Ear decomposition and balanced
neighborly simplicial manifolds

Hailun Zheng

Department of Mathematics
University of Michigan
Ann Arbor, MI, 48109, USA

`hailunz@umich.edu`

Abstract

We find the first non-octahedral balanced 2-neighborly 3-sphere and the balanced 2-neighborly triangulation of the lens space $L(3,1)$. Each construction has 16 vertices. We show that there exists a balanced 3-neighborly non-spherical 5-manifold with 18 vertices. We also show that the rank-selected subcomplexes of a balanced simplicial sphere do not necessarily have an ear decomposition.

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1 Introduction

A simplicial complex is called k-neighborly if every subset of vertices of size at most k is the set of vertices of one of its faces. Neighborly complexes, especially neighborly polytopes and spheres, are interesting objects to study. In the seminal work of McMullen [12] and Stanley [19], it was shown that in the class of polytopes and simplicial spheres of a fixed dimension and with a fixed number of vertices, the cyclic polytope simultaneously maximizes all the face numbers. The d-dimensional cyclic polytope is $\lfloor \frac{d}{2} \rfloor$-neighborly. Since then, many other classes of neighborly polytopes have been discovered. We refer to [4], [16] and [18] for examples and constructions of neighborly polytopes. Meanwhile, the notion of neighborliness was extended to other classes of objects: for instance, neighborly cubical polytopes were defined and studied in [8], [9], and [17], and neighborly centrally symmetric polytopes and spheres were studied in [1], [3], [7] and [14].

In this paper we discuss a similar notion for balanced simplicial complexes. Balanced complexes were defined by Stanley in [20], where they were called completely balanced. A $(d-1)$-dimensional simplicial complex is called balanced if its graph is d-colorable. For instance, the barycentric subdivision of regular CW complexes and order complexes are

balanced. We say that a balanced simplicial complex is *balanced k-neighborly* if every set of k or fewer vertices with *distinct* colors forms a face. For example, if Δ_1 and Δ_2 are balanced k-neighborly spheres, then the join $\Delta = \Delta_1 * \Delta_2$ is also a balanced neighborly k-sphere, and we call Δ *join-decomposable*. However, apart from the cross-polytopes, it is not known whether other *join-indecomposable* balanced k-neighborly polytopes or spheres exist. To the best of our knowledge, no examples of such objects appear in the current literature, even for $k = 2$. As for balanced 2-neighborly manifolds, one such construction that triangulates the sphere bundle is given in [11]; it is also a minimal balanced triangulation of the underlying topological space.

This more or less explains why so far there is even no plausible sharp upper bound conjecture for balanced spheres or manifolds. The goal of this paper is to partially remedy this situation by searching for balanced neighborly spheres and manifolds of lower dimensions. It turns out that even in the lower dimensional cases balanced neighborly spheres or manifolds with a given number of vertices do not always exist.

- The octahedral 3-sphere is the only balanced 2-neighborly 3-sphere with less than 16 vertices.

- There is a unique balanced 2-neighborly 4-sphere with 15 vertices, known as $^415_2^5$ in [10].

- There exists a balanced 3-neighborly non-spherical 5-manifold with 18 vertices.

- There are two constructions of balanced 2-neighborly 3-manifolds with 16 vertices; one triangulates the sphere, and the other triangulates the lens space $L(3,1)$.

In a different direction, it is also interesting to ask whether every rank-selected subcomplex of a balanced simplicial polytope or sphere has a convex ear decomposition. This statement, if true, would imply that rank-selected subcomplexes of balanced simplicial polytopes possess certain weak Lefschetz properties, see Theorem 3.9 in [22]. As a consequence, it would also provide an alternative proof of the balanced Generalized Lower Bound Theorem, see Theorem 3.3 and Remark 3.4 in [13]. We present an example giving a negative answer to this question for 3-dimensional spheres.

The structure of this manuscript is as follows. In Section 2, after reviewing basic definitions, we establish basic properties of balanced neighborly spheres; in particular, we prove that for some values of f_0, such spheres cannot exist. In Section 3 we discuss how to find balanced k-neighborly $(2k-1)$-manifolds from less neighborly balanced $(2k-2)$-spheres, for $k = 2, 3$. In Section 4, we construct a balanced 2-neighborly 3-sphere with 16 vertices. In Section 5, we present the balanced 2-neighborly triangulation of $L(3,1)$ with 16 vertices. In Section 6 we provide a way to construct balanced spheres whose rank-selected subcomplex does not have an ear decomposition.

2   Basic properties of balanced neighborly spheres

A *simplicial complex* Δ with vertex set V is a collection of subsets $\sigma \subseteq V$, called *faces*, that is closed under inclusion, and such that for every $v \in V$, $\{v\} \in \Delta$. For $\sigma \in \Delta$, let

$\dim \sigma := |\sigma| - 1$ and define the *dimension* of Δ, $\dim \Delta$, as the maximum dimension of the faces of Δ. A *facet* is a maximal face under inclusion. We say that a simplicial complex Δ is *pure* if all of its facets have the same dimension.

If Δ is a simplicial complex and σ is a face of Δ, the *star* of σ in Δ is $\mathrm{st}_\Delta \sigma := \{\tau \in \Delta : \sigma \cup \tau \in \Delta\}$. We also define the *link* of σ in Δ as $\mathrm{lk}_\Delta \sigma := \{\tau - \sigma \in \Delta : \sigma \subseteq \tau \in \Delta\}$, and the *deletion* of a subset of vertices W from Δ as $\Delta \backslash W := \{\sigma \in \Delta : \sigma \cap W = \varnothing\}$. If Δ_1 and Δ_2 are simplicial complexes on disjoint vertex sets, then the join of Δ_1 and Δ_2, denoted $\Delta_1 * \Delta_2$, is the simplicial complex with vertex set $V(\Delta_1) \cup V(\Delta_2)$ whose faces are $\{\sigma_1 \cup \sigma_2 : \sigma_1 \in \Delta_1, \sigma_2 \in \Delta_2\}$.

If Δ is a pure $(d-1)$-dimensional complex such that every $(d-2)$-dimensional face of Δ is contained in at most two facets, then the *boundary complex* of Δ consists of all $(d-2)$-dimensional faces that are contained in exactly one facet, as well as their subsets. A simplicial complex Δ is a *simplicial sphere* (resp. *simplicial ball*) if the geometric realization of Δ is homeomorphic to a sphere (resp. ball). The boundary complex of a simplicial d-ball is a simplicial $(d-1)$-sphere. A simplicial sphere is called *polytopal* if it is the boundary complex of a convex polytope. For instance, the boundary complex of an octahedron is a polytopal sphere; we will refer to it as an octahedral sphere.

For a fixed field or group $\mathbf{k}$, we say that Δ is a $(d-1)$-dimensional $\mathbf{k}$-*homology sphere* if $\tilde{H}_i(\mathrm{lk}_\Delta \sigma; \mathbf{k}) \cong \tilde{H}_i(\mathbb{S}^{d-1-|\sigma|}; \mathbf{k})$ for every face $\sigma \in \Delta$ (including the empty face) and $i \geqslant -1$. A *homology d-ball* (over $\mathbf{k}$) is a d-dimensional simplicial complex Δ such that (i) Δ has the same homology as the d-dimensional ball, (ii) for every face F, the link of F has the same homology as the $(d - |F|)$-dimensional ball or sphere, and (iii) the boundary complex $\partial \Delta$ is a homology $(d-1)$-sphere. The classes of simplicial $(d-1)$-spheres and homology $(d-1)$-spheres coincide when $d \leqslant 3$. From now on all homology are computed with coefficients in $\mathbb{Z}$ and we will omit it from our notation.

