On the weight of Berge-F-free hypergraphs

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Abstract

For a graph F, we say a hypergraph is a Berge-F if it can be obtained from F by replacing each edge of F with a hyperedge containing it. A hypergraph is Berge-F-free if it does not contain a subhypergraph that is a Berge-F. The weight of a non-uniform hypergraph $\mathcal{H}$ is the quantity $\sum_{h\in E(\mathcal{H})}|h|$.

Suppose $\mathcal{H}$ is a Berge-F-free hypergraph on n vertices. In this short note, we prove that as long as every edge of $\mathcal{H}$ has size at least the Ramsey number of F and at most $o(n)$, the weight of $\mathcal{H}$ is $o(n^2)$. This result is best possible in some sense. Along the way, we study other weight functions, and strengthen results of Gerbner and Palmer; and Grósz, Methuku and Tompkins.

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1 Introduction

Generalizing the notion of hypergraph cycles due to Berge, the authors Gerbner and Palmer [6] introduced so-called *Berge hypergraphs*. Given a graph F, we say that a

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hypergraph $\mathcal{H}$ is Berge-F if there is an injection $f : V(F) \to V(\mathcal{H})$ and bijection $f' : E(F) \to E(\mathcal{H})$ such that for every edge $uv \in E(F)$ we have $\{f(u), f(v)\} \subseteq f'(uv)$. Equivalently, $\mathcal{H}$ is Berge-F if we can embed a distinct graph edge into each hyperedge of $\mathcal{H}$ to obtain a copy of F. Note that for a fixed F there are many different hypergraphs that are Berge-F, and a fixed hypergraph $\mathcal{H}$ can be Berge-F for many different graphs F.

We say that a hypergraph is Berge-F-free if it does not contain a subhypergraph that is Berge-F. There are several results concerning the largest size of Berge-F-free hypergraphs, see e.g., [1, 3, 4, 5, 6, 7, 9, 10, 12, 13, 14, 11, 15]. For a short survey of extremal results for Berge hypergraphs see Subsection 5.2.2 in [8].

Most of these results deal with the uniform case, but some also examine non-uniform hypergraphs. Note that replacing a hyperedge with a larger hyperedge containing it never removes a copy of Berge-F, but may add a copy. Thus, to build a Berge-F-free hypergraph that maximizes the number of hyperedges, one picks small hyperedges. To make large hyperedges more attractive, one can assign a weight to each hyperedge that increases with the size of the hyperedge.

Győri [10] proved that if $\mathcal{H}$ is a Berge-triangle-free hypergraph, then $\sum_{h\in E(\mathcal{H})}(|h|-2) \leqslant n^2/8$ if n is large enough. Note that this result is about a multi-hypergraph $\mathcal{H}$, thus $\sum_{h\in E(\mathcal{H})}|h|$ can be arbitrarily large by taking a hyperedge of size 2 an arbitrary number of times. In [13], the authors showed that for a Berge-C_4-free multi-hypergraph $\mathcal{H}$ we have $\sum_{h\in E(\mathcal{H})}(|h|-3) \leqslant 12\sqrt{2}n^{3/2} + O(n)$ and they gave a construction of a Berge-C_4-free multi-hypergraph with approximately $n^{3/2}/8$ hyperedges. The upper bound was improved by Gerbner and Palmer [6] to $\sqrt{6}n^{3/2}/2$, while the lower bound was improved to $(1 + o(1))n^{3/2}/(3\sqrt{3})$. For arbitrary cycles, Győri and Lemons [14] proved that if $\mathcal{H}$ is either a Berge-C_{2k}-free or Berge-C_{2k+1}-free hypergraph on n vertices and every hyperedge in $\mathcal{H}$ has size at least $4k^2$, then $\sum_{h\in E(\mathcal{H})}|h| = O(n^{1+1/k})$.

Gerbner and Palmer [6] proved the following general result about Berge-F-free hypergraphs.

