A note on Hedetniemi's conjecture, Stahl's conjecture and the Poljak-Rödl function

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Abstract

We prove that $\min\{\chi(G), \chi(H)\} - \chi(G \times H)$ can be arbitrarily large, and that if Stahl's conjecture on the multichromatic number of Kneser graphs holds, then we can have $\chi(G\times H)/\min\{\chi(G), \chi(H)\} \leqslant 1/2+\epsilon$ for large values of $\min\{\chi(G), \chi(H)\}$.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

The *categorical product* $G \times H$ of graphs G and H has vertex set $V(G \times H) = \{(x, y) : x \in V(G), y \in V(H)\}$, in which two vertices (x, y) and (x', y') are adjacent if and only if $xx' \in E(G)$ and $yy' \in E(H)$. A proper colouring ϕ of G can be lifted to a proper colouring Φ of $G \times H$ defined as $\Phi(x, y) = \phi(x)$. So $\chi(G \times H) \leqslant \chi(G)$, and similarly $\chi(G\times H) \leqslant \chi(H)$. Hedetniemi conjectured in 1966 that $\chi(G\times H) = \min\{\chi(G), \chi(H)\}$ for all finite graphs G and H [6]. The conjecture received a lot of attention [7, 10, 13, 14] and remained open for more than half century. It is known that $\chi(G\times H) = \min\{\chi(G), \chi(H)\}$ whenever $\min\{\chi(G), \chi(H)\} \leqslant 4$ [1] and that the fractional version is true, i.e., for any graphs G and H, $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$ [15]. However, Shitov refuted this conjecture recently [12]. Yet, some problems concerning the chromatic number of product graphs remain open.

The *Poljak-Rödl function* $f : \mathbb{N} \to \mathbb{N}$ is defined by

$$f(n) = \min\{\chi(G \times H) : \chi(G), \chi(H) \geqslant n\}.$$

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Hedetniemi's conjecture is equivalent to the statement that $f(n) = n$ for all n. Shitov proved that for sufficiently large n, $f(n) \leqslant n - 1$. Still, very little is known about the behavior of the function $f(n)$. In particular, it is unknown whether $f(n)$ is bounded by a constant. However it is known that if $f(n)$ is bounded by a constant, then $f(n) \leqslant 9$ for all n (see [10, 14]). In this note, we prove the following facts.

Proposition 1.

(i) $\lim_{n\to\infty}(n - f(n)) = \infty$,

(ii) If Stahl's conjecture on the multichromatic number of Kneser graphs [11] holds, then $\limsup_{n\to\infty} f(n)/n \leqslant 1/2$.

Proposition 1 (i) will be proved in Section 2. Proposition 1 (ii) will be proved in Section 3, where a presentation of Stahl's conjecture is also given.

2 Discussion and extensions of Shitov's results

For a positive integer c, the *exponential graph* K_c^H has vertices all the mappings $f : V(H) \to \{1, 2, \ldots, c\}$, in which f, g are adjacent in K_c^H if $f(u) \neq g(v)$ for every edge $e = uv$ of H. It is well known and easy to verify that $\Phi(v, f) = f(v)$ is a proper c-colouring of $H \times K_c^H$. Thus the way to find counterexamples to Hedetniemi's conjecture is to find an integer c and a graph H such that both H and K_c^H have chromatic number larger than c.

The *lexicographic product* $G[H]$ of G and H is the graph with vertex set $V(G[H]) = \{(x, y) : x \in V(G), y \in V(H)\}$, in which two vertices (x, y) and (x', y') are adjacent if and only if $xx' \in E(G)$, or $x = x'$ and $yy' \in E(H)$.

Shitov's construction of counterexamples to Hedetniemi's conjecture is based on the following result.

Theorem 2 ([12], Claim 3)**.** *For any graph G with girth at least six, for all but finitely many values of q, we have* $\chi\left(K_c^{G[K_q]}\right) \geqslant c + 1$, *with* $c = \lceil 3.1q \rceil$[1].

Finding such a lower bound on chromatic numbers of some exponential graphs was the key part of Shitov's refutation of Hedetniemi's conjecture. Finding lexicographic products $G[K_q]$ with $\chi(G[K_q]) > c$ is standard theory. Indeed the fractional chromatic number $\chi_f(H)$ of a graph H is a standard lower bound for its chromatic number, and it is well known that $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$ (see [3]). Erdős' classic probabilistic proof [2] shows that there are graphs with girth at least 6 and fractional chromatic number at least 3.1. For such a graph G, we have $\chi(G[K_q]) \geqslant \lceil \chi_f(G[K_q]) \rceil = \lceil \chi_f(G) \cdot q \rceil \geqslant \lceil 3.1q \rceil$, and by Theorem 2, this yields a counterexample to Hedetniemi's conjecture.