Next we define a special structure that exists in some pure simplicial complexes.

Definition 1. An *ear decomposition* of a pure $(d-1)$-dimensional simplicial complex Δ is an ordered sequence $\Delta_1, \Delta_2, \cdots, \Delta_m$ of pure $(d-1)$-dimensional subcomplexes of Δ such that:

1. Δ_1 is a simplicial $(d-1)$-sphere, and for each $j = 2, 3, \cdots, m$, Δ_j is a simplicial $(d-1)$-ball.

2. For $2 \leqslant j \leqslant m$, $\Delta_j \cap (\cup_{i=1}^{j-1} \Delta_i) = \partial \Delta_j$.

3. $\cup_{i=1}^m \Delta_i = \Delta$.

We call Δ_1 the *initial complex*, and each Δ_j, $j \geqslant 2$, an *ear of this decompostion*. Notice that this definition is more general than Chari's original definition of a *convex ear decomposition*, see [2, Section 3.2], where the Δ_i's are required to be subcomplexes of the boundary complexes of polytopes. In particular, if a complex has no ear decomposition, then it has no convex ear decomposition. However, by the Steinitz theorem, all simplicial 2-spheres are polytopal, and hence also all simplicial 2-balls can be realized as subcomplexes of the boundary complexes of 3-dimensional polytopes. So for 2-dimensional

simplicial complexes, the notion of an ear decomposition coincides with that of a convex ear decomposition.

A $(d-1)$-dimensional simplicial complex Δ is called *balanced* if the graph of Δ is d-colorable, or equivalently, there is a coloring map $\kappa : V \to [d]$ such that $\kappa(x) \neq \kappa(y)$ for any edge $\{x, y\} \in \Delta$. Here $[d] = \{1, 2, \cdots, d\}$ is the set of colors. We denote by V_i the set of vertices of color i. A balanced simplicial complex is called *balanced k-neighborly* if every set of k or fewer vertices with distinct colors forms a face. We say e is a *missing colored edge* if $e \notin \Delta$ and the vertices of e have distinct colors. For $S \subseteq [d]$, the subcomplex $\Delta_S := \{F \in \Delta : \kappa(F) \subseteq S\}$ is called the *rank-selected subcomplex* of Δ. We also define the *flag f-vector* $(f_S(\Delta) : S \subseteq [d])$ and the *flag h-vector* $(h_S(\Delta) : S \subseteq [d])$ of Δ, respectively, by letting $f_S(\Delta) := \#\{F \in \Delta : \kappa(F) = S\}$, where $f_\varnothing(\Delta) = 1$, and $h_S(\Delta) := \sum_{T \subseteq S}(-1)^{\#S - \#T} f_T(\Delta)$. The usual f-numbers and h-numbers can be recovered from the relations $f_{i-1}(\Delta) = \sum_{\#S = i} f_S(\Delta)$ and $h_i(\Delta) = \sum_{\#S = i} h_S(\Delta)$.

In the remainder of this section, we establish some restrictions on the possible size of color sets of balanced neighborly spheres.

Lemma 2. *Let Δ be a balanced k-neighborly homology $(2k-1)$-sphere. Then Δ has the same number of vertices of each color. In particular, $f_0(\Delta) = 2k\ell$ for some $\ell \geqslant 2$.*

Proof. Let $W \subseteq [2k]$ be an arbitrary subset of the set of the colors with $|W| = k$. Since Δ is balanced k-neighborly, Δ_W is also balanced k-neighborly, and hence Δ_W is the join of k color sets of colors in W, each considered as a 0-dimensional complex. By the definition of the join and the flag h-numbers, we have $f_{U \cup \{i\}}(\Delta) = f_U(\Delta) f_{\{i\}}(\Delta)$ and hence $h_{U \cup \{i\}}(\Delta) = h_U(\Delta) h_{\{i\}}(\Delta)$ for all $i \in W$, $U \subset W$ and $i \notin U$. Therefore,

$$\prod_{i \in W}(|V_i| - 1) = \prod_{i \in W} h_{\{i\}}(\Delta) = h_W(\Delta) \overset{(*)}{=} h_{[2k] \setminus W}(\Delta) = \prod_{i \in [2k] \setminus W} h_{\{i\}}(\Delta) = \prod_{i \in [2k] \setminus W}(|V_i| - 1),$$
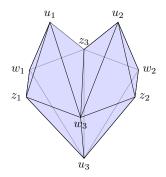
where $(*)$ follows from the Dehn-Sommerville relations. Since W is an arbitrary k-subset of $[2k]$, it follows that each color set in Δ must have the same size. $\qquad\square$

Remark 3. Lemma 2 not only holds for homology $(2k-1)$-spheres but also for orientable homology $(2k-1)$-manifolds. Indeed by replacing the flag h-numbers with the flag h''-numbers (see [6] for definition), Theorem 4.1 in [6] gives $h''_W(\Delta) = h''_{[2k] \setminus W}(\Delta)$, which further implies that $h_W(\Delta) = h_{[2k] \setminus W}(\Delta)$ since both W and $[2k] \setminus W$ are of size k. The rest of the proof is the same.

Unfortunately, the above lemma is not sufficient to tell whether a balanced k-neighborly homology $(2k-1)$-sphere or manifold with $2k\ell$ vertices can exist for given $k, \ell \geqslant 2$.

3   Balanced neighborly $(d-1)$-manifolds with $3d$ vertices

In this section, we consider balanced $\lfloor \frac{d}{2} \rfloor$-neighborly $(d-1)$-manifolds (for $d = 3, 4, 5$) with each color set of size 3. We begin with the following lemma.
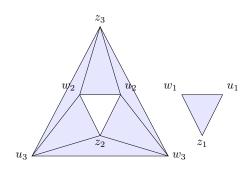
Figure 1: Left: triangulation of the vertex link $\mathrm{lk}_\Delta v_i$ for $v_i \in V_4$, where $\{u_1, u_2, u_3\}$, $\{w_1, w_2, w_3\}$ and $\{z_1, z_2, z_3\}$ are the three other color sets. Right: the complex Σ.

Lemma 4. *Let $d \geqslant 4$. If Δ is a balanced homology $(d-1)$-sphere and $V_d = \{v_1, v_2, v_3\}$ is the set of vertices of color d, then $\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$ is a homology $(d-2)$-ball for any $1 \leqslant i < j \leqslant 3$, and $\cap_{k=1}^3 \mathrm{lk}_\Delta v_k$ is a homology $(d-3)$-sphere.*

Proof. Let $\{i, j, k\} = [3]$ be distinct, $\Sigma = \mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$ and $\Gamma = \cap_{k=1}^3 \mathrm{lk}_\Delta v_k$. We first prove that Σ and Γ have the same homology as a simplicial $(d-2)$-ball and simplicial $(d-3)$-sphere respectively. Since each $(d-2)$-face of Δ is contained in exactly 2 facets, it follows that $\mathrm{lk}_\Delta v_i \cup \mathrm{lk}_\Delta v_j = \Delta_{[d-1]}$. By the Mayer-Vietoris sequence, for any $n \geqslant 0$,

$$\cdots \to H_{n+1}(\Delta_{[d-1]}) \to H_n(\Sigma) \to H_n(\mathrm{lk}_\Delta v_i) \oplus H_n(\mathrm{lk}_\Delta v_j) \to H_n(\Delta_{[d-1]}) \to \cdots . \quad (1)$$

Note that $\Delta_{[d-1]}$ is a deformation retract of Δ minus three points, hence $\beta_{d-2}(\Delta_{[d-1]}) = 2$ and $\beta_k(\Delta_{[d-1]}) = 0$ for $0 \leqslant k \leqslant d-3$. We conclude from (1) that $\beta_k(\Sigma) = 0$ for all $k \geqslant 0$. Since $\mathrm{lk}_\Delta v_k \cup \Sigma = \Delta_{[d-1]}$ and $\mathrm{lk}_\Delta v_k \cap \Sigma = \Gamma$, by the Mayer-Vietoris sequence we obtain

$$\cdots \to H_{n+1}(\Delta_{[d-1]}) \to H_n(\Gamma) \to H_n(\mathrm{lk}_\Delta v_k) \oplus H_n(\Sigma) \to H_n(\Delta_{[d-1]}) \to \cdots .$$

Hence $\beta_{d-3}(\Gamma) = 1$ and $\beta_k(\Gamma) = 0$ for $0 \leqslant k \leqslant d-4$.

