The maximum number of cycles
in a graph with fixed number of edges

Andrii Arman

School of Mathematics
Monash University
Melbourne, Australia

andrii.arman@monash.edu

Sergei Tsaturian

Department of Mathematics
University of Manitoba
Winnipeg, Canada

s.tsaturian@gmail.com

Abstract

The main problem considered in this paper is maximizing the number of cycles in a graph with given number of edges. In 2009, Király conjectured that there is constant c such that any graph with m edges has at most $c(1.4)^m$ cycles. In this paper, it is shown that for sufficiently large m, a graph with m edges has at most $(1.443)^m$ cycles. For sufficiently large m, examples of a graph with m edges and $(1.37)^m$ cycles are presented. For a graph with given number of vertices and edges an upper bound on the maximal number of cycles is given. Also, bounds tight up to a constant are presented for the maximum number of cycles in a multigraph with given number of edges, as well as in a multigraph with given number of vertices and edges.

Mathematics Subject Classifications: 05C35, 05C38

1 Introduction

Counting the number of cycles in a graph is a problem that was studied for different classes of the graphs: graphs with given cyclomatic number, planar graphs, 3-regular and 4-regular graphs, and many others. However, only a few general bounds for the number of cycles that use basic graph parameters are known. In this paper, bounds on the number of cycles in a graph as a function of the number of vertices and edges are presented.

Let $C(G)$ be the number of cycles in a graph G. In 1897, Ahrens [1] proved that for a graph G with n vertices, m edges and k components,

$$m - n + k \leqslant C(G) \leqslant 2^{m-n+k} - 1. \tag{1}$$

The lower bound in (1) is tight; for example, it is achieved by any disjoint union of cycles and trees. The tightness of the upper bound in (1) was shown by Mateti and Deo

[10] and the only graphs for which the upper bound is tight are the ones that can be obtained from subdivisions of K_3, K_4, $K_{3,3}$ and $K_4 - e$ by sequentially adding pendant edges. Aldred and Thomassen [2] improved the upper bound in (1) by showing that for a connected graph G,

$$C(G) \leqslant \frac{15}{16} 2^{m-n+1}. \tag{2}$$

Entringer and Slater [7] considered $C(G)$ for the class of connected graphs with fixed *cyclomatic number* $r = m - n + 1$. It follows from the results of [7] that there is a 3-regular connected graph G for which $C(G) > 2^{r-1}$. Shi [13] presented an example of an outer-planar 3-regular Hamiltonian graph G with $C(G) = 2^{r-1} + r - 1$.

Alt, Fuchs and Kriegel [3] and Aldred and Thomassen [2] studied $C(G)$ for the class of planar graphs, in terms of the number of vertices and in terms of the cyclomatic number, respectively. Arman, Gunderson and Tsaturian [5] studied $C(G)$ for the class of triangle-free graphs on n vertices, and recently, Morrison, Roberts and Scott [11] studied $C(G)$ for the class of H-free graphs, where H contains a colour-critical edge.

Király [9] investigated $C(G)$ for several classes of graphs: the union and the sum of two trees, 3-regular and 4-regular graphs, and graphs with average degree 4. Király also conjectured that there is a constant c, such that for any graph G that has m edges,

$$C(G) \leqslant c(1.4)^m.$$

We refer an interested reader to [4, 5, 14] for a more comprehensive literature review.

Motivated by Ahrens's bound (1) and by the work of Király, in this paper we further investigate $C(G)$ for two classes of graphs: those with n vertices and m edges, and those with m edges.

The main result of this paper is providing an estimate on $C(G)$ for graphs with given number of vertices and edges.

Theorem 1. *Let G be a multigraph with $n \geqslant 2$ vertices and m edges.*
If $\frac{m}{n-1} < 3$, then

$$C(G) < \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\frac{m}{n-1} \geqslant 3$, and $\lfloor \frac{m}{n-1} \rfloor = s$, $\alpha = \frac{m}{n-1} - s$, then

$$C(G) < \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^\alpha)^{n-1}.$$

A more compact, but slightly weaker bounds can be obtained as a corollary of Theorem 1:

$$C(G) \leqslant \begin{cases} \frac{3}{4}\Delta(G)(\frac{m}{n-1})^{n-1}, & \text{if } \frac{m}{n-1} \geqslant 3, \\ \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m, & \text{if } \frac{m}{n-1} < 3. \end{cases} \tag{3}$$

The bound in (3) is better than the one in (2) for graphs with sufficiently large number of edges and average degree at least 4.25. The main tool used to derive Theorem 1 is Lemma 4 that provides an estimate on the number of cycles that contain a given path.

For $m \in \mathbb{Z}^+$ let $C(m)$ be the maximum number of cycles in a graph with m edges. As a consequence of Theorem 1 we also obtain a bound for $C(m)$.

Corollary 2. *For any positive integer m,*

$$C(m) < 4.5(\sqrt[3]{3})^m.$$

Theorem 1 and Corollary 2 are proved in Section 2.

In Section 3, for m sufficiently large, a graph G with m edges is constructed, such that

$$C(G) \geqslant (2 + \sqrt{8})^{\frac{m}{5}-1} \geqslant 1.37^m.$$

Corollary 2 and the result of Section 3 imply that for m large enough,

$$1.37^m \leqslant C(m) \leqslant 1.443^m. \tag{4}$$

The lower and upper bound in (4) differ exponentially, which motivated us to question the sharpness of Lemma 4. Although this lemma might not be sharp for counting cycles in graphs, we show in Section 4 that Lemma 4 provides exponentially sharp bound on the number of cycles in multigraphs. It is shown (Theorem 8) that if G is a multigraph that has the most cycles among all loopless multigraphs with m edges, then

$$\frac{9}{10}(\sqrt[3]{3})^m \leqslant C(G) \leqslant 4.5(\sqrt[3]{3})^m.$$

Finally, in Appendix A it is shown that extremal graphs for the function $C(m)$ have bounded degrees. Namely, it is shown (Theorem 10) that if G is a graph with m edges with $C(G) = C(m)$, then the maximum degree of G is at most 11.

