Planar Ramsey graphs

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Submitted: Dec 5, 2018; Accepted: Sep 26, 2019; Published: Oct 11, 2019

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Abstract

We say that a graph H is *planar unavoidable* if there is a planar graph G such that any red/blue coloring of the edges of G contains a monochromatic copy of H, otherwise we say that H is *planar avoidable*. That is, H is planar unavoidable if there is a Ramsey graph for H that is planar. It follows from the Four-Color Theorem and a result of Gonçalves that if a graph is planar unavoidable then it is bipartite and outerplanar. We prove that the cycle on 4 vertices and any path are planar unavoidable. In addition, we prove that all trees of radius at most 2 are planar unavoidable and there are trees of radius 3 that are planar avoidable. We also address the planar unavoidable notion in more than two colors.

Mathematics Subject Classifications: 05C55,05C10,05D10

1 Introduction

Ramsey's theorem [16] claims that any graph is Ramsey in the class of all complete graphs, i.e., for any graph G and any number k of colors there is a sufficiently large complete graph

*Supported by Independent Research Fund Denmark, 8021-002498 AlgoGraph

such that in any coloring of its edges in k colors there is a monochromatic copy of G. In general for graphs G and H, we write $G \to_k H$ and say that G *k-arrows* H if any coloring of the edges of G in k colors contains a monochromatic copy of H. We write $G \to H$ and say that G *arrows* H if $k = 2$. There are classes of graphs that are Ramsey in their own class, meaning that for any graph H in a class $\mathcal{F}$ there is a graph $G \in \mathcal{F}$ such that $G \to H$. Examples of such classes include bipartite graphs, graphs with a given clique number, and graphs of a given odd girth, see [13, 14]. Here, we are concerned with Ramsey properties of the class of all planar graphs. We say that a planar graph H is *k-planar unavoidable* if there is a planar graph G such that $G \to_k H$, otherwise we call H *k-planar avoidable*. Similarly, we define *outerplanar unavoidable* and *outerplanar avoidable* graphs. When $k = 2$, we write *planar unavoidable* instead of 2-*planar unavoidable*, or, if clear from context, simply *unavoidable*. The complexity of the problem to edge-color planar graphs with a given number of colors so that there is no monochromatic copy of a given graph was addressed by Broersma et al. [3]. Clearly, if a graph H is planar unavoidable, then H has a relatively sparse Ramsey graph. The related problem of bounding local density of Ramsey graphs has been addressed for example in [17] and [11]. Let us also remark that there are several papers [10, 18, 19] on so-called "planar Ramsey numbers," which however address entirely different questions.

A result of Gonçalves [8] states that any planar graph can be edge-colored in two colors so that each color class is an outerplanar graph. Thus any planar unavoidable graph is outerplanar. The Four Color Theorem [2] implies that any planar graph is a union of two bipartite graphs. In general any graph that is 2^k-colorable for $k \in \mathbb{N}$ is a union of at most k bipartite graphs. In fact, encoding the colors by bit-vectors of length k, the bipartition classes of the i-th bipartite graph correspond to the bit-vectors with a 0, respectively a 1, in the i-th position.

This shows that any planar unavoidable graph is bipartite and outerplanar and thus gives necessary conditions for planar unavoidability.

Next we give several sufficient conditions. Here, a *generalized broom* is a union of a path and a star such that they share only the center of the star. In particular, any path is also a generalized broom.

Theorem 1. *If H is a cycle on 4 vertices, a tree of radius at most 2, or a generalized broom, then H is planar unavoidable. Moreover, if H is a path, then it is outerplanar unavoidable.*

The next result shows that not only odd cycles and non-outerplanar graphs are planar avoidable, but also some trees.

Theorem 2. *There is a planar avoidable tree of radius 3 and an outerplanar avoidable tree of radius 2.*

By Theorem 1 any planar avoidable tree has at least 8 vertices, while in the proof of Theorem 2 we give a planar avoidable tree on 106 vertices.

A result of Hakimi et al. [6], see also [1], states that any planar graph can be edge-decomposed into at most five star forests, i.e., if a graph H is k-planar unavoidable for some $k \geqslant 5$, then H is a star forest. In fact for every $k \geqslant 5$ the k-planar unavoidable graphs are precisely the star forests, since any forest of s stars with t edges each is contained monochromatically in any k-coloring of a forest of $(s-1)k+1$ stars with $(t-1)k+1$ edges each. Next we summarise our results for k-planar unavoidable graphs, for $k=3$ and 4.