Next, for any $\tau \in \Gamma$, we have $\mathrm{lk}_\Sigma \tau = \mathrm{lk}_{\mathrm{lk}_\Delta \tau} v_i \cap \mathrm{lk}_{\mathrm{lk}_\Delta \tau} v_j$ and $\mathrm{lk}_\Gamma \tau = \cap_{i=1}^3 \mathrm{lk}_{\mathrm{lk}_\Delta \tau} v_i$. Since $\mathrm{lk}_\Delta \tau$ is a balanced homology $(d-1-|\tau|)$-sphere, using the same argument as above, we may show that $\mathrm{lk}_\Sigma \tau$ and $\mathrm{lk}_\Gamma \tau$ have the same homology as a $(d-2-|\tau|)$-ball and $(d-3-|\tau|)$-sphere respectively. Therefore Γ is a homology $(d-3)$-sphere. Finally, for any interior face σ of Σ, $\mathrm{lk}_\Sigma \sigma = \mathrm{lk}_{\mathrm{lk}_\Delta v_i} \sigma = \mathrm{lk}_{\mathrm{lk}_\Delta v_j} \sigma$, and hence $\mathrm{lk}_\Sigma \sigma$ is a homology sphere. By definition we conclude that Σ is a homology $(d-2)$-ball. $\square$

Remark 5. The complex Γ in Lemma 4 is not balanced, since Γ is $(d-1)$-colorable instead of being $(d-2)$-colorable.

Proposition 6. *The only balanced 2-neighborly 3-manifold with 12 vertices triangulates the nonorientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$.*

Proof. Let Δ be a balanced 2-neighborly 3-manifold with 12 vertices. Its f-vector is $f(\Delta) = (1, 12, 54, 84, 42)$. By Lemma 2 and Remark 3, each color set of Δ has three

vertices. We let $V_4 = \{v_1, v_2, v_3\}$ be the set of vertices of color 4. Since Δ is balanced 2-neighborly, each $\mathrm{lk}_\Delta v_i$ is a 2-sphere with 9 vertices, its f-vector is $(1,9,21,14)$. Furthermore, the balancedness of Δ implies that every vertex $u \in \mathrm{lk}_\Delta v_i$ has $\deg_{\mathrm{lk}_\Delta v_i} u = 4$ or 6. If x is the number of vertices of degree 6 in $\mathrm{lk}_\Delta v_i$, then

$$4(9-x) + 6x = \sum_{u \in \mathrm{lk}_\Delta v_i} \deg(\mathrm{lk}_{\mathrm{lk}_\Delta v_i} u) = 2f_1(\mathrm{lk}_\Delta v_i) = 42,$$

and hence $x = 3$. A balanced 2-sphere with 9 vertices, 3 of which have degree 6, is unique up to isomorphism, as shown in Figure 1. Hence all vertex links in Δ are combinatorially isomorphic.

Since $f_i(\Delta_{[3]}) = f_i(\Delta) - \sum_{j=1}^3 f_{i-1}(\mathrm{lk}_\Delta v_j)$ for all $i \leqslant 2$, we have $f(\Delta_{[3]}) = (1, 9, 27, 21)$. Let $i, j, k \in [3]$ be distinct and $\Sigma := \mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$. Any facet F of Σ are 2-dimensional, for otherwise if F is an edge, then $\mathrm{lk}_\Delta F$ is either a 4-cycle or 6-cycle, where in both cases v_i and v_j share at least one common neighbor w, i.e., $F \cup \{w\} \in \Sigma$, a contradiction. Similarly, the facet cannot be 0-dimensional. Also the facets of Σ do not belong to $\mathrm{lk}_\Delta v_k$. Hence Σ is a pure 2-dimensional subcomplex of $\mathrm{lk}_\Delta v_i$ with 9 vertices and $f_2(\Delta_{[3]}) - f_2(\mathrm{lk}_\Delta v_k) = 7$ triangles.

On the other hand, for any $u \in \Sigma$, the vertex link $\mathrm{lk}_\Delta u$ is isomorphic to Figure 1 and $v_i, v_j \in \mathrm{lk}_\Delta u$. The intersection of links of arbitrary two vertices of the same color has the following property

$(*)$: $\quad \mathrm{lk}_\Sigma u = \mathrm{lk}_{\mathrm{lk}_\Delta u} v_i \cap \mathrm{lk}_{\mathrm{lk}_\Delta u} v_j$ is either an edge, or a path of length 3.

Since Σ is 2-dimensional, each connected component of Σ has at least 3 vertices. If there are three components, then Σ is the disjoint union of three triangles, contradicting that $f_2(\Sigma) = 7$. Otherwise, if Σ is connected, then by observation $(*)$ we have that Σ is a triangulated 2-manifold (with boundary). Since $21 = 3f_2(\Sigma) = \sum_{u \in \Sigma} f_1(\mathrm{lk}_\Sigma u)$ and by observation $(*)$ $f_1(\mathrm{lk}_\Sigma u) \in \{1, 3\}$, the links of three vertices in Σ are single edges, while the rest are paths of length 3. However, enumeration based on observation $(*)$ yields that there is no such complex Σ.

The last case is that Σ has two connected components. From observation $(*)$ we see that each component cannot have 4 or 5 vertices. If one component is the triangle, then the other component (as a 6-triangle subcomplex of the 9-vertex balanced 2-sphere) must be the triangulated annulus as shown in Figure 1. By symmetry $\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_k$ and $\mathrm{lk}_\Delta v_j \cap \mathrm{lk}_\Delta v_k$ are also isomorphic to Σ. In this way we determine $\Delta_{[3]} = \mathrm{lk}_\Delta v_i \cup \mathrm{lk}_\Delta v_j$: it is the union of three octahedral 2-spheres, each having a pair of antipodal facets $(F_1, F_2), (F_2, F_3)$ and (F_3, F_1), respectively. This also determines Δ, which is the balanced triangulation of the nonorientable $\mathbb{S}^2$-bundle over $\mathbb{S}^1$ known as $^3 12_2^{83}$ in [10]. $\qquad\square$

Remark 7. The balanced 2-neighborly manifold $^3 12_2^{83}$ is also known as BM_4 defined in [11]. In particular in [11, Proposition 6.9] it is shown that $^3 12_2^{83}$ is the only balanced 12-vertex 3-manifold with $\beta_1 \neq 0$. See [11] and [23] for extension in higher dimensional cases.

Next we characterize all balanced 3-spheres with each color sets of size 3.

Lemma 8. *Up to an isomorphism, there are four triangulations of balanced* 3-*spheres with each color set of size* 3.

