Fractional Domination Game

Csilla Bujtás*
Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia
csilla.bujtas@fmf.uni-lj.si

Zsolt Tuza†
Alfréd Rényi Institute of Mathematics
Budapest, Hungary;
Faculty of Information Technology
University of Pannonia
Veszprém, Hungary

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Abstract

Given a graph G, a real-valued function $f : V(G) \to [0,1]$ is a fractional dominating function if $\sum_{u \in N[v]} f(u) \geqslant 1$ holds for every vertex v and its closed neighborhood $N[v]$ in G. The aim is to minimize the sum $\sum_{v \in V(G)} f(v)$.

A different approach to graph domination is the domination game, introduced by Brešar et *al.* [SIAM J. Discrete Math. 24 (2010) 979–991]. It is played on a graph G by two players, namely Dominator and Staller, who take turns choosing a vertex such that at least one previously undominated vertex becomes dominated. The game is over when all vertices are dominated. Dominator wants to finish the game as soon as possible, while Staller wants to delay the end. Assuming that both players play optimally and Dominator starts, the length of the game on G is uniquely determined and is called the game domination number of G.

We introduce and study the fractional version of the domination game, where the moves are ruled by the condition of fractional domination. Here we prove a fundamental property of this new game, namely the fractional version of the so-called Continuation Principle. Moreover we present lower and upper bounds on the fractional game domination number of paths and cycles. These estimates are tight apart from a small additive constant. We also prove that the game domination number cannot be bounded above by any linear function of the fractional game domination number.

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1 Introduction

In this paper we introduce a new kind of competitive optimization game concerning graph domination.

We deal with finite undirected graphs $G = (V, E)$. A vertex $v \in V$ *dominates* itself and its neighbors; that is, exactly the vertices contained in the *closed neighborhood* $N[v]$ of v. A subset $D \subset V$ is called *dominating set* if every vertex of G is dominated by at least one vertex from D, i.e. $\bigcup_{v \in D} N[v] = V$. The smallest size of a dominating set in G is called the *domination number* and is denoted by $\gamma(G)$.

The *domination game*, introduced in [5], is a competitive optimization version of graph domination. It is played on a graph G by two players, namely Dominator and Staller, who take turns choosing a vertex such that at least one previously undominated vertex becomes dominated.[1] The game is over when all vertices are dominated, and the length of the game is the number of vertices chosen by the players. Dominator wants to finish the game as soon as possible, while Staller wants to delay the end. Assuming that both players play optimally and Dominator starts, the length of the game on G is uniquely determined; it is called the *game domination number* of G and is denoted by $\gamma_g(G)$. Analogously, the *Staller-start game domination number* of G, denoted by $\gamma'_g(G)$, is the length of the game under the same rules when Staller makes the first move.

Below we shall refer to these games as *integer games*, as opposed to their fractional versions which will be introduced.

Following [5], the domination game has been studied further in many papers, see e.g. [3, 4, 6, 7, 15, 18, 24, 25, 26, 27, 28, 29, 30]. The notion also inspired the introduction of the total domination game on graphs [8, 14, 19, 20, 21, 22, 23], transversal game [9, 10], disjoint domination game [12], connected domination game [2], and domination game on hypergraphs [11].

In Section 2, we define the fractional domination game and prove some inequalities for the related parameter. In Section 3, we prove the fractional analogue of the Continuation Principle. In Section 4, we prove lower and upper bounds on the fractional game domination number of paths and cycles; the gap is not greater than the additive constant $5/3$ in any case. Moreover, we determine the exact values for cycles and paths of order at most 6. The proofs are not trivial already for such small cycles as C_5 and C_6 that indicates the hardness of establishing the exact values for C_n and P_n in general. In the concluding Section 5, we propose some open problems and conjectures on the fractional domination game.

2 Fractional domination game

A *fractional dominating function* of $G = (V, E)$ is a real-valued function $f : V \to [0, 1]$ satisfying the inequality $\sum_{u \in N[v]} f(u) \geqslant 1$ for every vertex v in G. The minimum of the sum $\sum_{v \in V} f(v)$ over all such f is called the *fractional domination number* $\gamma^*(G)$ of G; this

[1]The condition turns out to be restrictive only for Staller.

graph invariant was introduced in [16, 17]. Observe that a dominating set corresponds to a fractional dominating function f where every vertex is assigned to either 0 or 1.

A function $d : V \to [0, 1]$ (omitting the local condition above on the vertices) is called a *partially dominating function*; we denote by $|d|$ the sum $\sum_{v\in V} d(v)$. Given a partially dominating function d, the associated *(domination) load function* is the function ℓ defined on V as

$$\ell(v) = \ell(v; d) = \min\{1, \sum_{u\in N[v]} d(u)\}$$

for every vertex v.

2.1 Definition

The *fractional domination game* starts with the all-0 load function $\ell(v) \equiv 0$, and is finished when the all-1 function $\ell(v) \equiv 1$ is reached. Dominator and Staller take turns making moves of weight 1 each, except possibly in the last move which may be smaller. A *move* is a sequence $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ of arbitrary length, with its *submoves* (v_{i_k}, w_k) ($k = 1, 2, \dots,$ possibly an infinite sequence) where $v_{i_1}, v_{i_2}, \dots$ are vertices of G; any number of repetitions is allowed. Here $w_1, w_2, \dots$ are real numbers from $(0, 1]$; it is required that

$$\sum_{k\geqslant 1} w_k = 1$$

in each move except the last one in which $\sum_{k\geqslant 1} w_k \leqslant 1$ must hold.

At the beginning the partially dominating function is $d_0 \equiv 0$ and also the load function is $\ell_0 \equiv 0$. After the i^{th} move ($i = 1, 2, \dots$) the values $d_i(v_j)$ are calculated to obtain the new partially dominating function d_i by the rule

$$d_i(v_j) = d_{i-1}(v_j) + \sum_{i_k=j} w_k$$

from which the corresponding load function ℓ_i is also derived. A move $(v_{i_1}, w_1), (v_{i_2}, w_2), \dots$ is legal if

($*$) For all $k = 1, 2, \dots,$ there exists a vertex $u \in N[v_{i_k}]$ with

$$\ell_{i-1}(u) + \sum_{u\in N[v_{i_s}],\ 1\leqslant s\leqslant k-1} w_s \ \leqslant\ 1 - w_k.$$

That is, in each submove there must exist a vertex whose load increases by exactly the weight in the submove.

The *value of the game* $\mathcal{G}$ is $|\mathcal{G}| = |d_q|$, provided that the all-1 load function is reached after a sequence of legal moves in the q^{th} move; i.e., $\ell_q \equiv 1$.