2   Main results

For $k \in \mathbb{Z}^+$, denote $\{i \in \mathbb{Z}; 1 \leqslant i \leqslant k\}$ by $[k]$, and for a set S, denote $\{T \subseteq S : |T| = k\}$ by $[S]^k$. A multigraph in this paper is an ordered triplet $G = (V, E, t)$, where V is a set of vertices, E is a set of edges and t is a function that maps E to $[V]^2$ ($t(e)$ indicates which vertices are incident to $e \in E$). Note that under this notation the edge set of a multigraph is indeed a set (not a multiset). The degree $deg_G(V)$ of a vertex $v \in V(G)$ is the number of edges incident to v. For two vertices $u, v \in V(G)$, denote by $E(u, v)$ the set of all edges between u and v. For a vertex $v \in V(G)$, denote by $N(v)$ the set of all vertices connected with v by at least one edge. Denote the average degree of a graph (or a multigraph) G by $d(G)$, the maximum degree by $\Delta(G)$, and the minimum degree by $\delta(G)$. A cycle in a multigraph G is a set of $k \geqslant 2$ distinct vertices and k distinct edges $\{v_1, e_1, v_2, e_2, \ldots, e_k, v_1\}$, where for each $i \in [k]$, $v_i \in V(G), e_i \in E(G)$ and any consecutive

vertex and edge are incident. As in the case of simple graphs, denote the number of cycles in a multigraph G by $C(G)$.

In order to prove Theorem 1, we introduce a counting Lemma 4 that bounds the number of cycles in a multigraph through a given path by essentially a number of ways to extend that path to a longer one. Lemma 4 is subsequently used to obtain Lemma 5 which provides an estimate on the number of cycles through a given vertex in a graph. We then use Lemma 5 in combination with induction to prove Theorem 1. Finally, we establish Corollary 2 as a quick consequence of Theorem 1.

Definitions 3. *Let G be a multigraph with n vertices.*
Let $P = v_1 e_1 v_2 \ldots e_{k-1} v_k$ be a path in G. Define

$$f(P) = \max\{deg_{G-\{v_2,\ldots,v_{k-1}\}}(v_k), 1\},$$

i.e. $f(P)$ is equal to the number of ways to select the next vertex on a cycle that starts with P, or equal to 1 if P cannot be extended to a cycle. Denote the number of cycles in G that contain P by $C(P)$.

For two paths P_1 and P_2 with the same starting vertex, write $P_1 \subseteq P_2$ if P_1 is a subpath of P_2 (i.e. $P_1 = v_1 e_1 v_2 \ldots e_{k-1} v_k, P_2 = v_1 e_1 v_2 \ldots e_{k-1} v_k e_k \ldots e_{\ell-1} v_\ell$). We write $P_1 \subset P_2$ if $P_1 \subseteq P_2$ and $P_1 \neq P_2$. In this case, define $f(P_1, P_2) = \prod_{j=k}^{\ell} f(v_1 e_1 \ldots v_j)$.

Lemma 4. *Let G be a multigraph with $n \geqslant 2$ vertices, $k \in [n]$, and $P = v_1 e_1 v_2 e_2 \ldots v_k$ be a path in G. Then*

$$C(P) \leqslant \max_{P': P \subseteq P'} f(P, P').$$

Proof. For a path $P = v_1 e_1 v_2 \ldots e_{k-1} v_k$ additionally define

$$F(P) = \{P' : P' \text{ is a path of form } P e_k v_{k+1}\},$$

i.e. $F(P)$ is the set of paths that extend P by exactly one edge.

Fix $n \geqslant 2$. Let G be a multigraph on n vertices. The proof is by downwards induction on k.
Base case. Let $k = n$. Let $P = v_1 e_1 \ldots v_n$ be a path in G, $C(P)$ in this case is equal to the number of edges between v_n and v_1 and $\max_{P \subseteq P'} f(P, P') = f(P)$. $f(P)$ is at least the number of edges between v_n and v_1, hence this proves the base case.
Inductive step. Let $\ell \in [n-1]$. Assume that the statement of the lemma holds for $k = \ell + 1$, and prove it for ℓ, i.e. let $P = v_1 e_1 \ldots v_\ell$, and we seek to bound $C(P)$.
Let s be the number of edges between v_ℓ and v_1. Then

$$C(P) = s + \sum_{P' \in F(P)} C(P').$$

If $F(P) = \emptyset$ then $C(P) = s \leqslant f(P)$ and hence the inductive step follows. Now assume that $F(P) \neq \emptyset$. For all possible choices of $P' \in F(P)$, according to inductive hypothesis,

$$C(P') \leqslant \max_{P'': P' \subseteq P''} f(P', P'').$$

Therefore,

$$
\begin{aligned}
C(P) &\leqslant s + (f(P) - s) \max_{P' \in F(P)} C(P') \\
&\leqslant s + (f(P) - s) \max_{P' \in F(P)} \max_{P'' : P' \subseteq P''} f(P', P'') \\
&\leqslant f(P) \max_{P' \in F(P)} \max_{P'' : P' \subseteq P''} f(P', P'') \\
&= \max_{P' \in F(P)} \max_{P'' : P' \subseteq P''} f(P) f(P', P'') \\
&= \max_{P' \in F(P)} \max_{P'' : P' \subseteq P''} f(P, P'') = \max_{P'' : P \subseteq P''} f(P, P'').
\end{aligned}
$$

