i-1}-n_{2,i})\right) \end{pmatrix}$$

$$= \frac{\ell}{q_i^* q_{i-1}}\binom{\frac{q_{i-1}}{q_i}(n_{1,i+1}-n_{1,i})-p_i^*(n_{1,i}-n_{1,i-1})}{-\frac{q_{i-1}}{q_i}(n_{2,i}-n_{2,i+1})+p_i^*(n_{2,i-1}-n_{2,i}))}$$

and

$$(\mathcal{B}^{\intercal})^{-1}\binom{0}{1} = \frac{q_i^*\det(\mathbf{n}_i,\mathbf{n}_{i+1})}{\ell(\det(\mathbf{n}_{i-1},\mathbf{n}_i)+\det(\mathbf{n}_i,\mathbf{n}_{i+1})-\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1}))}\binom{-(n_{1,i}-n_{1,i-1})}{n_{2,i-1}-n_{2,i}} = \frac{\ell}{q_{i-1}}\binom{n_{1,i-1}-n_{1,i}}{n_{2,i-1}-n_{2,i}}.$$

The minimal generators of the N-cone $(\sigma_i^*)^{\vee}$ are $\frac{\ell}{q_{i-1}}\binom{n_{1,i-1}-n_{1,i}}{n_{2,i-1}-n_{2,i}}$ and

$$q_i^*\mathbf{m}_{i-1} - p_i^*\mathbf{m}'_{i-1} = \frac{\ell}{q_{i-1}}\binom{\frac{q_{i-1}}{q_i}(n_{1,i+1}-n_{1,i})-p_i^*(n_{1,i}-n_{1,i-1})}{-\frac{q_{i-1}}{q_i}(n_{2,i}-n_{2,i+1})+p_i^*(n_{2,i-1}-n_{2,i}))} - \frac{\ell p_i^*}{q_{i-1}}\binom{n_{1,i-1}-n_{1,i}}{n_{2,i-1}-n_{2,i}} = \frac{\ell}{q_i}\binom{n_{1,i+1}-n_{1,i}}{n_{2,i+1}-n_{2,i}},$$

i.e., $(\sigma_i^*)^{\vee} = \tau_i \Rightarrow \sigma_i^* = \tau_i^{\vee}$. The proof of the equality $\sigma_i = (\tau_i^*)^{\vee}$ is similar. □

Proposition 7.3. $\Delta_Q = \Sigma_{Q^*}$ *and* $\Delta_{Q^*} = \Sigma_Q$.

Proof. Since $\tau_i^* = \varpi_{\mathbf{m}_i}$ (see (3.3)), Lemma 7.2 implies that $\varpi_{\mathbf{m}_i}^{\vee} = (\tau_i^*)^{\vee} = \sigma_i$ for all $i \in \{1,\ldots,\nu\}$. Hence, $\Sigma_{Q^*} = \Delta_Q$. (Alternatively, one may apply Theorem 3.8 for $\Delta = \Delta_Q$ and $D = -\ell K_{X(N,\Delta_Q)}$, because $P_{-\ell K_{X(N,\Delta_Q)}} = Q^*$.) Interchanging the roles of Q and Q^* we find $\Sigma_Q = \Delta_{Q^*}$ by the same arguments. □

Proposition 7.4. *The self-intersection number of the canonical divisor of* $X(N,\Delta_Q)$ *is*

$$K^2_{X(N,\Delta_Q)} = \frac{1}{\ell}\sharp\left(\partial Q^* \cap M\right). \tag{7.1}$$

Correspondingly, the self-intersection number of the canonical divisor of $X(M,\Delta_{Q^*})$ *is*

$$K^2_{X(M,\Delta_{Q^*})} = \frac{1}{\ell}\sharp\left(\partial Q \cap N\right). \tag{7.2}$$

Proof. Applying Proposition 1.21 for the ℓ-reflexive pair (Q^*, M) and formula (3.7) (for $P = Q^*$) we get

$$\ell\,\sharp\left(\partial Q^* \cap M\right) = 2\,\mathrm{area}_M(Q^*) = (-\ell K_{X(N,\Sigma_{Q^*})})^2 = (-\ell K_{X(N,\Delta_Q)})^2 = \ell^2 K^2_{X(N,\Delta_Q)}$$

which gives (7.1). The proof of (7.2) is similar. □

• **Passing to the minimal desingularizations.** Let $f : X(N,\widetilde{\Delta}_Q) \longrightarrow X(N,\Delta_Q)$ be the minimal desingularization of $X(N,\Delta_Q)$. Consider $\{C_i\,|i \in \{1,\ldots,\nu\}\}$, the regular and the negative-regular continued fraction expansions

$$\frac{q_i}{q_i - p_i} = \left[\!\left[b_1^{(i)}, b_2^{(i)}, \ldots, b_{s_i}^{(i)} \right]\!\right], \ \forall i \in I_{\Delta_Q}, \tag{7.3}$$

and

$$\frac{q_i}{p_i} = \left[\!\left[b_1^{*\,(i)}, b_2^{*\,(i)}, \ldots, b_{t_i}^{*\,(i)} \right]\!\right], \text{ with } \sum_{j=1}^{s_i} (b_j^{(i)} - 1) = \sum_{k=1}^{t_i} (b_k^{*\,(i)} - 1) = s_i + t_i - 1, \ \forall i \in I_{\Delta_Q}, \quad (7.4)$$

and $\mathrm{Hilb}_N(\sigma_i) = \left\{ \mathbf{u}_j^{(i)} \middle| \, j \in \{0, 1, \ldots, s_i + 1\} \right\}$ for all $i \in \{1, \ldots, \nu\}$, $\{\overline{C}_i \, | i \in \{1, \ldots, \nu\}\}$,

$$\left\{ E_j^{(i)} \middle| \, i \in I_{\Delta_Q}, \ j \in \{1, \ldots, s_i\} \right\}, \ \left\{ K(E^{(i)}) \middle| \, i \in I_{\Delta_Q} \right\}, \ \{ r_i \, | i \in \{1, \ldots, \nu\} \}$$

as in (4.1), (4.3), (4.4), (4.6), (4.7), and (4.2), respectively (where now $\Delta = \Delta_Q$). In the dual sense, consider the regular and the negative-regular continued fraction expansions

$$\frac{q_i^*}{q_i^* - p_i^*} = \left[\!\left[c_1^{*\,(i)}, c_2^{*\,(i)}, \ldots, c_{s_i^*}^{*\,(i)} \right]\!\right], \ \forall i \in I_{\Delta_{Q^*}},$$

and

$$\frac{q_i^*}{p_i^*} = \left[\!\left[c_1^{(i)}, c_2^{(i)}, \ldots, c_{t_i^*}^{(i)} \right]\!\right], \text{ with } \sum_{j=1}^{s_i^*} (c_j^{*\,(i)} - 1) = \sum_{k=1}^{t_i^*} (c_k^{(i)} - 1) = s_i^* + t_i^* - 1, \ \forall i \in I_{\Delta_{Q^*}},$$

attached to the minimal desingularization, say $\varphi : X(M, \widetilde{\Delta}_{Q^*}) \longrightarrow X(M, \Delta_{Q^*})$, of the surface $X(M, \Delta_{Q^*})$, as well as the other data $\{C_i^* \, | i \in \{1, \ldots, \nu\}\}$,

$$\mathrm{Hilb}_M(\sigma_i^*) = \left\{ \mathbf{u}_j^{*\,(i)} \middle| \, j \in \{0, 1, \ldots, s_i^* + 1\} \right\} \quad \text{for all} \quad i \in \{1, \ldots, \nu\},$$

$$\left\{ \overline{C}_i^* \, | i \in \{1, \ldots, \nu\} \right\}, \ \left\{ E_j^{*\,(i)} \middle| \, i \in I_{\Delta_{Q^*}}, \ j \in \{1, \ldots, s_i^*\} \right\},$$

$$\left\{ K(E^{*\,(i)}) \middle| \, i \in I_{\Delta_{Q^*}} \right\}, \ \{ r_i^* \, | i \in \{1, \ldots, \nu\} \},$$

which are defined analogously for $\Delta = \Delta_{Q^*}$. All the above accompanying data of f and φ will play a crucial role in what follows.

• **Noether's formula**. Since $H^j(X(N, \widetilde{\Delta}_Q), \mathcal{O}_{X(N, \widetilde{\Delta}_Q)})$ is trivial for $j = 1, 2$, the Euler-Poincaré characteristic

$$\chi(X(N, \widetilde{\Delta}_Q), \mathcal{O}_{X(N, \widetilde{\Delta}_Q)}) := \sum_{j=0}^{2} (-1)^j \dim_{\mathbb{C}} H^j(X(N, \widetilde{\Delta}_Q), \mathcal{O}_{X(N, \widetilde{\Delta}_Q)})$$

of the structure sheaf $\mathcal{O}_{X(N, \widetilde{\Delta}_Q)}$ equals 1. Thus, Noether's formula [41, p. 154]:

$$\chi(X(N, \widetilde{\Delta}_Q), \mathcal{O}_{X(N, \widetilde{\Delta}_Q)}) = \frac{1}{12} (K^2_{X(N, \widetilde{\Delta}_Q)} + e(X(N, \widetilde{\Delta}_Q)))$$

can be written as follows:

$$\boxed{K^2_{X(N, \widetilde{\Delta}_Q)} + e(X(N, \widetilde{\Delta}_Q)) = 12.} \quad (7.5)$$

• **Case 1.** $\boxed{\ell = 1.}$ In this case, $Q^* = Q^\circ$, $l_{F_i} = 1$ for all $i \in \{1, \dots, \nu\}$ (see Proposition 1.13), and by (5.2) we infer that

$$q_i = \gcd(q_i, p_i - 1),\ \forall i \in I_{\Delta_Q} \Rightarrow p_i = 1,\ s_i = q_i - 1,\ \forall i \in I_{\Delta_Q}.$$

Therefore $X(N, \Delta_Q)$ is either smooth (whenever $I_{\Delta_Q} = \varnothing$) or has only Gorenstein singularities (whenever $I_{\Delta_Q} \neq \varnothing$); cf. Proposition 2.15. Moreover, by Proposition 4.1 f is crepant.

Note 7.5 (Alternative proof of Theorem 1.16.)**.** Combining the fact that f is crepant with (7.1) the self-intersection number of the canonical divisor of $X(N, \widetilde{\Delta}_Q)$ equals

$$K^2_{X(N,\widetilde{\Delta}_Q)} = K^2_{X(N,\Delta_Q)} = \sharp\,(\partial Q^\circ \cap M)\,. \tag{7.6}$$

On the other hand, one computes the topological Euler characteristic of $X(N, \widetilde{\Delta}_Q)$ by 2.17 (ii) and (1.12):

$$e(X(N, \widetilde{\Delta}_Q)) = \nu + \sum_{i \in I_{\Delta_Q}} s_i = \nu + \sum_{i \in I_{\Delta_Q}} (q_i - 1) = \sum_{i=1}^{\nu} q_i = 2\,\mathrm{area}_N(Q) = \sharp\,(\partial Q \cap N)\,. \tag{7.7}$$

Formula (1.8) follows from (7.6), (7.7) and (7.5). Obviously,

$$\sharp(\partial Q^\circ \cap M) - K^2_{X(N,\widetilde{\Delta}_Q)} = 0 = e(X(N, \widetilde{\Delta}_Q)) - \sharp(\partial Q \cap N), \tag{7.8}$$

and, analogously,

$$\sharp(\partial Q \cap N) - K^2_{X(M,\widetilde{\Delta}_{Q^\circ})} = 0 = e(X(M, \widetilde{\Delta}_{Q^\circ})) - \sharp(\partial Q^\circ \cap M). \tag{7.9}$$

We shall hereafter call these two couples of differences occuring in (7.8) and (7.9) *characteristic differences* w.r.t. Q (and w.r.t. $Q^* = Q^\circ$, respectively). As we shall verify below in §8, these do not vanish whenever $\ell > 1$, and they have an interesting geometric interpretation. (See (8.4) and (8.8).)

• **Case 2.** $\boxed{\ell > 1.}$ In this case, $I_{\Delta_Q} = \{1, \dots, \nu\}$, and $X(N, \Delta_Q)$ has exactly ν singularities, all of which are non-Gorenstein singularities (see Proposition 2.15) because by hypothesis and by (5.2) we conclude

$$l_{F_i} = \frac{q_i}{\gcd(q_i, p_i - 1)} = \ell \geqslant 2 \Rightarrow p_i \geqslant 2,\ \forall i \in \{1, \dots, \nu\}\,. \tag{7.10}$$

Analogously, $I_{\Delta_{Q^*}} = \{1, \dots, \nu\}$, $p_i^* \geqslant 2$ for all $i \in \{1, \dots, \nu\}$, and $X(M, \Delta_{Q^*})$ has exactly ν (non-Gorenstein) singularities.

▶ *Proof of Theorem 1.27 for* $\ell > 1$. Passing from lattice N to lattice Λ_Q (and, respectively, from M to Λ_{Q^*}) we denote by $\widehat{f} : X(\Lambda_Q, \widehat{\Delta}_Q) \longrightarrow X(\Lambda_Q, \Delta_Q)$ (resp., by $\widehat{\varphi} : X(\Lambda_{Q^*}, \widehat{\Delta}_{Q^*}) \longrightarrow X(\Lambda_{Q^*}, \Delta_{Q^*})$) the minimal desingularization of the Gorenstein toric log del Pezzo surface $X(\Lambda_Q, \Delta_Q)$ (resp., of $X(\Lambda_{Q^*}, \Delta_{Q^*})$). Since $\mathrm{orb}_{\Lambda_Q}(\sigma_i)$ is either a nonsingular point (whenever σ_i is a basic Λ_Q-cone and $q_i = \ell$) or a Gorenstein cyclic quotient singularity (whenever σ_i is a non-basic Λ_Q-cone, necessarily of type $(1, \frac{q_i}{\ell})$ with[6] $\frac{q_i}{\ell} \in \{2,3,4\}$), formula (7.7) applied for the lattice Λ_Q and the refinement $\widehat{\Delta}_Q$ of the Λ_Q-fan Δ_Q gives

$$e(X(\Lambda_Q, \widehat{\Delta}_Q)) = \sum_{i=1}^{\nu} \frac{q_i}{\ell} \underset{(5.1)}{=} \sum_{i=1}^{\nu} \frac{2\,\mathrm{area}_N(T_{F_i})}{\ell} = \frac{2\,\mathrm{area}_N(Q)}{\ell} \underset{(1.12)}{=} \sharp(\partial Q \cap N). \tag{7.11}$$

On the other hand, since $\widehat{f}$ is crepant,

$$K^2_{X(\Lambda_Q, \widehat{\Delta}_Q)} = K^2_{X(\Lambda_Q, \Delta_Q)}. \tag{7.12}$$

Hence,

$$\begin{aligned} 12 &\underset{(1.8)}{=} \sharp(\partial Q \cap \Lambda_Q) + \sharp(\partial Q^\circ \cap \mathrm{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z})) = \sharp(\partial Q \cap \Lambda_Q) + \sharp(\partial Q^* \cap \Lambda_{Q^*}) \\ &= e(X(\Lambda_Q, \widehat{\Delta}_Q)) + K^2_{X(\Lambda_Q, \widehat{\Delta}_Q)} = \sharp(\partial Q \cap N) + K^2_{X(\Lambda_Q, \Delta_Q)} \text{ (by (7.11) and (7.12))} \\ &= \sharp(\partial Q \cap N) + \ell K^2_{X(N, \Delta_Q)} = \sharp(\partial Q \cap N) + \sharp(\partial Q^* \cap M) \quad \text{(by (6.4) and (7.1))}. \end{aligned}$$

One could, of course, use (1.8), $e(X(\Lambda_{Q^*}, \widehat{\Delta}_{Q*})) = \sharp(\partial Q^* \cap M)$, the fact that $\widehat{\varphi}$ is crepant (leading to $K^2_{X(\Lambda_{Q^*}, \widehat{\Delta}_{Q^*})} = K^2_{X(\Lambda_{Q^*}, \Delta_{Q^*})}$), (6.5) and (7.2), instead. Thus (1.15) is true. □

• **Consequences of Theorem 1.27**. Let ℓ be an integer $\geqslant 1$ and let (Q, N) be an ℓ-reflexive pair. Maintaining the notation introduced above, formula (1.15) gives significant information about Q, Q^*, $\sharp(\partial Q \cap N)$, $\sharp(\partial Q^* \cap M)$, ℓ, and the combinatorial triples of the corresponding fans Δ_Q, Δ_{Q^*}.