Theorem 3. *If H is k-planar unavoidable for $k \geqslant 3$, then H is a forest. If H is 4-planar unavoidable, then H is a caterpillar forest. There are 3- and 4-planar avoidable trees of radius 2.*

Moreover, there are 3- and 4-planar avoidable trees on 10 and 6 vertices, respectively.

We provide some definitions in Section 2. Sections 3, 4, and 5 contain the proofs of Theorems 1, 2, and 3 respectively. Finally Section 6 states some concluding remarks and open questions.

2 Definitions

We denote a complete graph, a path, and a cycle on n vertices by K_n, P_n, and C_n, respectively. A complete bipartite graph with parts of sizes m and n is denoted by $K_{m,n}$. For an integer k, $k \geqslant 2$, a *k-ary tree* is a rooted tree in which each vertex has at most k children. A *perfect* k-ary tree is a k-ary tree in which every non-leaf vertex has k children and all leaf vertices have the same distance from the root. For all other standard graph theoretic definitions, we refer the reader to the book of West [20].

Iterated Triangulation Tr(n):
An *iterated triangulation* is a plane graph $\mathrm{Tr}(n)$ defined as follows: $\mathrm{Tr}(0) = K_3$ is a triangle, $\mathrm{Tr}(i) \subseteq \mathrm{Tr}(i+1)$, $\mathrm{Tr}(i+1)$ is obtained from $\mathrm{Tr}(i)$ by inserting a vertex in each of the inner faces of $\mathrm{Tr}(i)$ and connecting this vertex with edges to all the vertices on the boundary of the respective face, see Figure 1. We see that $\mathrm{Tr}(i)$ is a triangulation and each triangle of $\mathrm{Tr}(i)$ bounds a face of $\mathrm{Tr}(j)$ for some $j \leqslant i$.

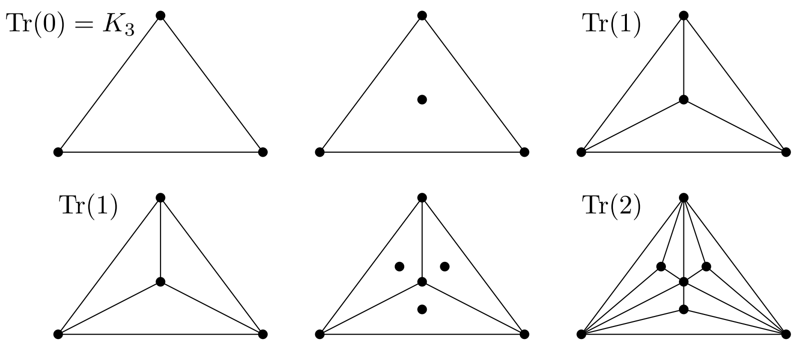

Figure 1: The iterative construction of Tr(2).

Universal outerplanar graph UOP(n):
A *universal outerplanar graph* $\mathrm{UOP}(n)$ is defined as follows: UOP(1) is a triangle. An

edge on the outer face is called an *outer* edge. For $k > 1$, UOP(k) is an outerplanar graph that is a supergraph of UOP$(k-1)$ obtained by introducing, for each outer edge $e = xy$, a new vertex v_e and new edges: $v_e x$ and $v_e y$. Then the set of outeredges of UOP(k) is $\{v_e x, v_e y : \ e = xy$ is an outeredge of UOP$(k-1)\}$. Observe that UOP$(n) \subseteq$ Tr$(n+1)$. In fact, starting with UOP(1) as an inner face of Tr(2) that is bounded by three inner edges, we find the construction sequence for UOP(n) as part of the construction sequence for Tr$(n+1)$.

Triangulated Grid Gr(n):
Let a *triangulated grid* be a graph $\text{Gr}(n) = (V, E)$ with $V = [n] \times [n]$ and $(k, j)(k', j') \in E$ if and only if either ($k = k'$ and $|j - j'| = 1$) or ($|k - k'| = 1$ and $j = j'$) or ($k = k' - 1$ and $j = j' - 1$) or ($k = k' + 1$ and $j = j' + 1$). We define left, right, top, and bottom sides of the grid as subsets of vertices $[n] \times \{1\}$, $[n] \times \{n\}$, $\{1\} \times [n]$, and $\{n\} \times [n]$ respectively.

