A degree sum condition on the order, the connectivity and the independence number for hamiltonicity

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Abstract

In [Graphs Combin. 24 (2008) 469–483], the third author and the fifth author conjectured that if G is a k-connected graph such that $\sigma_{k+1}(G) \geqslant |V(G)| + \kappa(G) + (k-2)(\alpha(G)-1)$, then G contains a Hamilton cycle, where $\sigma_{k+1}(G)$, $\kappa(G)$ and $\alpha(G)$ are the minimum degree sum of $k+1$ independent vertices, the connectivity and the independence number of G, respectively. In this paper, we settle this conjecture. The degree sum condition is best possible.

Mathematics Subject Classifications: 05C38, 05C45

1 Introduction

1.1 Degree sum condition for graphs with high connectivity to be hamiltonian

In this paper, we consider only finite undirected graphs without loops or multiple edges. For standard graph-theoretic terminology not explained, we refer the reader to [5].

A *Hamilton cycle* of a graph is a cycle containing all the vertices of the graph. A graph having a Hamilton cycle is called a *hamiltonian graph*. The hamiltonian problem has long been fundamental in graph theory. Many researchers have investigated sufficient conditions for a graph to be hamiltonian. In this paper, we deal with a degree-sum-type condition, which is one of the main stream of this study.

We introduce four invariants, including degree sum, which play important roles for the existence of a Hamilton cycle. Let G be a graph. The number of vertices of G is called its *order*, denoted by $n(G)$. A set X of vertices in G is called *an independent set in G* if no two vertices of X are adjacent in G. The *independence number* of G is the maximum cardinality of an independent set in G, denoted by $\alpha(G)$. For two distinct vertices $x, y \in V(G)$, the *local connectivity* $\kappa_G(x, y)$ is defined to be the maximum number of internally-disjoint paths connecting x and y in G. A graph G is *k-connected* if $\kappa_G(x, y) \geqslant k$ for any two distinct vertices $x, y \in V(G)$. The *connectivity* $\kappa(G)$ of G is the maximum value of k for which G is k-connected. We denote by $N_G(x)$ and $d_G(x)$ the neighborhood and the degree of a vertex x in G, respectively. If $\alpha(G) \geqslant k$, let

$$\sigma_k(G) := \min\Big\{\sum_{x\in X} d_G(x) \colon X \text{ is an independent set in } G \text{ with } |X| = k\Big\};$$

otherwise let $\sigma_k(G) := +\infty$. If the graph G is clear from the context, we simply write n, α, κ and σ_k instead of $n(G)$, $\alpha(G)$, $\kappa(G)$ and $\sigma_k(G)$, respectively.

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One of the main streams of the study of the hamiltonian problem is, as mentioned above, to consider degree-sum-type sufficient conditions for graphs to have a Hamilton cycle. We list some of them below.

Theorem 1. *Let G be a graph of order at least three. If G satisfies one of the following, then G is hamiltonian.*

(i) (Dirac [7]) *The minimum degree of G is at least $\frac{n}{2}$.*

(ii) (Ore [10]) $\sigma_2 \geqslant n$.

(iii) (Chvátal and Erdős [6]) $\alpha \leqslant \kappa$.

(iv) (Bondy [4]) *G is k-connected and* $\sigma_{k+1} > \frac{(k+1)(n-1)}{2}$.

(v) (Bauer, Broersma, Veldman and Li [2]) *G is 2-connected and* $\sigma_3 \geqslant n + \kappa$.

To be exact, Theorem 1 (iii) is not a degree-sum-type condition, but it is closely related. Bondy [3] showed that Theorem 1 (iii) implies (ii). From Theorem 1 (iii), it is natural to consider a σ_{k+1} condition for a k-connected graph. Bondy [4] gave a σ_{k+1} condition of Theorem 1 (iv).

In this paper, we give a much weaker σ_{k+1} condition than that of Theorem 1 (iv).

Theorem 2. *Let k be a positive integer and let G be a k-connected graph. If*

$$\sigma_{k+1} \geqslant n + \kappa + (k-2)(\alpha - 1),$$

then G is hamiltonian.

Theorem 2 was conjectured by Ozeki and Yamashita [12]. The case $k = 2$ of Theorem 2 coincides with Theorem 1 (v). The cases $k = 1$ and $k = 3$ were shown by Fraisse and Jung [8], and by Ozeki and Yamashita [12], respectively.

1.2 Sharpness of Theorem 2

In this subsection, we show that the σ_{k+1} condition in Theorem 2 is best possible in some senses.

We first discuss the lower bound of the σ_{k+1} condition. For an integer $l \geqslant 2$ and l vertex-disjoint graphs $H_1, \ldots, H_l$, we define the graph $H_1 \vee \cdots \vee H_l$ from the union of $H_1, \ldots, H_l$ by joining every vertex of H_i to every vertex of H_{i+1} for $1 \leqslant i \leqslant l-1$. Fix a positive integer k. Let κ, m and n be integers with $k \leqslant \kappa < m$ and $2m+1 \leqslant n \leqslant 3m - \kappa$. Let $G_1 = K_{n-2m} \vee \overline{K}_\kappa \vee \overline{K}_m \vee \overline{K}_{m-\kappa}$ (see Figure 1), where K_p denotes the complete graph of order p and $\overline{K}_p$ denotes the complement of K_p. Then $\alpha(G_1) = m + 1$, $\kappa(G_1) = \kappa$ and

$$\sigma_{k+1}(G_1) = (n - 2m - 1 + \kappa) + km = n(G_1) + \kappa(G_1) + (k-2)(\alpha(G_1) - 1) - 1.$$

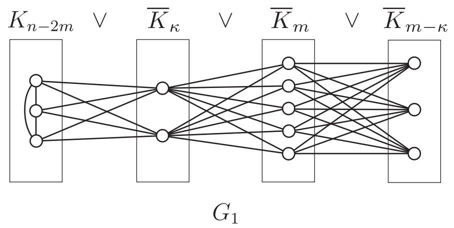

Figure 1: The case $n = 13, m = 5, \kappa = 2$ of the graph G_1.

Since deleting all the vertices in $\overline{K}_\kappa$ and those in $\overline{K}_{m-\kappa}$ breaks G_1 into $m+1$ components, we see that G_1 has no Hamilton cycle. Therefore, the σ_{k+1} condition in Theorem 2 is best possible.

We next discuss the relation between the coefficient of κ and that of $\alpha-1$. By Theorem 1 (iii), we may assume that $\alpha \geqslant \kappa + 1$. This implies that

$$n + \kappa + (k-2)(\alpha-1) \geqslant n + (1+\varepsilon)\kappa + (k-2-\varepsilon)(\alpha-1)$$

for arbitrarily $\varepsilon > 0$. Then one may expect that the σ_{k+1} condition in Theorem 2 can be replaced with "$n + (1+\varepsilon)\kappa + (k-2-\varepsilon)(\alpha-1)$" for some $\varepsilon > 0$. However, the graph G_1 as defined above shows that it is not true: For any $\varepsilon > 0$, there exist two integers m and κ such that $\varepsilon(m-\kappa) \geqslant 1$. If we construct the above graph G_1 from such integers m and κ, then we have

$$\begin{aligned}\sigma_{k+1}(G_1) &= n + (1+\varepsilon)\kappa + (k-2-\varepsilon)m - 1 + \varepsilon(m-\kappa)\\ &\geqslant n(G_1) + (1+\varepsilon)\kappa(G_1) + (k-2-\varepsilon)\big(\alpha(G_1)-1\big),\end{aligned}$$

but G_1 is not hamiltonian. This means that the coefficient 1 of κ and the coefficient $k-2$ of $\alpha-1$ are, in a sense, best possible.

1.3 Comparing Theorem 2 to other results

In this subsection, we compare Theorem 2 to Theorem 1 (iv) and to Theorem 3 (see below).

We first show that the σ_{k+1} condition of Theorem 2 is weaker than that of Theorem 1 (iv). Let G be a k-connected graph satisfying the σ_{k+1} condition of Theorem 1 (iv). Assume that $\alpha \geqslant (n+1)/2$. Let X be an independent set of order at least $(n+1)/2$. Then $|V(G) \setminus X| \leqslant (n-1)/2$ and $|V(G) \setminus X| \geqslant k$ since $V(G) \setminus X$ is a cut set. Hence $(n+1)/2 \geqslant k+1$, and we can take a subset Y of X with $|Y| = k+1$. Then $N_G(y) \subseteq V(G) \setminus X$ for $y \in Y$, and hence $\sum_{y \in Y} d_G(y) \leqslant (k+1)|V(G) \setminus X| \leqslant (k+1)(n-1)/2$. This contradicts the σ_{k+1} condition of Theorem 1 (iv). Therefore $n/2 \geqslant \alpha$. Moreover, by Theorem 1 (iii), we may assume that $\alpha \geqslant \kappa+1$. Therefore, the following inequality holds:

$$\begin{aligned}\sigma_{k+1} &> \frac{(k+1)(n-1)}{2}\\ &\geqslant n - 1 + \frac{(k-1)(2\alpha-1)}{2}\end{aligned}$$

$$\begin{aligned} &\geqslant n - 1 + (k-1)(\alpha - 1) \\ &\geqslant n + \kappa + (k-2)(\alpha - 1) - 1. \end{aligned}$$

Thus, the σ_{k+1} condition of Theorem 1 (iv) implies that of Theorem 2.

We next compare Theorem 2 to the following result of Ota.

Theorem 3 (Ota [11])**.** *Let G be a 2-connected graph. If $\sigma_{l+1} \geqslant n + l(l-1)$ for all integers l with $l \geqslant \kappa$, then G is hamiltonian.*

We mention about the reason to compare Theorem 2 to Theorem 3. Li [9] proved the following theorem as a corollary of Theorem 3.