Corollary 7.6 (Upper bound for the number of vertices). *We have*

$$\sharp(\mathrm{Vert}(Q)) = \sharp(\mathrm{Vert}(Q^*)) = \nu \leqslant 6.$$

First proof. Since the number of the vertices of Q (resp., of Q^*) does not change by passing from lattice N to lattice Λ_Q (resp., from M to Λ_{Q^*}), the claim is correct by Theorem 1.17.
Second proof. (1.15) directly implies

$$\begin{aligned} 12 &= \sharp(\partial Q \cap N) + \sharp(\partial Q^* \cap M) = \frac{1}{\ell}\left(2\,\mathrm{area}_N(Q) + 2\,\mathrm{area}_M(Q^*)\right) \\ &= \frac{2}{\ell}\left(\sum_{i=1}^{\nu} \mathrm{area}_N(T_{F_i}) + \sum_{i=1}^{\nu} \mathrm{area}_M(T_{F_i^*})\right) \\ &= \sum_{i=1}^{\nu} (\sharp(F_i \cap N) - 1) + \sum_{i=1}^{\nu} (\sharp(F_i^* \cap M) - 1) \geqslant 2\nu, \end{aligned}$$

i.e., $\sharp(\mathrm{Vert}(Q)) = \sharp(\mathrm{Vert}(Q^*)) = \nu \leqslant 6$. □

[6] The number of the lattice points lying in the interior of an edge of an 1-reflexive polygon is $\leqslant 3$. (See Figure 1.)

Corollary 7.7 (All possible values of $\sharp\,(\partial Q\cap N)$ and $\sharp(\partial Q^{*}\cap M)$)**.** *We have*

$$(\sharp\,(\partial Q\cap N)\,,\sharp(\partial Q^{*}\cap M))\in\{(3,9)\,,(4,8),(5,7),(6,6),(7,5),(8,4),(9,3)\}\,.$$

Proof. Since $\sharp\,(\partial Q\cap N)\geqslant 3$ and $\sharp(\partial Q^{*}\cap M)\geqslant 3$, this follows directly from (1.15). □

Corollary 7.8 ("Oddness"of ℓ)**.** *The index ℓ of Q is always odd.*

Proof. Suppose that the index ℓ of Q is *even*. By Corollary 1.22 $\sharp\,(\partial Q\cap N)$ has to be even. Therefore, by Corollary 7.7, $\sharp\,(\partial Q\cap N)\in\{4,6,8\}$. Taking into account that

$$\frac{2}{\ell}\,\mathrm{area}_N(Q)=\frac{1}{\ell}\left(\sum_{i=1}^{\nu}q_i\right)=\sum_{i=1}^{\nu}\gcd(q_i,p_i-1)=\sharp\,(\partial Q\cap N)\,, \tag{7.13}$$

we examine the three cases separately:

(i) If $\sharp\,(\partial Q\cap N)=4$, then $\nu\in\{3,4\}$ and by (7.13) $\exists i_{\bullet}\in\{1,\ldots,\nu\}:\gcd(q_{i_{\bullet}},p_{i_{\bullet}}-1)=1$. Since $\ell=q_{i_{\bullet}}$ is even, $p_{i_{\bullet}}$ is even $\geqslant 2$. This is impossible because $\gcd(p_{i_{\bullet}},q_{i_{\bullet}})=1$.

(ii) If $\sharp\,(\partial Q\cap N)=8$, then $\sharp(\partial Q^{*}\cap M)=4$, which is again impossible (by using the same argument as in case (i) but this time with Q^{*} in the place of Q).

(iii) If $\sharp\,(\partial Q\cap N)=6$, then $\nu\in\{3,4,5,6\}$. For $\nu\in\{4,5,6\}$ equality (7.13) informs us that there is an $i_{\bullet}\in\{1,\ldots,\nu\}:\gcd(q_{i_{\bullet}},p_{i_{\bullet}}-1)=1$, leading to contradiction (as in case (i)). It remains to see what happens for $\nu=3$ under the assumption that $\nexists i_{\bullet}\in\{1,2,3\}:$ $\gcd(q_{i_{\bullet}},p_{i_{\bullet}}-1)=1$. In this case we have necessarily

$$\gcd(q_1,p_1-1)=\gcd(q_2,p_2-1)=\gcd(q_3,p_3-1)=2 \text{ and } q_1=q_2=q_3=2\ell,$$

and consequently p_1,p_2 and p_3 are odd $\geqslant 3$. By [19, Lemma 6.2, pp. 232-233] we obtain

$$q_1q_2\mid\widehat{p}_1q_2+p_2q_1+q_3\Longrightarrow 2\ell\mid\widehat{p}_1+p_2+1.$$

Since the socius $\widehat{p}_1$ of p_1 is odd too, the last divisibility condition is impossible (because $\widehat{p}_1+p_2+1$ is an odd integer). By (i), (ii) and (iii) we conclude that ℓ is always odd. □

Note 7.9. (i) All possible values for the numbers of boundary lattice points described in Corollary 7.7 can be taken, as it is shown by examples 1.28.

(ii) If $\ell=3\lambda$, where λ is a positive odd integer, then it can be proven that Q has to be a lattice *hexagon*. (See [46, §2.5].)

Proposition 7.10. *For each $i\in\{1,\ldots,\nu\}$ we have*

$$q_i^{*}=\ell^2\left(\frac{1}{q_{i-1}}+\frac{1}{q_i}-\frac{1}{q_{i-1}q_i}\frac{\det\left(\mathbf{n}_{i-1},\mathbf{n}_{i+1}\right)}{\det(N)}\right) \tag{7.14}$$

and

$$q_i=\ell^2\left(\frac{1}{q_i^{*}}+\frac{1}{q_{i+1}^{*}}-\frac{1}{q_i^{*}q_{i+1}^{*}}\frac{\det\left(\mathbf{m}_i,\mathbf{m}_{i+2}\right)}{\det(M)}\right). \tag{7.15}$$

Proof. Since $\mathbf{m}_1, \ldots, \mathbf{m}_\nu$ and $\mathbf{n}_1, \ldots, \mathbf{n}_\nu$ are ordered anticlockwise, we have

$$\det(\mathbf{m}_{i-1}, \mathbf{m}_i) > 0, \quad \det(\mathbf{n}_{i-1}, \mathbf{n}_i) > 0, \quad \det(\mathbf{n}_i, \mathbf{n}_{i+1}) > 0,$$

and (5.1) (applied for the M-cone σ_i^*) gives

$$\begin{aligned} q_i^* &= \frac{\det(\mathbf{m}_{i-1}, \mathbf{m}_i)}{\det(M)} = \frac{\ell^2 \det(N)}{\det(\mathbf{n}_{i-1},\mathbf{n}_i)\det(\mathbf{n}_i,\mathbf{n}_{i+1})} \det\begin{pmatrix} n_{2,i-1}-n_{2,i} & n_{2,i}-n_{2,i+1} \\ n_{1,i}-n_{1,i-1} & n_{1,i+1}-n_{1,i} \end{pmatrix} \\ &= \frac{\ell^2 \det(N)}{\det(\mathbf{n}_{i-1},\mathbf{n}_i)\det(\mathbf{n}_i,\mathbf{n}_{i+1})} \left(\det(\mathbf{n}_i, \mathbf{n}_{i+1}) + \det(\mathbf{n}_{i-1}, \mathbf{n}_i) - \det(\mathbf{n}_{i-1}, \mathbf{n}_{i+1})\right) \\ &= \ell^2 \left(\frac{1}{\frac{\det(\mathbf{n}_{i-1},\mathbf{n}_i)}{\det(N)}} + \frac{1}{\frac{\det(\mathbf{n}_i,\mathbf{n}_{i+1})}{\det(N)}} - \frac{\frac{\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1})}{\det(N)}}{\frac{\det(\mathbf{n}_{i-1},\mathbf{n}_i)}{\det(N)} \frac{\det(\mathbf{n}_i,\mathbf{n}_{i+1})}{\det(N)}} \right) \\ &= \ell^2 \left(\frac{1}{q_{i-1}} + \frac{1}{q_i} - \frac{1}{q_{i-1}q_i} \frac{\det(\mathbf{n}_{i-1}, \mathbf{n}_{i+1})}{\det(N)} \right), \end{aligned}$$

for all $i \in \{1, \ldots, \nu\}$. The proof of equality (7.15) is similar. $\square$

Corollary 7.11 (Determinantal identities)**.** *The multiplicities $q_1, \ldots, q_\nu$ of the N-cones of Δ_Q satisfy the following identity:*

$$\sum_{i=1}^{\nu} \left(\frac{q_i}{\ell} + \frac{2\ell}{q_i} \right) = \sum_{i=1}^{\nu} \frac{\ell}{q_i q_{i+1}} \frac{\det(\mathbf{n}_i, \mathbf{n}_{i+2})}{\det(N)} + 12 \tag{7.16}$$

In dual terms, the multiplicities $q_1^*, \ldots, q_\nu^*$ of the M-cones of Δ_{Q^*} satisfy the identity:

$$\sum_{i=1}^{\nu} \left(\frac{q_i^*}{\ell} + \frac{2\ell}{q_i^*} \right) = \sum_{i=1}^{\nu} \frac{\ell}{q_i^* q_{i+1}^*} \frac{\det(\mathbf{m}_i, \mathbf{m}_{i+2})}{\det(M)} + 12 \tag{7.17}$$

Proof. Formula (1.15) can be rewritten via Proposition 7.10 in the form

$$\begin{aligned} 12 &= \sharp(\partial Q \cap N) + \sharp(\partial Q^* \cap M) = \frac{1}{\ell}\left(2\,\mathrm{area}_N(Q) + 2\,\mathrm{area}_M(Q^*)\right) \\ &= \frac{1}{\ell}\sum_{i=1}^{\nu} q_i + \frac{1}{\ell}\sum_{i=1}^{\nu} q_i^* = \frac{1}{\ell}\sum_{i=1}^{\nu} q_i + \sum_{i=1}^{\nu} \ell \left(\frac{2}{q_i} - \frac{1}{q_i q_{i+1}} \frac{\det(\mathbf{n}_i, \mathbf{n}_{i+2})}{\det(N)} \right). \end{aligned}$$

Hence, (7.16) is true. The proof of (7.17) is similar. $\square$

Corollary 7.12 (Dedekind sum identities)**.** *If $\ell > 1$, then the Dedekind sums of the pairs $(p_1, q_1), \ldots, (p_\nu, q_\nu)$ satisfy the identity:*

$$12 \left(\sum_{i=1}^{\nu} \mathrm{DS}(p_i, q_i) \right) = 12 - 3\nu + \sum_{i=1}^{\nu} \frac{1}{q_i q_{i+1}} \frac{\det(\mathbf{n}_i, \mathbf{n}_{i+2})}{\det(N)}. \tag{7.18}$$

In dual terms, the Dedekind sums of the pairs $(p_1^*, q_1^*), \ldots, (p_\nu^*, q_\nu^*)$ satisfy the identity:

$$12 \left(\sum_{i=1}^{\nu} \mathrm{DS}(p_i^*, q_i^*) \right) = 12 - 3\nu + \sum_{i=1}^{\nu} \frac{1}{q_i^* q_{i+1}^*} \frac{\det(\mathbf{m}_i, \mathbf{m}_{i+2})}{\det(M)}. \tag{7.19}$$

Proof. By (7.1) and (7.14) we obtain

$$K^2_{X(N,\Delta_Q)} = \frac{1}{\ell}\sharp\left(\partial Q^* \cap M\right) = \frac{1}{\ell^2}\sum_{i=1}^{\nu} q_i^* = \sum_{i=1}^{\nu}\left(\frac{2}{q_i} - \frac{1}{q_i q_{i+1}}\frac{\det\left(\mathbf{n}_i, \mathbf{n}_{i+2}\right)}{\det(N)}\right). \tag{7.20}$$

Formula (2.18) leads to another version of Noether's formula (see [18, Corollary 4.10, p. 99]):

$$K^2_{X(N,\Delta_Q)} = 12 - \nu + \sum_{i=1}^{\nu}\left(\frac{2}{q_i} - 12\mathrm{DS}(p_i, q_i) - 2\right). \tag{7.21}$$

(7.18) follows from (7.20) and (7.21). The proof of (7.19) is similar. □

• **Suyama's formula**. Let $N \subset \mathbb{R}^2$ be a lattice. If $\mathbf{v}_{1,\ldots,}\mathbf{v}_\nu \in N$ is a sequence of *primitive* lattice points with $\mathbf{v}_0 := \mathbf{v}_\nu$ and $\mathbf{v}_{\nu+1} := \mathbf{v}_1$, one denotes by

$$\mathrm{Rot}\left(\mathbf{v}_{1,\ldots,}\mathbf{v}_\nu\right) := \frac{1}{2\pi}\sum_{i=1}^{\nu}\int_{\mathrm{conv}(\{\mathbf{v}_i,\mathbf{v}_{i+1}\})}\frac{-\mathfrak{y}\ d\mathfrak{x} + \mathfrak{x}\ d\mathfrak{y}}{\mathfrak{x}^2 + \mathfrak{y}^2}$$

the *rotation* (or *winding*) *number* of $\mathbf{v}_{1,\ldots,}\mathbf{v}_\nu$ around $\mathbf{0}$. Suyama gave a nice formula in [67, Theorem 6, p. 854], by means of which one computes $\mathrm{Rot}(\mathbf{v}_{1,\ldots,}\mathbf{v}_\nu)$. Applying his formula for the (very special) sequence $\mathbf{n}_{1,\ldots,}\mathbf{n}_\nu$ of the vertices of Q for $\ell > 1$ (and taking into account the continued fraction expansion (7.3) of $\frac{q_i}{q_i - p_i}$ for all $i \in \{1, \ldots, \nu\}$ $(= I_\Delta)$) we obtain

$$\mathrm{Rot}\left(\mathbf{n}_{1,\ldots,}\mathbf{n}_\nu\right) = \tfrac{1}{12}\sum_{i=1}^{\nu}\left(3(s_i+1) - \sum_{j=1}^{s_i} b_j^{(i)} - \tfrac{1}{q_{i-1}q_i}\tfrac{\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1})}{\det(N)} - \tfrac{(q_i - p_i) + (q_i - \widehat{p}_i)}{q_i}\right). \tag{7.22}$$

Obviously, by construction,

$$\mathrm{Rot}\left(\mathbf{n}_{1,\ldots,}\mathbf{n}_\nu\right) = 1. \tag{7.23}$$

Proposition 7.13. *If* $\ell > 1$, *then* (7.22) *and* (7.23) *are equivalent to the known formula*

$$\sum_{i=1}^{\nu} r_i = 3\nu - 12 - \sum_{i=1}^{\nu}\sum_{j=1}^{s_i}(b_j^{(i)} - 3), \tag{7.24}$$

which follows from a generalised version of Noether's formula. (See [18, pp. 99-100].)