Proof. Let Δ be such a sphere and let $V_4 = \{v_1, v_2, v_3\}$. Each vertex link of Δ is a balanced 2-sphere with at most 9 vertices, hence it is either the octahedral sphere, the suspension of a 6-cycle, or the connected sum of two octahedral spheres. We denote these three 2-spheres as Σ_1, Σ_2 and Σ_3 respectively. By Lemma 4, $\Delta_{[3]}$ is the union of three triangulated 2-balls $B_i = \mathrm{lk}_\Delta v_j \cap \mathrm{lk}_\Delta v_k$, where $\{i, j, k\} = [3]$, glued along their common boundary complex c. Assume that $f_0(\mathrm{lk}_\Delta v_i) \leqslant f_0(\mathrm{lk}_\Delta v_j)$ when $i \leqslant j$. An easy counting leads to

$$f_0(\Delta_{[3]}) = f_0(c) + \sum_{i=1}^{3} f_0(B_i \backslash c) = 9, \quad f_0(\mathrm{lk}_\Delta v_i) = f_0(c) + f_0(B_j \backslash c) + f_0(B_k \backslash c) \in \{6, 8, 9\},$$
(2)

where $f_0(B_i \backslash c)$ counts the number of interior vertices of B_i. By the Dehn-Sommerville relations, the f-vector of any triangulated 3-manifold satisfies that $f_1 = f_3 + f_0$. Since every facet of Δ contains exactly one vertex from V_4, we have that $f_3(\Delta) = \sum_{i=1}^{3} f_2(\mathrm{lk}_\Delta v_i)$ and hence

$$f_1(\Delta) = \sum_{i=1}^{3} f_2(\mathrm{lk}_\Delta v_i) + f_0(\Delta) = \sum_{i=1}^{3} (2 f_0(\mathrm{lk}_\Delta v_i) - 4) + 12 = 2 \sum_{i=1}^{3} f_0(\mathrm{lk}_\Delta v_i) \leqslant 54,$$

we enumerate the combinatorial type of each $f_0(\mathrm{lk}_\Delta v_i)$ as follows:

Case 1: $\mathrm{lk}_\Delta v_1 \cong \Sigma_1$. It follows that $\mathrm{lk}_\Delta v_3$ is obtained from $\mathrm{lk}_\Delta v_2$ by a cross flip (see [5] for a reference). Since $f_0(\Delta_{[3]}) = 9$, either $\mathrm{lk}_\Delta v_2 \cong \Sigma_1$, $\mathrm{lk}_\Delta v_3 \cong \Sigma_3$, and the cross flip replaces a 2-face of $\mathrm{lk}_\Delta v_2$ with its complement in the octahedral sphere. Or $\mathrm{lk}_\Delta v_2 \cong \Sigma_2$, $\mathrm{lk}_\Delta v_3 \cong \Sigma_3$, and the cross flip replaces the union of three 2-faces of $\mathrm{lk}_\Delta v_2$ with its complement in the octahedral sphere. In the first case the 3-sphere is the connected sum of two octahedral 3-spheres, which we denote as S_1. In the second case we obtain a 3-sphere S_2 (with $\mathrm{lk}_{S_2} v_i \cong \Sigma_i$ for $i \in [3]$). Their f-vectors are $f(S_1) = (1, 12, 42, 60, 30)$ and $f(S_2) = (1, 12, 46, 68, 34)$.

Case 2: $\mathrm{lk}_\Delta v_1 \cong \Sigma_2$. In this case the number of missing colored edges in $\Delta_{[3]}$ is $\frac{9 \cdot 6}{2} - f_1(\Delta_{[3]}) = 27 - \sum_{i=1}^{3} f_0(\mathrm{lk}_\Delta v_i)$, which equals either $1,2$ or 3.
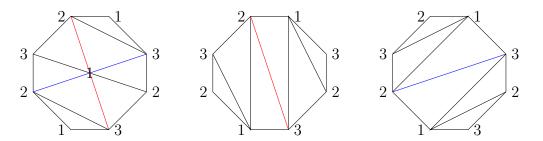
Subcase 1: $\mathrm{lk}_\Delta v_2 \cong \mathrm{lk}_\Delta v_3 \cong \Sigma_2$. By conditions (2), c is a 6-cycle and $\Delta_{[3]} \backslash c$ consists of three disjoint vertices of degree either 4 or 6. Note that in Σ_2 every pair vertices of degree 4 and degree 6 forms an edge. Since $\Delta_{[3]}$ has only three missing edges between vertices of different colors, the vertices in $\Delta_{[3]} \backslash c$ are of the same color and has degree 6. Hence $\Delta_{[3]}$ is the join of c and three disjoint vertices and Δ is the join of two 6-cycles. Denote this sphere as S_3; its f-vector is $(1, 12, 48, 72, 36)$.

Subcase 2: $\mathrm{lk}_\Delta v_i \cong \Sigma_i$ for $i = 2, 3$. Then c is a 7-cycle; furthermore, B_3 has no interior vertices and B_1, B_2 have a unique interior vertex b_1, b_2 respectively. Since $\Delta_{[3]}$ has two missing colored edges, and three vertices of degree 6 form an empty triangle in $\mathrm{lk}_\Delta v_3 \cong \Sigma_3$, WLOG assume that $\deg b_1 = 4$ and $\deg b_2 = 6$. The only vertex b_3 not connected to b_2 in B_2 must be the vertex of degree 6 in $\mathrm{lk}_\Delta v_1 \cong \Sigma_2$. Hence B_3 is the join

of b_3 with the path of length 5, which is $c \backslash b_3$. But then b_3 is also the degree-6 vertex in $\mathrm{lk}_\Delta v_2 \cong \Sigma_2$, and there is no way to triangulate B_1 such that it shares no common interior edge with B_2.

Subcase 3: $\mathrm{lk}_\Delta v_2 \cong \mathrm{lk}_\Delta v_3 \cong \Sigma_3$. Then c is an 8-cycle and only B_1 has an interior vertex a. Also $\Delta_{[3]}$ has one missing colored edge, so a is of degree 6. Three vertices of degree 6 in Σ_3 are of different colors, hence in $\mathrm{lk}_\Delta v_2$ and $\mathrm{lk}_\Delta v_3$ the other two vertices of degree 6 must be two pairs of antipodal vertices in $\mathrm{lk}_{\Delta_{[3]}} a$ (the other pair of antipodal vertices in $\mathrm{lk}_{\Delta_{[3]}} a$ is the missing edge in $\Delta_{[3]}$). In this way we recover $\mathrm{lk}_\Delta v_2 = B_1 \cup B_3, \mathrm{lk}_\Delta v_3 = B_1 \cup B_2$, where B_1, B_2, B_3 are the 2-balls shown in the figure below (from left to right); in particular, labels of the vertices represent the color and the blue (red resp.) edges form the missing triangle in $\mathrm{lk}_\Delta v_2$ ($\mathrm{lk}_\Delta v_3$ resp.). We call this 3-sphere S_4. Its f-vector is $(1, 12, 52, 80, 40)$.