Fish:
A graph G is called a *fish* and denoted $F_{x,y}$ if $V(G) = \{x, y\} \cup S$, where $S \cap \{x, y\} = \emptyset$, x and y are each adjacent to each vertex in S, S induces a path in G, and xy is an edge. We call S the set of *spine vertices*, $G[S]$ is called the *spine*, xs, ys are called *ribs*, $s \in S$, and the paths x, s, y of length 2 are called *double ribs*. Sometimes we say that a fish $F_{x,y}$ *hangs* on an edge xy. In an edge-colored fish, a double rib is called *bicolored* if there are different colors used on two edges of this double rib. We will call two double ribs x, s, y and x, s', y, with $s \neq s'$, $s, s' \in S$, *identically bicolored*, if the same color is used on both of the edges xs and xs', and a different color is used on both of the edges sy and $s'y$. Note that for any positive integers m and k and for any edge $xy \in E(\text{Tr}(m))$, there is a fish on xy in Tr$(m+k)$ with k spine vertices. Indeed, consider an inner face xyz in Tr(m). We can pick spine vertices $s_1, \ldots, s_k$ such that $s_i \in V(\text{Tr}(m+i) - \text{Tr}(m+i-1))$, $i \in \{1, \ldots, k\}$ and such that s_i is inserted in the face xys_{i-1} of Tr$(m+i-1)$, $i = 1, \ldots, k$, $s_0 = z$.

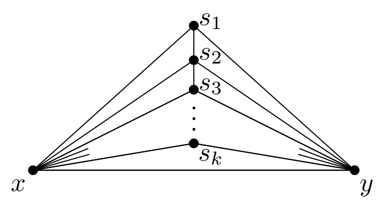

Figure 2: A fish $F_{x,y}$ with k spine vertices.

3 Proof of Theorem 1

Theorem 1 follows immediately from the following lemmas.

The following proof closely resembles the Hex-lemma [7].

Lemma 4. *Let G be a near-triangulation with outer cycle C, that is, G is a planar graph with outer face boundary C and each other face is bounded by a triangle. Let a, b, c, d be vertices on C in clockwise order dividing the edges of C in four paths $C(a, b)$, $C(b, c)$, $C(c, d)$, $C(d, a)$, respectively. If the edges of G are colored red and blue, then either there is a blue path from $C(a, b)$ to $C(c, d)$ or a red path from $C(b, c)$ to $C(d, a)$ (or both).*

Proof. Suppose there is no blue path from $C(a, b)$ to $C(c, d)$. Then the red graph contains a minimal edge-cut separating $C(a, b)$ and $C(c, d)$. A minimal edge-cut in G is a cycle C' in the dual graph G^*. This cycle C' must contain the vertex v^* corresponding to the outer face of G. Since G is a near-triangulation, it follows that any two consecutive edges of C' (except the two edges incident with v^*) correspond to two edges in G that are incident with the same vertex. Thus the edges in G corresponding to the edges in C' contain a red path path from $C(b, c)$ to $C(d, a)$. □

Corollary 5. *Any path is planar unavoidable, even in a class of planar graphs of bounded degrees (in fact of maximum degree at most 6).*

Proof. If the edges of the triangulated grid $\mathrm{Gr}(k)$ are colored red or blue, then there is a monochromatic P_k by Lemma 4, where the paths $C(a, b), C(b, c), C(c, d), C(d, a)$ correspond to the top, right, bottom, and the left sides of the grid. □

The above gives planar graphs of bounded maximum degree that arrow arbitrarily long paths, which however have large tree-width. Complementary, we can find planar graphs of tree-width 2 that also arrow arbitrarily long paths, where however the maximum degree is large.

Lemma 6. *Any path is outerplanar unavoidable. In particular, for any positive integer n, $\mathrm{UOP}(n^2) \to P_n$.*

Proof. We shall show that $\mathrm{UOP}(n^2) \to P_n$. Let $G = G_{n^2} = \mathrm{UOP}(n^2)$ and let it be edge-colored red and blue. We see that each edge of G is on the outer face of $G_i = \mathrm{UOP}(i)$ for some $i \leqslant n^2$, where $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_{n^2}$ as in the definition of the universal outerplanar graph. Consider the unique outerplanar embedding of G and for each edge e, consider G_i such that e is on the outer face of G_i. For a vertex let its rank be the least $i \in \{1, \dots, n^2\}$ for which it is in the vertex set of G_i. For each edge e, we define graphs $G(\mathrm{out}, e)$ and $G(\mathrm{in}, e)$ such that $G = G(\mathrm{out}, e) \cup G(\mathrm{in}, e)$, where $G(\mathrm{out}, e)$ and $G(\mathrm{in}, e)$ share only the edge e and no vertices except for the endvertices of e. We require in addition that $G(\mathrm{in}, e)$ contains G_1 as a subgraph, see Figure 3. Observe that among vertices of rank i in G there are two at distance i in G, $i = 1, \dots, n^2$.