Remarkably, replacing the condition $\chi_f(G) \geqslant 3.1$ by $\chi_f(G) \geqslant B$ for $B \gg 3.1$ readily gives counterexamples to Hedetniemi's conjecture where the chromatic number of at least

[1] Technically, Shitov refers to the "strong product" rather than the lexicographic product of graphs, but with K_q as a second factor, the strong product coincides with the lexicographic product (see [4]).

one factor is arbitrarily larger than the chromatic number of the product. Also, the proof of Theorem 2 only uses a small subgraph of $K_c^{G[K_q]}$. Therefore it is possible that Shitov's construction already gives examples that show that $\lim_{n\to\infty} f(n)/n = 0$. On the other hand, since $\chi_f(G[K_q]) > c$, the fractional version of Hedetniemi's conjecture [15] implies that $\chi_f(K_c^{G[K_q]}) = c$. Thus it is also reasonable to think that $\chi(K_c^{G[K_q]})/c$ may be bounded, and that the identity $\lim_{n\to\infty} f(n)/n = 0$, if true, can only be witnessed by a different construction.

Proof of Proposition 1 (i). Fix a positive integer d. We shall prove that if n is sufficiently large, then $f(n+d) \leqslant n$. Let G_d be a graph with girth at least 6 and fractional chromatic number at least $8d$. Then by Theorem 2, for sufficiently large q and $c = \lceil 3.1q \rceil$, we have $\chi\left(K_c^{G_d[K_q]}\right) \geqslant c+1$ while $\chi(G_d[K_q]) \geqslant 2cd$. Now consider the graph $K_{cd}^{G_d[K_q]}$. For $i = 0, 1, \ldots, d-1$, let Q_i be the subgraph of $K_{cd}^{G_d[K_q]}$ induced by the functions with image in $\{ic+1, ic+2, \ldots, ic+c\}$. Each Q_i is isomorphic to $K_c^{G_d[K_q]}$ and hence at least $c+1$ colours are needed for each copy. For $i \neq j$, each function in Q_i is adjacent to each function in Q_j. Hence, $\chi\left(K_{dc}^{G_d[K_q]}\right) \geqslant d(c+1)$. As $\chi(K_{dc}^{G_d[K_q]}) = dc$ and $\chi(G_d[K_q]) \geqslant 2cd \geqslant cd+d$, it follows that $f(dc+d) \leqslant dc$.

Thus for every d there exist infinitely many values of n (of the form $dc+d$) such that $n - f(n) \geqslant d$. It only remains to show that the gap between n and $f(n)$ will not close while going from one value of c to the next. Note that $c = \lceil 3.1q \rceil$, where q is any value above a fixed threshold, and $\lceil 3.1(q+1) \rceil - \lceil 3.1q \rceil \leqslant 4$. Thus it suffices to examine the values $n = dc + d + i$ where $i \leqslant 4d$, and we can suppose that $c \geqslant 5$. The graph $K_{cd+i}^{G_d[K_q]}$ contains a copy of $K_{cd}^{G_d[K_q]}$ induced by the functions with image in $\{1, 2, \ldots, cd\}$. For $j = cd+1, cd+2, \ldots, cd+i$, the constant functions g_j with image j are pairwise adjacent and each is adjacent to all the functions in $K_{cd}^{G_d[K_q]}$. Hence $\chi(K_{cd+i}^{G_d[K_q]}) \geqslant \chi(K_{cd}^{G_d[K_q]}) + i \geqslant cd+d+i$. For $i \leqslant (c-1)d$, we also have $\chi(G_d[K_q]) \geqslant cd+d+i$, so that $f(cd+d+i) \leqslant cd+i$. Altogether, the inequality $f(n+d) \leqslant n$ is established for all but finitely many values of n. Thus, $\lim_{n\to\infty} n - f(n) = \infty$. □

The gap between n and $f(n)$ proved in this section depends on the minimum number p of vertices of a girth 6 graph with fractional chromatic number at least $8d$. The best known upper bound for p to our knowledge is $p = O((d \log d)^4)$, which follows from a result of Krivelevich [8]. Using this result, one can show that for any $\epsilon > 0$, there is a constant a such that for sufficiently large n, $f(n) \leqslant n - a(\log n)^{1/4-\epsilon}$. Very recently, He and Wigderson [5] proved that for some $\epsilon \simeq 10^{-9}$, $f(n) < (1-\epsilon)n$ for sufficiently large n. The examples are again cases of Shitov's construction.