This proves that the statement of the lemma holds for ℓ, and therefore by induction it holds for all $k \in [n]$. □

Lemma 5. *Let G be a multigraph with $n \geqslant 3$ vertices and m edges, and let v_1 be a vertex in G of degree $\Delta(G)$.*

- *If $\frac{m}{n-1} \geqslant 3$, and $\lfloor \frac{m}{n-1} \rfloor = s$, $\frac{m}{n-1} - s = \alpha$, then there are at most $\frac{\Delta(G)}{2}(s^{1-\alpha}(s+1)^\alpha)^{n-1}$ cycles in G that contain v_1.*

- *If $\frac{m}{n-1} < 3$, then there are at most $\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m$ cycles in G that contain v_1.*

Proof. Let G be a multigraph with $n \geqslant 3$ vertices and m edges, and v_1 be a vertex with degree $\Delta(G)$.

For any edge $e = v_1 v_2$ incident to v_1, by Lemma 4, the number of cycles that contain e is at most

$$
\max_{P' : e \subseteq P'} f(e, P').
$$

Every cycle through v_1 contains two such edges, therefore the number of cycles that contain v_1 is at most

$$
\frac{\Delta}{2} \max_{e = v_1 v_2} \max_{P' : e \subseteq P'} f(e, P'). \tag{5}
$$

Let $e = v_1 v_2$ and $P' = v_1 v_2, \ldots v_t$ be an edge and a path that give the maximum in (5) with the smallest possible t. For $2 \leqslant i \leqslant t$ define $f_i = f(v_1 \ldots v_i)$, so

$$
\frac{\Delta}{2} \max_{e = v_1 v_2} \max_{P' : e \subseteq P'} f(e, P') = \frac{\Delta}{2} f_2 \cdot \ldots \cdot f_t.
$$

Then $f_t \geqslant 2$ (otherwise remove all $f_i = 1$ after the last $f_k \geqslant 2$ to obtain the smaller collection of vertices that gives maximum in (5)). Then for all $2 \leqslant i \leqslant t$,

$$
f_i = deg_{G - \{v_2, \ldots, v_{i-1}\}}(v_i).
$$

For $2 \leqslant i \leqslant t$, all the edge sets $\{v_i u \in E(G) : u \in V(G) \backslash \{v_2, \ldots, v_i\}\}$ are mutually disjoint, so $f_2 + \cdots + f_t \leqslant m$. Therefore,

$$
\frac{\Delta}{2} f_2 \cdot \ldots \cdot f_t \leqslant \frac{\Delta}{2} \cdot \max_{\substack{2 \leqslant t \leqslant n \\ x_2 + \ldots + x_t \leqslant m, \\ \forall i \in [2,t], x_i \in \mathbb{Z}^+}} \{x_2 \cdot x_3 \cdot \ldots \cdot x_t\}.
$$

So the number of cycles in G that contain v_1 is at most

$$\frac{\Delta}{2} \cdot \max_{\substack{2 \leqslant t \leqslant n \\ x_2 + \ldots + x_t \leqslant m, \\ \forall i \in [2,t], x_i \in \mathbb{Z}^+}} \{x_2 \cdot x_3 \cdot \ldots \cdot x_t\}. \tag{6}$$

For a fixed t the product $x_2 \ldots x_t$ in (6) obtains its maximum when x_is ($i \geqslant 2$) are as equal as possible (for all i, j $|x_i - x_j| \leqslant 1$), and their sum is equal to m. Let $\lfloor \frac{m}{n-1} \rfloor = s$, $\frac{m}{n-1} = s + \alpha$.
If $s \geqslant 3$ (which is equivalent to $\frac{m}{n-1} \geqslant 3$), let the maximum in (6) be achieved for some $t \leqslant n$ and let $x_2, \cdots, x_t$ be a collection of x_is that gives the maximum in (6). If $t < n$, then $s \geqslant 3$ implies that either for some $i \in [t]$, $x_i \geqslant 5$, or for two different $i, j \in [t]$, $x_i = x_j = 4$. In the first case replacing x_i by $x_i - 2$ and setting $x_{t+1} = 2$ gives a collection of x_i's with a bigger product. In the second case setting $x_i = x_j = 3$ and $x_{t+1} = 2$ increases the product of x_is. Hence, the maximum in (6) is achieved when $t = n$. For all $2 \leqslant i \leqslant n$, $x_i = s$ or $x_i = s + 1$. Then the number of cycles in G that pass through v_1 is at most

$$\frac{\Delta}{2} x_2 \ldots x_n = \frac{\Delta}{2} s^{(1-\alpha)(n-1)} (s+1)^{\alpha(n-1)} = \frac{\Delta}{2} (s^{1-\alpha}(s+1)^{\alpha})^{n-1}.$$

If $s < 3$, let the maximum of (6) be achieved for some $2 \leqslant t \leqslant n$ and let $x_2, \cdots, x_t$ be the collection of x_is that gives the maximum in (6). Recall that for all i, j $|x_i - x_j| \leqslant 1$. If for two different $i, j \in [t]$ $x_i = x_j > 3$, then $m > 6 + 3(t-2) = 3t$, and $s < 3$ implies that $t < n$. Replacing x_i by $x_i - 1$, x_j by $x_j - 1$ and setting $x_{t+1} = 2$ increases the product. Therefore, there is at most one i, such that $x_i = 4$. If there is i such that $x_i = 1$, then replacing any x_j ($j \neq i$) by $x_j + 1$ and deleting x_i increases the product. If for some $i, j, k \in [t]$ $x_i = x_j = x_k = 2$, then replacing x_i by 3, x_j by 3 and deleting x_k increases the product. Therefore, $\{x_2, \ldots, x_t\} \in \{\{3, 3, \ldots, 3, 2, 2\}, \{3, 3, \ldots, 3, 4\}, \{3, 3, \ldots, 3, 2\}, \{3, 3, \ldots, 3\}\}$. Then $x_2 \ldots x_t$ is at most $3^{\frac{m}{3}}$, so the number of cycles that pass through v_1 is at most

$$\frac{\Delta}{2} x_2 \ldots x_t \leqslant \frac{\Delta}{2} 3^{\frac{m}{3}}. \qquad \square$$