Theorem 4 (Li [9])**.** *Let k be a positive integer and let G be a k-connected graph. If $\sigma_{k+1} \geqslant n + (k-1)(\alpha - 1)$, then G is hamiltonian.*

Note that Theorem 2 is, assuming Theorem 1 (iii), an improvement of Theorem 4. Therefore we should show that Theorem 2 cannot be implied by Theorem 3.

Let κ, r, k, m be integers such that $4 \leqslant r$, $3 \leqslant k \leqslant \kappa - 2$ and $m = (k+1)(r-2) + 4$. Let $G_2 = K_1 \vee \overline{K}_\kappa \vee K_{\kappa+m-r} \vee (\overline{K}_m \vee K_r)$. Then $n(G_2) = 2\kappa + 2m + 1$, $\kappa(G_2) = \kappa$ and $\alpha(G_2) = \kappa + m$. Since

$$\kappa + k(\kappa + m) - (k+1)(\kappa + m - r + 1) = (k+1)(r-1) - m = k - 3 \geqslant 0,$$

it follows that

$$\begin{aligned} \sigma_{k+1}(G_2) &= \min\left\{\kappa + k(\kappa + m),\ (k+1)(\kappa + m - r + 1)\right\} \\ &= \kappa + k(\kappa + m) - (k-3) \\ &= (2\kappa + 2m + 1) + \kappa + (k-2)(\kappa + m - 1) \\ &= n(G_2) + \kappa(G_2) + (k-2)(\alpha(G_2) - 1). \end{aligned}$$

Hence the assumption of Theorem 2 holds. On the other hand, for $l = \alpha(G_2) - 1 = \kappa + m - 1$, we have

$$\begin{aligned} n(G_2) + l(l-1) - \sigma_{l+1}(G_2) &= (2\kappa + 2m + 1) + (\kappa + m - 1)(\kappa + m - 2) \\ &\quad - \left\{\kappa(\kappa + m - r + 1) + m(\kappa + m)\right\} \\ &= \kappa(r-2) - m + 3 \\ &= (\kappa - k - 1)(r-2) - 1 > 0. \end{aligned}$$

Hence the assumption of Theorem 3 does not hold. These yield that for the graph G_2, we can apply Theorem 2, but cannot apply Theorem 3.

2 Notation and lemmas

Let G be a graph and H be a subgraph of G, and let $x \in V(G)$ and $X \subseteq V(G)$. We denote by $N_G(X)$ the set of vertices in $V(G) \setminus X$ which are adjacent to some vertex in X. We define $N_H(x) := N_G(x) \cap V(H)$ and $d_H(x) := |N_H(x)|$. Furthermore, we define $N_H(X) := N_G(X) \cap V(H)$. We denote by $G[X]$ the subgraph of G induced by X, and let $G - X := G[V(G) \setminus X]$. If there is no fear of confusion, we often identify H with its vertex set $V(H)$. For example, we often write $G - H$, $G[H]$ and $X \cap H$ instead of $G - V(H)$, $G[V(H)]$ and $X \cap V(H)$. A path P is called an *H-path* if both end vertices of P are contained in H and all internal vertices are not contained in H. Note that each edge of H is an H-path. The *union* of two vertex-disjoint graphs H_1 and H_2 is denoted by $H_1 \cup H_2$.

Throughout this paper, we consider that each of cycles and paths has a fixed orientation. Let C be a cycle (or a path) in a graph G. For $x, y \in V(C)$, we denote by $C[x, y]$ the path from x to y along the orientation of C. The reverse sequence of $C[x, y]$ is denoted by $\overleftarrow{C}[y, x]$. We denote $C[x, y] - \{x, y\}$, $C[x, y] - \{x\}$ and $C[x, y] - \{y\}$ by $C(x, y)$, $C(x, y]$ and $C[x, y)$, respectively. We denote a path P from a vertex u to a vertex v by $P[u, v]$. For two vertex-disjoint paths $P[u, v]$ and $Q[x, y]$, if $v = x$ or $vx \in E(G)$, then $P[u, v]Q[x, y]$ is the path from u to y along P and Q. For $x \in V(C)$, we denote the successor and the predecessor of x on C by x^+ and x^-, respectively. (For the end vertices u, v of $P[u, v]$, u^- and v^+ do not exist.) For $X \subseteq V(C)$, we define $X^+ := \{x^+ : x \in X\}$ and $X^- := \{x^- : x \in X\}$.

In this paper, we extend the concept of *insertible*, introduced by Ainouche [1], which has been used for the proofs of the results on cycles. Let G be a graph, and H be a subgraph of G. Let $X(H) := \{x \in V(G - H) : xu, xv \in E(G) \text{ for some } uv \in E(H)\}$, and for $x \in V(G - H)$, let $I(x; H) := \{uv \in E(H) : xu, xv \in E(G)\}$. Let $Y(H) := \{y \in V(G - H) : d_H(y) \geqslant \alpha(G)\}$.

Lemma 5. *Let D be a cycle of a graph G. Let $Q_1, Q_2, \ldots, Q_k$ be vertex-disjoint paths in $G - D$, where Q_i is a path from a_i to b_i, $1 \leqslant i \leqslant k$, and let $Q := \bigcup_{i=1}^{k} Q_i$. If the following* (I) *and* (II) *hold, then $G[D \cup Q]$ is hamiltonian.*

(I) *$u \in X(D) \cup Y(Q_i(u, b_i] \cup D)$ for $u \in V(Q_i)$, $1 \leqslant i \leqslant k$.*

(II) *$I(x; D) \cap I(y; D) = \emptyset$ for $x \in V(Q_i)$ and $y \in V(Q_j)$, $1 \leqslant i < j \leqslant k$.*

Proof. We can easily see that $G[D \cup Q]$ contains a cycle D^* such that $V(D) \cup (X(D) \cap Q) \subseteq V(D^*)$. In fact, we can insert all vertices of $X(D) \cap Q_1$ into D by choosing the following $u_1, v_1 \in V(Q_1)$ and $w_1 w_1^+ \in E(D)$ inductively. Take the first vertex u_1 in $X(D) \cap Q_1$ along the orientation of Q_1, and let v_1 be the last vertex in $X(D) \cap Q_1$ such that $I(u_1; D) \cap I(v_1; D) \neq \emptyset$. Then we can insert all vertices of $Q_1[u_1, v_1]$ into D. To be exact, taking $w_1 w_1^+ \in I(u_1; D) \cap I(v_1; D)$, $D_1^1 := w_1 Q_1[u_1, v_1] D[w_1^+, w_1]$ is such a cycle. By the choice of u_1 and v_1, $w_1 w_1^+ \notin I(x; D)$ for all $x \in V(Q_1 - Q_1[u_1, v_1])$, and all vertices in $X(D) \cap V(Q_1 - Q_1[u_1, v_1])$ are contained in the path $Q_1 - Q_1[a_1, v_1]$. Moreover, note that

$E(D) \setminus \{w_1 w_1^+\} \subseteq E(D_1^1)$. Hence by repeating this argument, we can obtain a cycle D_1^* of $G[D \cup Q_1]$ such that $V(D) \cup (X(D) \cap Q_1) \subseteq V(D_1^*)$ and $E(D) \setminus \bigcup_{x \in V(Q_1)} I(x; D) \subseteq E(D_1^*)$. Then by (II), $I(x; D) \subseteq E(D_1^*)$ for all $x \in V(Q) \setminus V(Q_1)$. Therefore $G[D \cup Q]$ contains a cycle D^* such that $V(D) \cup (X(D) \cap Q) \subseteq V(D^*)$.

We choose a cycle C of $G[D \cup Q]$ containing all vertices in $V(D) \cup (X(D) \cap Q)$ so that $|C|$ is as large as possible. Now, we change the "base" cycle from D to C, and use the symbol $(\cdot)^+$ for the orientation of C. Suppose that $V(Q_i - C) \neq \emptyset$ for some i, $1 \leqslant i \leqslant k$. We may assume that $i = 1$. Let w be the last vertex in $V(Q_1 - C)$ along Q_1. Since C contains all vertices in $X(D) \cap Q_1$, it follows from (I) that $w \in Y(Q_1(w, b_1] \cup D)$, that is, $|N_G(w) \cap (Q_1(w, b_1] \cup D)| \geqslant \alpha(G)$. By the choice of w, we obtain $V(Q_1(w, b_1] \cup D) \subseteq V(C)$. Therefore $|N_C(w)^+ \cup \{w\}| \geqslant |N_G(w) \cap (Q_1(w, b_1] \cup D)| + 1 \geqslant \alpha(G) + 1$. This implies that $N_C(w)^+ \cup \{w\}$ is not an independent set in G. Hence $wz^+ \in E(G)$ for some $z \in N_C(w)$ or $z_1^+ z_2^+ \in E(G)$ for some distinct $z_1, z_2 \in N_C(w)$. In the former case, let $C' = wC[z^+, z]w$, and in the latter case, let $C' := w\overleftarrow{C}[z_1, z_2^+]C[z_1^+, z_2]w$. Then C' is a cycle of $G[D \cup Q]$ such that $V(C) \cup \{w\} = V(C')$, which contradicts the choice of C. Thus $V(Q)$ is contained in C, and hence C is a Hamilton cycle of $G[D \cup Q]$. $\square$

In the rest of this section, we fix the following notation. Let C be a longest cycle in a graph G, and H_0 be a component of $G - C$. For $u \in N_C(H_0)$, let $u' \in N_C(H_0) \setminus \{u\}$ be a vertex such that $C(u, u') \cap N_C(H_0) = \emptyset$, that is, u' is the successor of u in $N_C(H_0)$ along the orientation of C.

For $u \in N_C(H_0)$, a vertex $v \in C(u, u')$ is *insertible* if $v \in X(C[u', u]) \cup Y(C(v, u])$. A vertex in $C(u, u')$ is said to be *non-insertible* if it is not insertible.