Proof. By (7.14), Proposition 1.21 and (5.1) (applied for Q^* and q_i^*, respectively), and (7.1) we have

$$\tfrac{1}{q_{i-1}q_i}\tfrac{\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1})}{\det(N)} = -\tfrac{q_i^*}{\ell^2} + \tfrac{1}{q_{i-1}} + \tfrac{1}{q_i} \Rightarrow \sum_{i=1}^{\nu}\tfrac{1}{q_{i-1}q_i}\tfrac{\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1})}{\det(N)} = -K^2_{X(N,\Delta_Q)} + \sum_{i=1}^{\nu}\tfrac{2}{q_i}.$$

By [18, Proposition 4.8, p. 98] we know that

$$\sum_{i=1}^{\nu}\tfrac{2}{q_i} = K^2_{X(N,\Delta_Q)} + \sum_{i=1}^{\nu} r_i - \sum_{i=1}^{\nu}\tfrac{(q_i - p_i) + (q_i - \widehat{p}_i)}{q_i}$$

$$\Rightarrow \sum_{i=1}^{\nu}\left(-\tfrac{1}{q_{i-1}q_i}\tfrac{\det(\mathbf{n}_{i-1},\mathbf{n}_{i+1})}{\det(N)} - \tfrac{(q_i - p_i) + (q_i - \widehat{p}_i)}{q_i}\right) = \sum_{i=1}^{\nu} r_i.$$

Hence, (7.22) and (7.23) give

$$12=\sum_{i=1}^{\nu}\left(3(s_i+1)-\sum_{j=1}^{s_i}b_j^{(i)}\right)+\sum_{i=1}^{\nu}r_i\Rightarrow 3\nu-12-\sum_{i=1}^{\nu}\sum_{j=1}^{s_i}(b_j^{(i)}-3)=\sum_{i=1}^{\nu}r_i,$$

i.e., (7.24). □

• **Further interrelations of the data of both sides**. The duality established by the bijections (1.13) and (1.14) implies certain additional number-theoretic identities which involve the combinatorial triples of both sides.

Proposition 7.14. *If $\ell>1$, then for each $i\in\{1,\ldots,\nu\}$ we have*

$$q_i^*=\ell^2\left(\frac{q_{i-1}-\widehat{p}_{i-1}+1}{q_{i-1}}+\frac{q_i-p_i+1}{q_i}-r_i\right) \tag{7.25}$$

and

$$q_i=\ell^2\left(\frac{q_i^*-\widehat{p_i^*}+1}{q_i^*}+\frac{q_{i+1}^*-p_{i+1}^*+1}{q_{i+1}^*}-r_i^*\right). \tag{7.26}$$

Proof. Since $\sigma_i=\mathbb{R}_{\geqslant 0}\mathbf{n}_i+\mathbb{R}_{\geqslant 0}\mathbf{n}_{i+1}$ is a (p_i,q_i)-cone, there exist a basis matrix $\mathcal{B}$ of N and a matrix $\mathcal{M}_{\sigma_i}\in \mathrm{GL}_2(\mathbb{Z})$ such that

$$\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\sigma_i)=\Phi_{\mathcal{M}_{\sigma_i}}(\sigma_i^{\mathrm{st}})=\mathbb{R}_{\geqslant 0}\binom{1}{0}+\mathbb{R}_{\geqslant 0}\binom{p_i}{q_i},$$

where σ_i^{st} is the standard model of σ_i w.r.t. $\mathcal{B}$ (see Proposition 2.4 and Figure 12). $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}$ maps $\mathbf{n}_i$ onto $\binom{1}{0}$ and $\mathbf{n}_{i-1}$ onto a point $\binom{\mathfrak{n}_{1,i-1}}{\mathfrak{n}_{2,i-1}}\in\mathbb{Z}^2$, i.e., σ_{i-1} onto the $\mathbb{Z}^2$-cone $\mathbb{R}_{\geqslant 0}\binom{\mathfrak{n}_{1,i-1}}{\mathfrak{n}_{2,i-1}}+\mathbb{R}_{\geqslant 0}\binom{1}{0}$ with

$$-\mathfrak{n}_{2,i-1}=\det\begin{pmatrix}\mathfrak{n}_{1,i-1} & 1\\ \mathfrak{n}_{2,i-1} & 0\end{pmatrix}=\mathrm{mult}_N(\sigma_{i-1})=q_{i-1}\Rightarrow \mathfrak{n}_{2,i-1}=-q_{i-1}. \tag{7.27}$$

We observe that the point of $\partial\Theta^{\mathbf{cp}}_{\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\sigma_i)}\cap\mathbb{Z}^2$ (resp., of $\partial\Theta^{\mathbf{cp}}_{\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\sigma_{i-1})}\cap\mathbb{Z}^2$) closest to $\binom{1}{0}$ is $\binom{1}{1}$ (resp., $\frac{1}{q_{i-1}}\left(\binom{\mathfrak{n}_{1,i-1}}{\mathfrak{n}_{2,i-1}}+(q_{i-1}-\widehat{p}_{i-1})\binom{1}{0}\right)$. (Use (2.16) and (2.17) for the $\mathbb{Z}^2$-cones $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\sigma_i)$ and $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\sigma_{i-1})$, respectively.) By the linearity of $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}$ we infer that

$$\begin{pmatrix}\frac{1}{q_{i-1}}(\mathfrak{n}_{1,i-1}+q_{i-1}-\widehat{p}_{i-1})\\ -1\end{pmatrix}+\binom{1}{1}=r_i\binom{1}{0}\Rightarrow \mathfrak{n}_{1,i-1}=(r_i-2)\,q_{i-1}+\widehat{p}_{i-1}. \tag{7.28}$$

Using (7.27) and (7.28) we compute the multiplicity of $\tau_i=(\sigma_i^*)^\vee$:

$$\begin{aligned}q_i^*&=\mathrm{mult}_M(\sigma_i^*)=\mathrm{mult}_N(\tau_i)=\det\left(\frac{\ell}{q_i}\left(\binom{p_i}{q_i}-\binom{1}{0}\right),\frac{\ell}{q_{i-1}}\left(\binom{\mathfrak{n}_{1,i-1}}{\mathfrak{n}_{2,i-1}}-\binom{1}{0}\right)\right)\\ &=\det\begin{pmatrix}\frac{\ell}{q_i}(p_i-1) & \frac{\ell}{q_{i-1}}((r_i-2)q_{i-1}+\widehat{p}_{i-1}-1)\\ \ell & -\ell\end{pmatrix}\\ &=-\ell^2\left(\frac{\widehat{p}_{i-1}-1}{q_{i-1}}+\frac{p_i-1}{q_i}+(r_i-2)\right)\end{aligned}$$

and obtain (7.25). The proof of (7.26) is similar. □

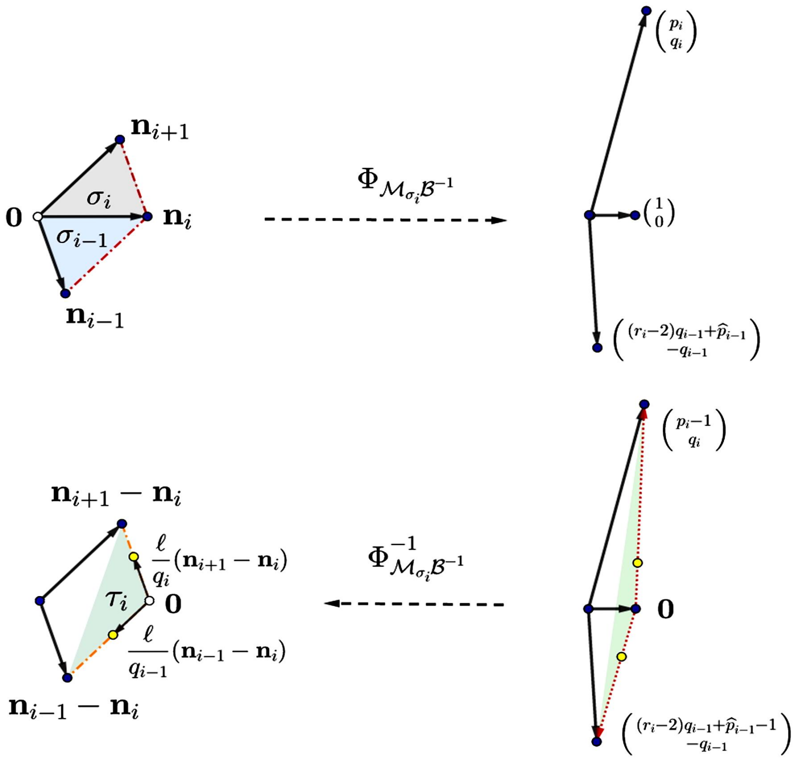


Figure 12: Transformation by $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}$ and by its inverse

Corollary 7.15 (Identities with combinatorial triples)**.** *If $\ell > 1$, then the combinatorial triples $(p_1, q_1, r_1), \dots, (p_\nu, q_\nu, r_\nu)$ of Δ_Q satisfy the identity:*

$$\sum_{i=1}^{\nu} \left(\frac{q_i}{\ell} + \frac{2\ell}{q_i} \right) = 12 - 2\ell\nu + \sum_{i=1}^{\nu} \ell \left(\frac{p_i + \widehat{p}_i}{q_i} + r_i \right), \tag{7.29}$$

and, in the dual sense, the combinatorial triples $(p_1^*, q_1^*, r_1^*), \dots, (p_\nu^*, q_\nu^*, r_\nu^*)$ of Δ_{Q^*} satisfy the identity:

$$\sum_{i=1}^{\nu} \left(\frac{q_i^*}{\ell} + \frac{2\ell}{q_i^*} \right) = 12 - 2\ell\nu + \sum_{i=1}^{\nu} \ell \left(\frac{p_i^* + \widehat{p_i^*}}{q_i^*} + r_i^* \right). \tag{7.30}$$

Proof. Formula (1.15) can be rewritten via (7.25) in the form

$$\begin{aligned} 12 = \sharp\,(\partial Q \cap N) + \sharp(\partial Q^* \cap M) &= \frac{1}{\ell} \sum_{i=1}^{\nu} q_i + \frac{1}{\ell} \sum_{i=1}^{\nu} q_i^* \\ &= \frac{1}{\ell} \sum_{i=1}^{\nu} q_i + \sum_{i=1}^{\nu} \ell \left(\frac{q_{i-1} - \widehat{p}_{i-1} + 1}{q_{i-1}} + \frac{q_i - p_i + 1}{q_i} - r_i \right) \\ &= \frac{1}{\ell} \sum_{i=1}^{\nu} q_i + \sum_{i=1}^{\nu} \ell \left(\frac{1 - \widehat{p}_i}{q_i} + \frac{1 - p_i}{q_i} - (r_i - 2) \right). \end{aligned}$$

Hence, (7.29) is true. The proof of (7.30) is similar. □

Finally, it remains to give the explicit number-theoretic description of the link between $p_i^*, \widehat{p_i}^*$ and the multiplicity q_i^*, provided that p_i and q_i are assumed to be known, and, respectively, of the link between $p_i, \widehat{p_i}$ and the multiplicity q_i, provided that p_i^* and q_i^* are assumed to be known.

Proposition 7.16. *Let ℓ be again > 1. For each $i \in \{1, \ldots, \nu\}$ consider the regular continued fraction expansion*

$$\frac{\ell}{\frac{\ell}{q_i}(p_i - 1)} = \frac{q_i}{p_i - 1} = \left[d_1^{(i)}, d_2^{(i)} \ldots, d_\rho^{(i)}\right] := d_1^{(i)} + \cfrac{1}{d_2^{(i)} + \cfrac{1}{\ddots \, d_{\rho-1}^{(i)} + \cfrac{1}{d_\rho^{(i)}}}}$$

of $\dfrac{\ell}{\frac{\ell}{q_i}(p_i - 1)}$ *and set*

$$\kappa_i := \begin{cases} \frac{-\varepsilon\ell}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_2^{(i)}, d_1^{(i)}\right]}, & \text{if } d_1^{(i)} \geqslant 2, \\ \frac{-\varepsilon\ell}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_3^{(i)}, d_2^{(i)}+1\right]}, & \text{if } d_1^{(i)} = 1, \end{cases} \quad \text{and} \quad \lambda_i := \begin{cases} \frac{-\frac{\varepsilon\ell}{q_i}(p_i-1)}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_3^{(i)}, d_2^{(i)}\right]}, & \text{if } d_2^{(i)} \geqslant 2, \\ \frac{-\frac{\varepsilon\ell}{q_i}(p_i-1)}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_4^{(i)}, d_3^{(i)}+1\right]}, & \text{if } d_2^{(i)} = 1, \end{cases}$$

with $\varepsilon = 1$ for ρ even and $\varepsilon = -1$ for ρ odd. Then $\kappa_i, \lambda_i \in \mathbb{Z}$ and

$$\kappa_i \frac{\ell}{q_i}(p_i - 1) - \lambda_i \ell = 1. \tag{7.31}$$

Denoting by $\mathfrak{z}_i$ the unique positive integer which is smaller than q_i^* and satisfies

$$\kappa_i \frac{\ell}{q_{i-1}} \left((r_i - 2)\, q_{i-1} + \widehat{p}_{i-1} - 1\right) + \lambda_i \ell \equiv \mathfrak{z}_i \,(\mathrm{mod}\, q_i^*)\,, \tag{7.32}$$

we obtain

$$\mathfrak{z}_i = \begin{cases} q_i^* \left(1 - \frac{1}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_2^{(i)}, d_1^{(i)}\right]}\right) - 1, & \text{if } \rho \text{ is odd and } d_1^{(i)} \geqslant 2, \\ q_i^* \left(1 - \frac{1}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_3^{(i)}, d_2^{(i)}+1\right]}\right) - 1, & \text{if } \rho \text{ is odd and } d_1^{(i)} = 1, \\ \frac{q_i^*}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_2^{(i)}, d_1^{(i)}\right]} - 1, & \text{if } \rho \text{ is even and } d_1^{(i)} \geqslant 2, \\ \frac{q_i^*}{\left[d_\rho^{(i)}, d_{\rho-1}^{(i)}, \ldots, d_3^{(i)}, d_2^{(i)}+1\right]} - 1, & \text{if } \rho \text{ is even and } d_1^{(i)} = 1, \end{cases} \tag{7.33}$$

and

$$\widehat{p_i^*} = q_i^* - \mathfrak{z}_i, \quad p_i^* = q_i^* - \widehat{\mathfrak{z}_i},$$

where $\widehat{\mathfrak{z}_i}$ is the socius of $\mathfrak{z}_i$ w.r.t. q_i^*.