Case 3: $\mathrm{lk}_\Delta v_1 \cong \Sigma_3$. This is not possible by Proposition 6. $\qquad\square$

Theorem 9. *There exists a unique balanced 2-neighborly homology 4-sphere $^4 15_2^5$ with each color set of size 3.*

Proof. Let Δ be such a sphere and let its color set $V_5 = \{v_1, v_2, v_3\}$. By Alexander Duality, $\tilde{H}_i(\Delta_{\{4,5\}}) \cong \tilde{H}_{3-i}(\Delta_{[3]})$. In particular, since $\Delta_{\{4,5\}}$ is balanced 2-neighborly, $\beta_2(\Delta_{[3]}) = \beta_1(\Delta_{\{4,5\}}) = 4$ and $\beta_1(\Delta_{[3]}) = 0$. Hence

$$f_2(\Delta_{[3]}) = (f_1 - f_0 + \chi)(\Delta_{[3]}) = \frac{9 \cdot 6}{2} - 9 + 5 = 23.$$

By double counting, $\sum_{i=1}^3 f_1(\mathrm{lk}_\Delta v_i) = \sum_{W = \{i,j,5\} \subseteq [5]} f_2(\Delta_W) = \binom{4}{2} f_2(\Delta_{[3]}) = 138$. By Proposition 6 and Lemma 8, $f_1(\mathrm{lk}_\Delta v_i) \in \{42, 46, 48, 52\}$, it follows that either $138 = 42 + 48 \cdot 2$, that is, $\mathrm{lk}_\Delta v_1 \cong S_1$ and $\mathrm{lk}_\Delta v_2, \mathrm{lk}_\Delta v_3 \cong S_3$; or $138 = 46 \cdot 3$ and $\mathrm{lk}_\Delta v_i \cong S_2$ for all i.

Consider the first case above. S_1 is the connected sum of two octahedral 3-spheres. For any 2-subset $W \subset [4]$, the induced subcomplex $(S_1)_W$ is the union of two 4-cycles glued along an edge, so $f_1((S_1)_W) = 7$. Similarly, S_3 is the join of two 6-cycles, so we have $(S_3)_W$ is either a 6-cycle or the bipartite graph $K_{3,3}$, i.e., $f_1((S_3)_W) = 6$ or 9. Hence

$$23 = f_2(\Delta_{W \cup \{5\}}) = \sum_{i=1}^3 f_1((\mathrm{lk}_\Delta v_i)_W) \in \{19, 22, 25\},$$

a contradiction.

Now we consider the second case, where all vertex links in Δ are isomorphic to S_2. Let $\Gamma = \mathrm{lk}_\Delta\, v_1 \cap \mathrm{lk}_\Delta\, v_2 \cap \mathrm{lk}_\Delta\, v_3$. The proof of Lemma 8 implies that for any vertex $p \notin V_5$, $\mathrm{lk}_\Gamma\, p = \mathrm{lk}_{\mathrm{lk}_\Delta\, p}\, v_1 \cap \mathrm{lk}_{\mathrm{lk}_\Delta\, p}\, v_2 \cap \mathrm{lk}_{\mathrm{lk}_\Delta\, p}\, v_3$ is a 5-cycle (as the boundary of the union of three 2-faces, where we apply the cross-flip). Hence Γ must be the boundary of the icosahedron. Since all $\mathrm{lk}_\Delta\, v_i$ are isomorphic, by Lemma 4 Γ divides the 3-sphere $\mathrm{lk}_\Delta\, v_1$ into two 3-balls, each having the same number of facets. If the facets of $\mathrm{lk}_\Delta\, v_1$ are labeled as in the link of vertex 1 in ${}^415_2^5$ (this is a vertex-transitive triangulation of 4-sphere whose vertex links are isomorphic to S_2, see [10]), then one such Γ is the intersection of vertices $1, 6, 8$ in ${}^415_2^5$; we rename it to Γ_1. In this case, $\mathrm{lk}_\Delta\, v_1 = B \cup_{\Gamma_1} B'$, where B, B' are isomorphic 3-balls. We check by sage [21] that all other subcomplexes in $\mathrm{lk}_\Delta\, v_1$ that are isomorphic to Γ_1 are of the form $\sigma(\Gamma_1)$, where σ is an element in the permutation group of $\mathrm{lk}_\Delta\, v_1$ (of order 8). So it suffices to consider just Γ_1. To reconstruct $\mathrm{lk}_\Delta\, v_2$ and hence Δ, note that $\mathrm{lk}_\Delta\, u_2$ has the decomposition $\mathrm{lk}_\Delta\, u_2 = B' \cup_{\Gamma_1} B'' \cong S_2$ for some 3-ball B'', and furthermore $B'' \cong B'$. To decide B'' it is equivalent to finding a balanced simplicial isomorphism $f : B' \to B''$ with $B' \cap B'' = B \cap B'' = \Gamma_1$ and $f(\Gamma_1) = \Gamma_1$; in other words, f is a permutation in $\mathrm{Aut}(\Gamma_1)$. We check by sage [21] that the links of vertex $6, 8$ in ${}^415_2^5$ are the only candidates for $\mathrm{lk}_\Delta\, v_2$. Hence $\Delta = {}^415_2^5$. Indeed Δ is balanced: the color sets are $\{1, 6, 8\}$, $\{2, 4, 9\}$, $\{3, 7, 11\}$, $\{5, 10, 15\}$ and $\{12, 13, 14\}$. $\square$

Theorem 10. *There exists a balanced 3-neighborly non-spherical 5-manifold with each color set of size 3.*

Proof. By Theorem 9, if such 5-manifold exists, then all vertex links are isomorphic to ${}^415_2^5$, which we denote as Γ. Based on the list of facets of Γ in [15], we take a color-preserving permutation $\sigma = (1, 6, 8)(2, 4, 9)(11, 3, 7)(10, 15, 5)(13, 14, 12)$. We choose σ in such a way that $\sigma \notin \mathrm{Aut}(\Gamma)$ and furthermore, $\Gamma \cap \sigma(\Gamma)$, $\Gamma \cap \sigma^2(\Gamma)$ and $\sigma(\Gamma) \cap \sigma^2(\Gamma)$ are isomorphic homology manifolds with no interior faces of dimension < 2 and with a common boundary C. By computer we check that $\Gamma \cup \sigma(\Gamma) \cup \sigma^2(\Gamma)$ is balanced 3-neighborly and C is the vertex-transitive 3-manifold ${}^315_1^{15}$ that triangulates $\mathbb{S}^3/Q$ as in [15]. Finally let $\Delta = (\Gamma * \{16\}) \cup (\sigma(\Gamma) * \{17\}) \cup (\sigma^2(\Gamma) * \{18\})$, where $\{16, 17, 18\}$ are the vertices of color 6. By sage [21] one verifies that all vertex links of Δ is isomorphic to Γ, which is known as a combinatorial 4-sphere (we say a simplicial complex is a combinatorial sphere if it is PL homeomorphic to the boundary of the simplex). Hence Δ is a combinatorial manifold that is balanced 3-neighborly. $\square$

Remark 11. The following properties of the balanced 3-neighborly 5-manifold found in the proof of Theorem 10 are verified by sage:

1. It is vertex-transitive and has the following generators of the automorphism group (of order 1080):

$$(2, 15)(4, 5)(9, 10)(12, 17)(13, 18)(14, 16), \quad (1, 2)(3, 15)(4, 6)(5, 7)(8, 9)(10, 11),$$
$$(1, 3)(6, 7)(8, 11)(12, 17)(13, 18)(14, 16), \quad (1, 13)(3, 18)(6, 14)(7, 16)(8, 12)(11, 17),$$
$$(1, 6, 8)(2, 3, 17)(4, 7, 18)(5, 15, 10)(9, 11, 16).$$

2. The homology groups of Δ are given by $(\mathbb{Z}, 0, \mathbb{Z}_2, 0, 0, \mathbb{Z})$.

3. The f-vector of Δ is $(1, 18, 135, 540, 1035, 918, 306)$.

Furthermore by the Dehn-Sommerville relations, any balanced 3-neighborly 5-manifold having 3 vertex in each color set also has the f-vector $(1, 18, 135, 540, 1035, 918, 306)$. Let Σ be such a complex, $\{v_1, v_2, v_3\}$ a color set, $M_i = \mathrm{lk}_\Sigma v_j \cap \mathrm{lk}_\Sigma v_k$ for $\{i, j, k\} = [3]$ and $N = M_1 \cap M_2 \cap M_3$. If F is an interior face of M_i, then $F \cup \{v_i\} \notin \Sigma$. Since Σ is balanced 3-neighborly, $\mathrm{lk}_\Sigma v_1 = M_2 \cup M_3 \cong M_1 \cup M_3 = \mathrm{lk}_\Sigma v_2$ and they have 102 facets, it follows that each M_i has no interior vertices or edges and with 51 facets. Also since all vertex links in Σ are isomorphic to $^4 15_2^5$, the same argument as in the proof of Theorem 9 implies that N is a 3-manifold whose vertex links are isomorphic to the boundary of the icosahedron; indeed $^3 15_1^{15}$ is one such example. We haven't checked if there exist other balanced 3-neighborly 5-manifolds. (It is not known if there exist 15-vertex non-vertex-transitive 3-manifolds whose vertex links are all isomorphic to the boundary of the icosahedron.)