Lemma 6. *There exists a non-insertible vertex in $C(u, u')$ for $u \in N_C(H_0)$.*

Proof. Let $u \in N_C(H_0)$, and suppose that every vertex in $C(u, u')$ is insertible. Let P be a C-path joining u and u' with $V(P) \cap V(H_0) \neq \emptyset$. Let $D := C[u', u]P[u, u']$ and $Q := C(u, u')$. Let $v \in V(Q)$. Since v is insertible, it follows that $v \in X(C[u', u]) \cup Y(C(v, u])$. Since $C[u', u]$ is a subpath of D, we have $v \in X(D) \cup Y(Q(v, u') \cup D)$. Hence, by Lemma 5, $G[D \cup Q]$ is hamiltonian, which contradicts the maximality of C. $\square$

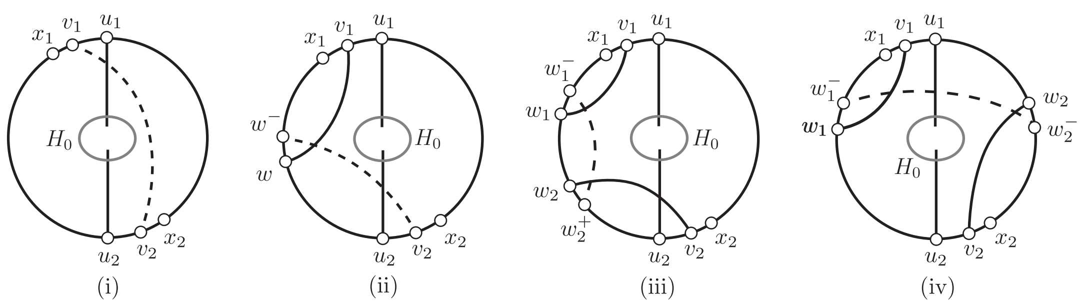

Figure 2: Lemma 7.

Lemma 7. *Let $u_1, u_2 \in N_C(H_0)$ with $u_1 \neq u_2$, and let x_i be the first non-insertible vertex along $C(u_i, u_i')$ for $i = 1, 2$. Then the following hold (see Figure 2).*

(i) *There exists no C-path joining $v_1 \in C(u_1, x_1]$ and $v_2 \in C(u_2, x_2]$. In particular, $x_1x_2 \notin E(G)$.*

(ii) *If there exists a C-path joining $v_1 \in C(u_1, x_1]$ and $w \in C(v_1, u_2]$, then there exists no C-path joining $v_2 \in C(u_2, x_2]$ and w^-.*

(iii) *If there exist a C-path joining $v_1 \in C(u_1, x_1]$ and $w_1 \in C(v_1, u_2)$ and a C-path joining $v_2 \in C(u_2, x_2]$ and $w_2 \in C[w_1, u_2)$, then there exists no C-path joining w_1^- and w_2^+.*

(iv) *If for each $i = 1, 2$, there exists a C-path joining $v_i \in C(u_i, x_i]$ and $w_i \in C(v_i, u_{3-i}]$, then there exists no C-path joining w_1^- and w_2^-.*

Proof. Let P_0 be a C-path which connects u_1 and u_2, and $V(P_0) \cap V(H_0) \neq \emptyset$. We first show (i) and (ii). Suppose that one of the following holds for some $v_1 \in C(u_1, x_1]$ and some $v_2 \in C(u_2, x_2]$: (a) There exists a C-path P_1 joining v_1 and v_2. (b) There exist disjoint C-paths P_2 joining v_l and w, and P_3 joining v_{3-l} and w^- for some $l = 1, 2$ and some $w \in C(v_l, u_{3-l}]$. We choose such vertices v_1 and v_2 so that $|C[u_1, v_1]| + |C[u_2, v_2]|$ is as small as possible. Without loss of generality, we may assume that $l = 1$ if (b) holds. Since $N_C(H_0) \cap \{v_1, v_2\} = \emptyset$, $(V(P_1) \cup V(P_2) \cup V(P_3)) \cap V(P_0) = \emptyset$. Therefore, we can define a cycle

$$D := \begin{cases} P_1[v_1, v_2]C[v_2, u_1]P_0[u_1, u_2]\overleftarrow{C}[u_2, v_1] & \text{if (a) holds,} \\ P_2[v_1, w]C[w, u_2]\overleftarrow{P_0}[u_2, u_1]\overleftarrow{C}[u_1, v_2]P_3[v_2, w^-]\overleftarrow{C}[w^-, v_1] & \text{if (b) holds.} \end{cases}$$

For $i = 1, 2$, let $Q_i := C(u_i, v_i)$ (possibly $V(Q_i) = \emptyset$). By the choice of x_i and Lemma 6, we have $N_C(H_0) \cap Q_i = \emptyset$, and hence the following statement (1) holds. By the choice of v_1 and v_2, we can obtain the following statements (2)–(5).

(1) $N_G(x) \cap P_0(u_1, u_2) = \emptyset$ for $x \in V(Q_1 \cup Q_2)$.

(2) $N_G(x) \cap (P_1(v_1, v_2) \cup P_2(v_1, w) \cup P_3(v_2, w^-)) = \emptyset$ for $x \in V(Q_1 \cup Q_2)$.

(3) $xy \notin E(G)$ for $x \in V(Q_1)$ and $y \in V(Q_2)$.

(4) $I(x; C) \cap I(y; C) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$.

(5) If (b) holds, then $w^-w \notin I(x; C)$ for $x \in V(Q_1 \cup Q_2)$.

Let $u \in V(Q_i)$ for some $i = 1, 2$. Note that each vertex of Q_i is insertible, that is, $u \in X(C[u_i', u_i]) \cup Y(C(u, u_i])$. We show that $u \in X(D) \cup Y(Q_i(u, v_i) \cup D)$. If $u \in X(C[u_i', u_i])$ and $V(Q_{3-i}) \neq \emptyset$, then the statements (3) and (5) yield that $u \in X(D)$. If $u \in X(C[u_i', u_i])$ and $V(Q_{3-i}) = \emptyset$, then the choice of v_1 and v_2 and the statement (5) yield that $u \in X(D)$. Suppose that $u \in Y(C(u, u_i])$. By (3), $N_G(u) \cap C(u, u_i] \subseteq N_G(u) \cap (Q_i(u, v_i) \cup D)$. This implies that $u \in Y(Q_i(u, v_i) \cup D)$. By (1), (2) and (4), $I(x; D) \cap I(y; D) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. Thus, by Lemma 5, $G[D \cup Q_1 \cup Q_2]$ is hamiltonian, which contradicts the maximality of C.

By using a similar argument as above, we can also show (iii) and (iv). We only denote the outline of the proof of (iii). Suppose that for some $v_1 \in C(u_1, x_1]$ and $v_2 \in C(u_2, x_2]$, there exist disjoint C-paths $P_1[v_1, w_1]$, $P_2[v_2, w_2]$ and $P_3[w_1^-, w_2^+]$ with $w_1 \in C(v_1, u_2)$ and $w_2 \in C[w_1, u_2)$. We choose such v_1 and v_2 so that $|C[u_1, v_1]| + |C[u_2, v_2]|$ is as small as possible. Let $Q_i := C(u_i, v_i)$ for $i = 1, 2$. Then by Lemma 7 (i), $xy \notin E(G)$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. By the choice of v_1 and v_2 and Lemma 7 (ii), $w_1 w_1^-, w_2 w_2^+ \notin I(x; C[v_1, u_1]) \cup I(y; C[v_2, u_2])$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. By Lemma 7 (i) and (ii), $I(x; C[v_1, u_2] \cup C[v_2, u_1]) \cap I(y; C[v_1, u_2] \cup C[v_2, u_1]) = \emptyset$ for $x \in V(Q_1)$ and $y \in V(Q_2)$. Hence by applying Lemma 5 as

$$D := P_1[v_1, w_1] C[w_1, w_2] \overleftarrow{P_2}[w_2, v_2] C[v_2, u_1] P_0[u_1, u_2] \overleftarrow{C}[u_2, w_2^+] \overleftarrow{P_3}[w_2^+, w_1^-] \overleftarrow{C}[w_1^-, v_1],$$

Q_1 and Q_2, we see that there exits a longer cycle than C, a contradiction. □

3 Proof of Theorem 2

Proof of Theorem 2. The cases $k = 1$, $k = 2$ and $k = 3$ were shown by Fraisse and Jung [8], by Bauer et al. [2] and by Ozeki and Yamashita [12], respectively. Therefore, we may assume that $k \geqslant 4$. Let G be a graph satisfying the assumption of Theorem 2. By Theorem 1 (iii), we may assume $\alpha(G) \geqslant \kappa(G) + 1$. Let C be a longest cycle in G. If C is a Hamilton cycle of G, then there is nothing to prove. Hence we may assume that $G - V(C) \neq \emptyset$. Set $H := G - V(C)$ and $x_0 \in V(H)$. Choose a longest cycle C and x_0 so that

$$d_C(x_0) \text{ is as large as possible.}$$

Let H_0 be the component of H such that $x_0 \in V(H_0)$. Set

$$U := N_C(H_0) := \{u_1, u_2, \ldots, u_m\}.$$

Note that $m \geqslant \kappa(G) \geqslant k$. Let u_i' be the vertex in $N_C(H_0) \setminus \{u_i\}$ such that $C(u_i, u_i') \cap N_C(H_0) = \emptyset$ for each $i \in [m]$, where $[m]$ means $\{1, 2, \ldots, m\}$. By Lemma 6, there exists a non-insertible vertex in $C(u_i, u_i')$. Let $x_i \in C(u_i, u_i')$ be the first non-insertible vertex along the orientation of C for each $i \in [m]$, and set

$$X := \{x_1, x_2, \ldots, x_m\}.$$

By Lemma 7 (i) and since $N_{H_0}(x_i) = \emptyset$ for $i \in [m]$, we obtain the following.

