On the Schur function expansion of a symmetric quasi-symmetric function

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Abstract

Egge, Loehr, and Warrington proved a formula for the Schur function expansion of a symmetric function in terms of its expansion in fundamental quasi-symmetric functions. Their formula involves the coefficients of a modified inverse Kostka matrix. Recently Garsia and Remmel gave a simpler reformulation of Egge, Loehr, and Warrington's result, with a new proof. We give here a simple proof of Garsia and Remmel's version, using a sign-reversing involution.

Mathematics Subject Classifications: 05E05

Egge, Loehr, and Warrington [1] proved a formula, involving the coefficients of a modified inverse Kostka matrix, for the Schur function expansion of a symmetric function in terms of its expansion as a linear combination of fundamental quasi-symmetric functions. We recall that for a composition $L = (L_1, \dots, L_k)$, the fundamental quasi-symmetric function F_L is defined by

$$F_L = \sum_{i_1,\dots,i_k} x_{i_1} \cdots x_{i_k},$$

where the sum is over all positive integers $i_1, \dots, i_k$ satisfying $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k$ and $i_j < i_{j+1}$ if $j \in \{L_1, L_1 + L_2, \dots, L_1 + L_2 + \cdots + L_{k-1}\}$.

Garsia and Remmel [2] gave a simpler reformulation of Egge, Loehr, and Warrington's result. For any composition L, we define the Schur function s_L by the Jacobi-Trudi determinant of complete symmetric functions: $s_L = \det(h_{L_i-i+j})$, where h_k is the complete symmetric function and $h_k = 0$ for $k < 0$. As explained below, for every composition L, s_L is either an ordinary Schur function (indexed by a partition), the negative of an ordinary Schur function, or zero.

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Garsia and Remmel's reformulation is that if f is symmetric and $f = \sum_L c_L F_L$, then $f = \sum_L c_L s_L$. (There will usually be some cancellation in this formula.)

We give here a short combinatorial proof of Garsia and Remmel's reformulation. By linearity, it is sufficient to prove the formula for the case in which f is a Schur function. We will show that for any partition λ, if $s_\lambda = \sum_L c_L F_L$ then $s_\lambda = \sum_L c_L s_L$.

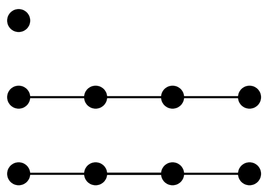
For example, if $\lambda = (4,1)$ then $s_\lambda = F_{(4,1)} + F_{(3,2)} + F_{(2,3)} + F_{(1,4)}$. We have $s_{(2,3)} = 0$ and $s_{(1,4)} = -s_{(3,2)}$, so

$$s_{(4,1)} + s_{(3,2)} + s_{(2,3)} + s_{(1,4)} = s_{(4,1)} + s_{(3,2)} + 0 - s_{(3,2)} = s_{(4,1)},$$

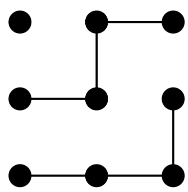
confirming the formula in this case.

Let $L = (L_1, \dots, L_k)$ be a composition. If $2 \leqslant i \leqslant n$ and $L_i \geqslant 2$, then we define the composition $L^{(i)}$ to be $(L_1, \dots, L_{i-2}, L_i - 1, L_{i-1} + 1, L_{i+1}, \dots, L_k)$. In other words, $L^{(i)}$ is obtained from L by replacing L_{i-1}, L_i with $L_i - 1, L_{i-1} + 1$, and leaving the other parts of L unchanged. It follows from the Jacobi-Trudi determinant that $s_{L^{(i)}} = -s_L$. Thus if $L^{(i)} = L$ then $s_L = 0$ and an easy induction argument shows that for any composition L, s_L is either 0 or $\pm s_\lambda$ for some partition λ. (This can also be shown by rearranging the rows of the Jacobi-Trudi determinant.)

The reduction of s_L to 0 or $\pm s_\lambda$ can be described graphically through the well-known "slinky rule," which we illustrate by example with $L = (1,4,4)$. We first draw a Ferrers diagram for L, with L_i dots in row i, for $i = 1, 2, \dots, k$ and connect the dots in each row with edges, forming chains:



(We are using English notation, so the first row is at the top.) Then for i from 2 to k, if the chain in row i extends beyond the dots in row $i-1$, we bend it upwards, along the preceding dots, but not going above row 1:



If we end up with the Ferrers diagram of a partition λ, then s_L is equal to $(-1)^v s_\lambda$, where v is the number of vertical steps in the final diagram. If a Ferrers diagram of a partition is not obtained, then $s_L = 0$. In our example with $L = (1,4,4)$, bending the second row corresponds to $s_{(1,4,4)} = -s_{(3,2,4)}$ and then bending the third row corresponds to $s_{(3,2,4)} = -s_{(3,3,3)}$, so $s_{(1,4,4)} = s_{(3,3,3)}$.