4 Balanced 2-neighborly 3-sphere with 16 vertices

In this section we provide a balanced 2-neighborly triangulation of the 3-sphere. The construction is motivated by Lemma 4.
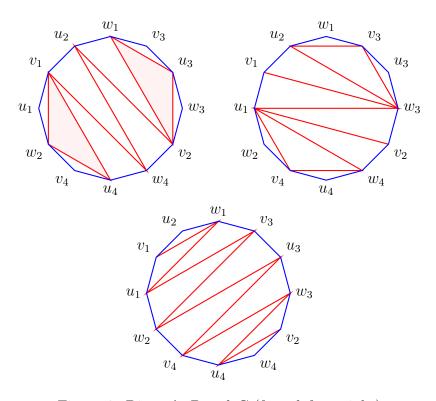

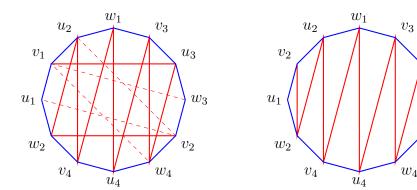Figure 2: Discs A, B and C (from left to right)

Figure 3: Left: disc D'. Right: disc D obtained after rearranging the boundary of D'.
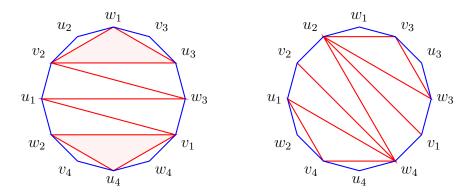

Figure 4: Left: disc A'. Right: disc B'. Notice that $\partial A' = \partial B' = \partial D$.

Construction 12. Assume $V_1 = \{u_1, u_2, u_3, u_4\}$, $V_2 = \{v_1, v_2, v_3, v_4\}$, $V_3 = \{w_1, w_2, w_3, w_4\}$ and $V_4 = \{z_1, z_2, z_3, z_4\}$ are the four color sets of a balanced 3-sphere Γ. We let $\mathrm{lk}_\Gamma z_1 = A \cup_{\partial A \sim \partial C} C$ and $\mathrm{lk}_\Gamma z_3 = B \cup_{\partial B \sim \partial C} C$, where A, B and C are triangulated 2-balls sharing the same boundary as shown in Figure 2. All possible edges that do not appear in A, B and C are shown in Figure 3 as solid red edges in disc D'. Notice that the dashed edges in D' are edges in discs A and B, so we may rearrange the boundary of D by switching the positions of vertices v_1 and v_2, and then replacing the edges containing v_1 or v_2 in $\partial D'$ by the dashed edges. In this way, we obtain a triangulation of a 12-gon D as shown in Figure 3. Furthermore, $\partial D \subseteq A \cup B$, and ∂D divides the sphere $= A \cup_{\partial A \sim \partial B} B$ into two discs A' and B' as shown in Figure 4.

We let $\mathrm{lk}_\Gamma z_2 = A' \cup_{\partial A' \sim \partial D} D$ and $\mathrm{lk}_\Gamma z_4 = B' \cup_{\partial B' \sim \partial D} D$. Since both $\mathrm{st}_\Gamma z_1 \cap \mathrm{st}_\Gamma z_3 = C$ and $\mathrm{st}_\Gamma z_2 \cap (\mathrm{st}_\Gamma z_1 \cup \mathrm{st}_\Gamma z_3) = A'$ are simplicial 2-balls, it follows that $\Sigma = \cup_{i=1}^3 \mathrm{st}_\Gamma z_i$ is a simplicial 3-ball. Furthermore, the boundary of Σ is exactly $\mathrm{lk}_\Gamma z_4$. Hence $\Gamma = \Sigma \cup \mathrm{st}_\Gamma z_4$ is indeed a balanced 2-neighborly 3-sphere.

Remark 13. Here we provide some properties of Γ in Construction 12.

1. $(A \cup B, C, D)$ is an ear decomposition of $\Gamma_{[3]}$.

2. The automorphism group of Γ has two generators

$$(u_1u_3u_2u_4)(v_1z_2v_2z_1)(v_3z_4v_4z_3)(w_1w_4w_2w_3),$$
$$(z_1v_1)(z_2v_2)(z_3v_3)(z_4v_4)(u_1w_1)(u_2w_2)(u_3w_3)(u_4w_4).$$

(The second generator is given by switching vertices of color 1 and 3, and color 2 and 4, but with the same subscript.) Hence $\mathrm{Aut}(\Gamma)$ has 8 elements.

3. The complex Γ given in Construction 12 is shellable. For $\mathrm{lk}_\Gamma z_1 = A \cup_{\partial A \sim \partial C} C$, there exist two shellings $c_1, \ldots, c_{10}, a_1, \ldots, a_{10}$ and $a'_1, \ldots, a'_{10}, c'_1, \ldots, c'_{10}$ such that for any $1 \leqslant i \leqslant 10$, c_i, c'_i are facets from C and a_i, a'_i are facets from A. Similarly, there exist two shellings $c_1, \ldots, c_{10}, b_1, \ldots, b_{10}$ and $b'_1, \ldots, b'_{10}, c'_1, \ldots, c'_{10}$ for $\mathrm{lk}_\Gamma z_3 = B \cup_{\partial B \sim \partial C} C$, where b_i, b'_i are facets from B. Then

$$a'_1 * z_1, \ldots, a'_{10} * z_1, c'_1 * z_1, \ldots, c'_{10} * z_1, c_1 * z_3, \ldots, c_{10} * z_3, b_1 * z_3, \ldots, b_{10} * z_3$$

gives a shelling of $\mathrm{st}_\Gamma z_1 \cup \mathrm{st}_\Gamma z_3$. We may extend this shelling into a complete shelling of Γ by constructing two similar shellings of $\mathrm{lk}_\Gamma z_2$ and $\mathrm{lk}_\Gamma z_4$. However, we tried some computer tests and failed to prove either polytopality or non-polytopality.

Remark 14. It is easy to see that if Δ_1 is a balanced 2-neighborly $(d_1 - 1)$-sphere and Δ_2 is a balanced 2-neighborly $(d_2 - 1)$-sphere, then $\Delta_1 * \Delta_2$ is a balanced 2-neighborly $(d_1 + d_2 - 1)$-sphere. Hence by taking joins, we find balanced 2-neighborly $(4k-1)$-spheres with $16k$ vertices for any $k \geqslant 1$.