Claim 8. $X \cup \{x_0\}$ *is an independent set, and hence* $|X| \leqslant \alpha(G) - 1$.

Define

$$D_0 := \emptyset \quad \text{and} \quad D_i := C(u_i, x_i) \quad \text{for each } i \in [m],$$

and $D := \bigcup_{i\in[m]} D_i$. By Claim 8 and the definition of U and X, we obtain

$$d_C(x_0) \leqslant |U| = |X| \leqslant \alpha(G) - 1. \tag{1}$$

Since $D_0 = \emptyset$ and x_i is non-insertible, we can see that

$$d_C(x_i) \leqslant |D_i| + \alpha(G) - 1 \quad \text{for } 0 \leqslant i \leqslant m. \tag{2}$$

By the definition of x_i, $N_{H_0}(x_i) = \emptyset$ for $i \in [m]$. By Lemma 7 (i), $N_H(x_i) \cap N_H(x_j) = \emptyset$ for $i, j \in [m]$ with $i \neq j$. Thus we obtain

$$\sum_{0\leqslant i\leqslant m} d_H(x_i) \leqslant |H| - 1. \tag{3}$$

In this paragraph, let i and j be distinct two integers in $[m]$, and set $C_i := C[x_i, u_j]$ and $C_j := C[x_j, u_i]$. By Lemma 7 (ii), we have $N_{C_i}(x_i)^- \cap N_{C_i}(x_j) = \emptyset$ and $N_{C_j}(x_j)^- \cap N_{C_j}(x_i) = \emptyset$. By Lemma 7 (i), $N_{C_i}(x_i)^- \cup N_{C_i}(x_j) \subseteq C_i \setminus D$, $N_{C_j}(x_j)^- \cup N_{C_j}(x_i) \subseteq C_j \setminus D$ and $N_{D_i}(x_j) = N_{D_j}(x_i) = \emptyset$. Thus, we obtain

$$d_C(x_i) + d_C(x_j) \leqslant |C| - \sum_{h\in[m]\setminus\{i,j\}} |D_h| \quad \text{for } i, j \in [m] \text{ with } i \neq j. \tag{4}$$

We will frequently use these upper bounds (1)–(4) on degree (sum) of vertices in $X \cup \{x_0\}$.

By replacing the labels x_2 and x_3 if necessary, we may assume that x_1, x_2 and x_3 appear in this order along the orientation of C. From this paragraph to the paragraph below Claim 9, the indices are taken modulo 3. From now on, for each $i \in [3]$, set

$$C_i := C[x_i, u_{i+1}]$$

and

$$W_i := \{w \in V(C_i) : w^+ \in N_{C_i}(x_i) \text{ and } w^- \in N_{C_i}(x_{i+1})\}$$

and $W := W_1 \cup W_2 \cup W_3$ (see Figure 3 (i)). Note that $W \cap (U \cup \{x_1, x_2, x_3\}) = \emptyset$, by the definition of C_i and W_i and by Lemma 7 (i). Furthermore, for $i \in [3]$, set

$$L_i := \left\{x_j \in X \setminus \{x_{i+1}\} : N_{C_i}(x_{i+1})^+ \cap D_j \neq \emptyset\right\}$$

and $L := L_1 \cup L_2 \cup L_3$ (see Figure 3 (ii)). By the definition and Lemma 7 (i),

$$W \cap L = \emptyset. \tag{5}$$

In the following proof, we will set suitable three vertices as x_1, x_2, x_3 if necessary. Note that W and L will be defined by them in each case. Moreover, note that the following claims which hold for x_1, x_2, x_3, in fact, hold for any x_i, x_j, x_k with respect to corresponding W and L.

By Lemma 7, we can improve Claim 8 as follows:

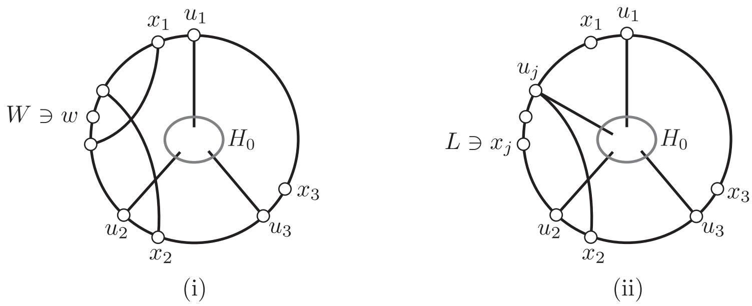

Figure 3: The definition of W and L.

Claim 9. *$X \cup W \cup \{x_0\}$ is an independent set.*

We now check the upper bound of $d_C(x_1) + d_C(x_2) + d_C(x_3)$. By Lemma 7(ii), $(N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+) \cap N_{C_i}(x_{i+2}) = \emptyset$ for each $i \in [3]$. Clearly, $N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+1})^+ = W_i$ and $N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+ \cup N_{C_i}(x_{i+2}) \subseteq C_i \cup \{u_{i+1}^+\}$. By Lemma 7(i), $(N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+2})) \cap D_j = \emptyset$ for each $i \in [3]$ and $j \in [m]$. Note that

$$\big|N_{C_i}(x_{i+1})^+ \cap \big(\bigcup_{j \in [m] \setminus \{i+1\}} D_j\big)\big| = |L_i|$$

for $i \in [3]$. Note also that $L \cap \{x_1, x_2, x_3\} = \emptyset$ and $W \cap L = \emptyset$ by (5). Therefore, for $i \in [3]$, the following inequality holds:

$$\begin{aligned}
&d_{C_i}(x_1) + d_{C_i}(x_2) + d_{C_i}(x_3)\\
&= |N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+ \cup N_{C_i}(x_{i+2})|\\
&\quad + |(N_{C_i}(x_i)^- \cup N_{C_i}(x_{i+1})^+) \cap N_{C_i}(x_{i+2})| + |N_{C_i}(x_i)^- \cap N_{C_i}(x_{i+1})^+|\\
&\leqslant |C_i| + |W_i| + 1 - \sum_{j \in [m]} |C_i \cap D_j| + |L_i|.
\end{aligned}$$

By Lemma 7 (i), we have $N_C(x_i) \cap D_j = \emptyset$ for $i, j \in [m]$, $i \neq j$, and hence

$$d_{D_i}(x_1) + d_{D_i}(x_2) + d_{D_i}(x_3) \leqslant |D_i|$$

for $i \in [3]$. Let I be a subset of $\{0, 1, \ldots, m\} \setminus \{1, 2, 3\}$ and let $L_I := L \cap \{x_i : i \in I\}$ (We will set a suitable subset I for each case.). Note that $|L \cap \{x_i\}| - |D_i| \leqslant 0$ for each $i \in [m] \setminus [3]$. Thus, we deduce

$$\begin{aligned}
d_C(x_1) + d_C(x_2) + d_C(x_3) &\leqslant \sum_{i=1}^{3} \big(|C_i| + |W_i| + |L_i| + 1 - \sum_{j \in [m]} |C_i \cap D_j| + |D_i|\big)\\
&= |C| + |W| + |L| - \sum_{i \in [m] \setminus [3]} |D_i| + 3\\
&\leqslant |C| + |W| + |L_I| - \sum_{i \in I} |D_i| + 3. \qquad (6)
\end{aligned}$$

Claim 10. $|W \cup L| \geqslant \kappa(G) - 2 \geqslant 2$.

Proof. Let I be a $(k-2)$-subset of $\{0, 1, \dots, m\} \setminus \{1, 2, 3\}$, where a *k-subset* is a subset of order k. Suppose that $|W| + |L_I| \leqslant \kappa(G) - 3$. By Claim 8, $\{x_i : i \in I\} \cup \{x_1, x_2, x_3\}$ is independent. By (6), we obtain

$$d_C(x_1) + d_C(x_2) + d_C(x_3) \leqslant |C| + \kappa(G) - \sum_{i \in I} |D_i|.$$

Therefore, this inequality and (2) and (3) yield that

$$\sum_{i=1}^{3} d_G(x_i) + \sum_{i \in I} d_G(x_i) \leqslant n + \kappa(G) + (k-2)(\alpha(G) - 1) - 1,$$

a contradiction. Therefore, by (5) and since $\kappa(G) \geqslant k \geqslant 4$, we obtain $|W \cup L| = |W| + |L| \geqslant |W| + |L_I| \geqslant \kappa(G) - 2 \geqslant 2$. □

Claim 11. $|X| \geqslant \kappa(G) + 1$.