Theorem 1 (Gerbner and Palmer [6]). *Let F be a graph and let $\mathcal{H}$ be a Berge-F-free hypergraph on n vertices. If every hyperedge in $\mathcal{H}$ has size at least $|V(F)|$, then $\sum_{h\in E(\mathcal{H})}|h| = O(n^2)$.*

We strengthen Theorem 1 in Theorem 3 by showing that the statement still holds if one replaces $|h|$ with $|h|^2$ in the above sum; moreover, our proof is much simpler compared to the proof of Gerbner and Palmer in [6]. For uniform hypergraphs, the above theorem states that for any graph F and Berge-F-free r-uniform n-vertex hypergraph $\mathcal{H}$ we have $|E(\mathcal{H})| = O(n^2)$ provided r is large enough. Grósz, Methuku and Tompkins showed that, in fact, $|E(\mathcal{H})| = o(n^2)$ for large enough r. This is stated more precisely in the following theorem. Given two graphs F and G, let $R(F, G)$ denote the 2-color Ramsey number of F and G. If $e \in E(F)$, then we write $F \setminus e$ for the graph with $V(F \setminus e) = V(F)$ and $E(F \setminus e) = E(F) \setminus \{e\}$.

Theorem 2 (Grósz, Methuku and Tompkins [9]). *Let F be a fixed graph and $e \in E(F)$. Let $\mathcal{H}$ be an r-uniform Berge-F-free hypergraph. If $r \geqslant R(F, F \setminus e)$, then $|E(\mathcal{H})| = o(n^2)$.*

We improve this theorem in Theorem 4. Let us return to non-uniform hypergraphs. So far, we have only added up the sizes of the hyperedges. Here we will change the weight function and consider first $\sum_{h\in E(\mathcal{H})} |h|^2$.

Theorem 3. *Let F be a fixed graph. Let $\mathcal{H}$ be a Berge-F-free hypergraph on n vertices such that every edge of $\mathcal{H}$ has size at least $|V(F)|$. Then*

$$\sum_{h\in E(\mathcal{H})} |h|^2 = O(n^2).$$

Furthermore, this result is trivially sharp as can be seen by considering any hypergraph with at least one edge of size $\Omega(n)$. Interestingly, the next theorem shows that either small or large edges are necessary for such a weighted sum to reach this upper bound.

Theorem 4. *Let F be a fixed graph and let $e \in E(F)$. Let $\mathcal{H}$ be a Berge-F-free hypergraph on n vertices such that every edge of $\mathcal{H}$ has size at least $R(F, F \setminus e)$ and at most $o(n)$. Then*

$$\sum_{h\in E(\mathcal{H})} |h|^2 = o(n^2).$$

Combining Theorem 3 and Theorem 4, we can show the sum of the sizes of the edges (i.e., the weight) of a Berge-F-free hypergraph is $o(n^2)$ provided all the hyperedges are large enough, presenting another improvement of Theorem 1 and Theorem 2. In fact, this follows from a much more general theorem (which is presented below) by setting $w(m) = m$.

Theorem 5. *Define $w : \mathbb{Z}_+ \to \mathbb{Z}_+$ to be any weight function such that $w(m) = o(m^2)$. Let F be a fixed graph and let $e \in E(F)$. If $\mathcal{H}$ be a Berge-F-free hypergraph on n vertices such that every edge of $\mathcal{H}$ has size at least $R(F, F \setminus e)$, then*

$$\sum_{h\in E(\mathcal{H})} w(|h|) = o(n^2).$$

Before we prove our results, we will comment on some of the specific conditions in Theorem 5.

Theorem 5 is best possible in the sense that one cannot take a larger weight function. Indeed if $w(m) = \Omega(m^2)$, then considering a single hyperedge of size n shows that the conclusion of Theorem 5 cannot hold. More precisely, this gives a Berge-F-free hypergraph $\mathcal{H}$ with $\sum_{h\in\mathcal{H}} w(|h|) \geqslant w(n)$. On the other hand, for many weight functions with $w(m) = \Omega(m^2)$, the bound $O(w(n))$ is an upper bound on the weight of Berge-F-free hypergraphs: Indeed, if the function $\max_{1\leqslant i\leqslant n} w(i)/i^2 = O\left(w(n)/n^2\right)$ (which is achieved e.g., if $w(m)/m^2$ is eventually non-decreasing in m), then using Theorem 3 we have

$$\sum_{h\in E(\mathcal{H})} w(|h|) = \sum_{h\in E(\mathcal{H})} \frac{w(|h|)}{|h|^2}|h|^2 = O\left(\frac{w(n)}{n^2}\right) \sum_{h\in E(\mathcal{H})} |h|^2 = O(w(n)).$$