Analogously to the integer game, also here Dominator wants a small $|\mathcal{G}|$, while Staller wants a large $|\mathcal{G}|$. To define the *fractional game domination number* $\gamma_g^*(G)$, assume that Dominator starts the game on G, and consider the set

$$D_G = \{a : \text{Dominator has a strategy which ensures } |\mathcal{G}| \leqslant a\}.$$

Now, the fractional game domination number is defined as

$$\gamma_g^*(G) = \inf D_G.$$

Assuming that Staller starts the game, the *Staller-start fractional game domination number* $\gamma_g^{*\prime}(G)$ is defined similarly.

Consider the set, analogous to D_G, which expresses the situation from Staller's point of view:

$$S_G = \{b : \text{Staller has a strategy which ensures } |\mathcal{G}| \geqslant b\}.$$

Proposition 1. *For every graph G,* $\inf(D_G) = \sup(S_G)$.

Proof. First, assume that there exist x and y such that $\inf(D_G) < x < y < \sup(S_G)$. By definition, $x \in D_G$ and hence, Dominator can ensure that, under any strategy of Staller, the value of the fractional domination game is at most x. Similarly, $y \in S_G$ and Staller has a strategy which ensures that, under any strategy of Dominator, the value of the game is at least y. This contradiction proves $\inf(D_G) \geqslant \sup(S_G)$.

On the other hand, by definition, if $z < \inf(D_G)$, Dominator does not have a strategy to achieve $|\mathcal{G}| \leqslant z$. Thus, for any strategy of Dominator, there is a strategy of Staller which results in $|\mathcal{G}| > z$. Then, $z \in S_G$ and $z \leqslant \sup(S_G)$. Since this holds for every $z < \inf D_G$, we have $\inf(D_G) \leqslant \sup(S_G)$. This completes the proof of the proposition. □

The above statement and proof clearly remains valid if D'_G and S'_G are defined in terms of the Staller-start fractional domination game. For this reason we have

$$\gamma_g^{*\prime}(G) = \inf(D'_G) = \sup(S'_G).$$

2.2 Example

Consider a fractional domination game on the 5-cycle $v_1v_2v_3v_4v_5$. If Dominator starts the game with the move $(v_1, \frac{1}{5}), (v_2, \frac{1}{5}), (v_3, \frac{1}{5}), (v_4, \frac{1}{5}), (v_5, \frac{1}{5})$, then $d_1(v_i) = \frac{1}{5}$ and $\ell_1(v_i) = \frac{3}{5}$ for every i. Suppose that Staller's reply is $(v_1, \frac{2}{5}), (v_5, \frac{2}{5}), (v_2, \frac{1}{5})$. This is a legal move and results in $\ell_2(v_1) = \ell_2(v_2) = \ell_2(v_4) = \ell_2(v_5) = 1$ and $\ell_2(v_3) = \frac{4}{5}$. In the next move, Dominator plays e.g. $(v_2, \frac{1}{5})$ and the game is finished with the value $|\mathcal{G}| = \frac{11}{5}$. Note that the permutation $(v_5, \frac{2}{5}), (v_2, \frac{1}{5}), (v_1, \frac{2}{5})$ of the submoves would not be legal in Staller's turn, since after performing the first two submoves the load would be strictly greater than $\frac{3}{5}$ for every $u \in N[v_1]$. It indicates that the reordering of submoves inside of a legal move might result in a move which is not legal.

2.3 Comparison of domination parameters

Proposition 2. *For any graph G, the following inequalities hold:*

(i) $\gamma^*(G) \leqslant \gamma_g^*(G) < 2\gamma^*(G)$;

(ii) $\gamma^*(G) \leqslant \gamma_g^{*\prime}(G) < 2\gamma^*(G) + 1$.

Proof. Observe that at the end of the fractional domination game, no matter whether Dominator or Staller starts the game, the partially dominating function d_q must be a dominating function. This establishes $\gamma^*(G) \leqslant \gamma_g^*(G)$ and $\gamma^*(G) \leqslant \gamma_g^{*\prime}(G)$. On the other hand, Dominator may fix an optimal fractional dominating function f and then he plays a submove (v_{i_j}, w_j) only if $d_{i-1}(v_{i_j}) + w_j \leqslant f(v_{i_j})$. If such a legal submove does not exists, then G is fully dominated and the game is finished. This proves that the total sum W_D of the weights w_j in Dominator's submoves is at most $|f| = \gamma^*(G)$. Now, assume that Dominator starts the game. Since the players take turns moving with weight 1, the total sum W_S of weights in Staller's submoves is at most $\lfloor W_D \rfloor$. But, if $W_D = \gamma^*(G)$ then Dominator finishes the game and W_S must be strictly smaller than W_D. This proves $\gamma_g^*(G) < 2\gamma^*(G)$. If Staller starts the game, we have $W_S \leqslant \lfloor W_D \rfloor + 1$, but $W_S = \gamma^*(G) + 1$ would mean that the game was finished by Staller with a move of value 1, moreover Dominator played total value $\gamma^*(G)$. These two conditions together are impossible under the assumed strategy of Dominator. This yields $\gamma_g^{*\prime}(G) < 2\gamma^*(G) + 1$. □

Proposition 3. (*i*) *There does not exist any universal constant C with*

$$\gamma_g(G) \leqslant C \cdot \gamma_g^*(G).$$

(*ii*) *If every block of G is a complete graph (and in particular if G is a tree), then*

$$\frac{\gamma_g(G) + 1}{2} \leqslant \gamma_g^*(G) < 2\gamma_g(G).$$

Proof. (*i*) It follows from the result of Alon [1] that, for any two positive constants d and ϵ, there exists a graph F of order n and of minimum degree at least $(1-\epsilon)d$ such that

$$\gamma(F) > (1-\epsilon)\frac{1 + \ln(d+1)}{d+1} n.$$

For this graph F, a fractional dominating function is obtained by assigning a weight of $\frac{1}{(1-\epsilon)d+1}$ to each vertex of G. Thus,

$$\gamma^*(F) \leqslant \frac{n}{(1-\epsilon)d+1} < \frac{\gamma(F)}{(1+\ln(d+1))(1-\epsilon)^2}.$$

This implies that $\frac{\gamma(G)}{\gamma^*(G)}$ can be arbitrarily large on the class of graphs. Furher, since $\gamma(G) \leqslant \gamma_g(G)$ holds for all graphs and $\gamma_g^*(G) < 2\gamma^*(G)$ is also true by Proposition 2(*i*), we have that

$$\frac{\gamma(G)}{2\gamma^*(G)} < \frac{\gamma_g(G)}{\gamma_g^*(G)}.$$

We conclude that there is no universal constant upper bound on $\frac{\gamma_g(G)}{\gamma_g^*(G)}$.