Proof. τ_i is mapped by $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}$ (with $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}$ as in the proof of Proposition 7.14) onto the $\mathbb{Z}^2$-cone

$$\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\tau_i) = \Phi_{(\mathcal{B}\mathcal{M}_{\sigma_i}^{-1})^{-1}}(\tau_i) = \mathbb{R}_{\geqslant 0}\binom{\frac{\ell}{q_{i-1}}((r_i-2)q_{i-1}+\widehat{p}_{i-1}-1)}{-\ell} + \mathbb{R}_{\geqslant 0}\binom{\frac{\ell}{q_i}(p_i-1)}{\ell} \tag{7.34}$$

which is the standard model of τ_i w.r.t. $\mathcal{B}\mathcal{M}_{\sigma_i}^{-1}$ with

$$\operatorname{mult}_{\mathbb{Z}^2}(\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\tau_i)) = \operatorname{mult}_N(\tau_i) = q_i^*.$$

(7.31) is valid by the definition of κ_i, λ_i (see [20, Remark 3.2, p. 217]). Assume that

$$\mathbb{R}_{\geqslant 0}\binom{\frac{\ell}{q_i}(p_i-1)}{\ell} + \mathbb{R}_{\geqslant 0}\binom{\frac{\ell}{q_{i-1}}((r_i-2)q_{i-1}+\widehat{p}_{i-1}-1)}{-\ell}$$

(defined by interchanging the ordering of the minimal generators of (7.34)) is a $(\mathfrak{z}_i, q_i^*)$-cone. By Proposition 2.4 $\mathfrak{z}_i$ has to be the unique positive integer which is smaller than q_i^* and satisfies (7.32). Using (7.31) and (7.25) we can write the left-hand side of (7.32) as follows:

$$\begin{aligned}&\kappa_i \frac{\ell}{q_{i-1}}\left((r_i-2)\,q_{i-1} + \widehat{p}_{i-1} - 1\right) + \kappa_i \frac{\ell}{q_i}(p_i-1) - 1 \\ &= \kappa_i \ell \left(\frac{\widehat{p}_{i-1}-1}{q_{i-1}} + \frac{p_i-1}{q_i} + (r_i-2)\right) - 1 = -\frac{\kappa_i q_i^*}{\ell} - 1.\end{aligned}$$

Thus, (7.33) is true and (7.34) is a $(\widehat{\mathfrak{z}}_i, q_i^*)$-cone (cf. Note 2.5 and the proof of Proposition 2.7). Since both $\tau_i = (\sigma_i^*)^\vee$ and $\Phi_{\mathcal{M}_{\sigma_i}\mathcal{B}^{-1}}(\tau_i)$ are $(q_i^* - p_i^*, q_i^*)$-cones (cf. Proposition 2.10), we have $q_i^* - p_i^* = \widehat{\mathfrak{z}}_i$. □

Note 7.17. Similarly, one shows that $p_i = q_i - \widehat{\mathfrak{z}_i^*}$ for all $i \in \{1, \ldots, \nu\}$, where $\mathfrak{z}_i^*$ is determined by the dual procedure, and $\widehat{\mathfrak{z}_i^*}$ is its socius w.r.t. q_i.

8 Geometric interpretation of the characteristic differences whenever $\ell > 1$

Throughout this section we assume that $\ell > 1$. By (7.10) we have $\operatorname{int}(T_{F_i}) \cap N \neq \varnothing$, $\forall i \in \{1, \ldots, \nu\}$, and $\operatorname{int}(Q) \cap N$ consists of at least $\nu + 1 \geqslant 4$ non-collinear lattice points. This means that

$$\boxed{\mathbf{I}(Q) := \operatorname{conv}\left(\operatorname{int}(Q) \cap N\right)}$$

is an N-polygon. Analogously,

$$\boxed{\mathbf{I}(Q^*) := \operatorname{conv}\left(\operatorname{int}(Q^*) \cap M\right)}$$

is an M-polygon. We wish to relate $\sharp\left(\partial(\mathbf{I}(Q^*)) \cap M\right)$ with $\sharp\left(\partial Q^* \cap M\right)$ and $\sharp\left(\partial Q \cap N\right)$.

Lemma 8.1. *The divisor* $f^{\star}(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)}$ *on* $X(N,\widetilde{\Delta}_Q)$ *is nef. Moreover, using the notation introduced in* (3.1),

$$P_{f^{\star}(-\ell K_{X(N,\Delta_Q)})+K_{X(N,\widetilde{\Delta}_Q)}} = \mathbf{I}(Q^*). \tag{8.1}$$

Proof. Since $\Sigma_{Q^*} = \Delta_Q$ and $-\ell K_{X(N,\Delta_Q)} = D_{Q^*}$, Theorem 3.9 (applied for the lattice M-polygon Q^*) implies that the pullback $f^{\star}(-\ell K_{X(N,\Delta_Q)}) = f^{\star}(D_{Q^*})$ of D_{Q^*} via f is the unique nef divisor on $X(N,\widetilde{\Delta}_Q)$ for which $Q^* = P_{D_{Q^*}} = P_{f^{\star}(D_{Q^*})}$. Hence,

$$Q^* = \left\{ \mathbf{x} \in \mathbb{R}^2 \middle| \, \langle \mathbf{x}, \mathbf{n}_\varrho \rangle \geqslant h_{f^{\star}(D_{Q^*})}(\mathbf{n}_\varrho), \forall \varrho \in \widetilde{\Delta}_Q\,(1) \right\}$$

and

$$\mathbf{I}(Q^*) = \left\{ \mathbf{x} \in \mathbb{R}^2 \middle| \, \langle \mathbf{x}, \mathbf{n}_\varrho \rangle > h_{f^{\star}(D_{Q^*})}(\mathbf{n}_\varrho), \forall \varrho \in \widetilde{\Delta}_Q\,(1) \right\}.$$

We define the function $h' : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by setting

$$h'(\mathbf{y}) := \min \left\{ \langle \mathbf{x}, \mathbf{y} \rangle \,\middle|\, \langle \mathbf{x}, \mathbf{n}_\varrho \rangle \geqslant h_{f^{\star}(D_{Q^*})}(\mathbf{n}_\varrho) + 1, \forall \varrho \in \widetilde{\Delta}_Q\,(1) \right\}, \ \forall \mathbf{y} \in \mathbb{R}^2,$$

(with $h'(\mathbf{n}_\varrho) = h_{f^{\star}(D_{Q^*})}(\mathbf{n}_\varrho) + 1, \forall \varrho \in \widetilde{\Delta}_Q\,(1)$.) This function is an upper convex $\widetilde{\Delta}_Q$-support function because $h_{f^{\star}(D_{Q^*})}$ is upper convex $\widetilde{\Delta}_Q$-support function (by the implication (viii)$\Rightarrow$(ii) in Theorem 3.2 for the divisor $f^{\star}(D_{Q^*})$) and $\widetilde{\Delta}_Q$ contains *only basic* N-cones. Thus, by (2.9) and (2.10) (and by the implication (ii)$\Rightarrow$(viii) in Theorem 3.2 for h'), h' determines a unique nef divisor $D_{h'} \in \mathrm{Div}^{\mathbb{T}}_{\mathrm{C}}(X(N,\widetilde{\Delta}_Q))$, namely

$$\begin{aligned} D_{h'} &= -\sum_{\varrho \in \widetilde{\Delta}_Q(1)} h'(\mathbf{n}_\varrho)\mathbf{V}_{\widetilde{\Delta}_Q}(\varrho) = -\sum_{\varrho \in \widetilde{\Delta}_Q(1)} h_{f^{\star}(D_{Q^*})}(\mathbf{n}_\varrho)\mathbf{V}_{\widetilde{\Delta}_Q}(\varrho) - \sum_{\varrho \in \widetilde{\Delta}_Q(1)} \mathbf{V}_{\widetilde{\Delta}_Q}(\varrho) \\ &= f^{\star}(D_{Q^*}) + K_{X(N,\widetilde{\Delta}_Q)} = f^{\star}(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)} \end{aligned}$$

(according to (2.12) for the N-fan $\widetilde{\Delta}_Q$). Since

$$\mathbf{I}(Q^*) = \left\{ \mathbf{x} \in \mathbb{R}^2 \middle| \, \langle \mathbf{x}, \mathbf{n}_\varrho \rangle \geqslant h'(\mathbf{n}_\varrho), \forall \varrho \in \widetilde{\Delta}_Q\,(1) \right\},$$

(8.1) is true. □

Note 8.2. An alternative proof of the neffity of $f^{\star}(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)}$ (from the point of view of intersection theory) comes from the fact that

$$f^{\star}(-\ell K_{X(N,\Delta_Q)}) \sim -\ell K_{X(N,\widetilde{\Delta}_Q)} + \ell \sum_{i=1}^{\nu} K(E^{(i)})$$

(cf. (4.7)), which gives

$$f^{\star}(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)} \sim (\ell - 1)(-K_{X(N,\widetilde{\Delta}_Q)}) + \ell \sum_{i=1}^{\nu} K(E^{(i)}).$$

Since $-\ell K_{X(N,\Delta_Q)} \in \mathrm{Div}_{\mathbb{C}}^{\mathbb{T}}(X(N,\Delta))$ is ample, the implication (i)⇒(v) in Theorem 3.3 (applied for the N-fan Δ_Q and the divisor $-\ell K_{X(N,\Delta_Q)}$) and [18, Lemma 4.7, pp. 97-98] inform us that for all $i \in \{1,\ldots,\nu\}$,

$$\begin{aligned}(-\ell K_{X(N,\Delta_Q)}) \cdot C_i &= \ell\left(C_{i-1}\cdot C_i + C_i^2 + C_i \cdot C_{i+1}\right)\\ &= \ell\left(-r_i + \tfrac{q_i-p_i+1}{q_i} + \tfrac{q_{i-1}-\widehat{p}_{i-1}+1}{q_{i-1}}\right)\\ &= \ell\left(-r_i + 2 - (\tfrac{p_i+1}{q_i} + \tfrac{\widehat{p}_{i-1}+1}{q_{i-1}})\right) > 0 \Rightarrow r_i \leqslant 1\end{aligned}$$

as it is $\frac{p_i+1}{q_i}, \frac{\widehat{p}_{i-1}+1}{q_{i-1}} \in (0,1] \cap \mathbb{Q}$. (Alternatively, by (7.25), $(-\ell K_{X(N,\Delta_Q)}) \cdot C_i = \frac{q_i^*}{\ell}$ equals $\gcd(q_i^*, p_i^*-1) \geqslant 1$.) Using [18, Lemma 4.3, pp. 93-94] we infer that

$$((\ell-1)(-K_{X(N,\widetilde{\Delta}_Q)})+\ell\sum_{i=1}^{\nu}(K(E^{(i)}))\cdot\overline{C}_i = (\ell-1)\cdot 2+(\ell-1)\,(-r_i)+\ell\cdot 2 = 4\ell-2+(\ell-1)\,(-r_i) > 0$$

because $\ell \geqslant 2$ and $-r_i \geqslant -1$ for all $i \in \{1,\ldots,\nu\}$. Furthermore, since each of $E_j^{(i)}$'s is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, adjunction formula and [18, Lemma 4.3, pp. 93-94] give

$$K_{X(N,\widetilde{\Delta}_Q)} \cdot E_j^{(i)} = K(E^{(i)}) \cdot E_j^{(i)} = -2 - (E_j^{(i)})^2 = b_j^{(i)} - 2,$$

i.e.,

$$\left((\ell-1)(-K_{X(N,\widetilde{\Delta}_Q)}) + \ell\sum_{i=1}^{\nu}(K(E^{(i)})\right) \cdot E_j^{(i)} = (1-\ell)\,(b_j^{(i)}-2) + \ell(b_j^{(i)}-2) = b_j^{(i)} - 2 \geqslant 0$$

for all $i \in \{1,\ldots,\nu\}$ and all $j \in \{1,\ldots,s_i\}$ (see (4.5)). From the implication (vii)⇒(viii) in Theorem 3.2 (applied for $\widetilde{\Delta}_Q$ and the Cartier divisor $f^\star(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)}$) we conclude that $f^\star(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)}$ is indeed nef.

Lemma 8.3. *The area of the M-polygon $\mathbf{I}(Q^*)$ is given by the formula*

$$\mathrm{area}_M(\mathbf{I}(Q^*)) = \frac{1}{2}\left((\ell-2)\sharp\,(\partial Q^* \cap M) + K^2_{X(N,\widetilde{\Delta}_Q)}\right). \tag{8.2}$$

Proof. Using formula (3.2) for the nef divisor $f^\star(-\ell K_{X(N,\Delta_Q)})+K_{X(N,\widetilde{\Delta}_Q)}$ we deduce from (7.1):

$$\begin{aligned}&2\,\mathrm{area}_M(P_{f^\star(-\ell K_{X(N,\Delta_Q)})+K_{X(N,\widetilde{\Delta}_Q)}}) = \left(f^\star(-\ell K_{X(N,\Delta_Q)}) + K_{X(N,\widetilde{\Delta}_Q)}\right)^2\\ &= f^\star(-\ell K_{X(N,\Delta_Q)})^2 + 2f^\star(-\ell K_{X(N,\Delta_Q)}) \cdot K_{X(N,\widetilde{\Delta}_Q)} + K^2_{X(N,\widetilde{\Delta}_Q)}\\ &= \ell^2 K^2_{X(N,\Delta_Q)} + 2f^\star(-\ell K_{X(N,\Delta_Q)}) \cdot K_{X(N,\widetilde{\Delta}_Q)} + K^2_{X(N,\widetilde{\Delta}_Q)}\\ &= \ell\,\sharp\,(\partial Q^* \cap M) + 2f^\star(-\ell K_{X(N,\Delta_Q)}) \cdot K_{X(N,\widetilde{\Delta}_Q)} + K^2_{X(N,\widetilde{\Delta}_Q)}.\end{aligned}$$

Since $\Sigma_{Q^*} = \Delta_Q$ and $-\ell K_{X(N,\Delta_Q)} = D_{Q^*}$, applying (3.8) for the lattice M-polygon Q^* we get

$$f^\star(-\ell K_{X(N,\Delta_Q)}) \cdot K_{X(N,\widetilde{\Delta}_Q)} = -\sharp\,(\partial Q^* \cap M)\,.$$

Hence,

$$2\,\mathrm{area}_M(P_{f^\star(-\ell K_{X(N,\Delta_Q)})+K_{X(N,\widetilde{\Delta}_Q)}}) = (\ell-2)\sharp\,(\partial Q^* \cap M) + K^2_{X(N,\widetilde{\Delta}_Q)}. \tag{8.3}$$

(8.2) follows from (8.3) and (8.1). ☐

Theorem 8.4. *The number of lattice points lying on the boundary of* $\mathbf{I}(Q^*)$ *is given by the formulae*

$$\boxed{\sharp\,(\partial Q^* \cap M) - K^2_{X(N,\widetilde{\Delta}_Q)} = \sharp\,(\partial(\mathbf{I}(Q^*)) \cap M) = e(X(N,\widetilde{\Delta}_Q)) - \sharp\,(\partial Q \cap N)\,.} \tag{8.4}$$

Proof. At first we apply Pick's formula (1.3) for the M-polygon $\mathbf{I}(Q^*)$:

$$\sharp\,(\mathbf{I}(Q^*) \cap M) = \mathrm{area}_M(\mathbf{I}(Q^*)) + \tfrac{1}{2}\sharp\,(\partial(\mathbf{I}(Q^*)) \cap M) + 1. \tag{8.5}$$