For an edge e in G_i with endvertex v of rank i, we define the following. Let $R(e)$ be a longest red path in G_i with last edge e and last vertex v. Let $B(e)$ be a longest blue path in $G(\mathrm{in}, e)$ with last vertex v. We shall write $e > e'$ for two edges of G if $|R(e)| > |R(e')|$, or $|R(e)| = |R(e')|$ and $|B(e)| > |B(e')|$.

Consider the edges in G_n. Suppose that the endvertices of each such edge belong to the same blue component. Then for any two vertices of rank n, there is a blue path joining

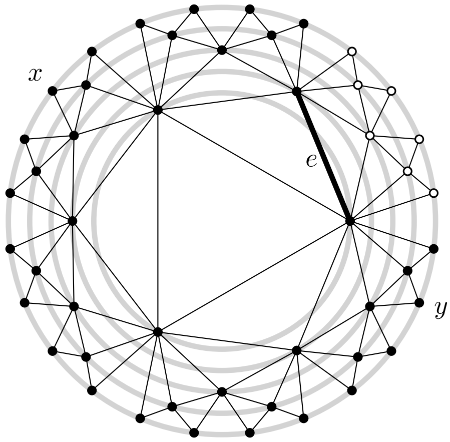

Figure 3: The universal outerplanar graph UOP(5). Vertices with the same rank lie on concentric circles. For the thick edge e, the vertices in $G(\text{in}, e)$ are shown in black. The two vertices x, y in UOP(5) have distance 5.

them. Since there are two such vertices at distance at least n in G, we see that there is a blue path on n edges, and we are done. So, assume that e_n is an outer edge of G_n such that its endvertices belong to different blue components of G. Assume we constructed a sequence of edges $e_n < e_{n+1} < \cdots < e_{n+i}$ of outer edges in $G_n, \ldots, G_{n+i}$ respectively such that the endvertices of each of these edges belong to different blue components of G. Consider $e = e_{n+i}$, we shall construct e_{n+i+1}. Let $e = uv$ with v of rank $n + i$ and let e', e'' be two adjacent outer edges of G_{n+i+1} that are incident to u and v, respectively. Let $e' = uw$ and $e'' = vw$ where w has rank $n + i + 1$. Note that either (u and w) or (v and w) are in different blue components in G, otherwise u and v would have been in the same blue component. See Figure 4 for illustrations.

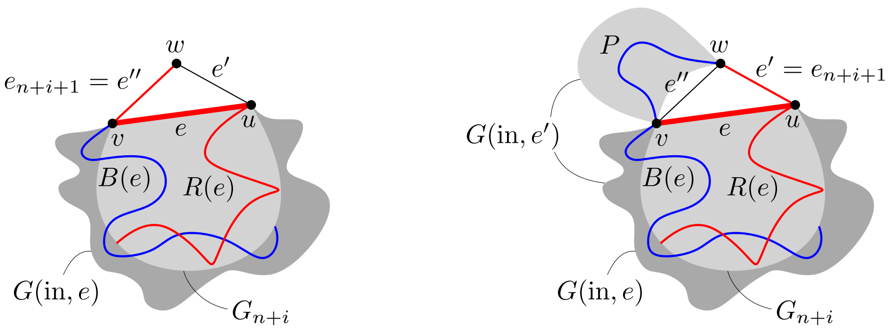

Figure 4: Illustrations of Case 1 (left) and Case 2 (right) in the proof of Lemma 6.

Case 1. v and w are in different blue components. Then $e'' = vw$ is red and the path $R(e) \cup vw$ is a red path in G_{n+i+1} of length $|R(e)| + 1$ ending in e'' at vertex w. Then let $e_{n+i+1} = e''$. We see that $e'' > e$.

Case 2. v and w are in the same blue component and u and w are in different blue components. Then $e' = uw$ is red and the path $(R(e) - uv) \cup uw$ is a red path of length $|R(e)|$ in G_{n+i+1} ending with e' at vertex w. Since v and w are in the same blue component, there is a blue path P of length q, $q \geqslant 1$, between them in $G(\text{out}, e'')$. The union of P and the blue path $B(e)$ ending at v in $G(\text{in}, e)$ forms a blue path ending at w in $G(\text{in}, e')$. Let $e_{n+i+1} = e'$. We see that $e' > e$.