(*ii*) Since $\gamma_g(G) \leqslant 2\gamma(G) - 1$ holds for every graph G and, as proved in [13], block graphs also satisfy $\gamma(G) = \gamma^*(G)$, the inequality chain

$$\frac{\gamma_g(G)+1}{2} \leqslant \gamma(G) = \gamma^*(G) \leqslant \gamma_g^*(G) < 2\gamma^*(G) = 2\gamma(G) \leqslant 2\gamma_g(G)$$

proves the statement. □

3 Continuation Principle

A monotone property of the fractional game domination number is expressed in the following idea, which provides a useful tool in simplifying several arguments. Given a graph G and a load function ℓ, let the fractional game ℓ-domination number be denoted by $\gamma_g^*(G|\ell) = \inf(D|\ell) = \sup(S|\ell)$, where $\inf(D|\ell)$ and $\sup(S|\ell)$ are defined analogously to D_G and S_G, for the game starting with a load function ℓ on G with the move of Dominator. The value $\gamma_g^{*\prime}(G|\ell)$ for the Staller-start case can be defined in the same way.

Theorem 4 (Fractional Continuation Principle). *If ℓ_1 and ℓ_2 are load functions on the graph G such that $\ell_1(v) \leqslant \ell_2(v)$ holds for every $v \in V(G)$, then $\gamma_g^*(G|\ell_1) \geqslant \gamma_g^*(G|\ell_2)$. Also, $\gamma_g^{*\prime}(G|\ell_1) \geqslant \gamma_g^{*\prime}(G|\ell_2)$.*

Proof. Assume that $\gamma_g^*(G|\ell_1) < \gamma_g^*(G|\ell_2)$ holds, and choose a real number x from the open interval $(\gamma_g^*(G|\ell_1), \gamma_g^*(G|\ell_2))$. We use a version of the imagination strategy [5] to derive a contradiction. First, define the following games:

Game 1: Dominator plays on $G|\ell_1$ applying a strategy which ensures that the length is at most $t_1 < x$.

Game 2: Staller plays on $G|\ell_2$ applying a strategy which ensures that the length is at least $t_2 > x$.

The moves will be copied between Games 1 and 2 (according to some rules) such that $\ell_1(v) \leqslant \ell_2(v)$ remains valid after each move.

If it is a Dominator-start game, first Dominator moves in Game 1, we copy this move, i.e. the submoves in the given order, into Game 2, where Staller replies. This move is then copied into Game 1. This continues until at least one of the two games finishes.

Under the condition $\ell_1(v) \leqslant \ell_2(v)$, Staller's any legal move in Game 2 will be legal in Game 1. Hence, Staller's moves can be repeated in Game 1 and the condition $\ell_1(v) \leqslant \ell_2(v)$ remains valid. On the other hand it may happen that a a submove (v_{i_k}, w_k) in Game 1 is not legal in Game 2. In this case, let s be the minimum vertex load in $N[v_{i_k}]$. Then, we first make the legal submove $(v_{i_k}, 1-s)$ in Game 2. After it, all the loads in the closed neighborhood of v_{i_k} reach 1 and $\ell_1(v) \leqslant \ell_2(v)$ still holds. The remaining weight $w_k - (1-s)$ can be used in one or more arbitrary legal submoves in Game 2. If, for some k, the entire weight $w_k - (1-s)$ cannot be distributed among the vertices in a legal way, then Game 2 terminates earlier than Game 1. Otherwise the sequence of weights of the submoves yields an increasing load sequence bounded above by 1 at each vertex, which thus converges to a value also in the case when Dominator applies an infinite number of submoves. In this way we obtain load functions satisfying $\ell_1 \leqslant \ell_2$ after the move. (The number of moves is always finite, bounded above by $2\gamma(G)$.)

By the condition $\ell_1(v) \leqslant \ell_2(v)$, Game 2 finishes no later than Game 1. That is, we have $|\mathcal{G}_2| \leqslant |\mathcal{G}_1|$ for the values of the games. By the strategies of the players, $|\mathcal{G}_1| \leqslant t_1$ and $|\mathcal{G}_2| \geqslant t_2$. These give the following contradiction:

$$x < t_2 \leqslant |\mathcal{G}_2| \leqslant |\mathcal{G}_1| \leqslant t_1 < x.$$

The same proof works also for the Staller-start fractional domination game, Staller making the first move in Game 2, which is copied into Game 1. □

One of the consequences of Theorem 4 is the following relation:

Theorem 5. *For every graph G we have $|\gamma_g^*(G) - \gamma_g^{*\prime}(G)| \leqslant 1$.*

Proof. We apply Theorem 4. Consider the Staller-start game. Whatever Staller moves first, she assigns total weight 1, and creates a situation which is at least as favorable for Dominator as the all-zero load at the beginning of the Dominator-start game. Then, by definition, Dominator can ensure that the game ends using at most $\gamma_g^*(G)$ further weight. This proves $\gamma_g^{*\prime}(G) \leqslant \gamma_g^*(G) + 1$.

Similarly, if Dominator starts, after his first move he is in at least as favorable position as with the all-zero load at the beginning of the Staller-start game. This proves the reverse inequality $\gamma_g^*(G) \leqslant \gamma_g^{*\prime}(G) + 1$. □

4 Cycles and paths

In this section we first estimate $\gamma_g^*(C_n)$ and $\gamma_g^{*\prime}(C_n)$ for each $n \geqslant 7$. The lower and upper bounds do not coincide but differ only by at most $\frac{4}{3}$ for each case. In the continuation, we prove similar lower and upper bounds for the fractional game domination numbers of paths. The section will be closed by proving exact values for very short cycles and paths.

4.1 Cycles

We introduce the notations $A(n)$, $B(n)$, $A'(n)$ and $B'(n)$ for every integer $n \geqslant 7$ as follows:

	$A(n)$	$B(n)$	$A'(n)$	$B'(n)$
$n \equiv 0 \pmod 4$	$\frac{n}{2} - \frac{4}{3} + \frac{2}{n}$	$\frac{n}{2}$	$\frac{n}{2} - \frac{2}{3}$	$\frac{n}{2}$
$n \equiv 1 \pmod 4$	$\frac{n}{2} - \frac{3}{2} + \frac{6}{n}$	$\frac{n}{2} - \frac{1}{6}$	$\frac{n}{2} - \frac{5}{6}$	$\frac{n}{2} + \frac{1}{2}$
$n \equiv 2 \pmod 4$	$\frac{n}{2} - 1 + \frac{2}{n}$	$\frac{n}{2} - \frac{1}{3}$	$\frac{n}{2} - 1$	$\frac{n}{2} + \frac{1}{3}$
$n \equiv 3 \pmod 4$	$\frac{n}{2} - \frac{7}{6} + \frac{2}{n}$	$\frac{n}{2} - \frac{1}{2}$	$\frac{n}{2} - \frac{1}{2}$	$\frac{n}{2} + \frac{1}{6}$

Theorem 6. *For every $n \geqslant 7$, we have*

$$A(n) \leqslant \gamma_g^*(C_n) \leqslant B(n), \qquad \text{and} \qquad A'(n) \leqslant \gamma_g^{*\prime}(C_n) \leqslant B'(n).$$

Proof. Let $C_n = v_1 \dots v_n$ with $n \geqslant 7$ and consider the indices cyclically, so $v_{n+1} = v_1$ and $v_0 = v_n$.