By (1.5) and (1.12) we obtain

$$\sharp\,(\mathrm{int}(Q^*) \cap M) = \mathrm{area}_M(Q^*) - \tfrac{1}{2}\sharp\,(\partial Q^* \cap M) + 1 = \tfrac{1}{2}\,(\ell-1)\,\sharp\,(\partial Q^* \cap M) + 1. \tag{8.6}$$

Obviously,

$$\sharp\,(\mathbf{I}(Q^*) \cap M) = \sharp\,(\mathrm{int}(Q^*) \cap M)\,. \tag{8.7}$$

The first of the equalities (8.4) follows from (8.5), (8.6), (8.7) and (8.2). The second one follows directly from (1.15) and (7.5). ☐

Note 8.5. (i) The second term in the left-hand side of (8.4) can be written (by (7.1) and [18, Corollary 4.6, p. 96]) as

$$\begin{aligned} -K^2_{X(N,\widetilde{\Delta}_Q)} &= -K^2_{X(N,\Delta_Q)} - \sum_{i=1}^{\nu} K(E^{(i)})^2 \\ &= -\frac{1}{\ell}\sharp\,(\partial Q^* \cap M) - \sum_{i=1}^{\nu}\left(\frac{p_i + \widehat{p}_i - 2}{q_i}\right) + \sum_{i=1}^{\nu}\sum_{j=1}^{s_i}(b_j^{(i)} - 2). \end{aligned}$$

(ii) *Dual formulae.* Interchanging the roles of the ℓ-reflexive pairs (Q^*, M) and (Q, N), and using the minimal desingularization $\varphi : X(M, \widetilde{\Delta}_{Q^*}) \longrightarrow X(M, \Delta_{Q^*})$ of $X(M, \Delta_{Q^*})$, we obtain

$$\boxed{\sharp\,(\partial Q \cap N) - K^2_{X(M,\widetilde{\Delta}_{Q^*})} = \sharp\,(\partial(\mathbf{I}(Q)) \cap N) = e(X(M,\widetilde{\Delta}_{Q^*})) - \sharp\,(\partial Q^* \cap M)} \tag{8.8}$$

with

$$-K^2_{X(M,\widetilde{\Delta}_{Q^*})} = -\frac{1}{\ell}\sharp\,(\partial Q \cap N) - \sum_{i=1}^{\nu}\left(\frac{p_i^* + \widehat{p_i^*} - 2}{q_i^*}\right) + \sum_{i=1}^{\nu}\sum_{j=1}^{s_i^*}({c_j^*}^{(i)} - 2).$$

(The numbers of lattice points counted in (8.4) and (8.8) are not necessarily equal. See example 8.7.)
(iii) If for each $i \in \{1, \ldots, \nu\}$ we denote by

$$\mathbf{I}(\Theta_{\tau_i}) := \mathrm{conv}(\mathrm{int}\,(\tau_i) \cap N) \subset \Theta_{\tau_i} \quad (\text{resp., } \mathbf{I}(\Theta_{\tau_i^*}) := \mathrm{conv}(\mathrm{int}\,(\tau_i^*) \cap M) \subset \Theta_{\tau_i^*})$$

the convex hull of the lattice points lying in the interior of the auxiliary cone τ_i (resp., of τ_i^*) and by $\partial\mathbf{I}(\Theta_{\tau_i})^{\mathbf{cp}} \subset \partial\Theta_{\tau_i}^{\mathbf{cp}}$ (resp., by $\partial\mathbf{I}(\Theta_{\tau_i^*})^{\mathbf{cp}} \subset \partial\Theta_{\tau_i^*}^{\mathbf{cp}}$) the part of the boundary of $\mathbf{I}(\Theta_{\tau_i})$ (resp., of $\mathbf{I}(\Theta_{\tau_i^*})$) containing only its compact edges, then

$$\mathrm{Vert}\,(\mathbf{I}(Q)) = \bigcup_{i=1}^{\nu} \mathrm{Vert}(\partial\mathbf{I}(\Theta_{\tau_i})^{\mathbf{cp}} + \mathbf{n}_i) \quad \text{and} \quad \mathrm{Vert}\,(\mathbf{I}(Q^*)) = \bigcup_{i=1}^{\nu} \mathrm{Vert}(\partial\mathbf{I}(\Theta_{\tau_i^*})^{\mathbf{cp}} + \mathbf{m}_i).$$

Moreover, setting

$$\mathfrak{K}_Q^{(i)} := \left\{ \mathbf{u} + \mathbf{n}_i \,\middle|\, \mathbf{u} \in \mathrm{Hilb}_N(\tau_i) \smallsetminus \left\{ \begin{array}{c} \text{the two minimal} \\ \text{generators of } \tau_i \end{array} \right\} \right\},$$
$$\mathfrak{K}_{Q^*}^{(i)} := \left\{ \mathbf{w} + \mathbf{m}_i \,\middle|\, \mathbf{w} \in \mathrm{Hilb}_M(\tau_i^*) \smallsetminus \left\{ \begin{array}{c} \text{the two minimal} \\ \text{generators of } \tau_i^* \end{array} \right\} \right\},$$

and denoting by $\mathbf{u}_{\mathrm{last}}^{(i)}$ (resp., by $\mathbf{w}_{\mathrm{last}}^{(i)}$) the last lattice point of $\mathfrak{K}_Q^{(i)}$ (resp., of $\mathfrak{K}_{Q^*}^{(i)}$) and by $\mathbf{u}_{\mathrm{first}}^{(i+1)}$ (resp., by $\mathbf{w}_{\mathrm{first}}^{(i+1)}$) the first lattice point of $\mathfrak{K}_Q^{(i+1)}$ (resp., of $\mathfrak{K}_{Q^*}^{(i+1)}$) w.r.t. the anticlockwise direction, then

$$\partial(\mathbf{I}(Q)) \cap N = \bigcup_{i=1}^{\nu} (\mathfrak{K}_Q^{(i)} \cup \mathfrak{L}_Q^{(i)}) \quad \text{and} \quad \partial(\mathbf{I}(Q^*)) \cap M = \bigcup_{i=1}^{\nu} (\mathfrak{K}_{Q^*}^{(i)} \cup \mathfrak{L}_{Q^*}^{(i)}),$$

where

$$\mathfrak{L}_Q^{(i)} := \begin{cases} \mathrm{int}(\mathrm{conv}(\{\mathbf{u}_{\mathrm{last}}^{(i)}, \mathbf{u}_{\mathrm{first}}^{(i+1)}\})) \cap N, & \text{if } \mathbf{u}_{\mathrm{last}}^{(i)} \neq \mathbf{u}_{\mathrm{first}}^{(i+1)}, \\ \varnothing, & \text{if } \mathbf{u}_{\mathrm{last}}^{(i)} = \mathbf{u}_{\mathrm{first}}^{(i+1)}, \end{cases}$$

and

$$\mathfrak{L}_{Q^*}^{(i)} := \begin{cases} \mathrm{int}(\mathrm{conv}(\{\mathbf{w}_{\mathrm{last}}^{(i)}, \mathbf{w}_{\mathrm{first}}^{(i+1)}\})) \cap M, & \text{if } \mathbf{w}_{\mathrm{last}}^{(i)} \neq \mathbf{w}_{\mathrm{first}}^{(i+1)}, \\ \varnothing, & \text{if } \mathbf{w}_{\mathrm{last}}^{(i)} = \mathbf{w}_{\mathrm{first}}^{(i+1)}. \end{cases}$$

Example 8.6. Let Q be the 5-reflexive $\mathbb{Z}^2$-pentagon of Figure 13 with vertices

$$\mathbf{n}_1 = \binom{3}{-10},\ \mathbf{n}_2 = \binom{1}{0},\ \mathbf{n}_3 = \binom{-1}{5},\ \mathbf{n}_4 = \binom{-2}{5},\ \mathbf{n}_5 = \binom{-1}{0},$$

(i.e., (1.20) for $\ell = 5$). Its dual Q^* has the vertices

$$\mathbf{m}_1 = \binom{-5}{-1},\ \mathbf{m}_2 = \binom{-5}{-2},\ \mathbf{m}_3 = \binom{0}{-1},\ \mathbf{m}_4 = \binom{5}{1},\ \mathbf{m}_5 = \binom{5}{2}.$$

For $\sigma_1 = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2$ we have $|\det(\mathbf{n}_1, \mathbf{n}_2)| = 10$, and since

$$(-3) \cdot 3 - 1 \cdot (-10) = 1 \quad \text{and} \quad -3 = (-3) \cdot 1 - 1 \cdot 0 = 7 (\mathrm{mod}\ \ 10),$$

Proposition 2.4 implies that σ_1 is of type $(7,10)$. Working similarly with all the other cones in Δ_Q, we conclude with the table:

i	2-dim. cones σ_i in Δ_Q	of type (p_i, q_i)	socius of p_i	$(-)$-continued fraction expansion	length
1	$\sigma_1 = \mathbb{R}_{\geqslant 0}\mathbf{n}_1 + \mathbb{R}_{\geqslant 0}\mathbf{n}_2$	$(7,10)$	$\widehat{p}_1 = 3$	$\frac{q_1}{q_1-p_1} = \frac{10}{10-7} = \frac{10}{3} = [\![4,2,2]\!]$	$s_1 = 3$
2	$\sigma_2 = \mathbb{R}_{\geqslant 0}\mathbf{n}_2 + \mathbb{R}_{\geqslant 0}\mathbf{n}_3$	$(4,5)$	$\widehat{p}_2 = 4$	$\frac{q_2}{q_2-p_2} = \frac{5}{5-4} = \frac{5}{1} = [\![5]\!]$	$s_2 = 1$
3	$\sigma_3 = \mathbb{R}_{\geqslant 0}\mathbf{n}_3 + \mathbb{R}_{\geqslant 0}\mathbf{n}_4$	$(2,5)$	$\widehat{p}_3 = 3$	$\frac{q_3}{q_3-p_3} = \frac{5}{5-2} = \frac{5}{3} = [\![2,3]\!]$	$s_3 = 2$
4	$\sigma_4 = \mathbb{R}_{\geqslant 0}\mathbf{n}_4 + \mathbb{R}_{\geqslant 0}\mathbf{n}_5$	$(3,5)$	$\widehat{p}_4 = 2$	$\frac{q_4}{q_4-p_4} = \frac{5}{5-3} = \frac{5}{2} = [\![3,2]\!]$	$s_4 = 2$
5	$\sigma_5 = \mathbb{R}_{\geqslant 0}\mathbf{n}_5 + \mathbb{R}_{\geqslant 0}\mathbf{n}_1$	$(7,10)$	$\widehat{p}_5 = 3$	$\frac{q_5}{q_5-p_5} = \frac{10}{10-7} = \frac{10}{3} = [\![4,2,2]\!]$	$s_5 = 3$

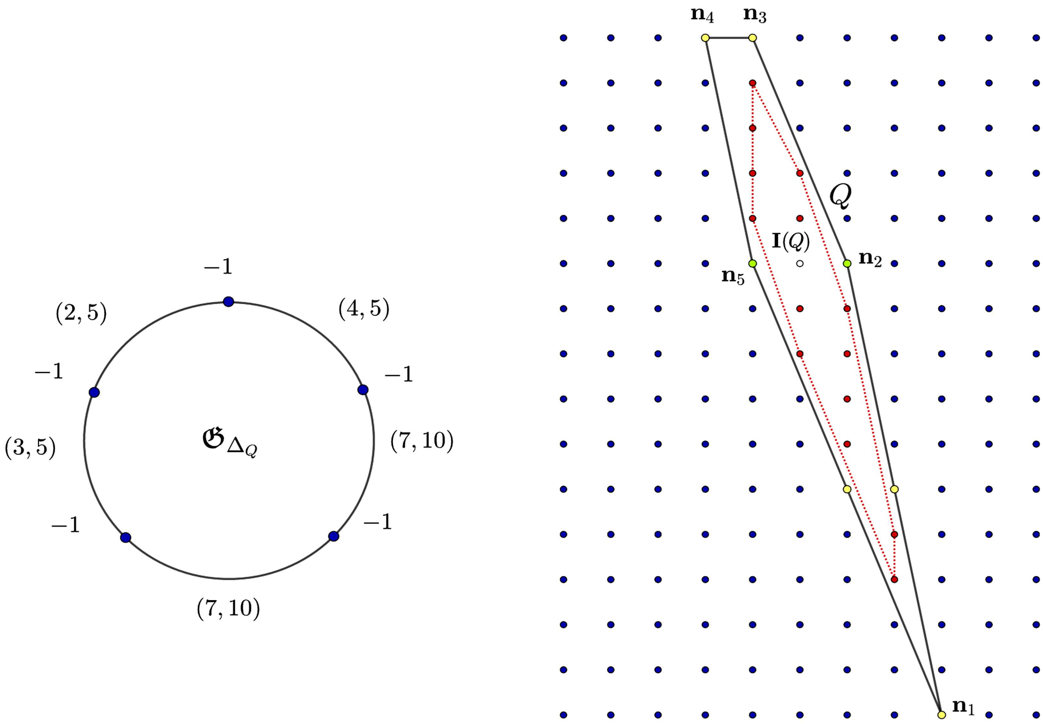


Figure 13: Q, $\mathbf{I}(Q)$ and $\mathfrak{G}_{\Delta_Q}$

Since $e(X(\mathbb{Z}^2, \widetilde{\Delta}_Q)) = \sum_{i=1}^{5}(s_i+1) = 2\cdot 4 + 2 + 2\cdot 3 = 16$, and

$$-K^2_{X(\mathbb{Z}^2,\widetilde{\Delta}_Q)} = -\frac{1}{5}\sharp\left(\partial Q^* \cap \mathbb{Z}^2\right) - \sum_{i=1}^{5}\left(\tfrac{p_i+\widehat{p}_i-2}{q_i}\right) + \sum_{i=1}^{5}\sum_{j=1}^{s_i}(b_j^{(i)}-2)$$
$$= -1 - 2\left(\tfrac{7+3-2}{10}\right) - \left(\tfrac{4+4-2}{5}\right) - 2\left(\tfrac{2+3-2}{5}\right) + 2 + 3 + 1 + 1 + 2 = 4,$$

we have

$$\sharp(\partial Q^* \cap \mathbb{Z}^2) - K^2_{X(\mathbb{Z}^2,\widetilde{\Delta}_Q)} = 5+4 = 9 = \sharp(\partial(\mathbf{I}(Q^*)) \cap \mathbb{Z}^2) = 16-7 = e(X(\mathbb{Z}^2, \widetilde{\Delta}_Q)) - \sharp(\partial Q \cap \mathbb{Z}^2).$$

In particular, $\mathbf{I}(Q^*) = \mathrm{conv}\left\{\binom{2}{1}, \binom{-4}{-1}, \binom{-1}{-1}, \binom{2}{0}, \binom{4}{1}\right\}$, and

$$\partial(\mathbf{I}(Q^*)) \cap \mathbb{Z}^2 = \left\{\binom{2}{1}, \binom{-1}{0}, \binom{-4}{-1}, \binom{-3}{-1}, \binom{-2}{-1}, \binom{-1}{-1}, \binom{2}{0}, \binom{4}{1}, \binom{3}{1}\right\}.$$