Proof of Theorem 1. The proof is by mathematical induction on n.
Base case. If $n = 2$, there is only one multigraph on n vertices with m edges – two vertices connected by m edges. In this case $s = \frac{m}{n-1} = m$, and G has $\max\{\binom{m}{2}, 0\}$ cycles, which is less than $\frac{3}{4} m (\sqrt[3]{3})^m$ (for the case $m < 3$), and less than $\frac{3}{4} m \cdot m$ (for the case $m \geqslant 3$).
Inductive step. Let $k \geqslant 3$ be an integer, and suppose that the statement of the theorem is proved for $n = k - 1$. Let G be a multigraph with k vertices, m edges and let v_1 be a vertex of maximal degree in G.
Case 1: Suppose that $\frac{m}{k-1} < 3$.
If $\Delta(G) \leqslant 2$, then every edge is contained in at most one cycle, and every cycle contains at least two edges, so the number of cycles in G is at most

$$\frac{m}{2} \leqslant \frac{3}{4} \Delta(G) \cdot (\sqrt[3]{3})^m.$$

If $\Delta(G) \geqslant 3$, then the multigraph $G - v_1$ has at most $m - 3$ edges, $\Delta(G - v_1) \leqslant \Delta(G)$ and $\frac{|E(G-v_1)|}{|V(G-v_1)|-1} \leqslant \frac{m}{k-1} < 3$, therefore, by inductive assumption, the number of cycles in $G - v_1$ is at most $\frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m-3}$. By Lemma 5, the number of cycles that contain v_1 is at most $\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m$, therefore the total number of cycles in G is at most

$$\frac{\Delta(G)}{2} \cdot (\sqrt[3]{3})^m + \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m-3} = \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^m.$$

Case 2: Now we consider the case $\frac{m}{k-1} \geqslant 3$.
Let $s = \lfloor \frac{m}{k-1} \rfloor$, $\alpha = \frac{m}{k-1} - \lfloor \frac{m}{k-1} \rfloor$. Note that $\Delta(G - v_1) \leqslant \Delta(G)$ and let

$$y = \frac{|E(G - v_1)|}{|V(G - v_1)| - 1} \leqslant \frac{m}{k - 1}.$$

Note that the function

$$f(x) = (\lfloor x \rfloor)^{1-x+\lfloor x \rfloor}(\lfloor x \rfloor + 1)^{x-\lfloor x \rfloor}$$

is non-decreasing on every interval $[a, a + 1], a \in \mathbb{Z}_{\geqslant 0}$ (and hence on $\mathbb{R}^+$), therefore

$$s^{1-\alpha}(s + 1)^\alpha \geqslant f(3) = 3. \tag{7}$$

If $y \geqslant 3$, then, by the induction hypothesis,

$$|E(G - v_1)| \leqslant \frac{3}{4}\Delta(G)((\lfloor y \rfloor)^{1-y+\lfloor y \rfloor}(\lfloor y \rfloor + 1)^{y-\lfloor y \rfloor})^{k-2}$$
$$\leqslant \frac{3}{4}\Delta(G)(s^{1-\alpha}(s + 1)^\alpha)^{k-2}.$$

If $y < 3$, then $|E(G - v_1)| < 3(k - 2)$, and by the induction hypothesis

$$|E(G - v_1)| \leqslant \frac{3}{4}\Delta(G)(\sqrt[3]{3})^{|E(G-v_1)|} < \frac{3}{4}\Delta(G)(\sqrt[3]{3})^{3(k-2)}$$
$$= \frac{3}{4}\Delta(G) \cdot 3^{k-2} \leqslant \frac{3}{4}\Delta(G)(s^{1-\alpha}(s + 1)^\alpha)^{k-2}.$$

Hence, for any y, $|E(G - v_1)| \leqslant \frac{3}{4}\Delta(G)(s^{1-\alpha}(s + 1)^\alpha)^{k-2}$, which together with Lemma 5 and (7) implies that

$$C(G) = \frac{3\Delta(G)}{4}(s^{1-\alpha}(s + 1)^\alpha)^{k-2} + \frac{\Delta(G)}{2}(s^{1-\alpha}(s + 1)^\alpha)^{k-1} \tag{8}$$
$$\leqslant \frac{3\Delta(G)}{4}(s^{1-\alpha}(s + 1)^\alpha)^{k-1}, \tag{9}$$

which proves the inductive step and hence the theorem. $\qquad\square$

Proof of Corollary 2. Let G be a graph with n vertices and m edges, such that $C(G) = C(m)$. First, prove the statement of the corollary for the case when $\Delta(G) \leqslant 6$. Suppose that $\frac{m}{n-1} \geqslant 3$. Let $f(s, \alpha) = (s^{1-\alpha}(s + 1)^\alpha)^{\frac{1}{s+\alpha}}$, then for any $s > 0$, $f(s, \alpha)$ is monotone

in α and $\displaystyle\max_{s\in\mathbb{Z}_+,\alpha\in[0,1)} f(s,\alpha) = \max_{s\in\mathbb{Z}_+} s^{\frac{1}{s}} = \sqrt[3]{3}$. This, together with Theorem 1, implies that for $s = \lfloor \frac{m}{n-1} \rfloor$ and $\alpha = \frac{m}{n-1} - \lfloor \frac{m}{n-1} \rfloor$

$$C(m) = C(G) < \frac{3}{4}\Delta(G)((s^{1-\alpha}(s+1)^\alpha)^{\frac{1}{s+\alpha}})^m \leqslant 4.5(\sqrt[3]{3})^m.$$