Lower bounds. We first prove $A(n) \leqslant \gamma_g^*(C_n)$. Consider a Dominator-start game on C_n and after each submove let $f(G)$ denote the sum of the updated loads of the vertices; that is, $f(G) = \sum_{i=1}^n \ell(v_i)$ where $\ell(v_i)$ is the load of the vertex v_i after the considered submove. The following strategy of Staller ensures that the game finishes with a value of at least $A(n)$.

(S1) If there are no two consecutive fully dominated vertices on the cycle, then Staller creates such a pair in her very first submove. First, she chooses an index i such that $\ell(v_i)+\ell(v_{i+1})$ is maximum and she plays v_i with a weight $w_s = 1-\min\{\ell(v_i), \ell(v_{i+1})\}$.

(S2) Otherwise, if the game is not over yet, there are three consecutive vertices v_{i-1}, v_i and v_{i+1} with $\ell(v_{i-1}) = \ell(v_i) = 1$ and $\ell(v_{i+1}) < 1$. Staller plays (v_i, w_s) with $w_s \leqslant 1 - \ell(v_{i+1})$ as her next submove.

Clearly, the game starts with $f(G) = 0$, it finishes with $f(G) = n$ and any submove of Dominator with a weight x may increase the sum $f(G)$ by at most $3x$. Note that (S1) can be applied at most once, namely it is the first submove of Staller, unless Dominator's first move either was the single submove $(v_{i_1}, 1)$ or consisted of just two submoves (v_{i_1}, w_1), $(v_{i_2}, 1 - w_1)$ with $i_2 = i_1 + 1$ or $i_2 = i_1 - 1$.

Suppose that Staller's first submove is (v_i, w_s), it applies (S1) and $\ell(v_i) \leqslant \ell(v_{i+1})$. This submove is of weight $w_s = 1 - \ell(v_i)$ and increases $f(G)$ by at most

$$3(1 - \ell(v_{i+1})) + 2(\ell(v_{i+1}) - \ell(v_i)) = w_s + [2 - (\ell(v_i) + \ell(v_{i+1}))].$$

In this case, we had $\sum_{j=1}^{n} \ell(v_j) = 3$ before the submove and, by our maximality condition, $\ell(v_i) + \ell(v_{i+1}) \geqslant \frac{6}{n}$ holds. Hence, when Staller plays v_i with a weight $x = w_s$, the sum $f(G)$ increases by at most $x+2-\frac{6}{n}$. From now on, Staller always plays according to (S2), and in any submove of weight x she increases $f(G)$ by only x. In particular, if her first move is complete (of total weight 1) then in that moment we have $f(G) = 6 - \frac{6}{n}$.

To obtain a lower bound on $\gamma_g^*(C_n)$, we write n in the form $n = 4k + 2 - \frac{6}{n} + r$ where k is an integer and the remainder r satisfies $0 \leqslant r < 4$; hence $r = i + \frac{6}{n} < i + 1$ if $n = 4j + 2 + i \geqslant 7$, for any $j \geqslant 0$ and $i = 0, 1, 2, 3$. Note that the game does not finish after the first two moves, because we have $f(G) = 6 - 6/n < n$ at that point. The above strategy of Staller ensures that, from the third move of the game, $f(G)$ increases by at most 4 in any two consecutive moves. Then the number of complete pairs of moves is not smaller than $k = \frac{1}{4}(n - 2 + \frac{6}{n} - r)$. Therefore, if $r \leqslant 3$, that means $n \equiv 0$, 2, or 3 (mod 4) in case of $n \geqslant 7$, the value of the game is at least

$$\frac{n - 2 + \frac{6}{n} - r}{2} + \frac{r}{3} = \frac{n}{2} - 1 + \frac{3}{n} - \frac{r}{6} = A(n).$$

On the other hand, if $3 < r < 4$, that is if $n \equiv 1 \pmod 4$, then $n - f(G) \geqslant r - 3$ holds before Staller's last move, hence the length of the game is at least

$$\frac{n - 2 + \frac{6}{n} - r}{2} + 1 + (r - 3) = 2k + r - 2 = \frac{n}{2} - 3 + \frac{3}{n} + \frac{r}{2} = A(n).$$

If Staller starts the game, we consider the same strategy (S1)-(S2) and the same function $f(G)$ as before. Here the first (sub)move of Staller is $(v_i, 1)$ that results in an increase 3 in $f(G)$. The further details are analogous to those for the Dominator-start game. Now we write $n = 4k + 3 + r'$. After Staller's first move still there are at least

$\frac{1}{4}(n-3-r')$ complete pairs of moves, after which we have $n - f(G) \geqslant r'$. Consequently the value of the game is at least

$$1 + \frac{n-3-r'}{2} + \frac{r'}{3} = \frac{n}{2} - \frac{1}{2} - \frac{r'}{6} = A'(n).$$

Upper bounds. To prove $\gamma_g^*(C_n) \leqslant B(n)$ and $\gamma_g^{*\prime}(C_n) \leqslant B'(n)$, we consider the function

$$F(G) = \left(\sum_{j=1}^{n} \ell(v_j)\right) - \left(\sum_{j=1}^{n} |\ell(v_j) - \ell(v_{j-1})|\right)$$

where $f(G) = \sum_{j=1}^{n} \ell(v_j)$ is the sum of the updated loads after the considered submove of the game, and $g(G) = \sum_{j=1}^{n} |\ell(v_j) - \ell(v_{j-1})|$ is the sum of the differences between the loads of the neighboring vertices.

We will establish two claims that give estimates on the change of $F(G)$ when Staller or Dominator moves respectively.

Claim 7. *Every submove (v_i, x) increases $F(G)$ by at least x.*

Proof. We consider three cases, all the further possible submoves can be replaced by two or three submoves of the following types.