We can continue in this manner until rank n^2, i.e., we create a desired sequence $e_n < \cdots < e_{n^2}$. Note that $|R(e_i)| \geqslant 1$ and $|B(e_i)| \geqslant 0$ for $i = n, \dots, n^2$, and $(|R(e_i)|, |B(e_i)|) \neq (|R(e_j)|, |B(e_j)|)$ whenever $i \neq j$. As there are exactly $n^2 - n + 1 = (n-1)n + 1$ edges in this sequence, there exists some $i \in \{n, \dots, n^2\}$ with $|R(e_i)| \geqslant n$ or $|B(e_i)| \geqslant n$, proving that there is a red or a blue path of length at least n. □

Lemma 7. *Any generalized broom is planar unavoidable.*

Proof. Let H be a union of P_{2k+1} and $K_{1,k}$ that share only their center vertices. Note that any generalized broom on at most k vertices is a subgraph of H. Let n be sufficiently large, say $n \geqslant 10k^2$. Consider $G = \text{Tr}(n(n+1))$ colored red and blue. Since $\text{UOP}(n^2) \subseteq \text{Tr}(n^2+1)$, we see, using Lemma 6, that there is monochromatic path P on edges $e_1, \dots, e_n$ in order in a two-edge colored $\text{Tr}(n^2+1)$, say P is red. Consider a set $\mathcal{F}$ of $n - 2k$ fishes hanging on $e_{k+1}, e_{k+2}, \dots, e_{n-k}$ respectively such that the spines of fishes from $\mathcal{F}$ are pairwise disjoint and each fish has at least $4k$ spine vertices. If at least one of these fishes contains a red star of size k centered at a vertex of P, we have a red H. Otherwise, each fish in $\mathcal{F}$ contains at least $2k$ blue double ribs. The union of blue subgraphs of fishes from $\mathcal{F}$ clearly contains a blue copy of H. □

Lemma 8. *Any tree of radius 2 is planar unavoidable.*

Proof. Let H be a perfect k-ary tree of radius 2 with $k \geqslant 2$. Consider $\text{Tr}(19k)$ together with a fixed edge coloring in red and blue.

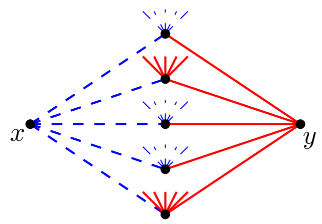

Figure 5: Two monochromatic stars of different colors with the same leaf-set.

Claim 9. *If there is a red star S_r and a blue star S_b on $2k$ edges in $\text{Tr}(n)$, $n \leqslant 18k$, such that the stars have the same leaf-set L, then there is a monochromatic copy of H in $\text{Tr}(n+k)$.*

Let x, y denote the centers of S_r and S_b, respectively. Each vertex $z \in L$ has at least $2k$ neighbors in $\text{Tr}(n+k)$ that are not neighbors of any vertex in $L - z$. Hence, by pigeonhole

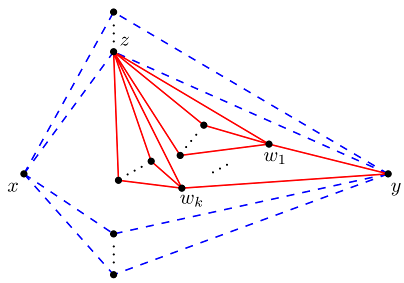

Figure 6: Part of a fish F with k blue double ribs, part of a fish $F_{x,s}$ with red double ribs, and part of a fish $F_{s,s'}$ between s and s' with monochromatic red double ribs.

principle z is the center of a monochromatic star on k edges, whose leaves have distance at least two to $L - z$, see Figure 5. At least k of these monochromatic stars are of the same color that together with either S_r or S_b form a monochromatic copy of H. This proves the Claim.

Now consider any two adjacent vertices x, y in $\mathrm{Tr}(n)$, $n \leqslant 12k$, and the set L of their at least $6k$ common neighbors in $\mathrm{Tr}(n + 6k)$. By the Claim, we may assume that fewer than $2k$ vertices of L have a red edge to x and a blue edge to y, and fewer than $2k$ vertices of L have a blue edge to x and a red edge to y. Each of the remaining at least $2k$ vertices in L has its edges to x and y in the same color, and by pigeonhole principle we may assume that for at least k of these vertices this the same color. We let $K(x, y)$ denote this monochromatic copy of $K_{2,k}$ in $\mathrm{Tr}(n + 6k)$.