(See Figure 14.) Analogously, one constructs the following table:

i	2-dim. cones σ_i^* in Δ_{Q^*}	of type (p_i^*, q_i^*)	socius of p_i^*	$(-)$-continued fraction expansion	length
1	$\sigma_1^* = \mathbb{R}_{\geqslant 0}\mathbf{m}_5 + \mathbb{R}_{\geqslant 0}\mathbf{m}_1$	$(2,5)$	$\widehat{p}_1^* = 3$	$\frac{q_1^*}{q_1^*-p_1^*} = \frac{5}{5-2} = \frac{5}{3} = [\![2,3]\!]$	$s_1^* = 2$
2	$\sigma_2^* = \mathbb{R}_{\geqslant 0}\mathbf{m}_1 + \mathbb{R}_{\geqslant 0}\mathbf{m}_2$	$(2,5)$	$\widehat{p}_2^* = 3$	$\frac{q_2^*}{q_2^*-p_2^*} = \frac{5}{5-2} = \frac{5}{3} = [\![2,3]\!]$	$s_2^* = 2$
3	$\sigma_3^* = \mathbb{R}_{\geqslant 0}\mathbf{m}_2 + \mathbb{R}_{\geqslant 0}\mathbf{m}_3$	$(3,5)$	$\widehat{p}_3^* = 2$	$\frac{q_3^*}{q_3^*-p_3^*} = \frac{5}{5-3} = \frac{5}{2} = [\![3,2]\!]$	$s_3^* = 2$
4	$\sigma_4^* = \mathbb{R}_{\geqslant 0}\mathbf{m}_3 + \mathbb{R}_{\geqslant 0}\mathbf{m}_4$	$(4,5)$	$\widehat{p}_4^* = 4$	$\frac{q_4^*}{q_4^*-p_4^*} = \frac{5}{5-4} = \frac{5}{1} = [\![5]\!]$	$s_4^* = 1$
5	$\sigma_5^* = \mathbb{R}_{\geqslant 0}\mathbf{m}_4 + \mathbb{R}_{\geqslant 0}\mathbf{m}_5$	$(2,5)$	$\widehat{p}_5^* = 3$	$\frac{q_5^*}{q_5^*-p_5^*} = \frac{5}{5-2} = \frac{5}{3} = [\![2,3]\!]$	$s_5^* = 2$

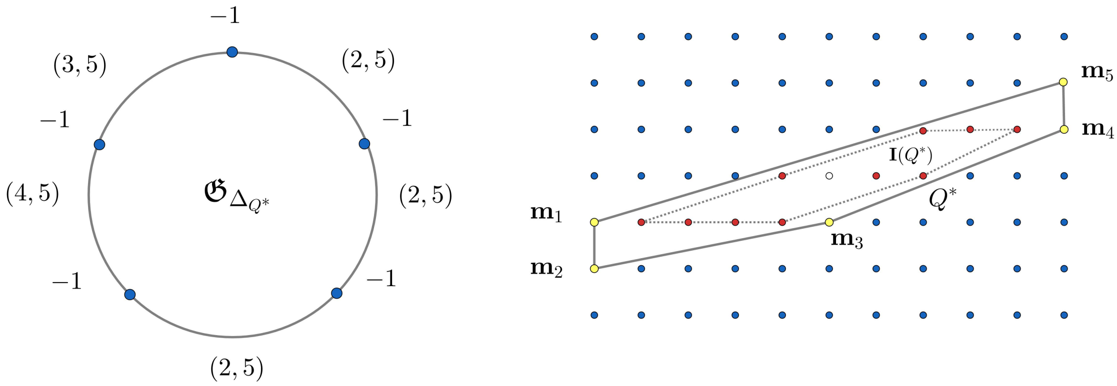


Figure 14: Q^*, $\mathbf{I}(Q^*)$ and $\mathfrak{G}_{\Delta_{Q^*}}$

Since $e(X(\mathbb{Z}^2, \widetilde{\Delta}_{Q^*})) = \sum_{i=1}^{5}(s_i^* + 1) = 4 \cdot 3 + 2 = 14$, and

$$-K^2_{X(\mathbb{Z}^2,\widetilde{\Delta}_{Q^*})} = -\frac{1}{5}\sharp\left(\partial Q \cap \mathbb{Z}^2\right) - \sum_{i=1}^{5}\left(\frac{p_i^* + \widehat{p}_i^* - 2}{q_i^*}\right) + \sum_{i=1}^{5}\sum_{j=1}^{s_i^*}(c_j^{*\,(i)} - 2)$$
$$= -\tfrac{7}{5} - 4\left(\tfrac{2+3-2}{5}\right) - \left(\tfrac{4+4-2}{5}\right) + 4 \cdot 1 + 3 = 2,$$

we have

$$\sharp(\partial Q \cap \mathbb{Z}^2) - K^2_{X(\mathbb{Z}^2,\widetilde{\Delta}_{Q^*})} = 7 + 2 = 9 = \sharp(\partial(\mathbf{I}(Q)) \cap \mathbb{Z}^2) = 14 - 5 = e(X(\mathbb{Z}^2, \widetilde{\Delta}_{Q^*})) - \sharp(\partial Q^* \cap \mathbb{Z}^2).$$

In particular, $\mathbf{I}(Q) = \mathrm{conv}\left\{ \binom{-1}{4}, \binom{-1}{1}, \binom{0}{-2}, \binom{2}{-7}, \binom{2}{-6}, \binom{1}{-1}, \binom{0}{2} \right\}$, and

$$\partial(\mathbf{I}(Q)) \cap \mathbb{Z}^2 = \left\{ \binom{-1}{4}, \binom{-1}{3}, \binom{-1}{2}, \binom{-1}{1}, \binom{0}{-2}, \binom{2}{-7}, \binom{2}{-6}, \binom{1}{-1}, \binom{0}{2} \right\}.$$

It is worth mentioning that, though each of $\mathbf{I}(Q^*)$ and $\mathbf{I}(Q)$ has 9 lattice points on its boundary, $\mathbf{I}(Q^*)$ is a $\mathbb{Z}^2$-pentagon while $\mathbf{I}(Q)$ is a $\mathbb{Z}^2$-heptagon, and

$$\tfrac{19}{2} = \mathrm{area}_{\mathbb{Z}^2}(\mathbf{I}(Q)) \neq \mathrm{area}_{\mathbb{Z}^2}(\mathbf{I}(Q^*)) = \tfrac{11}{2},\ 6 = \sharp(\mathrm{int}(\mathbf{I}(Q)) \cap \mathbb{Z}^2) \neq \sharp(\mathrm{int}(\mathbf{I}(Q^*)) \cap \mathbb{Z}^2) = 2.$$

In general, the area, as well as the precise location of the vertices and of the lattices points lying on the boundary or in the interior of $\mathbf{I}(Q)$ and $\mathbf{I}(Q^*)$ depend on the types of auxiliary cones. In our example, 7.1 gives

$$\begin{aligned}\tau_1 &= \mathbb{R}_{\geqslant 0}\tfrac{5}{q_5}(\mathbf{n}_5-\mathbf{n}_1)+\mathbb{R}_{\geqslant 0}\tfrac{5}{q_1}(\mathbf{n}_2-\mathbf{n}_1)\\ &= \mathbb{R}_{\geqslant 0}\tfrac{5}{10}\left(\binom{-1}{0}-\binom{3}{-10}\right)+\mathbb{R}_{\geqslant 0}\tfrac{5}{10}\left(\binom{1}{0}-\binom{3}{-10}\right)=\mathbb{R}_{\geqslant 0}\binom{-2}{5}+\mathbb{R}_{\geqslant 0}\binom{-1}{5}\end{aligned}$$

(and similarly for the other four auxiliary cones). From the table

i	auxiliary cones $\tau_i = (\sigma_i^*)^\vee$	of type $(q_i^* - p_i^*, q_i^*)$	$(-)$-continued fraction expansion	length
1	$\tau_1 = \mathbb{R}_{\geqslant 0}\binom{-2}{5}+\mathbb{R}_{\geqslant 0}\binom{-1}{5}$	$(3,5)$	$\frac{q_1^*}{p_1^*}=\frac{5}{2}=[\![3,2]\!]$	$t_1^*=2$
2	$\tau_1 = \mathbb{R}_{\geqslant 0}\binom{1}{5}+\mathbb{R}_{\geqslant 0}\binom{-2}{5}$	$(3,5)$	$\frac{q_2^*}{p_2^*}=\frac{5}{2}=[\![3,2]\!]$	$t_2^*=2$
3	$\tau_3 = \mathbb{R}_{\geqslant 0}\binom{2}{5}+\mathbb{R}_{\geqslant 0}\binom{-1}{0}$	$(2,5)$	$\frac{q_3^*}{p_3^*}=\frac{5}{3}=[\![2,3]\!]$	$t_3^*=2$
4	$\tau_4 = \mathbb{R}_{\geqslant 0}\binom{1}{0}+\mathbb{R}_{\geqslant 0}\binom{1}{-5}$	$(1,5)$	$\frac{q_4^*}{p_4^*}=\frac{5}{4}=[\![2,2,2,2]\!]$	$t_4^*=4$
5	$\tau_5 = \mathbb{R}_{\geqslant 0}\binom{-1}{5}+\mathbb{R}_{\geqslant 0}\binom{2}{-5}$	$(3,5)$	$\frac{q_5^*}{p_5^*}=\frac{5}{2}=[\![3,2]\!]$	$t_5^*=2$

we obtain again

$$\sharp(\partial(\mathbf{I}(Q))\cap\mathbb{Z}^2)=t_1^*+t_2^*+(t_3^*-1)+(t_4^*-1)+(t_5^*-1)=9$$

(with the 3 vertices subtracted in order to avoid counting lattice points twice). Correspondingly, from (and similarly for the other four auxiliary cones). From the table

i	auxiliary cones $\tau_i^* = \sigma_i^\vee$	of type $(q_i - p_i, q_i)$	$(-)$-continued fraction expansion	length
1	$\tau_1^* = \mathbb{R}_{\geqslant 0}\binom{10}{3}+\mathbb{R}_{\geqslant 0}\binom{0}{-1}$	$(3,10)$	$\frac{q_1}{p_1}=\frac{10}{7}=[\![2,2,4]\!]$	$t_1=3$
2	$\tau_2^* = \mathbb{R}_{\geqslant 0}\binom{0}{1}+\mathbb{R}_{\geqslant 0}\binom{5}{1}$	$(1,5)$	$\frac{q_2}{p_2}=\frac{5}{4}=[\![2,2,2,2]\!]$	$t_2=4$
3	$\tau_3^* = \mathbb{R}_{\geqslant 0}\binom{-5}{-1}+\mathbb{R}_{\geqslant 0}\binom{5}{2}$	$(3,5)$	$\frac{q_3}{p_3}=\frac{5}{2}=[\![3,2]\!]$	$t_3=2$
4	$\tau_4^* = \mathbb{R}_{\geqslant 0}\binom{-5}{-2}+\mathbb{R}_{\geqslant 0}\binom{0}{1}$	$(2,5)$	$\frac{q_4}{p_4}=\frac{5}{3}=[\![2,3]\!]$	$t_4=2$
5	$\tau_5^* = \mathbb{R}_{\geqslant 0}\binom{0}{-1}+\mathbb{R}_{\geqslant 0}\binom{-10}{-3}$	$(3,10)$	$\frac{q_5}{p_5}=\frac{10}{7}=[\![2,2,4]\!]$	$t_5=3$

we obtain $\mathfrak{L}_{Q^*}^{(i)}=\varnothing$ for all $i\in\{1,2,3,4,5\}$ and

$$\sharp(\partial(\mathbf{I}(Q^*))\cap\mathbb{Z}^2)=\sum_{i=1}^{5}(t_i-1)\underset{(7.4)}{=}\sum_{i=1}^{5}\sum_{j=1}^{s_i}(b_j^{(i)}-2)=9.$$

Example 8.7. Taking the 13-reflexive $\mathbb{Z}^2$-quadrilateral

$$Q=\ \operatorname{conv}\left\{\binom{3}{-13},\binom{-1}{13},\binom{-3}{13},\binom{1}{-13}\right\}\ \text{ which has }\ Q^*=\ \operatorname{conv}\left\{\binom{-13}{-2},\binom{0}{-1},\binom{13}{2},\binom{0}{1}\right\}$$

as its dual (i.e., (1.18) and (1.19) for $\ell=13$), we see easily that $X(\mathbb{Z}^2,\Delta_Q)$ has two cyclic quotient singularities of type $(3,26)$ and two cyclic quotient singularities of type $(17,26)$, and that $X(\mathbb{Z}^2,\Delta_{Q^*})$ has two cyclic quotient singularities of type $(11,13)$ and two cyclic quotient singularities of type $(7,13)$. (Cf. the WVE^{2}C-graphs in Figures 7 and 8.) Since

$$\tfrac{26}{26-3}=\tfrac{26}{23}=[\![2,2,2,2,2,2,2,3,2]\!]\,,\qquad \tfrac{26}{26-17}=\tfrac{26}{9}=[\![3,9]\!]\,,$$

$$\tfrac{13}{13-11}=\tfrac{13}{2}=[\![7,2]\!]\,,\qquad \tfrac{13}{13-7}=\tfrac{13}{6}=[\![3,2,2,2,2,2]\!]\,,$$

formulae (8.4) and (8.8) give

$$\sharp(\partial(\mathbf{I}(Q^*))\cap\mathbb{Z}^2)=e(X(\mathbb{Z}^2,\widetilde{\Delta}_Q))-\sharp(\partial Q\cap\mathbb{Z}^2)=2\cdot 10+2\cdot 3-8=18,\ \text{and}$$

$$\sharp(\partial(\mathbf{I}(Q))\cap\mathbb{Z}^2)=e(X(\mathbb{Z}^2,\widetilde{\Delta}_{Q^*}))-\sharp(\partial Q^*\cap\mathbb{Z}^2)=2\cdot 3+2\cdot 7-4=16,$$

respectively, i.e., $\sharp(\partial(\mathbf{I}(Q))\cap\mathbb{Z}^2)\neq\sharp(\partial(\mathbf{I}(Q^*))\cap\mathbb{Z}^2)$.