- If the submove (v_i, x) dominates three vertices, that is $\ell(v_j) \leqslant 1 - x$ holds for $j = i-1$, i and $i+1$ before the submove, then $f(G)$ is increased by $3x$. On the other hand, the difference $|\ell(v_k) - \ell(v_{k-1})|$ might change by at most x and only for $k = i-1$ and $k = i+2$. Consequently, $g(G)$ changes by at most $2x$ and $F(G)$ increases by at least $3x - 2x = x$.
- Assume that (v_i, x) dominates exactly two vertices, i.e. one from the loads $\ell(v_{i-1})$, $\ell(v_i)$, $\ell(v_{i+1})$ already equals 1 and the remaining two loads are at most $1 - x$. Then, the submove increases $f(G)$ by $2x$. Concerning the value of $g(G)$, we have three subcases. If $\ell(v_i) = 1$ before the submove, then each of $|\ell(v_i) - \ell(v_{i-1})|$ and $|\ell(v_{i+1}) - \ell(v_i)|$ decreases by x, and each of $|\ell(v_{i-1}) - \ell(v_{i-2})|$ and $|\ell(v_{i+2}) - \ell(v_{i+1})|$ may increase or decrease by at most x. Therefore, $g(G)$ cannot get larger. In the second subcase, assume that $\ell(v_{i-1}) = 1$ before the submove and $\ell(v_i), \ell(v_{i+1}) \leqslant 1 - x$. Then, $|\ell(v_k) - \ell(v_{k-1})|$ does not change for $k = i-1$ and $i+1$, it decreases by x for $k = i$ and might either decrease or increase by at most x for $k = i+2$. As follows, $g(G)$ cannot increase. The third subcase, when $\ell(v_{i+1}) = 1$, is symmetric to the second one. Consequently, $F(G) = f(G) - g(G)$ increases by at least $2x$ in either subcase.
- If the submove (v_i, x) dominates only one vertex, $f(G)$ increases by x. Further, if $\ell(v_{i-1}) \leqslant 1 - x$ and $\ell(v_i) = \ell(v_{i+1}) = 1$ were true before the submove, then $|\ell(v_k) - \ell(v_{k-1})|$ decreases by x for $k = i$, and it might change by at most x for $k = i-1$, while it remains the same for all the other k. Hence, $g(G)$ does not increase and $F(G)$ increases by at least x. For the remaining subcases, namely if $\ell(v_i) \leqslant 1 - x$ or $\ell(v_{i+1}) \leqslant 1 - x$, the statement can be proved similarly. (In the former, $F(G)$ increases by $3x$.) (□)

Claim 8. *Having a cycle C_n together with any load function ℓ before Dominator's submove, there can be played a submove (or a sequence of submoves) with a weight of x such that $F(G)$ increases by at least $3x$.*

Proof. The following four cases together cover all possibilities for the load function ℓ. In all cases, we assume that $x > 0$ and $\epsilon > 0$ and that x is always choosen such that the total weight played in the move does not exceed 1.

- If $\ell(v_i) \leqslant 1 - \epsilon$ for every $i \in [n]$, Dominator plays the submoves $(v_1, x), \ldots, (v_n, x)$ with a weight of $x \leqslant \frac{\epsilon}{3}$. By these submoves $f(G)$ increases by exactly $3nx$ and $g(G)$ remains unchanged. Therefore, $F(G)$ increases by exactly $3nx$.

  Assuming that $\epsilon' = 1 - \max \ell(v_i)$ and Dominator has distributed total weight w until now during his current move ($0 \leqslant w < 1$), there are two possibilities. If $\epsilon' n/3 \geqslant 1 - w$, then he can choose $\epsilon = (1-w)/n$ and this submove completes the move. Otherwise he can choose $\epsilon = \epsilon'/3$, and this is a legal submove which decreases the number of vertices whose domination load is smaller than 1.

  Similar case distinctions for choosing the proper value of ϵ apply also for the next three types of steps; we omit the details.

- If there is an index i such that $\ell(v_i) = 1$ and $\ell(v_j) \leqslant 1 - \epsilon$ holds for $j = i+1,\ i+2,\ i+3$, Dominator may play the submove (v_{i+2}, x) with $x \leqslant \epsilon$. Then, $f(G)$ increases by $3x$. The difference $|\ell(v_j) - \ell(v_{j-1})|$ decreases by x if $j = i+1$; it might increase or decrease by at most x if $j = i+4$; it does not change if $j \neq i+1$ and $j \neq i+4$. Hence, $F(G)$ increases by at least $3x$.

- If there is an index i such that $\ell(v_i) = \ell(v_{i+3}) = 1$ and $\max\{\ell(v_{i+1}), \ell_(v_{i+2})\} = 1 - \epsilon$, Dominator may play (v_{i+1}, x) with $x \leqslant \epsilon$. After this submove, $f(G)$ increases by $2x$, $g(G)$ decreases by $2x$ and, consequently, $F(G)$ increases by $4x$.

- If there is an index i such that $\ell(v_i) = \ell(v_{i+2}) = 1$ and $\ell(v_{i+1}) = 1 - \epsilon$, Dominator may play (v_{i+1}, x) with $x \leqslant \epsilon$. Then, $f(G)$ increases by x, $g(G)$ decreases by $2x$ and, consequently, $F(G)$ increases by $3x$. (□)

No matter which player starts the fractional domination game, the value of $F(G)$ equals 0 at the beginning (as it is computed from the all-0 load function) and equals n at the end of the game. For the Dominator-start game, using the notation $r = n \pmod 4$, Claims 7 and 8 give the following upper bound:

$$\gamma_g^*(C_n) \leqslant \left\lfloor \frac{n}{4} \right\rfloor \cdot 2 + \frac{r}{3} = \frac{n}{2} - \frac{r}{6} = B(n).$$

If Staller starts the game, we can refer to the same statements and use the same notation. If $r = 0$, we get $\gamma_g^{*\prime}(C_n) \leqslant \frac{n}{2} = B'(n)$. If $r > 0$, we have

$$\gamma_g^{*\prime}(C_n) \leqslant 1 + \left\lfloor \frac{n-1}{4} \right\rfloor \cdot 2 + \frac{r-1}{3} = \frac{n-r}{2} + 1 + \frac{r-1}{3} = \frac{n}{2} - \frac{r}{6} + \frac{2}{3} = B'(n).$$

These prove the upper bounds stated in the theorem. □

4.2 Paths

To give a similar estimation on the fractional game domination number of paths, we introduce the following notations for every $n \geqslant 6$. Remark that $D(n) - C(n) \leqslant 5/3$ and $D'(n) - C'(n) \leqslant 5/3$ holds in each case.

