An Edmonds–Gallai-Type Decomposition for the j-Restricted k-Matching Problem

Yanjun Li
Krannert School of Management
Purdue University
West Lafayette, IN, USA
li14@purdue.edu

Jácint Szabó
Google Switzerland
Zürich, Switzerland
jacintsz@google.com

Submitted: Apr 29, 2018; Accepted: Oct 28, 2019; Published: Jan 10, 2020

Abstract

Given a non-negative integer j and a positive integer k, a j-restricted k-matching in a simple undirected graph is a k-matching, so that each of its connected components has at least $j+1$ edges. The maximum non-negative node weighted j-restricted k-matching problem was recently studied by Li who gave a polynomial-time algorithm and a min-max theorem for $0 \leqslant j < k$, and also proved the NP-hardness of the problem with unit node weights and $2 \leqslant k \leqslant j$. In this paper we derive an Edmonds–Gallai-type decomposition theorem for the j-restricted k-matching problem with $0 \leqslant j < k$, using the analogous decomposition for k-piece packings given by Janata, Loebl and Szabó, and give an alternative proof to the min-max theorem of Li.

Mathematics Subject Classifications: 05C70

1 Introduction

In this paper all graphs are simple and undirected. Given a set $\mathcal{F}$ of graphs, an $\mathcal{F}$-*packing* of a graph G is a subgraph M of G such that each connected component of M is isomorphic to a member of $\mathcal{F}$. An $\mathcal{F}$-packing M is called *maximal* (resp. *maximum*) if there is no $\mathcal{F}$-packing M' with $V(M) \subsetneq V(M')$ (resp. $|V(M)| < |V(M')|$). An $\mathcal{F}$-packing M is *perfect* if $V(M) = V(G)$. The $\mathcal{F}$-packing problem is to find a maximum $\mathcal{F}$-packing of G.

Several polynomial $\mathcal{F}$-packing problems are known in the case $K_2 \in \mathcal{F}$. For instance, we get a polynomial packing problem if $\mathcal{F}$ consists of K_2 and a finite set of hypomatchable graphs [2, 3, 4, 9]. In all known polynomial $\mathcal{F}$-packing problems with $K_2 \in \mathcal{F}$ it holds that each maximal $\mathcal{F}$-packing is maximum too; those node sets which can be covered by an $\mathcal{F}$-packing form a matroid, and the analogue of the classical Edmonds–Gallai decomposition theorem for matchings (see [6, 7, 5, 16]) holds.

The first polynomial $\mathcal{F}$-packing problem with $K_2 \notin \mathcal{F}$ was considered by Kaneko [11], who presented a Tutte-type characterization of graphs having a perfect packing by *long paths*, that is, by paths of length at least 2.

A shorter proof for Kaneko's theorem and a min-max formula was subsequently found by Kano, Katona and Király [12] but polynomiality remained open. The long path packing problem was generalized by Hartvigsen, Hell and Szabó [8] by introducing the *k-piece packing problem*, that is, the $\mathcal{F}$-packing problem where $\mathcal{F}$ consists of all *connected graphs with highest degree exactly k*. Such a graph is called a *k-piece*. Note that a 1-piece is just K_2, thus the 1-piece packing problem is the classical matching problem. The 2-piece packing problem is equivalent to the long path packing problem because a 2-piece is either a long path or a circuit C of length at least 3 so deleting an edge from C results in a long path. The main result of [8] is a polynomial algorithm for finding a maximum k-piece packing. Later, Janata, Loebl and Szabó [10] gave a canonical Edmonds–Gallai-type decomposition for the k-piece packing problem, showed that maximal and maximum packings do not coincide, and actually the maximal packings have a nicer structure than the maximum ones.

As another generalization of matchings, Li [14] introduced *j-restricted k-matchings*. For an integer $k > 0$, a *k-matching* of G is a subgraph M of G with no isolated node and degrees at most k. For two integers $j \geqslant 0$ and $k > 0$, a *j-restricted k-matching* of G is a k-matching whose each connected component has more than j edges [14]. Obviously, k-matchings are equal to 0-restricted k-matchings. Moreover, the $(k-1)$-restricted k-matching problem is exactly the maximum matching problem for $k = 1$ and the long path packing problem for $k = 2$.

Given non-negative weights on the nodes of G, the *maximum non-negative node weighted j-restricted k-matching problem* is to find a j-restricted k-matching of G such that the total weight of the nodes covered is maximized. Note that, contrary to the usual analysis of k-matchings, here we are interested in the weight of covered nodes, not edges. In [14], a polynomial-time algorithm composed of a min-cost max-flow algorithm and an alternating tree algorithm was proposed for solving the above problem with $0 \leqslant j < k$, and the algorithm was proved valid by showing a min-max theorem (Theorem 6 in this paper). In contrast, the maximum unit node weight j-restricted k-matching problem with $2 \leqslant k \leqslant j$ is proved to be NP-hard in [14].