Remark 8.8. Another method to compute $\sharp\,(\partial(\mathbf{I}(Q^*))\cap M)$ is to apply Theorem 3.9 for the normal fan

$$\Sigma_{\mathbf{I}(Q^*)}:=\{\text{ the } N\text{-cones } \{\varpi_{\mathfrak{v}}^{\vee}\mid\mathfrak{v}\in\operatorname{Vert}(\mathbf{I}(Q^*))\}\text{ together with their faces}\}$$

of $\mathbf{I}(Q^*)$, where $\varpi_{\mathfrak{v}}:=\{\lambda(\mathbf{x}-\mathfrak{v})\mid\lambda\in\mathbb{R}_{\geqslant 0},\ \mathbf{x}\in\mathbf{I}(Q^*)\}$ for all $\mathfrak{v}\in\operatorname{Vert}(\mathbf{I}(Q^*))$, and to work with the minimal desingularization, say

$$\vartheta: X(N,\widetilde{\Sigma_{\mathbf{I}(Q^*)}})\longrightarrow X(N,\Sigma_{\mathbf{I}(Q^*)}) \tag{8.9}$$

of $X(N,\Sigma_{\mathbf{I}(Q^*)})$. If $F\in\operatorname{Edg}(\mathbf{I}(Q^*))$ and $\boldsymbol{\eta}_F\in N\smallsetminus\{\mathbf{0}\}$ is the (primitive) inward-pointing normal of F, then it is easy to see that $h_{\mathbf{I}(Q^*)}(\boldsymbol{\eta}_F)=h_{Q^*}(\boldsymbol{\eta}_F)+1$, where h_{Q^*} and $h_{\mathbf{I}(Q^*)}$ are the support functions of Q^* and $\mathbf{I}(Q^*)$, respectively (cf. (3.4)). Moreoever, $X(N,\Sigma_{\mathbf{I}(Q^*)})$ has at worst Gorenstein singularities, and (8.9) is crepant (with $\widetilde{\Delta}_Q=\widetilde{\Sigma_{Q^*}}$ being a refinement of $\widetilde{\Sigma_{\mathbf{I}(Q^*)}}$, $\{\boldsymbol{\eta}_F|F\in\operatorname{Edg}(\mathbf{I}(Q^*))\}\subset\bigcup_{i=1}^{\nu}\operatorname{Vert}(\partial\Theta_{\sigma_i}^{\mathbf{cp}})$ and $\operatorname{Vert}(\mathbf{I}(Q^*))$ and $\operatorname{Edg}(\mathbf{I}(Q^*))$ exactly computable via [20, §3] applied for ${\tau_i}^*=\sigma_i^{\vee}$ for all $i\in\{1,\ldots,\nu\}$). Now writing $\mathbf{I}(Q^*)$ in the form

$$\mathbf{I}(Q^*)=\bigcap_{F\in\operatorname{Edg}(\mathbf{I}(Q^*))}\left\{\mathbf{x}\in\mathbb{R}^2\middle|\,\langle\mathbf{x},\boldsymbol{\eta}_F\rangle\geqslant h_{\mathbf{I}(Q^*)}(\boldsymbol{\eta}_F)\right\}$$

and denoting by $D_{\mathbf{I}(Q^*)}:=-\sum_{F\in\operatorname{Edg}(\mathbf{I}(Q^*))}h_{\mathbf{I}(Q^*)}(\boldsymbol{\eta}_F)\mathbf{V}_{\Sigma_{\mathbf{I}(Q^*)}}(\mathbb{R}_{\geqslant 0}\boldsymbol{\eta}_F)\in\operatorname{Div}_{\mathbb{C}}^{\mathbb{T}}(X(N,\Sigma_{\mathbf{I}(Q^*)}))$ the distinguished ample divisor on $X(N,\Sigma_{\mathbf{I}(Q^*)})$, we obtain the following:

Proposition 8.9. *The number of lattice points lying on the boundary of* $\mathbf{I}(Q^*)$ *is given by the formulae*

$$\sharp\,(\partial(\mathbf{I}(Q^*))\cap M)=-D_{\mathbf{I}(Q^*)}\cdot K_{X(N,\Sigma_{\mathbf{I}(Q^*)})}=\sum_{F\in\operatorname{Edg}(\mathbf{I}(Q^*))}\Big(D_{\mathbf{I}(Q^*)}\cdot\mathbf{V}_{\Sigma_{\mathbf{I}(Q^*)}}(\mathbb{R}_{\geqslant 0}\boldsymbol{\eta}_F)\Big).$$

Proof. Applying formula (3.8) of Theorem 3.9 (for $P=\mathbf{I}(Q^*)$ and (8.9)) we deduce that

$$\sharp\,(\partial(\mathbf{I}(Q^*))\cap M)=-\vartheta^{\star}(D_{\mathbf{I}(Q^*)})\cdot K_{X(N,\widetilde{\Sigma_{\mathbf{I}(Q^*)}})}=-D_{\mathbf{I}(Q^*)}\cdot K_{X(N,\Sigma_{\mathbf{I}(Q^*)})},$$

because $K_{X(N,\widetilde{\Sigma_{\mathbf{I}(Q^*)}})}\sim\vartheta^{\star}(K_{X(N,\Sigma_{\mathbf{I}(Q^*)})})$. (See (2.14).) The second formula follows from 2.17 (i). □

9 Families of combinatorial mirror pairs in the lowest dimension

Batyrev's combinatorial mirror symmetry construction [3] is completely efficient whenever the "ambient spaces" are toric Fano varieties with at worst Gorenstein singularities of (complex) dimension $\geqslant 4$ or at least of dimension 3. In the latter case, the general members of the linear system defined by their anticanonical divisors are *K3-surfaces*. In the *lowest* dimension 2 (i.e., when the "ambient spaces" are Gorenstein log del Pezzo surfaces), the corresponding general members are *elliptic curves*. The generalisation (in dimension 2) which takes place by passing from Gorenstein log del Pezzo surfaces (defined by 1-reflexive polygons) to log del Pezzo surfaces defined by ℓ-reflexive polygons leaves little room for the determination of "combinatorial mirrors", and as yet only up to *homeomorphism*: The corresponding general members are smooth projective curves with Hodge diamond having (as unique non-trivial number) their *genus* (also called *sectional genus*) at the left and at the right corner. This genus is > 1 whenever $\ell > 1$.

Definition 9.1. Let (Q, N) be an ℓ-reflexive pair and $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. Since the Cartier divisor $-\ell K_{X(N,\Delta_Q)}$ is very ample on $X(N, \Delta_Q)$ (with $\Delta_Q = \Sigma_{Q^*}$) the complete linear system $\left|-\ell K_{X(N,\Delta_Q)}\right|$ induces the closed embedding

$$\mathbb{T}_N \hookrightarrow X(N, \Delta_Q) \hookrightarrow \mathbb{P}_{\mathbb{C}}^{\sharp(Q^* \cap M)-1}$$

with (the composition mapping)

$$\mathbb{T}_N \ni t \longmapsto [\cdots : z_{\mathbf{m}} : \ldots]_{\mathbf{m} \in Q^* \cap M} \in \mathbb{P}_{\mathbb{C}}^{\sharp(Q^* \cap M)-1}, \; z_{\mathbf{m}} := \mathbf{e}(\mathbf{m})(t),$$

where $\mathbf{e}(\mathbf{m}) : \mathbb{T}_N \to \mathbb{C}^{\times}$ is the character associated with the lattice point $\mathbf{m}$, for all $\mathbf{m} \in Q^* \cap M$. The image of $X(N, \Delta_Q)$ in $\mathbb{P}_{\mathbb{C}}^{\sharp(Q^* \cap M)-1}$ can be viewed as the projective variety $\mathrm{Proj}(\mathsf{S}_{Q^*})$, where

$$\mathsf{S}_{Q^*} := \mathbb{C}[C(Q^*) \cap (M \times \mathbb{Z})] = \bigoplus_{\kappa=0}^{\infty} \left(\bigoplus_{\mathbf{m} \in Q^* \cap M} \mathbb{C} \cdot \mathbf{e}(\mathbf{m}) \xi^{\kappa} \right)$$

(with $C(Q^*) := \{(\lambda y_1, \lambda y_2, \lambda) \,|\, \lambda \in \mathbb{R}_{\geqslant 0} \text{ and } (y_1, y_2) \in Q^*\}$) is the semigroup algebra which is naturally graded by setting $\deg(\mathbf{e}(\mathbf{m})\xi^{\kappa}) := \kappa$. (For a detailed exposition see [17, Theorem 2.3.1, p. 75; Proposition 5.4.7, pp. 237-238; Theorem 5.4.8, pp. 239-240, and Theorem 7.1.13, pp. 325-326].) Hyperplanes $\mathcal{H} \subset \mathbb{P}_{\mathbb{C}}^{\sharp(Q^* \cap M)-1}$ give curves $\mathrm{Proj}(\mathsf{S}_{Q^*}) \cap \mathcal{H}$ which are linearly equivalent to $-\ell K_{X(N,\Delta_Q)}$. For *generic* hyperplanes $\mathcal{H}$'s the intersection $\mathcal{C}_Q := \mathrm{Proj}(\mathsf{S}_{Q^*}) \cap \mathcal{H}$ is (by Bertini's Theorem) a smooth connected projective curve in the non-singular locus of $\mathrm{Proj}(\mathsf{S}_{Q^*}) \cong X(N, \Delta_Q)$. The genus $g(\mathcal{C}_Q)$ of $\mathcal{C}_Q$ is called *the sectional genus* of $X(N, \Delta_Q)$ and will be denoted simply as g_Q.

Lemma 9.2. *The sectional genus of* $X(N, \Delta_Q)(= X(N, \Sigma_{Q^*}))$ *is*

$$g_Q = \frac{1}{2}(\ell - 1)\sharp(\partial Q^* \cap M) + 1. \tag{9.1}$$

Proof. By [17, Proposition 10.5.8, p. 509], $g_Q = \sharp(\mathrm{int}(Q^*) \cap M)$. So it suffices to apply (8.6). ☐

Remark 9.3. The $\mathbb{C}$-vector space of the global sections of the canonical sheaf over $\mathcal{C}_Q$ is

$$H^0(\mathcal{C}_Q, \omega_{\mathcal{C}_Q}) \cong H^0(X(N, \Sigma_{\mathbf{I}(Q^*)}), \mathcal{O}_{X(N,\Sigma_{\mathbf{I}(Q^*)})}(D_{\mathbf{I}(Q^*)}))$$

and has dimension $h^0(\mathcal{C}_Q, \omega_{\mathcal{C}_Q}) := \dim_{\mathbb{C}}(H^0(\mathcal{C}_Q, \omega_{\mathcal{C}_Q})) = \dim_{\mathbb{C}}(H^1(\mathcal{C}_Q, \mathcal{O}_{\mathcal{C}_Q})) = g_Q$ (by adjunction). Moreover,

$$\mathcal{C}_Q^2 = (-\ell K_{X(N,\Delta_Q)})^2 = \ell^2 K^2_{X(N,\Delta_Q)} \underset{(7.1)}{=} \ell\, \sharp(\partial Q^* \cap M).$$

Definition 9.4. Let (Q,N), (Q^*,M) be ℓ-reflexive pairs (with $M := \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$). We shall say that (Q,N) *has the topological mirror property* if for any general member $\mathcal{C}_Q$ of the linear system $\left|-\ell K_{X(N,\Delta_Q)}\right|$ and any general member $\mathcal{C}_{Q^*}$ of the linear system $\left|-\ell K_{X(M,\Delta_{Q^*})}\right|$ we have

$$g_Q := g(\mathcal{C}_Q) = g(\mathcal{C}_{Q^*}) =: g_{Q^*}.$$

In this case, we shall say that $(\mathcal{C}_Q, \mathcal{C}_{Q^*})$ is a *combinatorial mirror pair* and we may think of $\mathcal{C}_Q$ as *combinatorial mirror partner* of $\mathcal{C}_{Q^*}$ and vice versa.

Note 9.5. If $\ell > 1$, then by the Twelve-Point Theorem 1.27 and by (9.1) the equality $g_Q = g_{Q^*}$ implies

$$\left.\begin{array}{l} \sharp(\partial Q \cap N) = \sharp(\partial Q^* \cap M) \\ \sharp(\partial Q \cap N) + \sharp(\partial Q^* \cap M) = 12 \end{array}\right\} \Rightarrow \sharp(\partial Q \cap N) = \sharp(\partial Q^* \cap M) = 6. \qquad (9.2)$$

And conversely, from (9.2) we get obviously $g_Q = g_{Q^*}$.

Proposition 9.6. *Let ℓ be an odd integer $\geqslant 3$. Then the families of ℓ-reflexive pairs $(Q, \mathbb{Z}^2)$ constructed by the $\mathbb{Z}^2$-polygons Q of the following tables have the topological mirror property.*

No.	The $\mathbb{Z}^2$-triangles	under the restrictions
(i)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{3}, \binom{-3\ell}{-5}\}$	$\ell \geqslant 7,\ 3 \nmid \ell$ and $5 \nmid \ell$
(ii)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{5}, \binom{-3\ell}{-8}\}$	$\ell \geqslant 7,\ 5 \nmid \ell$ and $13 \nmid \ell$
(iii)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{7}, \binom{-3\ell}{-11}\}$	$\ell \geqslant 5$ and $j \nmid \ell,\ \forall j \in \{3,7,11\}$
(iv)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{9}, \binom{-3\ell}{-14}\}$	$\ell \geqslant 11$ and $j \nmid \ell,\ \forall j \in \{3,5,7\}$
(v)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{11}, \binom{-3\ell}{-17}\}$	$\ell \geqslant 13$ and $j \nmid \ell,\ \forall j \in \{3,5,7,11,17\}$
(vi)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{13}, \binom{-3\ell}{-20}\}$	$\ell \geqslant 17$ and $j \nmid \ell,\ \forall j \in \{3,5,7,11,13\}$
(vii)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{15}, \binom{-3\ell}{-23}\}$	$\ell \geqslant 11$ and $j \nmid \ell,\ \forall j \in \{3,5,7,23\}$
(viii)	$\mathrm{conv}\{\binom{0}{1}, \binom{2\ell}{17}, \binom{-3\ell}{-26}\}$	$\ell \geqslant 11$ and $j \nmid \ell,\ \forall j \in \{3,5,13\}$
...	......	......

No.	The $\mathbb{Z}^2$-quadrilaterals	under the restrictions
(i)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{2},\binom{0}{1},\binom{-2\ell}{-3}\}$	$\ell \geqslant 5$ and $3 \nmid \ell$
(ii)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{3},\binom{0}{1},\binom{-2\ell}{-5}\}$	$\ell \geqslant 7,\ 3 \nmid \ell$ and $5 \nmid \ell$
(iii)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{4},\binom{0}{1},\binom{-2\ell}{-7}\}$	$\ell \geqslant 11$ and $j \nmid \ell,\ \forall j \in \{3,5,7\}$
(iv)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{5},\binom{0}{1},\binom{-2\ell}{-9}\}$	$\ell \geqslant 7,\ 3 \nmid \ell$ and $5 \nmid \ell$
(v)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{6},\binom{0}{1},\binom{-2\ell}{-11}\}$	$\ell \geqslant 13$ and $j \nmid \ell,\ \forall j \in \{3,5,7,11\}$
...	......	......

No.	The $\mathbb{Z}^2$-pentagons	under the restrictions
(i)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{1},\binom{\ell}{3},\binom{0}{1},\binom{-\ell}{-2}\}$	$\ell \geqslant 5$ and $3 \nmid \ell$
(ii)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{2},\binom{\ell}{4},\binom{0}{1},\binom{-\ell}{-3}\}$	$\ell \geqslant 7,\ 3 \nmid \ell$ and $5 \nmid \ell$
(iii)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{3},\binom{\ell}{5},\binom{0}{1},\binom{-\ell}{-4}\}$	$\ell \geqslant 11$ and $j \nmid \ell,\ \forall j \in \{3,5,7\}$
(iv)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{4},\binom{\ell}{6},\binom{0}{1},\binom{-\ell}{-5}\}$	$\ell \geqslant 11$ and $j \nmid \ell,\ \forall j \in \{3,5,7\}$
(v)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{5},\binom{\ell}{7},\binom{0}{1},\binom{-\ell}{-6}\}$	$\ell \geqslant 13$ and $j \nmid \ell,\ \forall j \in \{3,5,7,11\}$
...	......	......