Finally, consider two adjacent vertices x, y in $\mathrm{Tr}(0)$. Say that $K(x, y) \subset \mathrm{Tr}(6k)$ is blue. For each vertex z in $K(x, y) - \{x, y\}$ consider a different inner face of $\mathrm{Tr}(6k)$ bounded by z, y, and a third vertex. Each such face of $\mathrm{Tr}(6k)$ together with all vertices of $\mathrm{Tr}(18k)$ in its interior gives a copy of $\mathrm{Tr}(12k)$ in $\mathrm{Tr}(18k)$. If for every such vertex z we find a monochromatic $K(z, a)$ in its copy of $\mathrm{Tr}(12k)$ which is blue for some a, then there is a blue copy of H, as desired. So assume that for at least one vertex z in $K(x, y) - \{x, y\}$ all monochromatic $K(z, a)$ for for all a are red; see Figure 6. Then in particular $K(z, y) \subseteq \mathrm{Tr}(12k)$ is red with vertices z, y and $w_1, \dots, w_k$. Moreover, for each $i = 1, \dots, k$ the monochromatic $K(z, w_i) \subset \mathrm{Tr}(18k)$ is red. However, this gives a red copy of H rooted at y; see Figure 6. $\square$

Lemma 10. *A cycle C_4 is planar unavoidable. For $n \geqslant 16$, $\mathrm{Tr}(n) \to C_4$.*

Proof. Consider the graph G consisting of a fish $F_{x,y}$ hanging on edge xy with 15 spine vertices $s_1, \dots, s_{15}$, and a vertex of degree three in each face of $F_{x,y}$ bounded by two spine vertices; see the left part of Figure 7. Note that $G \subset \mathrm{Tr}(15)$. Consider any fixed edge-coloring of G in red and blue.

First we claim that $F_{x,y}$ contains a monochromatic C_4 or a monochromatic inner face f such that any two vertices u, v of f have a common neighbor w in $F_{x,y}$, not in f, such that edges uw and vw have the same color. To this end, consider the spine vertices $s_1, \dots, s_{15}$ and the corresponding double ribs x, s_i, y, $i = 1, \dots, 15$. If $F_{x,y}$ contains no monochromatic C_4, at most two double ribs are monochromatic – one red and one blue.

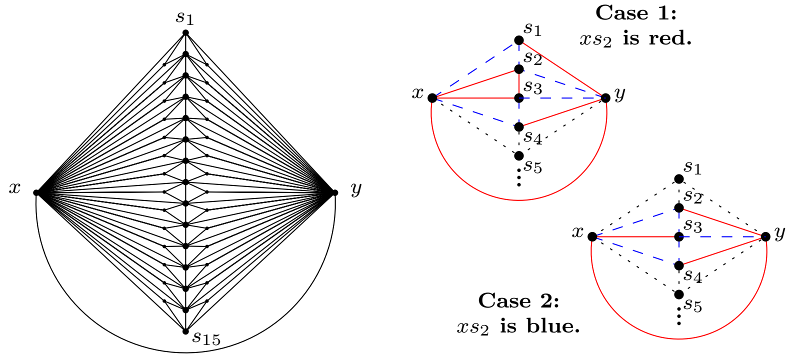

Figure 7: Left: A planar graph G with $G \to C_4$. Right: Illustrations for the two cases in the proof of Lemma 10.

Hence there are five consecutive spine vertices $s_i, \dots, s_{i+4}$ whose double ribs are bicolored. Assume, without loss of generality, that $i = 1$. Further assume that the edges xy and xs_3 are red, so the edge s_3y is blue.

Case 1: xs_2 is red. Then s_2y is blue. If the spine edge s_2s_3 is blue, then s_2, s_3, y bound an inner face f with the desired properties ensured by the vertex x that sends red edges to f. So we may assume that s_2s_3 is red. For the same reason, if xs_1 is also red, then also s_1s_2 is red, giving a red C_4 with vertices x, s_1, s_2, s_3. So we may assume that xs_1 is blue and hence s_1y is red. Now if s_1s_2 is red, there is a red C_4 with vertices x, y, s_1, s_2. So we may assume that s_1s_2 is blue.

Symmetrically, we may assume that xs_4 is blue, hence s_4y is red, and s_3s_4 is blue. But now s_2, s_3, x bound an inner face with the desired properties; see the right part of Figure 7.

Case 2: xs_2 is blue. Then s_2y is red. Now if s_2s_3 is red, we have a red C_4 with vertices x, y, s_2, s_3. So we may assume that s_2s_3 is blue. By symmetry we may also assume that xs_4 is blue, hence s_4y is red, and s_3s_4 is blue. But then we have a blue C_4 with vertices x, s_2, s_3, s_4; see the right part of Figure 7.

This proves the claim that $F_{x,y}$ contains a monochromatic C_4 or a monochromatic inner face f, say in red, such that any two vertices of f are joined by a blue P_3 in $F_{x,y}$. In the former case we are done. In the latter case note that as f is all red, any two vertices of f are also joined by a red P_3 in $F_{x,y}$. Now consider the vertex z in $G - F_{x,y}$ whose three neighbors are the vertices of f. As two of the three edges incident to z have the same color, there are two vertices in f that are joined by two distinct but identically colored P_3's in G. That is, there is a monochromatic copy of C_4 in G. □

To close this section, let us prove Theorem 1.