There is a simple but essential relation between k-piece packings and j-restricted k-matchings, namely that every k-piece is a j-restricted k-matching for every $0 \leqslant j < k$. This connection has many important implications. The most prominent example is the fact that the *critical* graphs with respect to the j-restricted k-matching problem are also critical with respect to the k-piece packing problem (the role of critical graphs will be clear from the Edmonds–Gallai-type decomposition Theorems 3 and 10). This connection makes it possible to translate the analysis on k-piece packings to j-restricted k-matchings, and to prove analogous results.

Exploiting this relationship, in this paper we give an alternative proof to Theorem 6 of Li [14]. In addition, we prove two new results on the j-restricted k-matching problem. Theorem 3 is an Edmonds–Gallai-type decomposition, and Theorem 4 is a characterization of the maximal j-restricted k-matchings. Both proofs are based on the analogous results on k-piece packings [10].

The $k = 1$ case is the classical matching problem, for which our results are well known theorems. Thus in this paper the focus will be on the $k \geqslant 2$ case. However, for the sake of completeness, the general $k \geqslant 1$ case will be treated as a whole.

After formulating the main results and the min-max Theorem 6 of Li [14] in Section 2, we review the k-piece packing problem and associated concepts and results from [8, 10] in Section 3. From these results we then derive Theorem 3 in Section 4, and Theorem 4 in Section 5. In Section 5 we give the alternative proof to Theorem 6, as well. Finally, we conclude the paper with open questions in Section 6.

2 Main results

We need some notations to state our main results, Theorems 3 and 4. For a simple, undirected graph G we denote by $c(G)$ the number of connected components (shortly, components) of G, and by $\Delta(G)$ the largest degree of G. For $X \subseteq V(G)$, let $G[X]$ denote the subgraph induced by X; let $\Gamma(X)$ denote the set of nodes not belonging to X but adjacent to a node in X; and let $G - X$ denote the subgraph of G induced by the nodes of G not in X. $G - \{v\}$ is simply written as $G - v$ for $v \in V(G)$. Similarly, for node or edge sets S and T we sometimes use the shorthand $S - T$ for $S \setminus T$ and $S + T$ for $S \cup T$. An edge is said to *enter* X if exactly one end node of the edge is contained in X. A node set $X \subseteq V(G)$ is said to be *covered* (resp. *missed*) by a subgraph M of G if $X \subseteq V(M)$ (resp. $X \cap V(M) = \varnothing$).

Definition 1. A connected graph G is *hypomatchable* if $G - v$ has a perfect matching for every $v \in V(G)$.

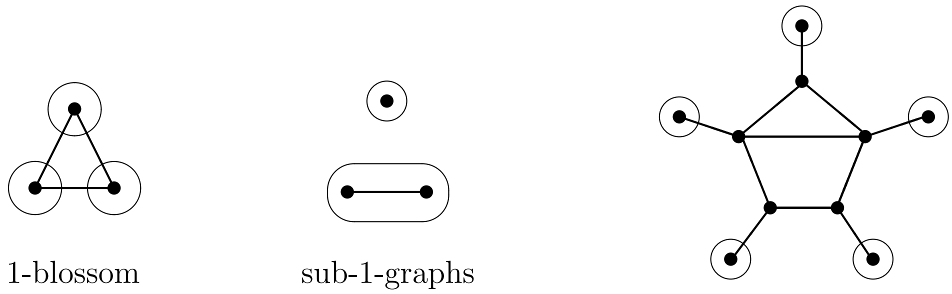
Hereafter we always assume that k and j are integers satisfying $1 \leqslant k$ and $0 \leqslant j < k$.

Definition 2. For an integer $k \geqslant 1$ a connected graph G is a *k-blossom* if there exists a hypomatchable graph F with $|V(F)| \geqslant 3$, such that $V(G) = V(F) \cup \{z_1^v, \dots, z_{k-1}^v : v \in V(F)\}$ and $E(G) = E(F) \cup \{vz_1^v, \dots, vz_{k-1}^v : v \in V(F)\}$.

A connected graph G is a *sub-j-graph* if $|E(G)| \leqslant j$.

Thus a k-blossom is obtained from F by adding $k - 1$ pendant edges together with their end nodes to every node of F. For a k-blossom G, where $k \geqslant 2$, every degree-1 node of G is called a *tip*; every node of a 1-blossom is called a *tip*; and every sub-j-graph itself is called a *tip*. See Figure 1 for some examples of k-blossoms and sub-j-graphs.

One of our main results is the following.



1-blossom sub-1-graphs 2-blossom

Figure 1: k-blossoms and sub-j-graphs. Tips are circled.