Note that in Theorem 5, the smallest possible size of edges allowed in $\mathcal{H}$ must grow with the forbidden graph F: Indeed, let r be an integer and assume $r \mid n$. Let a vertex set on n vertices be partitioned into n/r singletons and n/r sets of size $r-1$. Let $\mathcal{H}$ be the r-uniform hypergraph consisting of all the edges that contain one singleton and one $(r-1)$-set. Then it is easily verified that $\mathcal{H}$ is an r-uniform Berge-K_r-free hypergraph, but $\sum_{h \in E(\mathcal{H})} |h| = n^2/r$. In fact, it was shown by Grósz, Methuku and Tompkins [9] that there are $(\omega(F)-1)^2$-uniform Berge-F-free hypergraphs with $\Omega(n^2)$ edges, where $\omega(F)$ denotes the clique number of F. It is an interesting open problem to determine the smallest uniformity when $\Omega(n^2)$ drops to $o(n^2)$.

It is also worth noting that the bound $o(n^2)$ in Theorem 5 is close to being best possible: Erdős, Frankl and Rödl [2] constructed r-uniform hypergraphs with more than $n^{2-\varepsilon}$ hyperedges for any ε, and with the property that there are no 3 hyperedges on $3(r-1)$ vertices. Observe that a Berge-triangle is on at most $3(r-1)$ vertices, hence those hypergraphs are also Berge-triangle-free.

Notation. In the rest of the paper, we use the following notation. For a set S of vertices, let $\Gamma(S)$ denote the graph whose edge-set is the set of all the pairs contained in S. For a hypergraph $\mathcal{H}$, its 2-shadow is the graph whose edge-set is $\Gamma(\mathcal{H}) := \cup_{h \in \mathcal{H}} \Gamma(\{h\})$, i.e. all the edges contained in at least one hyperedge of $\mathcal{H}$.

2 Proofs

In this section we will use the following easy consequence of Turán's theorem: if G is F-free, then there exists a constant $\alpha = \alpha(F)$ such that the number of edges in G is at most $(1-\alpha)\binom{|V(G)|}{2}$.

2.1 Proof of Theorem 3

We will say an edge in $\Gamma(\mathcal{H})$ is *blue* if it is contained in at most $|E(F)|-1$ hyperedges of $\mathcal{H}$.

Claim 6. *Every copy of F in $\Gamma(\mathcal{H})$ contains a blue edge.*

Proof. Consider a copy of F in $\Gamma(\mathcal{H})$. If there is no blue edge in F then every edge of F is contained in at least $|E(F)|$ hyperedges of $\mathcal{H}$ by definition, so one can greedily choose different hyperedges representing the edges of F. Thus we have a Berge-F in $\mathcal{H}$, a contradiction. □

The following claim bounds the number of blue edges in a hyperedge of $\mathcal{H}$ from below.

Claim 7. *Let $h \in E(\mathcal{H})$ be a hyperedge. Then there exists a constant $\beta = \beta(F) > 0$, such that the number of blue edges in $\Gamma(h)$ is at least $\beta\binom{|h|}{2}$.*

Proof. The graph $\Gamma(h)$ is a clique on $|h| \geqslant |V(F)|$ vertices, and by Claim 6, the set of blue edges in $\Gamma(h)$ form a graph, the complement of which is F-free. Thus, Turán's theorem guarantees a constant $\beta = \beta(F) > 0$ such that there are at least $\beta\binom{|h|}{2}$ blue edges in $\Gamma(h)$. □