3 Stahl's conjecture

In the proof of Proposition 1(i), based on the fact that $\chi(K_c^{G_d[K_q]}) \geqslant c+1$, we have shown that $\chi(K_{cd}^{G_d[K_q]}) \geqslant cd+d$. In this section, we show that if a special case of a conjecture of Stahl on the multichromatic number of Kneser graphs is true, then $\chi(K_{cd}^{G_d[K_q]})$ is much larger.

Consider a proper colouring ϕ of the graph $K_{cd}^{G_d[K_q]}$ with x colours. Let A be a subset of $\{1,\dots,cd\}$ of cardinality c. Let R_A be the subgraph of $K_{cd}^{G_d[K_q]}$ induced by the functions with image contained in A. Then R_A is isomorphic to $K_c^{G_d[K_q]}$, so ϕ uses at least $c+1$ colours on R_A. Let $\psi(A) \subseteq \{1,\dots,x\}$ be a subset of exactly $c+1$ colours used by ϕ on R_A. We have $\psi(A)$ disjoint from $\psi(B)$ whenever A is disjoint from B, because R_A is totally joined to R_B in $K_{cd}^{G_d[K_q]}$. This property can be formulated in terms of homomorphisms of Kneser graphs. Recall that the vertices of the Kneser graph $K(m,n)$ are the n-subsets of $\{1,\dots,m\}$, and two of these are joined by an edge whenever they are disjoint. Thus the colouring $\phi : K_{cd}^{G_d[K_q]} \to K_x$ induces a homomorphism $\psi : K(cd,c) \to K(x,c+1)$. The question is how large does x need to be for such a homomorphism to exist.

Stahl's conjecture deals with the latter question. For an integer n, the *n-th multichromatic number* $\chi_n(H)$ of a graph H is the least integer m such that H admits a homomorphism to $K(m,n)$. In particular $\chi_1(H) = \chi(H)$. Lovász [9] proved that $\chi_1(K(m,n)) = \chi(K(m,n)) = m-2n+2$. Stahl [11] investigated the general multichromatic numbers of Kneser graphs, and observed the following.

a. For $1 \leqslant k \leqslant n$, $\chi_k(K(m,n)) = m-2(n-k)$,

b. $\chi_{kn}(K(m,n)) = km$,

c. $\chi_{k+k'}(K(m,n)) \leqslant \chi_k(K(m,n)) + \chi_{k'}(K(m,n))$.

Based on this he conjectured the following.

Conjecture 3 ([11])**.** If $k = an+b$, $a \geqslant 1, 0 \leqslant b \leqslant n-1$, then for $m \geqslant 2n$,

$$\chi_k(K(m,n)) = \chi_{an}(K(m,n)) + \chi_b(K(m,n)) = (a+1)m - 2(n-b).$$

Proof of Proposition 1 (ii). For a fixed d, let G_d have girth at least 6 and fractional chromatic number at least $8d$. For any q above a given threshold q_d and for $c = \lceil 3.1q \rceil$, we have $\chi(G_d[K_q]) \geqslant 2cd$ and $\chi\left(K_{cd}^{G_d[K_q]}\right) \geqslant \chi_{c+1}(K(cd,c))$, as explained in the first three paragraphs of this section. If Stahl's conjecture holds, then $\chi\left(K_{cd}^{G_d[K_q]}\right) \geqslant 2cd - 2c + 2$. Since f is monotonic, this gives $f(2cd-2c+2) \leqslant cd$. Therefore

$$n \in [2(c-4)d - 2(c-4) + 2, 2cd - 2c + 2] \text{ implies } \frac{f(n)}{n} \leqslant \frac{cd}{2(c-4)d - 2(c-4) + 2}.$$

The intervals $[2(c-4)d - 2(c-4)+2, 2cd-2c+2], c \in \mathbb{N}$ cover all but a finite part of $\mathbb{N}$. Hence

$$\limsup \frac{f(n)}{n} \leqslant \lim_{c\to\infty} \frac{cd}{2(c-4)d - 2(c-4) + 2} = \frac{d}{2d-2}.$$

Since this holds for arbitrarily large d, $\limsup \frac{f(n)}{n} \leqslant \frac{1}{2}$. $\square$

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