If $\frac{m}{n-1} < 3$ then, by Theorem 1,

$$C(m) = C(G) < \frac{3}{4}\Delta(G)(\sqrt[3]{3})^m \leqslant 4.5(\sqrt[3]{3})^m.$$

The proof for the case $\Delta(G) \geqslant 7$ goes by induction on m. Let v be a vertex of degree at least 7 in G, then there exists an edge e, incident to v, such that there are at most $\frac{2}{7}C(G)$ cycles in G that contain e. Hence $C(G) \leqslant \frac{5}{7}C(G-e)$, and by inductive hypothesis, $C(G-e) \leqslant 4.5(\sqrt[3]{3})^{m-1}$. Therefore

$$C(G) \leqslant \frac{5}{7}4.5(\sqrt[3]{3})^{m-1} \leqslant 4.5(\sqrt[3]{3})^m. \qquad \square$$

In Appendix A it is shown that extremal graphs for the function $C(m)$ have bounded degrees. Theorem 10 states that if G is a graph with m edges with $C(G) = C(m)$, then the maximum degree of G is at most 11.

3   Example of a graph with $(1.37)^m$ cycles

In this section we present an example of a graph with m edges that has at least $(1.37)^m$ cycles. The previous best known bound is due to Alt, Fuchs and Kriegel [3] who provided an example of a planar graph with 1.34^m cycles.

For $n \geqslant 1$ let H_n be the graph on $2n+2$ vertices with

$$V(H_n) = \{u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots v_{n+1}\} \qquad \text{and}$$

$$E(H_n) = \{u_i v_j : i, j \in [n+1], |i-j| \leqslant 1\} \cup \{u_i u_{i+1} : i \in [n]\} \cup \{v_i v_{i+1} : i \in [n]\}.$$
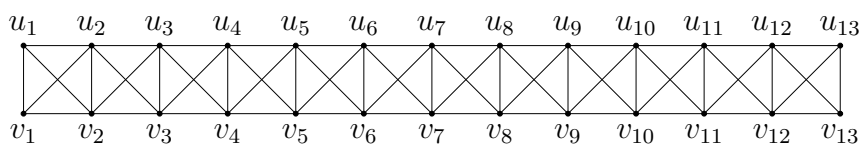

Figure 1: Graph H_{12}.

For $n \geqslant 1$ denote by $P(n)$ the number of paths from the vertex u_1 to the vertex u_{n+1} in H_n. Note that $P(n)$ is also equal to the number of paths from u_1 to v_{n+1} in H_n.

Claim 6. *For all $n \geqslant 2$*

$$P(n) = 4P(n-1) + 4P(n-2).$$

Proof sketch. The proof of the claim relies on an inductive argument and an observation that each path from u_1 to u_{n+1} in H_n corresponds to exactly one of the following paths:

- path from u_1 to u_n in H_{n-1} followed by the path $u_n u_{n+1}$ or by the path $u_n v_{n+1} u_{n+1}$.

- path from u_1 to v_n in H_{n-1} followed by the path $v_n u_{n+1}$ or by the path $v_n v_{n+1} u_{n+1}$.

- path from u_1 to u_{n-1} in H_{n-2} followed by the path $u_{n-1} u_n v_{n+1} v_n u_{n+1}$ or by the path $u_{n-1} v_n v_{n+1} u_n u_{n+1}$.

- path from u_1 to v_{n-1} in H_{n-2} followed by the path $v_{n-1} u_n v_{n+1} v_n u_{n+1}$ or by the path $v_{n-1} v_n v_{n+1} u_n u_{n+1}$. $\qquad\square$

Solving the recurrence relation leads to the inequality

$$P(n) \geqslant (2 + 2\sqrt{2})^n.$$

Define the graph G_n by identifying vertices u_1 and u_{n+1} in H_n. Then G_n has $2n+1$ vertices, $m = 5n + 1$ edges and

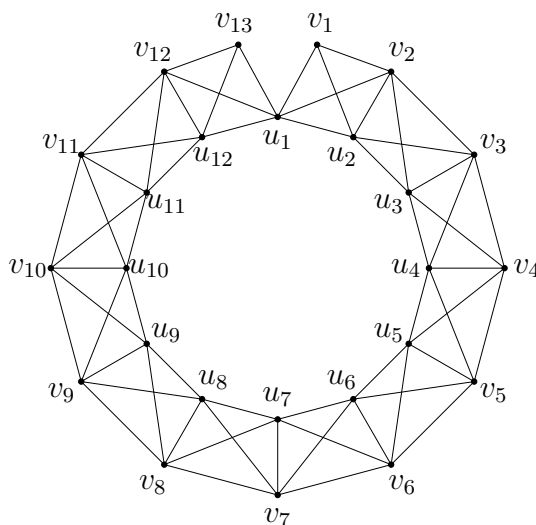$$C(G_n) \geqslant P(n) \geqslant (2 + 2\sqrt{2})^n.$$

Figure 2: G_{12} with 25 vertices and 61 edges.