	$C(n)$	$D(n)$	$C'(n)$	$D'(n)$
$n \equiv 0 \pmod 4$	$\frac{n}{2} - 1$	$\frac{n}{2} + \frac{2}{3}$	$\frac{n}{2} - \frac{1}{3}$	$\frac{n}{2} + \frac{4}{3}$
$n \equiv 1 \pmod 4$	$\frac{n}{2} - \frac{1}{2}$	$\frac{n}{2} + \frac{1}{2}$	$\frac{n}{2} - \frac{1}{2}$	$\frac{n}{2} + \frac{7}{6}$
$n \equiv 2 \pmod 4$	$\frac{n}{2} - \frac{2}{3}$	$\frac{n}{2} + 1$	$\frac{n}{2}$	$\frac{n}{2} + 1$
$n \equiv 3 \pmod 4$	$\frac{n}{2} - \frac{5}{6}$	$\frac{n}{2} + \frac{5}{6}$	$\frac{n}{2} - \frac{1}{6}$	$\frac{n}{2} + \frac{3}{2}$

Theorem 9. *For every $n \geqslant 7$, we have*

$$C(n) \leqslant \gamma_g^*(P_n) \leqslant D(n), \qquad \text{and} \qquad C'(n) \leqslant \gamma_g^{*\prime}(P_n) \leqslant D'(n).$$

Proof. Consider the path $P_n = v_1 \dots v_n$ with $n \geqslant 7$.

Lower bounds. In a fractional domination game on P_n, after each submove let $f(G)$ denote the sum of the updated loads of the vertices; that is, $f(G) = \sum_{i=1}^{n} \ell(v_i)$ where $\ell(v_i)$ is the load of the vertex after the considered submove. No matter which player starts the game, Staller may apply a strategy which is similar to (S1)-(S2). In her first submove she plays (v_1, x) with $x = 1 - \ell(v_1) \geqslant 1$. This increases $f(G)$ by at most $2x$ and results in a load of 1 on v_1 (and also on v_2). After that, in each submove she can choose a legal submove with a weight x such that $f(G)$ increases by exactly x. It is also clear that any submove (v_i, x) of Dominator increases $f(G)$ by at most $3x$. With this strategy of Staller, after the first complete pair of moves $f(G) \leqslant 5$ holds. Since we have $f(G) = n$ at the end of the game, putting $r = n - 1 \pmod 4$ the following lower bound is obtained:

$$\gamma_g^*(P_n) \geqslant \frac{n-1-r}{2} + \frac{r}{3} = \frac{n}{2} - \frac{r}{6} - \frac{1}{2} = C(n).$$

In the Staller-start game we write $n = 4k + 2 + r'$. Staller can start with the move $(v_1, 1)$ and apply (S2) in all later moves. Then we have

$$\gamma_g^{*\prime}(P_n) \geqslant 1 + \frac{n-2-r'}{2} + \frac{r'}{3} = \frac{n}{2} - \frac{r'}{6} = C'(n).$$

Upper bounds. After every submove of a fractional domination game on P_n, we consider the function

$$H(G) = \left(\sum_{j=1}^{n} \ell(v_j) \right) - \left((1 - \ell(v_1)) + (1 - \ell(v_n)) + \sum_{j=2}^{n} |\ell(v_j) - \ell(v_{j-1})| \right)$$

where $\ell(v)$ denotes the updated load of the vertex v as earlier. The first term $\sum_{j=1}^{n} \ell(v_j)$ will be denoted by $f(G)$ and the second term by $h(G)$. Hence, $H(G) = f(G) - h(G)$ holds at any point of the game. Remark that $H(G)$ equals -2 at the beginning and equals n when the game is over. Further, the value of $h(G)$ is always equal to $g(G^+)$ where G^+ is the $(n+1)$-cycle obtained from P_n by keeping the loads of the vertices and inserting a vertex v_{n+1} with a load 1 and joining it to v_1 and v_n. Therefore, Claims 7 and 8 remain valid for the fractional domination game on a path P_n, too. Consequently, we have the following upper bound if it is a Dominator-start game and $r = n + 2 \pmod 4$:

$$\gamma_g^*(P_n) \leqslant \left\lfloor \frac{n+2}{4} \right\rfloor \cdot 2 + \frac{r}{3} = \frac{n}{2} + 1 - \frac{r}{6} = D(n).$$

If Staller starts the game, let us write $r' = n + 1 \pmod 4$. After Staller's first move we surely have $H(G) \geqslant -1$. After that, if Dominator plays the strategy above (increasing $H(G)$ by at least 3 in each complete move), there follow at most $\frac{1}{4}(n + 1 - r')$ complete pairs of moves, ending with $n - f(G) \leqslant r' \leqslant 3$. Then Dominator can terminate the game by assigning at most $r'/3$ further weight. Consequently we have

$$\gamma_g^{*\prime}(P_n) \leqslant 1 + \frac{n+1-r'}{2} + \frac{r'}{3} = \frac{n}{2} - \frac{r'}{6} + \frac{3}{2} = D'(n).$$

This finishes the proof of the theorem. □

4.3 Small cycles and paths

In this subsection we determine the exact values for the fractional game domination numbers of small cycles and paths. The following table summarizes the results of Propositions 10, 11 and 12.

	$\gamma_g^*(C_n)$	$\gamma_g^{*\prime}(C_n)$	$\gamma_g^*(P_n)$	$\gamma_g^{*\prime}(P_n)$
$n = 2$	–	–	1	1
$n = 3$	1	1	1	2
$n = 4$	$\frac{3}{2}$	2	2	2
$n = 5$	$\frac{11}{5}$	2	$\frac{5}{2}$	3
$n = 6$	$\frac{5}{2}$	2	3	3

Proposition 10. *We have*

$$\gamma_g^*(P_2) = \gamma_g^{*\prime}(P_2) = \gamma_g^*(C_3) = \gamma_g^{*\prime}(C_3) = \gamma_g^*(P_3) = 1,$$

$$\gamma_g^*(C_4) = \frac{3}{2}, \quad \textit{and} \quad \gamma_g^{*\prime}(P_3) = \gamma_g^{*\prime}(C_4) = \gamma_g^*(P_4) = \gamma_g^{*\prime}(P_4) = 2.$$

Proof. It is clear that d is a (minimal) fractional dominating function of a 3-cycle $v_1v_2v_3$ if and only if $d(v_1)+d(v_2)+d(v_3)=1$. Hence, $\gamma_g^*(C_3)=\gamma_g^{*\prime}(C_3)=1$.