Proof of Theorem 1. By Lemma 10 the cycle C_4 is planar unavoidable. By Lemma 8 any tree of radius at most 2 is planar unavoidable. By Lemma 7 any generalized broom is planar unavoidable. By Corollary 5 any path is planar unavoidable. By Lemma 6 any path is outerplanar unavoidable. □

4 Proof of Theorem 2

Nash-Williams [12] proved that any graph G can be edge-decomposed into at most k forests if and only if every subgraph $H = (V_H, E_H)$ of G satisfies $|E_H| \leqslant k(|V_H| - 1)$. Using Euler's formula, it is easy to verify that this inequality holds for $k = 3$ when G is planar, for $k = 2$ when G is outerplanar, and for $k = 2$ when G is planar and bipartite. As each forest can be oriented with out-degree at most 1 at each vertex, we obtain the following corollary, which also follows from [5].

Lemma 11 (Nash-Williams [12], Hakimi [5])**.**

- *Any planar graph admits an orientation with out-degree at most* 3 *at each vertex.*
- *Any outerplanar graph admits an orientation with out-degree at most* 2 *at each vertex.*
- *Any bipartite planar graph admits an orientation with out-degree at most* 2 *at each vertex.*

Let T_1 be a tree of radius 3 with root r and all vertices of distance $0, 1, 2$ to r having degree 5. See Figure 8. Let G be a planar graph. Let V_1, V_2, V_3 be a partition of $V(G)$ such that each V_i induces a linear forest in G, $i = 1, 2, 3$, such a partition exists by a result of Poh [15]. Further, consider an orientation of G with out-degree at most 3 at each vertex, given by Lemma 11. For $i = 1, 2, 3$ color the edges in $G[V_i]$ alternately red and blue along the paths in $G[V_i]$. For each remaining directed edge uv of G we have $u \in V_i$ and $v \in V_j$ for some $i \neq j$. Color uv red if $i < j$ and blue if $i > j$. See Figure 8. Assume that there is a monochromatic copy of T_1, say red. Since the out-degree of each vertex in G is at most 3, we see that each non-leaf vertex of T_1 has at least two incoming edges. Due to the color alternation in each $G[V_i]$, at least one of the two incoming edges has its two endvertices in distinct parts. In particular, the root r is in V_2 or in V_3. Then at least one vertex at distance 1 or 2 from r is in V_1. However, the vertices of V_1 have in-degree at most 1 in the red graph, a contradiction.

Let T_2 be a tree of radius 2 with root r and all vertices of distance $0, 1$ to r having degree 4. See Figure 8. Similarly, let G be an outerplanar graph. Let V_1, V_2 be a partition of $V(G)$ such that each V_i induces a linear forest in G, $i = 1, 2$, such a partition exists by a result of Cowen et al. [4]. Further, consider an orientation of G with out-degree at most 2 at each vertex, given by Lemma 11. For $i = 1, 2$ color the edges in $G[V_i]$ alternately red and blue along the paths in $G[V_i]$. For each remaining directed edge uv of G we have $u \in V_i$ and $v \in V_j$ for some $i \neq j$. Color uv red if $i < j$ and blue if $i > j$. See Figure 8. Assume that there is a red copy of T_2. Since the out-degree of each vertex in G is at most 2, each non-leaf vertex of T_2 has two incoming edges. Thus the root r is in V_2 and at least one of its neighbors is in V_1, a contradiction.

The trees T_1 and T_2 have 106 and 17 vertices, respectively, and are illustrated in Figure 8. We know that every planar avoidable tree has at least 8 vertices since it has radius at least three and it is not a generalized broom.