For an integer m let graph G be obtained from $G_{\lfloor \frac{m-1}{5} \rfloor}$ by adding $(m - 5\lfloor \frac{m-1}{5} \rfloor - 1)$ edges. Then G has m edges and for m large enough

$$C(G) \geqslant C(G_{\lfloor \frac{m-1}{5} \rfloor}) \geqslant (2 + 2\sqrt{2})^{\lfloor \frac{m-1}{5} \rfloor} \geqslant (2 + 2\sqrt{2})^{\frac{m}{5} - 1} > 1.37^m.$$

4 Maximum number of cycles in multigraphs

The problems of maximizing the number of cycles with fixed number of edges or fixed average degree can also be considered for multigraphs. Using the techniques presented in this paper, the authors can prove the following two results.

Theorem 7. *Let G be a multigraph that has the maximum number of cycles among all the multigraphs with $n \geqslant 2$ vertices and $m \geqslant 3$ edges. Let $\lfloor \frac{m}{n-1} \rfloor = s$, $\alpha = \frac{m}{n-1} - s$. If $\frac{m}{n-1} \geqslant 3$, then*

$$\frac{8}{27}s(s^{1-\alpha}(s+1)^{\alpha})^{n-1} \leqslant C(G) \leqslant \frac{3}{4}\Delta(G)(s^{1-\alpha}(s+1)^{\alpha})^{n-1}.$$

If $\frac{m}{n-1} \leqslant 3$, then

$$4(\sqrt[3]{3})^{m-4} \leqslant C(G) < \frac{3}{4}\Delta(G) \cdot (\sqrt[3]{3})^{m}.$$

The upper bounds in Theorem 7 follow from Theorem 1. For the lower bounds, define $C_{n,m}$ to be the multigraph obtained from the cycle C_n by replacing each of some $m - \lfloor \frac{m}{n} \rfloor n$ consecutive edges with edges of multiplicity $\lfloor \frac{m}{n} \rfloor + 1$ and the rest $\lfloor \frac{m}{n} \rfloor n - m + n$ edges with edges of multiplicity $\lfloor \frac{m}{n} \rfloor$. The lower bound in the first case is achieved by the graph $C_{n,m}$. The lower bound in the second case is achieved by the graph $C_{\lfloor \frac{m+1}{3} \rfloor, m}$ with extra $n - \lfloor \frac{m+1}{3} \rfloor$ isolated vertices. For a detailed proof of Theorem 7 we refer an interested reader to [4, 14].

Theorem 8. *Let G be a multigraph with $m \geqslant 3$ edges that has the maximum number of cycles among all the multigraphs with m edges. Then*

$$\frac{9}{10}(\sqrt[3]{3})^{m} < 4(\sqrt[3]{3})^{m-4} \leqslant C(G) \leqslant 4.5(\sqrt[3]{3})^{m}$$

The upper bound in Theorem 8 can be obtained by repeating the argument of Corollary 2 and a version of Theorem 10, modified for multigraphs. The example for the lower bound is the same as for the second case of Theorem 7.

Theorems 7 and 8 answer both questions for multigraphs up to a constant factor. The authors believe that for $m \geqslant 9$ the graph $C_{\lfloor \frac{m+1}{3} \rfloor, m}$ has the most cycles among all multigraphs with m edges.

5 Concluding remarks

Theorem 1 gives an upper bound for the number of cycles in a graph G with n vertices and m edges. For a graph G with n vertices and average degree $d \geqslant 6$, Theorem 1 and AM-GM imply

$$C(G) \leqslant 3\Delta(G) \left(\frac{d}{2}\right)^{n}.$$

For $d = \Omega(\ln n)$, let G be a random graph $G(n, p)$ with $p = \frac{d}{n-1}$. Glebov and Krivelevich [8] proved that the number of hamiltonian cycles in G is a.a.s. at least $\left(\frac{d}{e}\right)^n (1 + o(1))^n$. It was shown in [4, 14] that, provided $d \to \infty$, a random $G(n, m)$ graph has $\left(\frac{d}{e}\right)^n (1 + o(1))^n$ cycles in expectation. Therefore, if G is a graph with the maximal number of cycles among all graphs with n vertices and average degree d (with $d \to \infty$), then for n large enough

$$\left(\frac{d}{e}\right)^n (1 + o(1))^n \leqslant C(G) \leqslant (1 + o(1))^n \left(\frac{d}{2}\right)^n.$$

In an earlier version of this paper, we conjectured that $C(G) \leqslant \left((1 + o(1))\frac{d}{e}\right)^n$. This was shown to be true by Noel [12], who used Bregman's theorem to show that

$$C(G) \leqslant (d + 1)!^{n/(d+1)} = (1 + o(1))^n (d/e)^n.$$

We note that the bound in Theorem 1 is exponentially better than the one of Noel for the case when $d < 9.2411$.

As mentioned in the introduction, Theorem 2 and the result of Section 3 imply that

$$1.37^m \leqslant C(m) \leqslant 1.443^m.$$

Király [9] conjectured that $C(m) < 1.4^m$. The upper bound in Corollary 2 is $4.5(\sqrt[3]{3})^m$, which inspired the following conjecture.

Conjecture 9. For sufficiently large m, there exists a graph G with m edges and at least $(1 + o(1))^m (\sqrt[3]{3})^m$ cycles.

Acknowledgements

Added in proof:

After submission of this paper Dvořák, Morrison, Noel, Norin and Postle [6] have substantially improved Theorem 1 and disproved Conjecture 9. Their proof is based on bounding the number of paths between two vertices in a graph by a function of degree sequence that is much more intricate, compared to the one used in this paper.