Using Claim 7, we have

$$\sum_{h\in E(\mathcal{H})} \beta\binom{|h|}{2} \leqslant \sum_{h\in E(\mathcal{H})} \#\{\text{Blue edges in } \Gamma(h)\}. \tag{1}$$

On the other hand, we have

$$\sum_{h\in E(\mathcal{H})} \#\{\text{Blue edges in } \Gamma(h)\} \leqslant \#\{\text{Blue edges in } \Gamma(\mathcal{H})\} \cdot (|E(F)|-1) = O(n^2). \tag{2}$$

Indeed each blue edge is counted at most $|E(F)|-1$ times in the summation as it is contained in at most $|E(F)|-1$ hyperedges of $\mathcal{H}$. Then combining equations (1) and (2), we have $\sum_{h\in E(\mathcal{H})} \beta\binom{|h|}{2} = O(n^2)$ for constant β, which implies that $\sum_{h\in E(\mathcal{H})} |h|^2 \leqslant \sum_{h\in E(\mathcal{H})} 4\binom{|h|}{2} = O(n^2)$, completing the proof.

2.2 Proof of Theorem 4

If F has one or fewer edges, the statement is trivial, so we will assume F has at least two edges throughout the rest of the proof. Here we follow an argument similar to Grósz, Methuku and Tompkins [9] but with some important changes. We wish to apply the graph removal lemma to the 2-shadow of a hypergraph $\mathcal{H}$. To this end, we prove the following claim.

Claim 8. *The number of copies of F in $\Gamma(\mathcal{H})$ is $o(n^{|V(F)|})$.*

Proof. Any copy of F in $\Gamma(\mathcal{H})$ has at least two edges (and therefore at least three vertices) in some hyperedge of $\mathcal{H}$, otherwise the hyperedges containing the edges of F would form a Berge-F. Thus we have the following upper bound:

$$\#\{F\text{-copies in } \Gamma(\mathcal{H})\} \leqslant \sum_{h\in\mathcal{H}} \binom{|h|}{3} n^{|V(F)|-3} \binom{\binom{|V(F)|}{2}}{|E(F)|} \leqslant n^{|V(F)|-3} \binom{\binom{|V(F)|}{2}}{|E(F)|} \sum_{h\in\mathcal{H}} |h|^3.$$

Indeed, there are $\binom{|h|}{3}$ ways to select three vertices from a hyperedge $h \in \mathcal{H}$ and there are at most $n^{|V(F)|-3}$ ways to select the remaining $|V(F)|-3$ vertices to form a set of $|V(F)|$ vertices. The number of copies of F in this set is bounded by $\binom{\binom{|V(F)|}{2}}{|E(F)|}$.

By our assumption, $|h| = o(n)$, and by Theorem 3, we have $\sum_{h\in\mathcal{H}} |h|^2 = O(n^2)$, so $\sum_{h\in\mathcal{H}} |h|^3 = \sum_{h\in\mathcal{H}} |h| \cdot |h|^2 = o(n) \sum_{h\in\mathcal{H}} |h|^2 = o(n^3)$. Therefore, the number of copies of F in $\Gamma(\mathcal{H})$ is $o(n^{|V(F)|})$, proving the claim. □

By Claim 8 and the graph removal lemma, there is a set $\mathcal{R}$ of $o(n^2)$ edges in $\Gamma(\mathcal{H})$ such that every copy of F in the 2-shadow of $\mathcal{H}$ contains an edge of $\mathcal{R}$. We will call an edge in the 2-shadow of $\mathcal{H}$ *special* if it is contained in $\mathcal{R}$ and is contained in at most $|E(F)|-1$ hyperedges. Note that the special edges here play a similar but slightly different role than the blue edges in the proof of Theorem 3. Let $\mathcal{R}_s$ be the set of all the special edges. Of course, $\mathcal{R}_s \subseteq \mathcal{R}$.

Recall that $e \in E(F)$, and $R(F, F\setminus e)$ denotes the Ramsey number of F versus $F\setminus e$.