Question 15. Let $d \geqslant 4$ and $m \geqslant 5$ be arbitrary integers. Is there a balanced 2-neighborly simplicial $(d-1)$-sphere all of whose color sets have the same size m? Is there a polytopal sphere with these properties?

5 Balanced 2-neighborly $L(3,1)$ with 16 vertices

In this section we present our first construction of a balanced 2-neighborly lens space $L(3,1)$ with 16 vertices. We denote it by Δ. Each color set of Δ has four vertices.

Construction 16. Denote the color sets of Δ by $V_1 = \{u_1, u_2, u_3, u_4\}$, $V_2 = \{v_1, v_2, v_3, v_4\}$, $V_3 = \{w_1, w_2, w_3, w_4\}$ and $V_4 = \{z_1, z_2, z_3, z_4\}$.

In Figure 5 we illustrate the construction of the vertex links $\mathrm{lk}_\Delta z_i$ for $i = 1, \ldots, 4$. All these links are realized as cylinders. Two links $\mathrm{lk}_\Delta z_1$ and $\mathrm{lk}_\Delta z_2$ share the same top and bottom, which are triangulated hexagons spanned by vertices $\{u_i, v_i, w_i : i = 1, 3\}$ and $\{u_i, v_i, w_i : i = 2, 4\}$, respectively. To construct $\mathrm{lk}_\Delta z_3$ from $\mathrm{lk}_\Delta z_1$, we switch the positions of vertices u_3, v_3, w_3 with vertices u_4, v_4, w_4 respectively and form a new cylinder. The new top and bottom hexagons contain the 2-faces $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$. Similarly, we construct the link $\mathrm{lk}_\Delta z_4$ from $\mathrm{lk}_\Delta z_2$ by switching the positions of vertices u_3, v_3, w_3 with vertices u_4, v_4, w_4 and letting $\{u_1, v_1, w_1\}$ and $\{u_2, v_2, w_2\}$ be the 2-faces that appear in the triangulation of the top and bottom hexagons. It follows that $\mathrm{lk}_\Delta z_3$ and $\mathrm{lk}_\Delta z_4$ also share the same top and bottom.
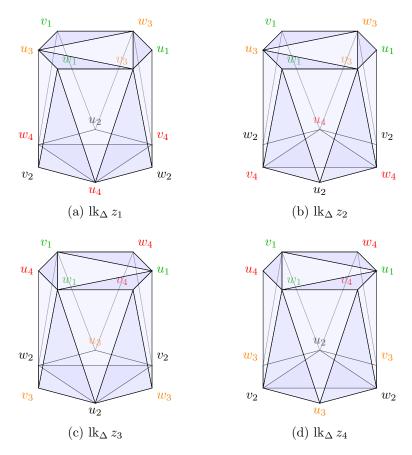
(a) $\mathrm{lk}_\Delta\, z_1$

(b) $\mathrm{lk}_\Delta\, z_2$

(c) $\mathrm{lk}_\Delta\, z_3$

(d) $\mathrm{lk}_\Delta\, z_4$

Figure 5: Four vertex links of Δ

Now since Δ is balanced 2-neighborly, by our construction, it only remains to show that Δ triangulates the lens space $L(3,1)$. The geometric realizations of $\mathrm{st}_\Delta\, z_1$ and $\mathrm{st}_\Delta\, z_2$ are filled cylinders that share top and bottom. So their union $A := \mathrm{st}_\Delta\, z_1 \cup \mathrm{st}_\Delta\, z_2$ is a filled torus (that is, a genus-1 handlebody); so is the union $B := \mathrm{st}_\Delta\, z_3 \cup \mathrm{st}_\Delta\, z_4$. Note that these two handlebodies have identical boundary complexes, thus they provide a Heegaard splitting of a lens space.

To identify which lens space Δ triangulates, we need to determine the homeomorphism $\phi : \partial A \to \partial B$. Consider two generators γ, δ of $\pi_1(A \cap B) = \pi_1(\partial A)$, where γ is the 6-cycle $(u_3, v_1, w_3, u_1, v_3, w_1)$ and δ is the 4-cycle (u_1, w_2, u_4, w_3). In particular, δ is also a generator of $\pi_1(A)$. From the construction we see that $\phi(\gamma)$ is a loop running around the equator of ∂B thrice and the meridian of ∂B once. Also $\phi(\delta)$ runs around the equator of ∂B twice and the meridian of ∂B once. Hence it is indeed the lens space $L(3,1)$.

Remark 17. Our construction Δ has the following properties:

1. All vertex links are combinatorially equivalent.

2. From Figure 5 we see $\mathrm{lk}_\Delta\, z_i \cap \mathrm{lk}_\Delta\, z_j$ has two connected components when $\{i, j\} = \{1, 2\}$ or $\{3, 4\}$ (they are the top and bottom hexagons as shown in Figure 2); and it

has three connected components when $i \in \{1,2\}$ and $j \in \{3,4\}$ (each component is the union of two facets along the side of the cylinders). In general, the intersection of two vertex links, where the vertices are of the same color, always has at least two connected components.

3. There are three group actions on the vertices of Δ:

   (a) Fix the subscript and rotate the corresponding vertices of color 1, 2 and 3 respectively. The generator is given by $(u_1 v_1 w_1)(u_2 v_2 w_2)(u_3 v_3 w_3)$.

   (b) Rotate vertices of the same color. The generator is

   $$(u_1 u_3 u_2 u_4)(v_1 v_3 v_2 v_4)(w_1 w_3 w_2 w_4)(z_1 z_3 z_2 z_4).$$

   (c) Exchange $\mathrm{lk}_\Delta z_1$ and $\mathrm{lk}_\Delta z_2$, $\mathrm{lk}_\Delta z_3$ and $\mathrm{lk}_\Delta z_4$, by exchanging v_i and w_i (or u_i and w_i, u_i and v_i) for all $i \in [4]$. The generators are

   $$(z_1 z_2)(z_3 z_4)(v_1 w_1)(v_2 w_2)(v_3 w_3)(v_4 w_4),$$
   $$(z_1 z_2)(z_3 z_4)(u_1 w_1)(u_2 w_2)(u_3 w_3)(u_4 w_4) \text{ and}$$
   $$(z_1 z_2)(z_3 z_4)(u_1 v_1)(u_2 v_2)(u_3 v_3)(u_4 v_4).$$

The automorphism group of Δ is of size 96.

Proposition 18. *The complex Δ is a balanced vertex minimal triangulation of $L(3,1)$.*

Proof. By Proposition 6.1 in [11], each color set of Δ is of size at least 3. If there are exactly three vertices v_1, v_2, v_3 of color 1 in Δ, apply the Mayer-Vietoris sequence on the triple $(\mathrm{st}_\Delta v_1 \cup \mathrm{st}_\Delta v_2, \mathrm{st}_\Delta v_3, \Delta)$ and we obtain that

$$0 = H_1(\mathrm{lk}_\Delta v_3) \to H_1(\mathrm{st}_\Delta v_1 \cup \mathrm{st}_\Delta v_2) \oplus H_1(\mathrm{st}_\Delta v_3) \to H_1(\Delta) \to H_0(\mathrm{lk}_\Delta v_3) = 0.$$

Hence $H_1(\mathrm{st}_\Delta v_1 \cup \mathrm{st}_\Delta v_2) \cong H_1(\Delta) = \mathbb{Z}/3\mathbb{Z}$. However, this is impossible since $H_1(\mathrm{st}_\Delta v_1 \cup \mathrm{st}_\Delta v_2) \cong H_0(\mathrm{st}_\Delta v_1 \cap \mathrm{st}_\Delta v_2)$, which cannot be $\mathbb{Z}/3\mathbb{Z}$. $\square$

The same argument as above also shows that the balanced triangulation of any lens space $L(p,q)$ with $p > 1$ must have at least 16 vertices.