During the preparation of this paper, Morrison, Roberts and Scott [11] have considered the problem of maximising $C(G)$ for the class of H-free graphs. For the class of H-free graphs they obtain bounds that are exponentially better than the ones in Theorem 1, provided m is close to the Turán number of H.

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Appendix A Maximal degree of graphs with $C(m)$ cycles

Recall that, for $m \in \mathbb{Z}^+$, $C(m)$ is the maximum number of cycles in a graph with m edges.

Theorem 10. *If G is a graph with m edges such that $C(G) = C(m)$, then $\Delta(G) \leqslant 11$.*

The proof of Theorem 10 relies on the following two lemmas.

Lemma 11. *Let $k \geqslant 6$ be a positive integer. For $1 \leqslant i < j \leqslant k$, let $w_{i,j}$ be a non-negative real number, and let $S = \sum_{1 \leqslant i < j \leqslant k} w_{i,j}$. Then there exists a 6-element set $D \subseteq [k]$ such that*

$$\sum_{\substack{1 \leqslant i < j \leqslant k \\ i \notin D, j \notin D}} w_{i,j} \geqslant \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.$$

Proof. The proof relies on an averaging argument. For each $i \in [k]$ set $w_i = \sum_{j \in [k], j \neq i} w_{i,j}$. Note that

$$\sum_{i \in [k]} w_i = 2S.$$

Let X be a collection of all 6-element subsets of $[k]$. For $D \in X$ let

$$S(D) = \sum_{\substack{1 \leqslant i < j \leqslant k \\ i \notin D, j \notin D}} w_{i,j}$$

$$= S - \sum_{i \in D} w_i + \sum_{i,j \in D, i < j} w_{i,j}.$$

Let $\overline{S(D)}$ be the average of $S(D)$ over all $D \in X$, then

$$\overline{S(D)} = \frac{\sum_{D \in X} \left(S - \sum_{i \in D} w_i + \sum_{i,j \in D, i < j} w_{i,j}\right)}{\binom{k}{6}}$$

$$= S - \frac{\binom{k-1}{5} \cdot 2S}{\binom{k}{6}} + \frac{\binom{k-2}{4} \cdot S}{\binom{k}{6}}$$

$$= \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S.$$

There exists $D \in X$, such that $S(D) \geqslant \overline{S(D)}$, i.e.

$$\sum_{\substack{1 \leqslant i < j \leqslant k \\ i \notin D, j \notin D}} w_{i,j} \geqslant \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \qquad \square$$

Lemma 12. *Let $k \geqslant 2$ be a positive integer. For $1 \leqslant i < j \leqslant k$, let $w_{i,j}$ be a non-negative real number, and let $S = \sum_{1 \leqslant i < j \leqslant k} w_{i,j}$. Then there exists a partition $A_1 \cup A_2 \cup A_3 \cup A_4 = [k]$, such that*

$$\sum_{\substack{1 \leqslant l < m \leqslant 4}} \sum_{\substack{i \in A_l \\ j \in A_m}} w_{i,j} \geqslant \left(\frac{3k^2 - 4}{4k(k-1)} \right) S.$$

Proof. For all $l \in [4]$ let $a_l = \lfloor \frac{k+l-1}{4} \rfloor$ (note that $a_1 + a_2 + a_3 + a_4 = k$). Let X be the collection of all ordered quadruples (A_1, A_2, A_3, A_4), such that $\pi = A_1 \cup A_2 \cup A_3 \cup A_4$ is a partition of $[k]$ and for all $l \in [4]$, $|A_l| = a_l$. Note that

$$|X| = \frac{k!}{a_1! a_2! a_3! a_4!}.$$

For $p = (A_1, A_2, A_3, A_4) \in X$ define

$$S(p) = \sum_{\substack{1 \leqslant l < m \leqslant 4}} \sum_{\substack{i \in A_l \\ j \in A_m}} w_{i,j} = S - \sum_{l \in [4]} \sum_{\substack{i < j \\ i,j \in A_l}} w_{i,j}.$$

Let $\overline{S(p)}$ be the average of $S(p)$ over all possible choices of p.

$$\overline{S(p)} = \frac{\sum_{p \in X} (S - \sum_{l \in [4]} \sum_{i,j \in A_l, i<j} w_{i,j})}{|X|}$$

$$= S - \frac{\sum_{l \in [4]} \sum_{1 \leqslant i < j \leqslant k} \sum_{p \in X : i,j \in A_l} w_{i,j}}{|X|}$$

Note that for any choice of $l \in [4]$ and any choice of i, j, such that $1 \leqslant i < j \leqslant k$ there are exactly

$$\frac{(k-2)!(a_l)(a_l - 1)}{a_1! a_2! a_3! a_4!}$$

quadruples $p \in X$, such that $i, j \in A_l$. Then,

$$\overline{S(p)} = S - (\sum_{l \in [4]} \sum_{1 \leqslant i < j \leqslant k} \frac{(k-2)!(a_l)(a_l - 1)}{a_1! a_2! a_3! a_4!} w_{i,j}) / |X|$$

$$= S - (\sum_{l \in [4]} \frac{(k-2)!(a_l)(a_l - 1)}{a_1! a_2! a_3! a_4!} \cdot S) \cdot \frac{1}{|X|}$$

$$= S - (\sum_{l \in [4]} \frac{\lfloor \frac{k+l-1}{4} \rfloor (\lfloor \frac{k+l-1}{4} \rfloor - 1)}{k(k-1)}) \cdot S$$

$$\geqslant S \left(1 - \frac{(k-2)^2}{4k(k-1)} \right).$$

There exists a $p = (A_1, A_2, A_3, A_4) \in X$, such that $S(p) \geqslant \overline{S(p)}$, therefore the partition $A_1 \cup A_2 \cup A_3 \cup A_4$ satisfies the statement of Lemma 12. □

Proof of Theorem 10. Let m be a positive integer and G be a graph with m edges. To prove Theorem 10, it is sufficient to prove that if $\Delta(G) \geqslant 12$, then there is a graph H with m edges and with $C(H) > C(G)$.