If Dominator starts on the 4-cycle $v_1v_2v_3v_4$ with the move $(v_1,\frac{1}{4}),(v_2,\frac{1}{4}),(v_3,\frac{1}{4}),(v_4,\frac{1}{4})$, the load function ℓ assigns $\frac{3}{4}$ to each vertex. Then, if Staller plays with a total weight x in her submoves, it increases the current $\sum_{j=1}^4 \ell(v_i)=3$ by at least $3\cdot\frac{1}{4}+(x-\frac{1}{4})$. Thus, $x\leqslant\frac{1}{2}$ which shows that Dominator has a startegy to ensure that the value of the game is at most $\frac{3}{2}$. On the other hand, after any first move of Dominator we have $\sum_{j=1}^4 \ell(v_i)=3$ and there are two consecutive vertices with $\ell(v_i)+\ell(v_{i+1})\geqslant\frac{6}{4}$. If Staller plays (v_i,x) with $x=1-\min\{\ell(v_i),\ell(v_{i+1})\}$, the sum $\sum_{j=1}^4 \ell(v_i)$ is increased by at most $x+(2-(\ell(v_i)+\ell(v_{i+1})))\leqslant x+\frac{1}{2}$. All the further submoves of Staller can be chosen such that any submove of weight x' increases $\sum_{j=1}^4 \ell(v_i)$ by exactly x'. This proves $\gamma_g^*(C_4)\geqslant\frac{3}{2}$ and we may conclude that $\gamma_g^*(C_4)=\frac{3}{2}$.

If Staller starts on the same 4-cycle, she may play $(v_1,1)$ as the first move. Then, the load of v_3 remains zero and Dominator has to put a further weight of 1 to finish the game. That is, Staller's strategy ensures $\gamma_g^{*\prime}(C_4)\geqslant 2$. On the other hand, any first move of Staller results in $\sum_{j=1}^4 \ell(v_i)=3$ and hence, Dominator can always finish the game by using a weight of at most $\sum_{j=1}^4(1-\ell(v_i))=1$. This gives $\gamma_g^{*\prime}(C_4)=2$.

Using similar arguments, one can prove that the fractional game domination numbers of P_2, P_3 and P_4 are the values stated in the proposition. □

Proposition 11. *We have* $\gamma_g^*(C_5)=11/5$, $\gamma_g^*(C_6)=5/2$, *and* $\gamma_g^{*\prime}(C_5)=\gamma_g^{*\prime}(C_6)=2$.

Proof. Let us begin with the Staller-start versions. If Staller's first move is $(v_1,1)$ then v_3 remains completely undominated, hence Dominator cannot finish the game using weight smaller than 1. On the other hand after any first move $(v_{i_1},w_1),(v_{i_2},w_2),\ldots$ of Staller, Dominator can reach fractional domination by playing $(v_{i_1+3},w_1),(v_{i_2+3},w_2),\ldots$ on both C_5 and C_6. Hence the value of the game is 2.

In the Dominator-start game let us show first, how Dominator can ensure that the claimed upper bounds are valid. For both $n=5$ and $n=6$ he assigns weight $1/n$ to each vertex. After that, all domination loads are equal to 3/5 or 1/2, respectively.

Then, on C_5 the first 2/5 weights of Staller increase the total load $f(G)=\sum_{i=1}^5 \ell(v_i)$ from $3=15/5$ to 21/5, and her next 3/5 weights increase it further to 24/5 at least. Hence Dominator needs no more than 1/5 in his next move to finish the game, and $\gamma_g^*(C_5)\leqslant 11/5$ follows.

On C_6, however, Dominator's response is analogous to that in the Staller-start game, rather than to the case of C_5. Suppose that the first 1/2 weights of Staller are included as submoves (v_{i_1},w_1), (v_{i_2},w_2), …, (v_{i_k},w_k). Then no matter how the other 1/2 weights are distributed among Staller's further submoves, Dominator can finish the game with the submoves (v_{i_1+3},w_1), (v_{i_2+3},w_2), …, (v_{i_k+3},w_k). This yields $\gamma_g^*(C_6)\leqslant 5/2$.

Somehow the proof of the lower bound is easier for $n=6$, so let us consider C_6 first. To simplify notation, let us denote $\ell_i=\ell(v_i)$, hence after the first move of Dominator we have $\ell_1+\cdots+\ell_6=3$ and at that moment the average value of the pairwise sums

$\ell_i + \ell_{i+1}$ $(i = 1, \ldots, 6)$ is 1. Suppose without loss of generality that $\ell_5 + \ell_6 \leqslant 1$. Since any submove of Dominator increased the load of exactly one from v_1 and v_4, we have $\ell_1 + \ell_4 = 1$. Thus, it is legal for Staller to play $(v_3, 1 - \ell_4)$ and (v_2, ℓ_4). After this move we still have $\ell_5 + \ell_6 \leqslant 1$ and $\min(\ell_5, \ell_6) \leqslant 1/2$. Hence, Dominator cannot finish the game with a value which is smaller than $5/2$.

Using analogous notation for $n = 5$, after the first move of Dominator the average value of the ℓ_i is $3/5$ and that of the sums $\ell_i + \ell_{i+1}$ $(i = 1, \ldots, 5)$ is $6/5$. Suppose that ℓ_5 is the smallest current load, that is $\ell_5 = 3/5 - y$, for some $y \geqslant 0$. If $\ell_1 + \ell_4 \leqslant 6/5 + y$, then Staller distributes a weight at least $4/5 - y$ between v_2 and v_3 with legal submoves, e.g. playing $(v_3, 1 - \ell_4)$ and $(v_2, \max(\ell_4 - y - 1/5, 0))$. It is still legal for Staller to distribute $\min(y + 1/5, \ell_4)$ weight between ℓ_1 and ℓ_4 to complete her move, keeping $\ell_5 \leqslant 4/5$. Hence Dominator cannot finish the game before $11/5$.

Otherwise, if $\ell_1 + \ell_4 = 6/5 + y + z$ for some $z > 0$, then $\ell_2 + \ell_3 = 6/5 - z$. Hence one of ℓ_1 and ℓ_4, say ℓ_4, is at most $3/5 + y/2 + z/2$; and one of ℓ_2 and ℓ_3 is at most $3/5 - z/2$. First, the weight $2/5 + z/2$ is assigned to v_2, this keeps ℓ_4 unchanged. Second, the weight $2/5 - y/2 - z/2$ is assigned to v_3. Third, the weight $1/5 + y/2$ is distributed between v_1 and v_4 in an arbitrary way. This completes the move of Staller, and leaves ℓ_5 still not larger than $4/5$. Hence the value of the game is at least $11/5$. □

Proposition 12. *We have* $\gamma_g^*(P_5) = 5/2$ *and* $\gamma_g^{*\prime}(P_5) = \gamma_g^{*\prime}(P_6) = \gamma_g^*(P_6) = 3$.