Claim 9. *Let $h \in E(\mathcal{H})$ be an arbitrary hyperedge. Then any subset $S \subseteq h$ of size $R(F, F \setminus e)$ contains a special edge (i.e., $\Gamma(S) \cap \mathcal{R}_s \neq \emptyset$).*

Proof. Assume by contradiction that there is a set $S \subseteq h$ of size $R(F, F \setminus e)$ which contains no special edge. In other words, every edge of $\mathcal{R}$ contained in S is in at least $|E(F)|$ hyperedges. By the definition of $\mathcal{R}$, $\Gamma(S) \setminus \mathcal{R}$ cannot contain a copy of F. Applying Ramsey's theorem with the edges of $\Gamma(S) \setminus \mathcal{R}$ colored with the first color and those in $\Gamma(S) \cap \mathcal{R}$ colored with the second, we obtain that $\Gamma(S) \cap \mathcal{R}$ must contain a copy of $F \setminus e$. Let $\hat{e}$ be an edge contained in S whose addition would complete this copy of F. The other edges of this copy of F are each contained in at least $|E(F)|$ hyperedges of $\mathcal{H}$. Thus we can select greedily $|E(F)|$ different hyperedges of $\mathcal{H}$ to represent the edges in this copy of F: h itself for $\hat{e}$, and $|E(F)| - 1$ other hyperedges for the rest of the edges of F. These hyperedges form a Berge-F in $\mathcal{H}$, a contradiction. □

Now we provide a lower bound on the number of special edges contained in a hyperedge of $\mathcal{H}$.

Claim 10. *Let $h \in \mathcal{H}$ be a hyperedge. Then there is a constant $\gamma = \gamma(F)$ such that*

$$|\Gamma(h) \cap \mathcal{R}_s| \geqslant \gamma \binom{|h|}{2}.$$

Proof. Claim 9 implies that $\Gamma(h) \setminus \mathcal{R}_s$ does not contain a complete graph on $R(F, F \setminus e)$ vertices. So by Turán's theorem there is a constant $\gamma = \gamma(F)$ such that $\Gamma(h) \setminus \mathcal{R}_s$ contains at most $(1-\gamma)\binom{|h|}{2}$ edges. So $\Gamma(h) \cap \mathcal{R}_s$ contains at least $\gamma\binom{|h|}{2}$ edges, as desired. □

Now since $\mathcal{R}_s \subseteq \mathcal{R}$, we have $|\mathcal{R}_s| = o(n^2)$. This fact together with Claim 10 implies the following.

$$\sum_{h \in \mathcal{H}} \gamma \binom{|h|}{2} \leqslant \sum_{h \in \mathcal{H}} |\Gamma(h) \cap \mathcal{R}_s| \leqslant |\mathcal{R}_s| \left(|E(F)| - 1\right) = o(n^2).$$

Indeed, the sum $\sum_{h \in \mathcal{H}} |\Gamma(\{h\}) \cap \mathcal{R}_s|$ counts each edge of $\mathcal{R}_s$ at most $|E(F)| - 1$ times.

2.3 Proof of Theorem 5

Since $w(m) = o(m^2)$ and w is defined only on $\mathbb{Z}_+$, there are only finitely many values of m such that $w(m) > m^2$, and thus $w(m) = O(m^2)$. Let C be a constant such that $w(m) \leqslant Cm^2$ for all $m \in \mathbb{Z}_+$. Theorem 4 implies that

$$\sum_{h \in E(\mathcal{H}) : |h| \leqslant n^{1/2}} w(|h|) \leqslant \sum_{h \in E(\mathcal{H}) : |h| \leqslant n^{1/2}} C|h|^2 = o(n^2), \tag{3}$$

simply because $n^{1/2} = o(n)$. Now since $w(m) = o(m^2)$, Theorem 3 implies that

$$\sum_{h \in E(\mathcal{H}) : |h| > n^{1/2}} w(|h|) = \sum_{h \in E(\mathcal{H}) : |h| > n^{1/2}} o(|h|^2) = o\left(\sum_{h \in E(\mathcal{H}) : |h| > n^{1/2}} |h|^2 \right) = o(n^2). \tag{4}$$

So adding up (3) and (4), the proof is complete.

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