Proof. Let s and t be distinct two integers in $[m]$. By (4), we have

$$d_C(x_s) + d_C(x_t) \leqslant |C| - \sum_{i \in [m] \setminus \{s,t\}} |D_i|.$$

Let I be a $(k+1)$-subset of $\{0, 1, \dots, m\}$ such that $\{0, s, t\} \subseteq I$. By Claim 8, $\{x_i : i \in I\}$ is an independent set. By (2), we deduce

$$\sum_{i \in I \setminus \{0,s,t\}} d_C(x_i) \leqslant \sum_{i \in I \setminus \{0,s,t\}} |D_i| + (k-2)(\alpha(G) - 1).$$

By (3), we obtain $\sum_{i \in I} d_H(x_i) \leqslant |H| - 1$. Thus, it follows from the above three inequalities that

$$\sum_{i \in I} d_G(x_i) \leqslant n + (k-2)(\alpha(G) - 1) - 1 + d_C(x_0).$$

Since $\sigma_{k+1}(G) \geqslant n + \kappa(G) + (k-2)(\alpha(G) - 1)$, we have $|X| \geqslant d_C(x_0) \geqslant \kappa(G) + 1$. □

Let S be a cut set with $|S| = \kappa(G)$. By Claim 11, there exists an integer $l \in [m]$ such that $C[u_l, u_l') \cap S = \emptyset$. Hence all vertices in $C[u_l, u_l')$ are contained in some component of $G - S$. Let

$$V_1 \text{ be the component of } G - S \text{ such that } C[u_l, u_l') \subseteq V_1$$

and

$$V_2 := G - (S \cup V_1).$$

By Lemma 7 (i), we obtain

$$d_C(x_l) \leqslant |C \cap (V_1 \cup S)| - \Big| \bigcup_{i \in [m] \setminus \{l\}} D_i \cap (V_1 \cup S) \Big| - |X \cap (V_1 \cup S)|. \tag{7}$$

Claim 12. *$D \cup X \cup W \cup H \subseteq V_1 \cup S$. In particular, $x_0 \in V_1 \cup S$.*

Proof. We first show that $D \cup X \cup W \subseteq V_1 \cup S$. Suppose not. Then, for some $h \in [m] \setminus \{l\}$, there exists a vertex $v \in \big(D_h \cup \{x_h\} \cup (W \cap C(x_h, u'_h))\big) \cap V_2$. Choose v so that $v = x_h$ if possible. Note that if $v \notin D_h$ then $x_h \notin N_G(v)$ by Lemma 7 (ii); if $v \in D_h$ and $x_h \in N_G(v)$ then $x_h \in S$ by the choice of v. Since $v \in V_2$, it follows from Lemma 7 (i) and (ii) that

$$d_C(v) \leqslant |C \cap (V_2 \cup S)| - \Big| \bigcup_{i \in [m] \setminus \{h\}} D_i \cap (V_2 \cup S) \Big| - |X \cap (V_2 \cup S)| + |\{x_h\} \cap S|.$$

Let I be a k-subset of $\{0, 1, \ldots, m\} \setminus \{h\}$ such that $\{0, l\} \subseteq I$. By Claim 8 and Lemma 7 (i) and (ii), $\{x_i : i \in I\} \cup \{v\}$ is independent. By the above inequality and (1) and (7), we obtain

$$\begin{aligned} d_C(x_l) + d_C(v) &\leqslant |C \cap (V_1 \cup V_2 \cup S)| + |C \cap S| - \Big| \bigcup_{i \in [m] \setminus \{l,h\}} D_i \cap (V_1 \cup V_2 \cup S) \Big| \\ &\quad - |X \cap (V_1 \cup V_2 \cup S)| - |X \cap S| + |\{x_h\} \cap S| \\ &\leqslant |C| + |C \cap S| - \sum_{i \in [m] \setminus \{l,h\}} |D_i| - |X| \\ &\leqslant |C| + \kappa(G) - \sum_{i \in I \setminus \{0,l\}} |D_i| - d_C(x_0). \end{aligned}$$

On the other hand, (2) yields that

$$\sum_{i \in I \setminus \{0,l\}} d_C(x_i) \leqslant \sum_{i \in I \setminus \{0,l\}} |D_i| + (k-2)(\alpha(G) - 1).$$

By the above two inequalities, we deduce

$$\sum_{i \in I} d_C(x_i) + d_C(v) \leqslant |C| + \kappa(G) + (k-2)(\alpha(G) - 1).$$

Recall that $\{x_i : i \in I\} \cup \{v\}$ is independent, in particular, $x_0 \notin \bigcup_{i \in I} N_H(x_i) \cup N_H(v)$. Since $N_H(x_i) \cap N_H(x_j) = \emptyset$ for $i, j \in I$ with $i \neq j$ and $\big(\bigcup_{i \in I} N_H(x_i)\big) \cap N_H(v) = \emptyset$ by Lemma 7 (i) and (ii), it follows that $\sum_{i \in I} d_H(x_i) + d_H(v) \leqslant |H| - 1$. Combining this inequality with the above inequality, we get $\sum_{i \in I} d_G(x_i) + d_G(v) \leqslant n + \kappa(G) + (k-2)(\alpha(G) - 1) - 1$, a contradiction.

We next show that $H - H_0 \subseteq V_1 \cup S$. Suppose not. Then, there exists a vertex $y \in (H - H_0) \cap V_2$. Let H_y be the component of H with $y \in V(H_y)$. If $N_C(H_y) \cap (D_h \cup \{x_h\}) \neq \emptyset$ for some $h \in [m] \setminus \{l\}$, then let $M := \{0, 1, \ldots, m\} \setminus \{h\}$ and by Lemma 7 (i),

$$d_C(y) \leqslant |C \cap (V_2 \cup S)| - \Big| \bigcup_{i \in [m] \setminus \{h\}} D_i \cap (V_2 \cup S) \Big| - |X \cap (V_2 \cup S)| + |\{x_h\} \cap S|;$$

if $N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset$ for all $i \in [m] \setminus \{l\}$, then let $M := \{0, 1, \ldots, m\}$ and by this assumption,

$$d_C(y) \leqslant |C \cap (V_2 \cup S)| - \Big|\bigcup_{i\in[m]} D_i \cap (V_2 \cup S)\Big| - |X \cap (V_2 \cup S)|.$$

Let I be a k-subset of M such that $\{0, l\} \subseteq I$. By the same argument as above, we obtain

$$\sum_{i\in I} d_C(x_i) + d_C(y) \leqslant |C| + |C \cap S| + (k-2)(\alpha(G) - 1).$$

If $N_C(H_y) \cap (D_h \cup \{x_h\}) \neq \emptyset$ for some $h \in [m] \setminus \{l\}$, then Lemma 7 (i) and (ii) yield that $\big(\bigcup_{i\in I\setminus\{l\}} N_H(x_i)\big) \cap V(H_y) = \emptyset$; otherwise, since $H_0 \neq H_y$ and $N_C(H_y) \cap (D_i \cup \{x_i\}) = \emptyset$ for all $i \in [m] \setminus \{l\}$, the same conclusion holds. In particular, $y \notin \bigcup_{i\in I\setminus\{l\}} N_G(x_i)$. Since $x_l \in V_1$ and $y \in V_2$, we have $x_l y \notin E(G)$ and $N_H(x_l) \cap N_H(y) \subseteq H \cap S$. Therefore, we obtain $\{x_i : i \in I\} \cup \{y\}$ is independent, and

$$\sum_{i\in I} d_H(x_i) + d_H(y) \leqslant |H| + |H \cap S| - |\{x_0, y\}| = |H| + |H \cap S| - 2.$$

Combining the above two inequalities, $\sum_{i\in I} d_G(x_i) + d_G(y) \leqslant n + \kappa(G) + (k-2)(\alpha(G) - 1) - 2$, a contradiction.

We finally show that $H_0 \subseteq V_1 \cup S$. Suppose not, that is, there exists a vertex $y_0 \in H_0 \cap V_2$. Then

$$d_G(y_0) \leqslant |U \cap (V_2 \cup S)| + |H_0| - 1.$$

Since $u_l \in V_1$, we have $H_0 \cap S \neq \emptyset$. Note that by the above argument, $X \subseteq V_1 \cup S$. Therefore, by Claim 11, $|X \cap V_1| = |X| - |X \cap S| \geqslant \kappa(G) + 1 - (|S| - |H_0 \cap S|) \geqslant \kappa(G) + 1 - (\kappa(G) - 1) = 2$. Hence there exists a vertex $x_s \in (X \cap V_1) \setminus \{x_l\}$. Let I be a k-subset of $[m]$ such that $\{l, s\} \subseteq I$. Then $\{x_i : i \in I\} \cup \{y_0\}$ is an independent set of order $k+1$. By Lemma 7 (i), we have $N_C(x_l)^- \cap (U \setminus \{u_l\}) = \emptyset$ and $N_C(x_s)^- \cap (U \setminus \{u_s\}) = \emptyset$. Since $\{x_l, x_s\} \subseteq V_1$, it follows that $(N_C(x_l) \cup N_C(x_s)) \cap (U \cap V_2) = \emptyset$. Therefore, we can improve (4) as follows:

$$d_C(x_l) + d_C(x_s) \leqslant |C| - \sum_{i\in I\setminus\{l,s\}} |D_i| - |U \cap V_2|.$$

By (2),

$$\sum_{i\in I\setminus\{l,s\}} d_C(x_i) \leqslant \sum_{i\in I\setminus\{l,s\}} |D_i| + (k-2)(\alpha(G) - 1).$$

By the definition of x_i, we clearly have $N_{H_0}(x_i) = \emptyset$ for $i \in I$. Hence we improve (3) as follows:

$$\sum_{i\in I} d_H(x_i) \leqslant |H| - |H_0|.$$

Hence, by the above four inequalities, we deduce $\sum_{i\in I} d_G(x_i) + d_G(y_0) \leqslant n + \kappa(G) + (k-2)(\alpha(G) - 1) - 1$, a contradiction. □

By Claim 12,

$$\text{there exists an integer } r \text{ such that } C(x_r, u'_r] \cap V_2 \neq \emptyset,$$

say

$$v_2 \in C(x_r, u'_r] \cap V_2.$$

Choose r and v_2 so that $v_2 \neq u'_r$ if possible. Note that

$$d_G(v_2) \leqslant |V_2 \cup S| - 1. \tag{8}$$

By Lemma 7 and Claim 12, we can improve (3) as follows:

$$\sum_{0 \leqslant i \leqslant m} d_H(x_i) + \sum_{w \in W} d_H(w) \leqslant |H| - |\{x_0\}| = |H \cap (V_1 \cup S)| - 1. \tag{9}$$

By Claim 12 and (1), we can improve (7) as follows:

$$\begin{aligned} d_C(x_l) &\leqslant |C \cap (V_1 \cup S)| - \sum_{i \in [m] \setminus \{l\}} |D_i| - |X| \\ &\leqslant |C \cap (V_1 \cup S)| - \sum_{i \in [m] \setminus \{l\}} |D_i| - d_C(x_0). \end{aligned} \tag{10}$$

Claim 13. $d_C(x_0) = |U| = |X| = \alpha(G) - 1$. *In particular,* $N_C(x_0) = U$.