Let $\Delta(G) \geqslant 12$ and u be a vertex of maximal degree in G. Let $N(u) = \{u_1, u_2, \ldots, u_k\}$ be the neighbourhood of u (note that $k \geqslant 12$). For $1 \leqslant i < j \leqslant k$, define $w_{i,j}$ to be the number of paths from the vertex u_i to the vertex u_j in the graph $G - u$. Then the number of cycles in graph G that pass through vertex u is $S = \sum_{1 \leqslant i < j \leqslant k} w_{i,j}$. By Lemma 11, there is a 6-element set $D = \{i_1, i_2, \ldots, i_6\}$, such that

$$\sum_{\substack{1 \leqslant i < j \leqslant k \\ i \notin D, j \notin D}} w_{i,j} \geqslant \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{10}$$

Suppose, upon re-indexing, that $D = \{k - 5, k - 4, \ldots, k - 1, k\}$. Applying Lemma 12 to the collection of real numbers $w_{i,j}$ with $1 \leqslant i < j \leqslant k - 6$ gives a partition $A_1 \cup A_2 \cup A_3 \cup A_4 = [k - 6]$ with

$$\sum_{1 \leqslant l < m \leqslant 4} \sum_{\substack{i \in A_l \\ j \in A_m}} w_{i,j} \geqslant \left(\frac{3(k-6)^2 - 4}{4(k-6)(k-7)}\right) \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S. \tag{11}$$

For $i \in [4]$, let $U_i = \{u_j \; : \; j \in A_i\}$. Construct a graph H by deleting u and all of the edges incident to u, adding four new vertices v_1, v_2, v_3, v_4, then for all $1 \leqslant i \leqslant 4$ adding edges from v_i to each vertex of U_i, and for all $1 \leqslant i < j \leqslant 4$ adding edges $v_i v_j$ (see Figure 3). Then $|E(H)| = |E(G)|$.
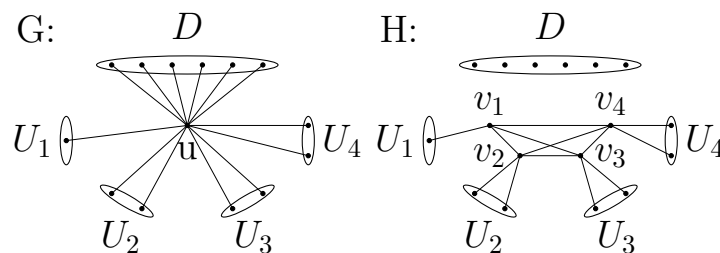

Figure 3: Constructing graph H from G.

To count the number of cycles in H, note the following:

- Every cycle in G that does not pass through the vertex u is still a cycle in H. There are $C(G) - S$ such cycles.

- Let C be a cycle in G that for some $1 \leqslant i < j \leqslant k - 6$ contains a path $u_i u u_j$. If for some $l \in [4]$ u_i and u_j are in the same class U_l, then C corresponds to the cycle in H that uses the path $u_i v_l u_j$ instead of $u_i u u_j$. In the case if $u_i \in U_l$ and $u_j \in U_m$ for some $1 \leqslant l < m \leqslant 4$, cycle C corresponds to the cycle that uses the path $u_i v_l v_m u_j$ instead of $u_i u u_j$. By (10), there are at least

$$\left(1 - \frac{6(2k-7)}{k(k-1)}\right) S$$

cycles in G that use path $u_i u u_j$ with $u_i, u_j \in N(u) \backslash D$.

- Every cycle in G that for some $i \in A_l$ and $j \in A_m$ with $l \neq m$ contains a path $u_i u u_j$ gives rise to additional 4 cycles in H(except the one containing $u_i v_l v_m u_j$). For example, if $l = 1$, $m = 2$ the four new cycles contain paths $u_i v_1 v_3 v_2 u_j$, $u_i v_1 v_4 v_2 u_j$, $u_i v_1 v_3 v_4 v_2 u_j$ and $u_i v_1 v_4 v_3 v_2 u_j$ instead of $u_i u u_j$. According to (11), there are at least

$$\left(\frac{3(k-6)^2 - 4}{4(k-6)(k-7)}\right) \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S = \left(\frac{3k^2 - 36k + 104}{4k(k-1)}\right) S$$

cycles in G that for some $i \in A_l$ and $j \in A_m$ with $l \neq m$ pass through a path $u_i u u_j$.

- There are 7 new cycles in H spanned by the vertices v_1, v_2, v_3, v_4.

By all of the observations above, the number of cycles in H is

$$
\begin{aligned}
C(H) &\geqslant C(G) - S + \left(1 - \frac{6(2k-7)}{k(k-1)}\right) S + 4 \left(\frac{3k^2 - 36k + 104}{4k(k-1)}\right) S + 7 \\
&> C(G) + 7 + S \left(\frac{3(k-4)(k-12)}{k(k-1)}\right) \\
&> C(G).
\end{aligned}
$$

Therefore, H has more cycles than G. $\qquad \square$

Note, that for $m = 7$ the graphs that have the most cycles are K_4 plus an edge and K_4 with one edge replaced by a path of length two. In the first case minimum degree is one, in the second case – two.

The authors can also prove the following theorem (for the proof, see [4]).

Theorem 13. *If $m > 7$ and G is a connected graph with $C(G) = C(m)$, then $\delta(G) \geqslant 3$.*