Proof. Consider first the path $P_5 = v_1v_2v_3v_4v_5$. Since $N[v_1] \subset N[v_2]$ and $N[v_5] \subset N[v_4]$, we may assume by the Continuation Principle that Dominator never plays v_1 or v_5 and also that Staller does not play v_2 or v_4 when playing v_1 or v_5, respectively, is also legal.

If Dominator starts the game and plays $(v_2, 1/2), (v_4, 1/2)$ as his first move, we have $\sum \ell_i = 3$. In Staller's turn, this sum of the loads increases by at least $3/2$. Hence, Dominator may ensure that the game finishes with a value of $|\mathcal{G}| \leqslant 5/2$. This proves $\gamma_g^*(P_5) \leqslant 5/2$. For the lower bound, consider the following strategy of Staller. Assume that Dominator assigns y_2, y_3 and y_4 to v_2, v_3 and v_4, respectively, in his first turn. Then, $y_2 + y_3 + y_4 = 1$ and, by symmetry, we may suppose that $y_2 \leqslant y_4$. Let Staller's first submove be $(v_3, 1 - y_3 - y_2)$. This yields a load function with values $\ell_2 = \ell_3 = \ell_4 = 1$, $\ell_1 = y_2$ and $\ell_5 = y_4$. Since v_1 and v_5 have no common neighbor, the sum of the further weights assigned to the vertices in the game is $(1 - y_2) + (1 - y_4)$. Thus, we have

$$\gamma_g^*(P_5) \geqslant 1 + (1 - y_3 - y_2) + (2 - y_2 - y_4) = 4 - (y_2 + y_3 + y_4) - y_2 = 3 - y_2 \geqslant 5/2\,,$$

where the last inequality follows from the assumption $y_2 \leqslant y_4$ and from $y_2 + y_4 \leqslant 1$. We may conclude $\gamma_g^*(P_5) = 5/2$.

If Staller starts the game by playing $(v_3, 1)$, the load remains zero on the two leaves and, therefore, the sum of the further weights assigned to the vertices in the game is 2. This strategy of Staller shows $\gamma_g^{*\prime}(P_5) \geqslant 3$. To prove the other direction, assume that Staller assigns weights y_1, y_3, and y_5 to the vertices v_1, v_3 and v_5 respectively. Then, Dominator replies with the legal move $(v_2, 1 - y_1), (v_4, y_1)$. This results in the following loads: $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1$ and $\ell_5 = y_1 + y_5 = 1 - y_3$. Thus, Dominator can ensure that the game finishes with a value $|\mathcal{G}| \leqslant 1 + 1 + y_3 \leqslant 3$. Hence, we have $\gamma_g^{*\prime}(P_5) = 3$ as required. It also follows that the unique best strategy for Staller is to start with the move $(v_3, 1)$.

Consider now $P_6 = v_1v_2v_3v_4v_5v_6$. Similarly as above, we may assume that Dominator never plays v_1 or v_6 and Staller does not play v_2 and v_5, respectively, if playing v_1 and v_6 is also legal. If Dominator assigns y_i to v_i $(i = 2, 3, 4, 5)$ in his first move of the original game, where $y_2 + y_3 + y_4 + y_5 = 1$, then Staller can reply with the submoves $(v_3, y_4 + y_5)$ and $(v_4, y_2 + y_3)$. This move does not increase the loads y_2 of v_1 and y_5 of v_6, hence the value of the game will be $2 + (2 - y_2 - y_5) \geqslant 3$. On the other hand, Dominator can begin the game with the submoves $(v_2, 1/2)$ and $(v_5, 1/2)$, which yields uniform load 1/2 on all vertices. No matter how Staller replies in her first move, Dominator can finish the game by placing the largest feasible weights on v_2 and v_5 because both are at most 1/2. It follows that $\gamma_g^*(P_6) = 3$.

In the Staller-start game the lower bound $\gamma_g^{*\prime}(P_6) \geqslant 3$ is easily achieved by the first move $(v_3, 1)$ because it leaves zero loads on v_1, v_5, v_6. Assume that Staller plays (v_i, y_i) for $i = 1, 3, 4, 6$ in her first move; here $y_1 + y_3 + y_4 + y_6 = 1$. An efficient reply by Dominator is then (v_2, y_4), (v_3, y_6), (v_4, y_1), (v_5, y_3). After this move the load sequence is $y_1 + y_4$, 1, 1, 1, 1, $y_3 + y_6$. Hence the game terminates within $2 + (1 - y_1 - y_4) + (1 - y_3 - y_6) = 3$, so that $\gamma_g^{*\prime}(P_6) \leqslant 3$. □

5 Concluding remarks and open problems

We close this paper with three conjectures and a further problem. Although all of them are formulated for the fractional domination game where the first move is performed by Dominator, we conjecture that the analogues of Conjectures 13–15 hold for the Staller-start fractional domination game as well. In particular, we conjecture that $\gamma_g^{*\prime}(G) \leqslant 3n/5$ holds for every isolate-free graph G if its order is large enough.[2]

Conjecture 13. If each of the first $2k - 1$ $(k \geqslant 1)$ moves was an integer move, i.e. of the form $(v_{i_1}, 1)$, then Staller has an integer move in the $(2k)^{\text{th}}$ turn, which is optimal in the fractional game.

This means that fractional moves would be advantageous for Dominator only. If true then this conjecture implies the following weaker one.

Conjecture 14. For every graph G, $\gamma_g^*(G) \leqslant \gamma_g(G)$.

We also guess an explicit upper bound on the fractional game domination number in terms of the number of vertices. Its validity would follow if both our Conjecture 14 and also the 3/5-conjecture $\gamma_g(G) \leqslant 3n/5$ hold true for every isolate-free graph G, the latter conjectured in [24]; but probably a direct approach would be more promising.

Conjecture 15. For every isolate-free graph G, $\gamma_g^*(G) \leqslant 3n/5$.

[2]For small n a little anomaly occurs with P_3 that has $\gamma_g^{*\prime} = 2$, hence the best we can expect in geneal is $\gamma_g^{*\prime}(G) \leqslant (3n + 1)/5$.

Although there are infinitely many connected examples where $\gamma_g(G) = 3n/5$ holds in the integer game, so far we have not found a graph whose fractional game domination number is at least $3n/5$. Therefore, the following question arises.

Problem 16. Is there a constant $c < 3/5$ such that $\gamma_g^*(G) \leqslant c \cdot n$ holds for every isolate-free G, or at least for every tree of order n?

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