Proof. We first show that $d_C(w) \leqslant d_C(x_0)$ for each $w \in W$. Let $w \in W$. Without loss of generality, we may assume that $w \in W_1$. Then by applying Lemma 5 as $Q_1 := D_1$, $Q_2 := D_2$ and

$$D := x_1 C[w^+, u_2] P[u_2, u_1] \overleftarrow{C}[u_1, x_2] \overleftarrow{C}[w^-, x_1],$$

where $P[u_2, u_1]$ is a C-path passing through a vertex of H_0, we can obtain a cycle C' such that $V(C) \setminus \{w\} \subseteq V(C')$ and $V(C') \cap V(H_0) \neq \emptyset$ (note that (I) and (II) of Lemma 5 hold, by Lemma 7 (i) and (ii)). Note that $|C'| = |C|$ by the maximality of $|C|$. Note also that $d_{C'}(w) \geqslant d_C(w)$. By the choice of C and x_0, we have $d_{C'}(w) \leqslant d_C(x_0)$, and hence $d_C(w) \leqslant d_C(x_0)$.

We next show that $d_C(x_0) = |U| = |X| = \alpha(G) - 1$. By (1), it suffices to prove that $d_C(x_0) \geqslant \alpha(G) - 1$. Suppose that $d_C(x_0) \leqslant \alpha(G) - 2$. In this proof, we assume $x_l = x_1$.

We divide the proof into two cases.

Case 1. $|W| \geqslant \kappa(G) + k - 4$.

By the assumption of Case 1 and by Claim 12, we obtain

$$\begin{aligned} |(W \cup \{x_0, x_1, x_2, x_3\}) \cap V_1| &= |W \cup \{x_0, x_1, x_2, x_3\}| - |(W \cup \{x_0, x_1, x_2, x_3\}) \cap S| \\ &\geqslant (\kappa(G) + k - 4 + 4) - \kappa(G) = k. \end{aligned}$$

Let W' be a k-subset of $(W \cup \{x_0, x_1, x_2, x_3\}) \cap V_1$ such that $x_1 \in W'$. Since $W' \subseteq V_1$ and $v_2 \in V_2$, and by Claim 9, $W' \cup \{v_2\}$ is independent. Since $d_C(w) \leqslant d_C(x_0)$ for each $w \in W$, it follows from (10) that

$$d_C(x_1) \leqslant |C \cap (V_1 \cup S)| - \sum_{i \in \{2,3\}} |D_i| - d_C(w_0),$$

where $w_0 \in W' \setminus \{x_1, x_2, x_3\}$ (note that $|W'| = k \geqslant 4$). By (1) and (2),

$$\begin{aligned}\sum_{x \in W' \setminus \{x_1, w_0\}} d_C(x) &= \sum_{x \in W' \cap \{x_2, x_3\}} d_C(x) + \sum_{w \in W' \setminus \{w_0, x_1, x_2, x_3\}} d_C(w) \\ &\leqslant \sum_{i \in \{2,3\}} |D_i| + (k-2)(\alpha(G) - 1).\end{aligned}$$

By the above two inequalities, we obtain

$$\sum_{w \in W'} d_C(w) \leqslant |C \cap (V_1 \cup S)| + (k-2)(\alpha(G) - 1).$$

Therefore, since $\sum_{w \in W'} d_H(w) \leqslant |H \cap (V_1 \cup S)| - 1$ by (9), it follows that

$$\sum_{w \in W'} d_G(w) \leqslant |V_1 \cup S| + (k-2)(\alpha(G) - 1) - 1.$$

Summing this inequality and (8) yields that $\sum_{w \in W'} d_G(w) + d_G(v_2) \leqslant n + \kappa(G) + (k-2)(\alpha(G) - 1) - 2$, a contradiction.

Case 2. $|W| \leqslant \kappa(G) + k - 5$.

By Claim 10, we can take a $(k-3)$-subset Z of $W \cup L$ so that $|W \cap Z|$ is as large as possible. Let $W^* := Z \cap W$, $L^* := Z \cap L$ and $I^* := \{i : x_i \in L^*\}$. By Claim 9, $Z \cup \{x_0, x_1, x_2, x_3\}$ is independent. By (6), we have

$$d_C(x_1) + d_C(x_2) + d_C(x_3) \leqslant |C| + |W| + |L^*| - \sum_{i \in I^*} |D_i| + 3.$$

On the other hand, since $d_C(w) \leqslant d_C(x_0)$ for $w \in W$ and $d_C(x_0) \leqslant \alpha(G) - 2$, it follows from (2) that

$$\begin{aligned}\sum_{w \in W^* \cup \{x_0\}} d_C(w) + \sum_{i \in I^*} d_C(x_i) &\leqslant (|W^*| + 1)(\alpha(G) - 2) + \sum_{i \in I^*} |D_i| + |L^*|(\alpha(G) - 1) \\ &= (k-2)(\alpha(G) - 1) - |W^*| - 1 + \sum_{i \in I^*} |D_i|.\end{aligned}$$

Thus, we deduce

$$\sum_{v \in Z \cup \{x_0, x_1, x_2, x_3\}} d_C(v) \leqslant |C| + |W| + |L^*| - |W^*| + (k-2)(\alpha(G) - 1) + 2.$$

If $W \subseteq Z$, then $W = W^*$ and $|L^*| \leqslant |Z| = k-3$, and hence

$$\begin{aligned}\sum_{v \in Z \cup \{x_0,x_1,x_2,x_3\}} d_C(v) &\leqslant |C| + |L^*| + (k-2)(\alpha(G)-1) + 2 \\ &\leqslant |C| + (k-3) + (k-2)(\alpha(G)-1) + 2 \\ &\leqslant |C| + \kappa(G) + (k-2)(\alpha(G)-1) - 1;\end{aligned}$$

otherwise, $L^* = \emptyset$ and $|W^*| = |Z| = k-3$, and so by the assumption of Case 2,

$$\begin{aligned}\sum_{v \in Z \cup \{x_0,x_1,x_2,x_3\}} d_C(v) &\leqslant |C| + |W| - |W^*| + (k-2)(\alpha(G)-1) + 2 \\ &\leqslant |C| + \kappa(G) + k - 5 - (k-3) + (k-2)(\alpha(G)-1) + 2 \\ &= |C| + \kappa(G) + (k-2)(\alpha(G)-1).\end{aligned}$$

Therefore, by (9), we have $\sum_{v \in Z \cup \{x_0,x_1,x_2,x_3\}} d_G(v) \leqslant n + \kappa(G) + (k-2)(\alpha(G)-1) - 1$, a contradiction. $\square$

Claim 14. *$W \subseteq X$.*

Proof. If $W \setminus X \neq \emptyset$, then by Claim 9, we have $|X| \leqslant \alpha(G) - 2$, which contradicts Claim 13. $\square$

Note that $W^- \subseteq U$ by Claim 14 and Lemma 7 (i) (see Figure 4).

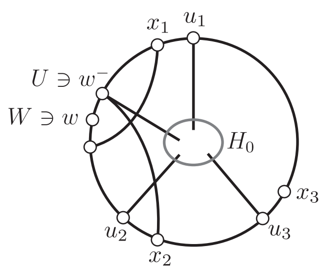

Figure 4: $W^- \subseteq U$.

Claim 15. *If $u_s \in N_C(x_t)$ for some $s, t \in [m]$, then $N_C(x_s) \cap C[u_t, u_s] \subseteq U$.*

Proof. Suppose that there exists a vertex $z \in N_C(x_s) \cap C[u_t, u_s]$ such that $z \notin U$ (see Figure 5 (i)). Since $z \notin U$, it follows from Lemma 7 (i) that $z^+ \notin X$. By Claim 13, $X \cup \{x_0, z^+\}$ is not an independent set. Hence $z^+ \in N_C(x_h)$ for some $x_h \in X \cup \{x_0\}$. Since x_s is a non-insertible vertex, it follows that $x_h \neq x_s$. Let z_s be the vertex in $C(u_s, x_s]$ such that $z \in N_G(z_s)$ and $z \notin N_G(v)$ for all $v \in C(u_s, z_s)$. By Lemma 7 (ii), we obtain $x_h \notin C[u'_s, z]$. Therefore, $x_h \in C(z, u_s] \cup \{x_0\}$. If $x_h \in C(z, u_s]$, then we let z_h be the vertex in $C(u_h, x_h]$ such that $z^+ \in N_G(z_h)$ and $z^+ \notin N_G(v)$ for all $v \in C(u_h, z_h)$. We define the cycle C^* as follows (see Figure 5 (ii) and (iii)):

$$C^* = \begin{cases} z_s \overleftarrow{C}[z, x_t]\overleftarrow{C}[u_s, z_h]C[z^+, u_h]x_0\overleftarrow{C}[u_t, z_s] & \text{if } x_h \in C(z, u_s], \\ z_s \overleftarrow{C}[z, x_t]\overleftarrow{C}[u_s, z^+]x_h\overleftarrow{C}[u_t, z_s] & \text{if } x_h = x_0. \end{cases}$$

Then, by a similar argument in the proof of Lemma 7, we can obtain a longer cycle than C by inserting all vertices of $V(C \setminus C^*)$ into C^*. This contradicts that C is longest. $\square$

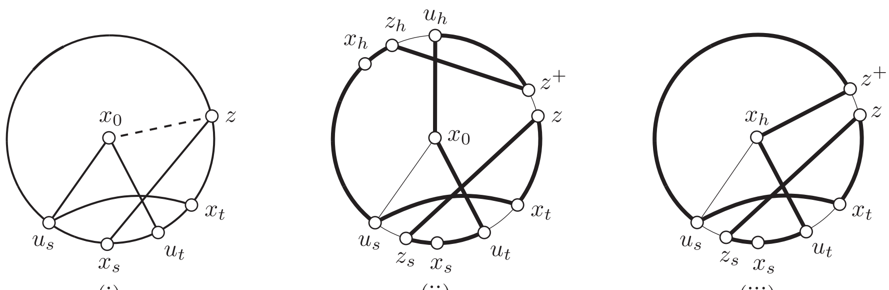

Figure 5: (i) A vertex $z \in N_C(x_s) \cap C[u_t, u_s]$ such that $z \notin U$, (ii) the cycle C^* in the case $x_h \in C(z, u_s]$ and (iii) the case $x_h = x_0$.