The expansion of a Schur function into fundamental quasi-symmetric functions is a well-known consequence of Richard Stanley's theory of P-partitions [3, Theorem 7.19.7,

p. 361]. A *descent* of a standard tableau T is an integer i such that $i+1$ appears in a lower row in T (in English notation) than i. Let the descents of the standard tableau T with entries $1, \ldots, n$ be $d_1 < d_2 < \cdots < d_s$. The *descent composition* of T, which we denote by $C(T)$, is the composition $(d_1, d_2 - d_1, \ldots, d_s - d_{s-1}, n - d_s)$ of n. Then for any partition λ, we have

$$s_\lambda = \sum_T F_{C(T)},$$

where the sum is over all standard tableaux T of shape λ. So we need to prove that

$$s_\lambda = \sum_T s_{C(T)}. \tag{1}$$

There is a unique standard tableau of shape λ with descent composition λ, called the *superstandard tableau*. It has entries $1, 2, \ldots, \lambda_1$ in the first row, entries $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$ in the second row, and so on. If T is superstandard of shape λ then $C(T) = \lambda$, so $s_{C(T)} = s_\lambda$.

We will define a shape-preserving involution θ on standard but not superstandard tableaux, with the property that $s_{C(\theta(T))} = -s_{C(T)}$. This property implies that if $\theta(T) = T$ then $s_{C(T)} = 0$. So in the sum on the right side of (1) everything cancels except the term corresponding to the superstandard tableau of shape λ, which contributes s_λ, thus proving (1).

Let T be a standard tableau with descent set $S = \{d_1 < d_2 < \cdots < d_s\}$. We define the *ith run* of T, for i from 1 to $s+1$, to be the skew subtableau of T consisting of the elements $d_{i-1} + 1, d_{i-1} + 2, \ldots, d_i$, where we set $d_0 = 0$ and $d_{s+1} = n$. Thus the number of elements in the ith run of T is the ith part in the descent composition of T.

For example, in the following tableau, the elements of the first run, 1, 2, 3, are colored red, the elements of the second run, 4, 5, 6, 7, are colored green, and the elements of the third run, 8, 9, are colored blue.

1	2	3	6	7	9
4	5				
8					

We define the involution θ first for tableaux with exactly two runs. If the tableau T has two runs then the shape of T has two parts. There are $\lambda_1 - \lambda_2 + 1$ standard tableaux of shape (λ_1, λ_2) with two runs. Each such tableau is uniquely determined by an integer j with $\lambda_2 \leqslant j \leqslant \lambda_1$ for which the first run contains $1, 2, \ldots, j$, all in the first row, and the second run contains $j+1$, $j+2$, $\ldots$, $\lambda_1 + \lambda_2$, with $j+1, j+2, \ldots, j+\lambda_2$ in the second row and $j + \lambda_2 + 1, \ldots, \lambda_1 + \lambda_2$ in the first row. Let T_j be this tableau, where $\lambda = (\lambda_1, \lambda_2)$ is fixed. Then the descent composition for T_j is $(j, n-j)$, where $n = |\lambda| = \lambda_1 + \lambda_2$.

The superstandard tableau of shape λ is T_{λ_1}. For $\lambda_2 \leqslant j \leqslant \lambda_1 - 1$, we define $\theta(T_j)$ to be the tableau with descent composition $(n-j-1, j+1)$; i.e., $\theta(T_j) = T_{n-j-1}$. To show that this is a valid definition, we must show that $\lambda_2 \leqslant n - j - 1 \leqslant \lambda_1 - 1$. For the first inequality we have $(n-j-1) - \lambda_2 = (\lambda_1 + \lambda_2 - j - 1) - \lambda_2 = (\lambda_1 - 1) - j \geqslant 0$, and for the second inequality we have $(\lambda_1 - 1) - (n - j - 1) = (\lambda_1 - 1) - (\lambda_1 + \lambda_2 - j - 1) = j - \lambda_2 \geqslant 0$. Thus θ is well-defined and $s_{C(\theta(T_j))} = -s_{C(T_j)}$.

For example, if $\lambda = (4, 2)$ and $j = 3$ then T_3 is

1	2	3	6
4	5		

with descent composition $(3, 3)$, and $\theta(T_3) = T_2$ is

1	2	5	6
3	4		

with descent composition $(2, 4)$.

Next, we define θ for tableaux T of arbitrary shape λ in which the first row is not $1, 2, \ldots, \lambda_1$. Here we apply θ as defined above to the first two runs of T and leave the rest of T unchanged. So, for example, θ applied to

1	2	3	6	8	9
4	5	7			

gives

1	2	5	6	8	9
3	4	7			

Note that we may extend θ as just defined in an obvious way to tableaux with any distinct entries, not necessarily $1, 2, \ldots, n$.

In the general case, suppose that the first k rows of T constitute a superstandard tableau but the first $k + 1$ rows do not. (So T must have at least $k + 2$ rows.) Then to compute $\theta(T)$ we leave the first k rows unchanged and apply θ to the subtableau of T consisting of rows $k + 1$, $k + 2$, ... It is clear that θ has the desired property: for every non-superstandard tableau T, we have $s_{C(\theta(T))} = -s_{C(T)}$.

For example, suppose that T is the standard tableau

1	2	3	4	5
6	7	9		
8				

with descent composition $C(T) = (5, 2, 2)$. The first row of T is superstandard but the first two rows are not. So $\theta(T)$ is

1	2	3	4	5
6	8	9		
7				

with descent composition $(5, 1, 3)$, and $s_{(5,2,2)} = -s_{(5,1,3)}$.

References

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[3] R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, 1999.