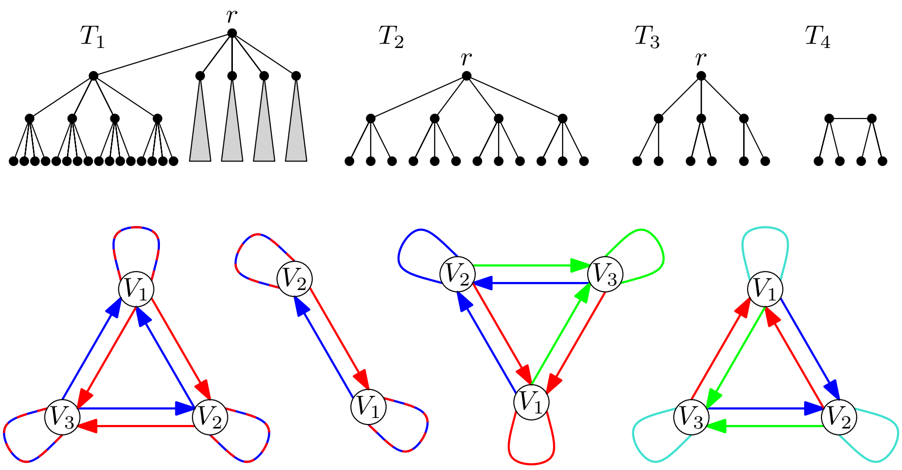

Figure 8: Illustrations of trees $T_1, \dots, T_4$ defined in the proofs of Theorem 2 and 3: T_1 is planar avoidable with 2 colors. T_2 is avoidable with 2 colors in the class of outerplanar graphs. T_3 is planar avoidable with 3 colors. T_4 is planar avoidable with 4 colors. The colorings below illustrate patterns of how to color any planar (outerplanar) graph on basis of a partition V_1, V_2, V_3 (V_1, V_2) of the vertices, and an orientation of the edges between the parts.

5 Proof of Theorem 3

A result of Nash-Williams [12] implies that any planar graph can be edge-decomposed into at most three forests. Thus any graph H that is not a forest is 3-planar avoidable. Another result of Gonçalves [9] states that any planar graph can be edge-colored in four colors so that each color class is a forest of caterpillars. Thus any graph H that is not a caterpillar forest is 4-planar avoidable.

For the remainder of the proof let G be any planar graph. Let V_1, V_2, V_3 be a partition of the vertex set $V(G)$ so that $G[V_i]$ is a linear forest [15]. We shall define two colorings c_3 and c_4 of the edges of G with three and four colors, respectively. To this end, consider the bipartite subgraphs B_1, B_2, B_3 of G with partitions (V_2, V_3), (V_1, V_3), (V_1, V_2), and containing all edges of G between respective parts. For each $i = 1, 2, 3$ orient the edges of B_i so that the out-degree at each vertex is no more than 2. (Such an orientation exists by Lemma 11.)

Coloring c_3: For $i = 1, 2, 3$, color all edges in $G[V_i]$ and all edges of G that are oriented incoming at a vertex of V_i in color i.

Coloring c_4: For $i = 1, 2, 3$, color all edges of G that are oriented incoming at a vertex of V_i in color i. Further, color all edges in $G[V_1]$, $G[V_2]$, $G[V_3]$ in color 4.

Next we show that a tree T_3 of radius 2 with root r and all vertices of distance $0, 1$ to r of degree at least 3 (see Figure 8) is 3-planar avoidable. We claim that c_3 does not contain a monochromatic copy of T_3. In fact, if v is any vertex with at least three incident edges of the same color i, then v must be a vertex in V_i. However, $G[V_i]$ has maximum degree at most 2, while the vertices of degree at least 3 in T_3 induce a subgraph of maximum degree at least 3. Hence there is no monochromatic copy of T_3 in G under coloring c_3.

Finally, we show that a symmetric double star T_4 on 6 vertices, i.e, a tree with two adjacent vertices of degree 3 and four leaves (see Figure 8) is 4-planar avoidable. We claim that c_4 does not contain a monochromatic copy of T_4. First, color 4 is a disjoint union of paths, and thus there is no copy of T_4 in color 4. For color $i \in \{1, 2, 3\}$ we see that, as before, only vertices in V_i may have three incident edges of color i. However, as V_i is an independent set in the subgraph of color i, there is no copy of T_4 in that subgraph. Hence there is no monochromatic copy of T_4 in G under coloring c_4, as desired. □

Let us remark that coloring c_3 shows that every graph H in which the vertices of degree at least 3 induce a subgraph of maximum degree at least 3 is 3-planar avoidable. Similarly, coloring c_4 shows that every graph H with an odd-length path whose two endvertices have degree at least three each, is 4-planar avoidable.

6 Conclusions

In this paper we initiated the study of Ramsey properties of planar graphs. When two colors are considered, only some outerplanar bipartite graphs are unavoidable and even some trees are avoidable. We showed that C_4 is unavoidable. The following questions remain open:

1. Are other even cycles unavoidable?

2. What is the smallest number of vertices in an avoidable tree?

All of our positive results, showing that some graphs are unavoidable, use the fact that the iterated triangulation $\mathrm{Tr}(n)$ arrows these graphs.

3. Is it true that for each planar unavoidable graph H there is $n = n(H)$ such that $\mathrm{Tr}(n) \rightarrow G$?

Acknowledgments

The authors would like to thank the anonymous referees for their careful reading and helpful comments, which led to an improved presentation of our results.

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