Notice that for each vertex $x_i \in W \cup L$, there exists $j \in [m] \setminus \{i\}$ such that $u_i \in N_C(x_j)$, and hence Claim 15 implies that $N_C(x_i) \cap C[u_j, u_i] \subseteq U$.

We divide the rest of the proof into two cases.

Case 1. $v_2 \notin U$.

Let $Y := N_G(v_2) \cap X$, and let $\gamma := |X| - \kappa(G) - 1$. Note that $x_l \notin Y$ since $x_l \in V_1$.

Claim 16. $|Y| \geqslant \gamma + 3$.

Proof. Suppose that $|Y| \leqslant \gamma + 2$. By the assumption of Case 1 and Claim 13, we have $x_0 v_2 \notin E(G)$. Since $|X \cup \{x_0\}| \geqslant k + \gamma + 2$ and $|Y| \leqslant \gamma + 2$, there exists a set I of k integers such that $\{x_0, x_l\} \subseteq \{x_i : i \in I\} \subseteq (X \cup \{x_0\}) \setminus Y$. Then $\{x_i : i \in I\} \cup \{v_2\}$ is independent. Therefore, it follows from (2) and (10) that

$$\sum_{i \in I} d_C(x_i) \leqslant |C \cap (V_1 \cup S)| + (k-2)(\alpha(G) - 1).$$

Hence, by this inequality and (8) and (9), we obtain

$$\sum_{i \in I} d_G(x_i) + d_G(v_2) \leqslant n + \kappa(G) + (k-2)(\alpha(G) - 1) - 2,$$

a contradiction. $\square$

In the rest of Case 1, we assume that $l = 1$. If $u'_r \neq u_1$, then let $r = 2$ and $u_3 = u'_2$; otherwise, let $r = 3$ and let u_2 be the vertex with $u'_2 = u_3$. By Claim 14, we obtain $Y \cup W \cup L \subseteq X \setminus \{x_1\}$. Therefore, by Claims 10 and 16 and by the definition of γ, we obtain

$$\begin{aligned}|Y \cap (W \cup L)| &= |Y| + |W \cup L| - |Y \cup (W \cup L)| \\ &\geqslant \gamma + 3 + \kappa(G) - 2 - |X \setminus \{x_1\}| \\ &= \gamma + 3 + \kappa(G) - 2 - ((\kappa(G) + \gamma + 1) - 1) = 1.\end{aligned}$$

Hence there exists a vertex $x_h \in Y \cap (W \cup L)$, that is, $v_2 \in N_C(x_h) \setminus U$. Note that if $x_h \in L$ then by the definition of L, $u_h \in N_C(\{x_1, x_2, x_3\})$; if $x_h \in W$ then by the definition of W and Claim 14, $x_h^- = u_h \in N_C(\{x_1, x_2, x_3\})$ (see Figure 4 and the paragraph below Claim 15). Since $C(x_2, x_3) \cap X = \emptyset$ and $C(x_3, x_1) \cap X = \emptyset$ if $r = 3$, either $u_h \in N_C(x_1)$ and $u_h \in C(x_3, u_1)$ or $u_h \in N_C(x_2)$ and $u_h \in C(x_1, u_2)$ holds (especially, if $r = 3$ then the latter case holds).

If $r = 2$ and $u_h \in N_C(x_1)$, then $v_2 \in C[u_1, u_h]$ (see Figure 6 (i)). If $r = 2$ and $u_h \in N_C(x_2)$, then $v_2 \in C[u_2, u_h]$ (see Figure 6 (ii)). If $r = 3$, then $u_h \in N_C(x_2)$ and $v_2 \in C[u_2, u_h]$ (see Figure 6 (iii)). In each case, we obtain a contradiction to Claim 15.

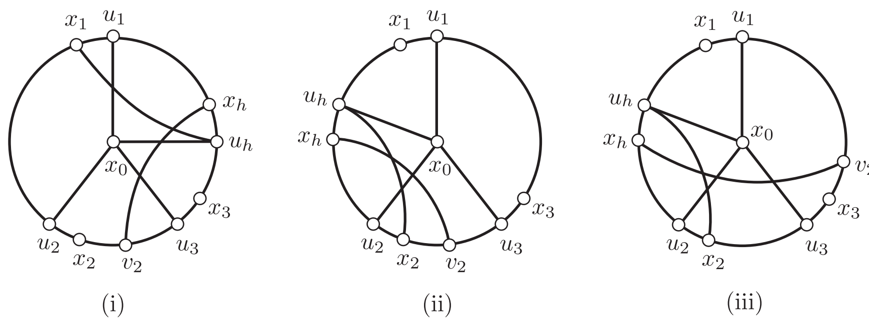

Figure 6: (i) The case $r = 2$ and $u_h \in N_C(x_1)$, (ii) the case $r = 2$ and $u_h \in N_C(x_2)$, and (iii) the case $r = 3$.

Case 2. $v_2 \in U$.

We first show that $N_C(x_i) \cap (U \setminus \{u_i\}) \neq \emptyset$ for each $x_i \in X$. For $x_i \in X$, let x'_i and x''_i be the successors of x_i and x'_i in X along the orientation of C, respectively. Let $x_1 = x_i$, $x_2 = x'_i$ and $x_3 = x''_i$. Then by Claim 10, it follows that $W \cup L \neq \emptyset$. By the definition of x'_i and x''_i, and Claim 14, we have $W_1 = W_2 = \emptyset$ (note that $W \cap \{x_1, x_2, x_3\} = \emptyset$). By the definitions of x'_i, x''_i, L_1 and L_2, we also have $L_1 = L_2 = \emptyset$. Thus $W_3 \cup L_3 \neq \emptyset$. By Lemma 7 (i) and since $W \cup L \subseteq X$, this implies that $N_C(x_i) \cap (U \setminus \{u_i\}) \neq \emptyset$.

We rename $x_i \in X$ for $i \geqslant 1$ as follows (see Figure 7 (i)): Rename an arbitrary vertex of X as x_1 (but we will re-choose x_1 later). For $i \geqslant 1$, we rename $x_{i+1} \in X$ so that $u_{i+1} \in N_C(x_i) \cap (U \setminus \{u_i\})$ and $|C[u_{i+1}, x_i)|$ is as small as possible. Let h be the minimum integer such that $x_{h+1} \in C(x_h, x_1]$. Note that this choice implies $h \geqslant 2$. We rename h

vertices in X as $\{x_1, x_2, \ldots, x_h\}$ as above (Note that the order is in opposite direction of C.), and $m-h$ vertices in $X \setminus \{x_1, x_2, \ldots, x_h\}$ as $\{x_{h+1}, x_{h+2}, \ldots, x_m\}$ arbitrarily. Set

$$A_1 := A_{h+1} := C[x_1, x_h) \quad \text{and} \quad A_i := C[x_i, x_{i-1}) \quad \text{for } 2 \leqslant i \leqslant h$$

(see Figure 7 (ii)).

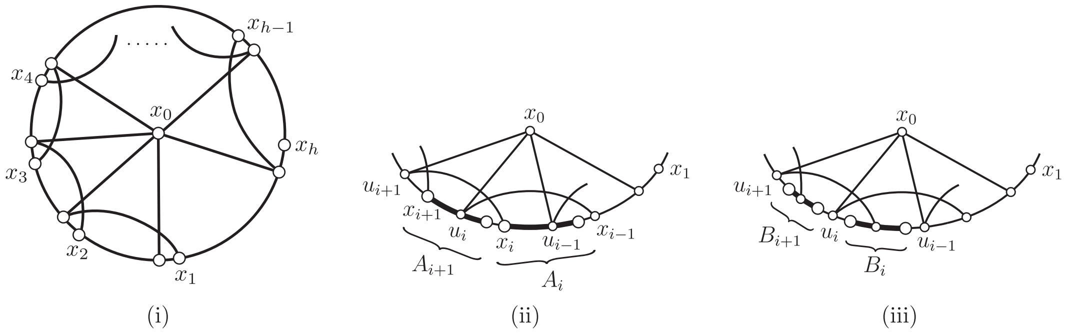

Figure 7: The definition: (i) $x_1, x_2, \ldots, x_h$, (ii) A_i, and (iii) B_i.

We divide the proof of Case 2 according to whether $h \leqslant k$ or $h \geqslant k+1$.

Subcase 2.1. $h \leqslant k$.