6 Balanced spheres and ear decomposition

In this section our goal is to construct a balanced 3-sphere whose rank-selected subcomplexes do not have ear decompositions. The motivation is from the balanced 2-neighborly construction of $L(3,1)$ in Section 5. Indeed, we want to construct a balanced 3-dimensional complex Δ so that 1) each vertex link is a 2-sphere; 2) for a fixed color set $V_4 = \{v_1, \cdots, v_k\}$, the intersection of any two vertex links $\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_j$ always has at least two connected components (as the property listed in Remark 17); and 3) $\cup_{i=1}^4 \mathrm{st}_\Delta v_i$ is 3-ball, which together with the condition 1) guarantees that Δ is a 3-sphere.

In the following we take $k = 5$ and give such a construction. Figure 6 illustrates the links $\mathrm{lk}_\Delta v_1, \cdots, \mathrm{lk}_\Delta v_4$. Every label represents the color of the vertex. Also each connected component of $\mathrm{lk}_\Delta v_1 \cap \mathrm{lk}_\Delta v_2$ is colored in green, $\mathrm{lk}_\Delta v_i \cap \mathrm{lk}_\Delta v_3$ is colored in blue for $i = 1, 2$, and $\mathrm{lk}_\Delta v_j \cap \mathrm{lk}_\Delta v_4$ is colored in pink for $j = 1, 2, 3$. Immediately we check that all these intersections of vertex links have 2 or 3 connected components.

(a) $\mathrm{lk}_\Delta v_1$ and $\mathrm{lk}_\Delta v_2$ (b) $\mathrm{lk}_\Delta v_3$ (c) $\mathrm{lk}_\Delta v_4$
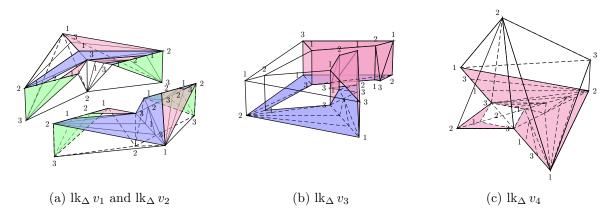
Figure 6: Four vertex links as triangulated 2-spheres. For simplicity's sake, we omit some diagonal edges in the quadrilaterals in (b), and some labels of vertices in (c).
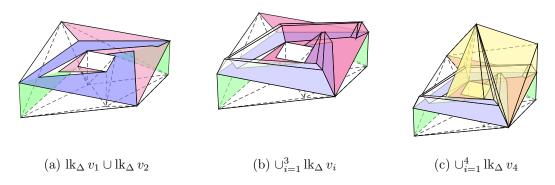

(a) $\mathrm{lk}_\Delta v_1 \cup \mathrm{lk}_\Delta v_2$ (b) $\cup_{i=1}^3 \mathrm{lk}_\Delta v_i$ (c) $\cup_{i=1}^4 \mathrm{lk}_\Delta v_4$

Figure 7: how the links are glued together.

Figure 7 shows how $\Delta \backslash V_4$ is formed from these links. First we glue $\mathrm{lk}_\Delta v_1$ and $\mathrm{lk}_\Delta v_2$ along two green triangles. The resulting complex $\mathrm{lk}_\Delta v_1 \cup \mathrm{lk}_\Delta v_2$ is shown in Figure 7a. Then we place $\mathrm{lk}_\Delta v_3$ on top of $\mathrm{lk}_\Delta v_1 \cup \mathrm{lk}_\Delta v_2$. As we see from Figure 7b, the boundary complex of $\cup_{i=1}^3 \mathrm{st}_\Delta v_i$ is a triangulated torus. Finally, we place $\mathrm{lk}_\Delta v_4$ on top of $\cup_{i=1}^3 \mathrm{lk}_\Delta v_i$ so that $\mathrm{st}_\Delta v_4$ "covers the 1-dimensional hole" in $\cup_{i=1}^3 \mathrm{st}_\Delta v_i$, see Figure 7c. We denote the subspace of $\mathbb{R}^3$ enclosed by $\mathrm{lk}_\Delta v_i$ as S_i for $1 \leqslant i \leqslant 4$, and let $S_5 := \cup_{i<4} S_i$. From our construction it follows that the boundary complex of S_5 is a 2-sphere; we let it be $\mathrm{lk}_\Delta v_5$.

Indeed Δ is a 3-sphere since Δ is the union of two 3-balls S_5 and $\mathrm{st}_\Delta\, v_5$ glued along their common boundary $\mathrm{lk}_\Delta\, v_5$.

Since each $\mathrm{lk}_\Delta\, v_i \cap \mathrm{lk}_\Delta\, v_j$ has at least two connected components for $1 \leqslant i \neq j \leqslant 4$, the Mayer-Vietoris sequence implies that $S_i \cup S_j$ is not contractible for all $1 \leqslant i \neq j \leqslant 4$. A similar inspection of $\mathrm{lk}_\Delta\, v_i \cup \mathrm{lk}_\Delta\, v_j \cup \mathrm{lk}_\Delta\, v_k$ also implies that the boundary complexes of $S_i \cup S_j \cup S_k$'s cannot be triangulated 2-spheres for distinct $1 \leqslant i, j, k \leqslant 4$.

Proposition 19. *Not all rank-selected subcomplexes of balanced simplicial spheres have ear decompositions.*

Proof. Consider the complex Δ constructed above. We denote the union of interior faces of a complex τ by $\mathrm{int}\,\tau$. Suppose $\Delta \backslash V_4$ has an ear decomposition $(\Gamma_1, \Gamma_2, \cdots, \Gamma_k)$. Since $|V_4| = 5$ and $\beta_2(\Delta \backslash V_4) = 4$, k must be 4. Notice first that $\cup_{i \leqslant 4} \mathrm{lk}_\Delta\, v_i$ divides $\mathbb{R}^3$ into five subspaces, namely, $S_1, \cdots, S_4$ and the complement of S_5, each having $\mathrm{lk}_\Delta\, v_i$ as the boundary complex for $1 \leqslant i \leqslant 5$ respectively. The complex Γ_1 is the union of 2-balls B_1, B_2 with $\partial B_1 = \partial B_2 = \Gamma_1 \cap \Gamma_2$. By the Jordan theorem, $B_1 \cup \Gamma_2$ is a triangulated 2-sphere that separates $\mathbb{R}^3$ into two connected components. Hence the bounded component must be either $S_i \cup S_j$ or $S_i \cup S_j \cup S_k$ for some $1 \leqslant i, j, k \leqslant 4$. (We may assume that it is not S_i, since otherwise we may consider the 2-sphere $\cup_{i \leqslant 3} \Gamma_i - \cup_{1 \leqslant i \neq j \leqslant 3} \mathrm{int}(\Gamma_i \cap \Gamma_j)$ instead of $\Gamma_1 \cup \Gamma_2 - \mathrm{int}(\Gamma_1 \cap \Gamma_2)$, where the subset enclosed by this sphere in $\mathbb{R}^3$ cannot be S_i anymore.) This contradicts the fact that the boundaries of $S_i \cup S_j$ or $S_i \cup S_j \cup S_k$ are not 2-spheres. $\square$

Remark 20. One can think of all the figures illustrated above as projections of a subcomplex of $\Delta - \mathrm{st}_\Delta\, v_5$ onto $\mathbb{R}^3$. However, we do not know whether the complex provided in this section can be realized as the boundary of a 4-polytope.

Acknowledgements

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