No.	The $\mathbb{Z}^2$-hexagons	under the restrictions
(i)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{1},\binom{\ell}{2},\binom{0}{1},\binom{-\ell}{-1},\binom{-\ell}{-2}\}$	—
(ii)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{2},\binom{\ell}{3},\binom{0}{1},\binom{-\ell}{-2},\binom{-\ell}{-3}\}$	$\ell \geqslant 7$ and $3 \nmid \ell$
(iii)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{3},\binom{\ell}{4},\binom{0}{1},\binom{-\ell}{-3},\binom{-\ell}{-4}\}$	$\ell \geqslant 13$ and $3 \nmid \ell$
(iv)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{4},\binom{\ell}{5},\binom{0}{1},\binom{-\ell}{-4},\binom{-\ell}{-5}\}$	$\ell \geqslant 21$ and $5 \nmid \ell$
(v)	$\mathrm{conv}\{\binom{0}{-1},\binom{\ell}{5},\binom{\ell}{6},\binom{0}{1},\binom{-\ell}{-5},\binom{-\ell}{-6}\}$	$\ell \geqslant 31,\ 3 \nmid \ell$ and $5 \nmid \ell$
...	......	......

(The sectional genus equals $3\ell - 2$. The tables are to be continued by following the same pattern: One increases gradually the ordinates of the corresponding vertices, as well as the lower bounds for ℓ, and excludes suitable primes from being divisors of ℓ.)

Proof. It is straightforward to check that the number of lattice points lying on the boundary of each of these $\mathbb{Z}^2$-polygons equals 6. $\square$

Note 9.7. (i) There are lots of examples of ℓ-reflexive pairs $(Q, \mathbb{Z}^2)$ which have the topological mirror property but they are not self-dual. For instance, for the 5-reflexive $\mathbb{Z}^2$-triangle (from the third row of the first table in Proposition 9.6)

$$Q := \text{conv}\left\{\binom{0}{1}, \binom{10}{7}, \binom{-15}{-11}\right\} \quad \text{with} \quad Q^* = \text{conv}\left\{\binom{3}{-5}, \binom{-18}{25}, \binom{4}{-5}\right\},$$

we have $[Q]_{\mathbb{Z}^2} = [Q^*]_{\mathbb{Z}^2}$. On the other hand, for the 11-reflexive $\mathbb{Z}^2$-triangle (from the fourth row of the first table in Proposition 9.6)

$$Q := \text{conv}\left\{\binom{0}{1}, \binom{22}{9}, \binom{-33}{-14}\right\} \quad \text{with} \quad Q^* = \text{conv}\left\{\binom{4}{-11}, \binom{-23}{55}, \binom{5}{-11}\right\},$$

we have

$$[Q]_{\mathbb{Z}^2} \neq [Q^*]_{\mathbb{Z}^2} = [Q^{*\prime}]_{\mathbb{Z}^2},$$

where $Q^{*\prime} := \text{conv}\left\{\binom{0}{1}, \binom{22}{17}, \binom{-33}{-26}\right\}$ (from the eighth row of the first table in Proposition 9.6).

(ii) Setting

$$\text{RP}_\nu(\ell; N)_{\text{t.m.p.}} := \left\{ [Q]_N \in \text{RP}(\ell; N)_{\text{t.m.p.}} \,|\, \sharp(\text{Vert}(Q)) = \nu \right\}, \quad \text{for} \quad \nu \in \{3, 4, 5, 6\},$$

where

$$\text{RP}(\ell; N)_{\text{t.m.p.}} := \left\{ [Q]_N \in \text{RP}(\ell; N) \,|\, (Q, N) \text{ has the topological mirror property} \right\},$$

we find via the database [9] that the number $\sharp(\text{RP}(\ell; N)_{\text{t.m.p.}})$ is by no means negligible:

ℓ	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
$\sharp(\text{RP}_3(\ell; N)_{\text{t.m.p.}})$	5	0	1	2	0	6	8	0	12	14	0	18	5	0	24	26
$\sharp(\text{RP}_4(\ell; N)_{\text{t.m.p.}})$	7	0	1	3	0	7	9	0	13	15	0	19	5	0	25	27
$\sharp(\text{RP}_5(\ell; N)_{\text{t.m.p.}})$	3	0	1	2	0	4	5	0	7	8	0	10	5	0	13	14
$\sharp(\text{RP}_6(\ell; N)_{\text{t.m.p.}})$	1	1	1	2	1	2	3	1	3	4	2	4	3	2	5	6
$\sharp(\text{RP}(\ell; N)_{\text{t.m.p.}})$	16	1	4	9	1	19	25	1	35	41	2	51	18	2	67	73
sectional genus	1	7	13	19	25	31	37	43	49	55	61	67	73	79	85	91

In fact, $\sharp(\text{RP}(\ell; N)_{\text{t.m.p.}})$ can take relative high values, as we see from the following table for the biggest 10 values of $\ell < 200$ with $j \nmid \ell$, $\forall j \in \{3, 5, 7, 11, 13\}$.

ℓ	157	163	167	173	179	181	191	193	197	199
$\sharp(\text{RP}(\ell; N)_{\text{t.m.p.}})$	409	425	435	451	467	473	499	505	515	521
sectional genus	469	487	499	517	535	541	571	577	589	595

10 Concluding remarks and questions

(i) About the role of[7] $\mathbf{I}(Q)$ (i.e., of the convex hull of interior lattice points of an *arbitrary* lattice polygon Q) for the description of geometric properties of curves on the toric compact surface defined by Q the reader is referred to Koelman [49, Chapters 2-4], Schicho [65, §3], Castryck [11, §2-§3], and Castryck & Cools [12], [13]. In our case, we can assume that the curves $\mathcal{C}_Q$ are nothing but Zariski closures $\overline{Z_{\mathfrak{f}}}$ of affine hypersurfaces $Z_{\mathfrak{f}} \subset \mathbb{T}_N$ for Laurent polynomials $\mathfrak{f}$ having Q^* as their Newton polygon. It would be interesting, for reflexive ℓ-polygons Q, to investigate if (beyond the topological equivalence) there is a deeper relation between (e.g., certain complex structures on) $\overline{Z_{\mathfrak{f}}} \subset X(N, \Delta_Q)$ and $\overline{Z_{\mathfrak{g}}} \subset X(M, \Delta_{Q^*})$ on the "other side". For given combinatorial mirror partners (as defined in 9.4) what would be the connection between their "strict" mirrors (which turn out to be particular 3-dimensional Landau–Ginzburg models) from the point of view of the *homological mirror symmetry* for curves of high genus? (Cf. Efimov [22].)

(ii) Let $Q \subset \mathbb{R}^d$ be a d-dimensional (1-)reflexive lattice *polytope* w.r.t. a lattice N (of rank d), $Q^\circ \subset \mathbb{R}^d$ its polar w.r.t. $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, and

$$\left\{ \begin{array}{c} i\text{-dimensional} \\ \text{faces of } Q \end{array} \right\} \ni F \longmapsto F^\circ := \{ \mathbf{x} \in Q^\circ \,|\, \langle \mathbf{x}, \mathbf{y} \rangle = -1, \forall \mathbf{y} \in F \} \in \left\{ \begin{array}{c} (d-1-i)\text{-dimensional} \\ \text{faces of } Q^\circ \end{array} \right\}$$

the bijection induced by the polarity. Furthermore, let us denote by $\mathrm{Vol}_N(F)$ the normalised volume of F w.r.t. the lattice N and by $\mathrm{Vol}_M(F^\circ)$ the normalised volume of F° w.r.t. M. The following generalisations of the Twelve-Point formula (1.8) in dimensions $d \geqslant 3$ are known: If $d = 3$, then

$$\boxed{\sum_{F \text{ edges of } Q} \mathrm{Vol}_N(F) \cdot \mathrm{Vol}_M(F^\circ) = 24.} \tag{10.1}$$

(See, e.g., [4, Part A, Theorem 7.2.1], [33, Theorem 5.1.16], [29, Theorem 1.1], and [6, Corollary 5.4].) If $d = 4$, then

$$\boxed{12\,(\sharp\,(\partial Q \cap N) + \sharp\,(\partial Q^\circ \cap M)) = 2(\mathrm{Vol}_N(Q) + \mathrm{Vol}_M(Q^\circ)) - \sum_{\substack{F \text{ faces of } Q \\ \text{with } \dim(F) \in \{1,2\}}} \mathrm{Vol}_N(F) \cdot \mathrm{Vol}_M(F^\circ)} \tag{10.2}$$

(See [6, Corollary 5.6].) If $d \geqslant 5$,

$$\mathrm{Ehr}_N(Q; k) := \sharp(kQ \cap N) \in \mathbb{Q}[k]$$

is the *Ehrhart polynomial* of Q, and

$$\mathfrak{Ehr}_N(Q; t) := \sum_{k=0}^{\infty} \mathrm{Ehr}_N(Q; k) t^k = \frac{1}{(1-t)^{d+1}} \left(\sum_{j=0}^{d} \psi_j(Q) t^j \right)$$

[7] $\mathbf{I}(Q)$ is often called *the adjoint polygon of* Q.

its *Ehrhart series*, then the so-called *stringy Libgober-Wood identity* (applied by Batyrev & Schaller in [6, Theorem 5.2]) gives

$$\sum_{j=0}^{d} \psi_j(Q)\,(2j-d)^2 = \frac{1}{3}\left(d\,\mathrm{Vol}_N(Q) + \sum_{\substack{F \text{ faces of } Q \\ \text{with } \dim(F)=d-2}} 2(\mathrm{Vol}_N(F)\cdot \mathrm{Vol}_M(F^\circ)) \right), \tag{10.3}$$

i.e., a formula which is no longer symmetric w.r.t. to Q and Q°. In particular, if Q happens to be a *smooth*[8] (also known as *Delzant*) polytope and $d \geqslant 3$, we have

$$\sum_{F \text{ edges of } Q} \mathrm{Vol}_N(F) = 12 f_2 + (5-3d)\, f_1, \tag{10.4}$$

where

$$\mathbf{f} = (f_0, f_1, \ldots, f_d)$$

is the **f**-vector of Q. (See Godinho, von Heymann & Sabatini [29, Theorem 1.2].)

(iii) Let $Q \subset \mathbb{R}^d$ be a d-dimensional ℓ-reflexive *polytope*[9] w.r.t. a lattice N (of rank d), and $Q^\circ \subset \mathbb{R}^d$ its polar w.r.t. $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. If we assume that $\ell > 1$, is it possible to generalise Theorems 6.8 and 6.9, as well as the formulae in **(ii)** and other properties (as those described in §7-§8 in the $d = 2$ case) for Q and its dual $Q^* := \ell Q^\circ$ *whenever* $d \geqslant 3$? It should be clear from the outset that there are certain particularities, restrictions and limitations (with some of them already mentioned in [46, §3]) which have to be taken into account in order to deal with realistic conjectures: For instance,

(a) in contrast to what happens in dimension $d = 2$ (see Corollary 7.8), already in dimension $d = 3$ there are ℓ-reflexive polytopes also for every *even* integer $\ell \geqslant 2$.

(b) Theorem 6.8 and formula (10.1) fail (in general) to hold in dimension $d = 3$. An appropriate modification is believed to be the following:

Conjecture A. ([46, §3.5]) *Suppose that* $d = 3$, $\Lambda_{\mathrm{Edg}(Q)}$ (resp., $\Lambda_{\mathrm{Edg}(Q^*)}$) *is the sublattice of* N (resp., *of* M) *generated by the set* $\mathrm{Edg}(Q)$ *of the edges of* Q (resp., *by the set* $\mathrm{Edg}(Q^*)$ *of the edges of* Q^*) *and that* $(Q, \Lambda_{\mathrm{Edg}(Q)})$ *is an* 1*-reflexive pair. Then* $(Q^*, \Lambda_{\mathrm{Edg}(Q^*)})$ (which is to be identified with $(Q^\circ, \mathrm{Hom}_{\mathbb{Z}}(\Lambda_{\mathrm{Edg}(Q)}, \mathbb{Z}))$) *is an* 1*-reflexive pair too, and* (10.1) *is true* (if one replaces in it F° by F^*).

(c) The corresponding modification of Theorem 6.8 for $d \geqslant 4$ gives[10]:

[8] See [17, Definition 2.4.2 (b) and Theorem 2.4.3, p. 87].

[9] This means that Q has the origin in its (strict) interior, all the vertices of Q are *primitive* w.r.t. N, and the *local indices* of Q w.r.t. all facets of Q (defined in analogy to 1.10 (ii)) are equal to ℓ.

[10] If Q happens to be smooth, then one could use in the Cojecture B formula (10.4) instead of (10.2) and (10.3).

Conjecture B. *Suppose that* $d \geqslant 4$, $\Lambda_{\mathcal{F}_{d-2}(Q)}$ (resp., $\Lambda_{\mathcal{F}_{d-2}(Q^*)}$) *is the sublattice of* N (resp., *of* M) *generated by the set* $\mathcal{F}_{d-2}(Q)$ *of the faces of* Q (resp., *by the set* $\mathcal{F}_{d-2}(Q^*)$ *of the faces of* Q^*) *of codimension* 2, *and that* $(Q, \Lambda_{\mathcal{F}_{d-2}(Q)})$ *is an* 1*-reflexive pair. Then* $(Q^*, \Lambda_{\mathcal{F}_{d-2}(Q^*)})$ *is an* 1*-reflexive pair too, formula* (10.2) *is true for* $d = 4$ (if one replaces in it F° by F^* and Q° by Q^*), *and formula* (10.3) *is true for* $d \geqslant 5$ (for both Q and Q^*).

(d) Since the "cyclic covering trick" of Theorem 6.5 is independent of the dimension (and is a standard tool for reducing log terminal and log canonical singularities of a $\mathbb{Q}$-Gorenstein variety, to canonical and, respectively, log canonical singularities of index 1, cf. [51, Proposition 4-5-3, pp. 186-191]), in order to tackle the above conjectures, one should come up with analogues of Lemma 6.7, Theorem 6.9, and Proposition 6.13, being valid in dimension $d \geqslant 3$. If $d \geqslant 3$, the singularities of $X(N, \Delta_Q)$ are not necessarily isolated, and one has to construct carefully a suitable stratification of the singular locus. In addition, even the nature of singularities may differ (as it is known that in dimensions $\geqslant 3$ there exist toric singularities which are not quotient singularities). Nevertheless, toric singularities are "relatively mild" singularities and it seems to be not very difficult to deal with them. On the other hand, the analogues of (6.4) in high dimensions should relate various (*usual, orbifold or stringy*) *Chern classes* of $X(N, \Delta_Q)$ and $X(\Lambda_{\mathcal{F}_{d-2}(Q)}, \Delta_Q)$. (Furthermore, it would be desirable if one could keep all the required arguments independent of particular desingularizations of $X(N, \Delta_Q)$.)

(iv) Recently, log del Pezzo surfaces have also attracted increasing interest in the framework of the so-called *homological mirror symmetry for Fano varieties* in dimension $d = 2$. (See, e.g., [1], [15] and [47], and the references therein.) It was proposed that log del Pezzo surfaces with cyclic quotient singularities admit $\mathbb{Q}$-Gorenstein toric degenerations corresponding (under mirror symmetry) to maximally mutable Laurent polynomials in two variables, and that the quantum period of such a surface coincides with the classical period of its mirror partner. Thus, *the combinatorics of mutation* and *toric deformations* (which are closely related to geometric properties of LDP-polygons[11]) play an important role in the conception of this new approach. It comes into question whether the toric log del Pezzo surfaces associated with *ℓ-reflexive* polygons (perhaps with *prescribed* singularities) are of particular value for these investigations.

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