By the definition of $\{x_1, \ldots, x_h\}$, we have

$$N_{A_{i+1}}(x_i) \cap U \subseteq \{u_i\} \quad \text{for } 1 \leqslant i \leqslant h. \tag{11}$$

By Claim 15 and (11), we obtain

$$N_{C \setminus A_i}(x_i) \subseteq \big(U \setminus (A_i \cup A_{i+1})\big) \cup D_i \cup \{u_i\} \quad \text{for } 2 \leqslant i \leqslant h. \tag{12}$$

By Lemma 7 (i) and (ii), $N_{A_i}(x_i)^- \cap N_{A_i}(x_1) = \emptyset$ for $2 \leqslant i \leqslant h$. By Lemma 7 (i), we have $N_{A_i}(x_i)^- \cup N_{A_i}(x_1) \subseteq A_i \setminus D$ for $3 \leqslant i \leqslant h$. Thus, it follows from (12) that for $3 \leqslant i \leqslant h$

$$d_C(x_i) \leqslant \big(|U| - |(A_i \cup A_{i+1}\big) \cap U| + |D_i| + 1) + \big(|A_i| - |A_i \cap D| - d_{A_i}(x_1)\big).$$

By Lemma 7 (i) and (11), we have $N_{A_2}(x_2)^- \cup N_{A_2}(x_1) \subseteq \big(A_2 \setminus (U \cup D)\big) \cup D_1 \cup \{u_1\}$. Thus, by (12), we have

$$\begin{aligned} d_C(x_2) \leqslant &\big(|U| - |(A_2 \cup A_3) \cap U| + |D_2| + 1\big) \\ &+ \big(|A_2| - |A_2 \cap U| - |A_2 \cap D| + |D_1| + 1 - d_{A_2}(x_1)\big). \end{aligned}$$

Since $|A_1 \cap X| = |A_1 \cap U|$, it follows from Lemma 7 (i) that

$$d_{A_1}(x_1) \leqslant |A_1| - |A_1 \cap D| - |A_1 \cap X| = |A_1| - |A_1 \cap D| - |A_1 \cap U|.$$

By Claim 13, $d_C(x_0) = |U| = \alpha(G) - 1$. Thus, since $h \leqslant k$, we obtain

$$\begin{aligned}\sum_{0\leqslant i\leqslant h} d_C(x_i) &\leqslant \sum_{1\leqslant i\leqslant h} |A_i| + h|U| - 2\sum_{1\leqslant i\leqslant h} |A_i \cap U| + h + \sum_{1\leqslant i\leqslant h} |D_i| - \sum_{1\leqslant i\leqslant h} |A_i \cap D| \\ &= |C| + (h-2)|U| + h + \sum_{1\leqslant i\leqslant h} |D_i| - |D| \\ &\leqslant |C| + k + (h-2)(\alpha(G)-1) + \sum_{1\leqslant i\leqslant h} |D_i| - |D|.\end{aligned}$$

Let I be a $(k+1)$-subset of $\{0, 1, \ldots, m\}$ such that $\{0, 1, \ldots, h\} \subseteq I$. By Claim 8, $\{x_i : i \in I\}$ is independent. By the above inequality and (2), we have

$$\sum_{i\in I} d_C(x_i) \leqslant |C| + k + (k-2)(\alpha(G)-1).$$

Hence, by (3), we obtain $\sum_{i\in I} d_G(x_i) \leqslant |G| + \kappa(G) + (k-2)(\alpha(G)-1) - 1$, a contradiction.

Subcase 2.2. $h \geqslant k+1$.

We first set

$$U_1 := \{u_i \in U : x_i \in X \cap V_1\}.$$

Choose x_1 so that $A_2 \cap U_1 = \emptyset$ if possible.

By the assumption of Case 2 and the choice of r and v_2 (see the paragraph below the proof of Claim 12), we have $V_2 \cap \bigcup_{i=1}^{m} C(x_i, u_i') = \emptyset$. Hence, it follows from Claims 12 and 13 that $V_2 \subseteq N_C(x_0)$. Since $x_0 \in V_1 \cup S$ by Claim 12, this implies that $x_0 \in S$.

Claim 17. $|X \cap V_1| \leqslant k-1$.

Proof. Suppose that $|X \cap V_1| \geqslant k$. Let I be a k-subset of $[m]$ such that $I \subseteq \{i : x_i \in X \cap V_1\}$. Then $\{x_i : i \in I\} \cup \{v_2\}$ is independent. Let s and t be distinct integers in I. Since $\{x_s, x_t\} \subseteq V_1$ and $D \subseteq V_1 \cup S$, the similar argument as that of (4) implies that

$$d_C(x_s) + d_C(x_t) \leqslant |C \cap (V_1 \cup S)| - \sum_{i\in I\setminus\{s,t\}} |D_i|.$$

By (2) and (9), we have $\sum_{i\in I\setminus\{s,t\}} d_C(x_i) \leqslant \sum_{i\in I\setminus\{s,t\}} |D_i| + (k-2)(\alpha(G)-1)$ and $\sum_{i\in I} d_H(x_i) \leqslant |H \cap (V_1 \cup S)| - 1$, respectively. On the other hand, we obtain $d_G(v_2) \leqslant |V_2 \cup S| - 1$. By these four inequalities, $\sum_{i\in I} d_G(x_i) + d_G(v_2) \leqslant n + \kappa(G) + (k-2)(\alpha(G)-1) - 2$, a contradiction. Therefore $|X \cap V_1| \leqslant k-1$. $\square$

By Claim 17, we have $|U_1| \leqslant k-1$. Therefore, by the assumption of Subcase 2.2 and the choice of x_1, we obtain $A_2 \cap U_1 = \emptyset$, and hence we can take a k-subset I of $\{2, 3, \ldots, h\}$ such that $\{i : A_{i+1} \cap U_1 \neq \emptyset\} \subseteq I$. Let

$$X_I := \{x_i : i \in I\}.$$

By Claim 8, $X_I \cup \{x_0\}$ is independent. Set

$$B_1 := B_{h+1} := C(u_1, u_h) \quad \text{and} \quad B_i := C(u_i, u_{i-1}) \quad \text{for } 2 \leqslant i \leqslant h$$

(see Figure 7 (iii)). Then, since $|C[u_i, u_i')| \geqslant 2$ for $i \in [m] \setminus I$, the following inequality holds:

$$\begin{aligned} |C| &\geqslant \sum_{i\in I} |B_i \cup \{u_i\}| + 2\big(|U| - \sum_{i\in I} |(B_i \cup \{u_i\}) \cap U|\big) \\ &= \sum_{i\in I} |B_i| + 2\big(|U| - \sum_{i\in I} |B_i \cap U|\big) - k. \end{aligned} \tag{13}$$

By the definition of $\{x_1, \dots, x_h\}$, we have $N_{B_{i+1}}(x_i) \cap U = \emptyset$ for $1 \leqslant i \leqslant h$. If $x_i \in X_I \cap S$, then it follows from Lemma 7 (i) and Claim 15 that

$$\begin{aligned} d_C(x_i) &\leqslant \big(|U| - |B_i \cap U| - |B_{i+1} \cap U|\big) + \big(|B_i| - |\{x_i\}| - |(B_i \cap U)^+|\big) \\ &= |U| + |B_i| - 2|B_i \cap U| - |B_{i+1} \cap U| - 1. \end{aligned}$$

If $x_i \in X_I \cap V_1$, then by Lemma 7 (i) and Claim 15,

$$\begin{aligned} d_C(x_i) &\leqslant \big(|U| - |B_i \cap U| - |B_{i+1} \cap U| - |(U \cap V_2) \setminus (B_i \cup B_{i+1})|\big) \\ &\quad + \big(|B_i| - |\{x_i\}| - |(B_i \cap U)^+| - |(U \cap V_2) \cap B_i|\big) \\ &= |U| + |B_i| - 2|B_i \cap U| - |B_{i+1} \cap U| - 1 - |(U \cap V_2) \setminus B_{i+1}|. \end{aligned}$$

Since $U \cap V_2 \neq \emptyset$, we obtain $|(U \cap V_2) \setminus B_{i+1}| \geqslant 1$ for all $i \in I$ except for at most one, and hence

$$\sum_{i\in I\,:\,x_i \in X_I \cap V_1} |(U \cap V_2) \setminus B_{i+1}| \geqslant |X_I \cap V_1| - 1.$$

By the choice of I, we have

$$|U_1| = \sum_{i\in I} |A_{i+1} \cap U_1| = \sum_{i\in I} |B_{i+1} \cap U_1| + |X_I \cap V_1| \leqslant \sum_{i\in I} |B_{i+1} \cap U| + |X_I \cap V_1|.$$

On the other hand, since $x_0 \in S$, it follows from Claim 12 that

$$|U_1| = |X \cap V_1| = |X \setminus S| \geqslant |X| - (\kappa(G) - 1),$$

and hance

$$\sum_{i\in I} |B_{i+1} \cap U| + |X_I \cap V_1| \geqslant |U_1| \geqslant |X| - (\kappa(G) - 1). \tag{14}$$

Moreover, by Claim 13, $d_C(x_0) = |U| = |X| = \alpha(G) - 1$. Thus, we deduce

$$\sum_{i\in I\cup\{0\}} d_C(x_i) \leqslant (k+1)|U| + \sum_{i\in I} |B_i| - 2\sum_{i\in I} |B_i \cap U|$$

$$
\begin{aligned}
&- \sum_{i\in I} |B_{i+1} \cap U| - k - (|X_I \cap V_1| - 1) \\
&= \Big(\sum_{i\in I} |B_i| + 2\big(|U| - \sum_{i\in I} |B_i \cap U|\big) - k\Big) + (k-1)|U| \\
&\quad - \Big(\sum_{i\in I} |B_{i+1} \cap U| + |X_I \cap V_1|\Big) + 1,
\end{aligned}
$$

and hence, by (13) and (14),

$$
\begin{aligned}
\sum_{i\in I\cup\{0\}} d_C(x_i) &\leqslant |C| + (k-1)|U| - (|X| - \kappa(G) + 1) + 1 \\
&= |C| + \kappa(G) + (k-2)(\alpha(G) - 1).
\end{aligned}
$$

Hence, by (3), we obtain $\sum_{i\in I\cup\{0\}} d_G(x_i) \leqslant |G| + \kappa(G) + (k-2)(\alpha(G)-1) - 1$, a contradiction. □

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