Theorem 3. [Edmonds–Gallai-type decomposition for j-restricted k-matchings] *For a graph G and integers $0 \leqslant j < k$, let*

$$U(G) = \{v \in V(G) : \text{ } v \text{ is missed by a maximal } j\text{-restricted } k\text{-matching of } G\},$$

$D = \{v : |U(G - v)| < |U(G)|\}$, $A = \Gamma(D)$ *and* $C = V(G) \setminus (D \cup A)$. *Then*

1. *every component of $G[D]$ is either a k-blossom or a sub-j-graph,*
2. *for all $\varnothing \neq A' \subseteq A$, the number of the components of $G[D]$ that are adjacent to A' is at least $k|A'| + 1$,*
3. *$G[C]$ has a perfect j-restricted k-matching, and*
4. *a j-restricted k-matching M of G is maximal if and only if*
   (a) *exactly $k|A|$ components of $G[D]$ are entered by an edge of M and these components are completely covered by M,*
   (b) *for every component H of $G[D]$ not entered by M, $M[H]$ is a maximal j-restricted k-matching of H, and*
   (c) *$M[C]$ is a perfect j-restricted k-matching of $G[C]$.*

We will prove Theorem 3 in Section 4 by deriving it from the Edmonds–Gallai-type decomposition for k-piece packings (Theorem 10, proved in [10]). After the proof we try to explain why this non-trivial definition of the canonical set D is required, and thus why Theorem 3 is not a direct generalization of the classical Edmonds–Gallai-theorem.

It is a well known fact in matching theory that those node sets which can be covered by a matching form a matroid. In the j-restricted k-matching problem, maximal and maximum j-restricted k-matchings do not coincide, thus this matroidal property holds only in the following weaker form.

Theorem 4. *There exists a partition π on $V(G)$ and a matroid $\mathcal{P}$ on π such that the node sets of the maximal j-restricted k-matchings are exactly the node sets of the form $\bigcup\{X : X \in \pi'\}$ where $\pi' \subseteq \pi$ is a base of $\mathcal{P}$.*

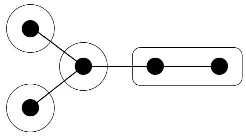


Figure 2: A graph with the matroidal partition π, $j = 1, k = 2$

Example 5. Figure 2 shows a graph with the partition π as in Theorem 4, for $j = 1$ and $k = 2$ (it even works for any $k \geqslant 2$). In this graph the node sets coverable by j-restricted k-matchings do not form a matroid.

The analogue of Theorem 4 for k-piece packings was proved in [10].

A Berge-type characterization 6 of j-restricted k-matchings with maximum node weight was proved by Li [14], based on a polynomial time alternating tree algorithm. In this paper we will derive it by analyzing the maximum weight bases of the matroid $\mathcal{P}$ above. Assume that a non-negative weight function $w : V(G) \to \mathbb{R}_+$ is given. [14] defines the **deficiency weight** of a k-blossom or sub-j-graph G as

1. $w(G) = \sum\{w(v) : v \in V(G)\}$ if G is a sub-j-graph,
2. $w(G) = \min\{w(v) : v$ is a tip of $G\}$ if G is a k-blossom.

Let $(j,k)\text{-gal}_t(G)$ denote the number of k-blossom and sub-j-graph components H of a graph G with $w(H) \geqslant t$. (The rationale of this definition and the notation "gal" will be clear later.)

Theorem 6. [14][Weighted j-restricted k-matchings] *Let G be a graph with n nodes, and $w : V(G) \to \mathbb{R}_+$ non-negative node weights. Then the maximum total weight of a j-restricted k-matching of G is*

$$\sum\{w(v) : v \in V(G)\} - \max \sum_{i=1}^{n} (t_i - t_{i-1}) \left((j,k)\text{-gal}_{t_i}(G - A_i) - k|A_i|\right),$$

where the max *is taken over all sequences of node sets $V(G) \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$ and $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n$.*

The analogue of Theorem 6 for k-piece packings was proved in [8]. We will prove Theorems 4 and 6 in Section 5 by analyzing the matroid $\mathcal{P}$ in Theorem 4.

3 k-piece packings

In this section we collect the relevant notions and results on k-piece packings from [8, 10]. In the rest of the paper k is a fixed positive integer.

A *k-piece* is a connected graph G with $\Delta(G) = k$. The *k-piece packing problem* is, given a graph G, to find a maximum k-piece packing of G. The main result of [8] is a polynomial-time algorithm for the k-piece packing problem. Moreover, from the algorithm, the graphs with a perfect k-piece packing were characterized, and a min-max theorem for the number of nodes in a maximum k-piece packing was derived.

It was revealed in [8] that k-galaxies play a critical role in solving the k-piece packing problem.

Definition 7. For a graph G we denote $I_G = G[\{v \in V(G) : \ \deg_G(v) \geqslant k\}]$.

Definition 8. For an integer $k \geqslant 1$ the connected graph G is a *k-galaxy* if it satisfies the following properties:

- each component of I_G is a hypomatchable graph, and
- for each $v \in V(I_G)$, there are exactly $k-1$ edges between v and $V(G) \setminus V(I_G)$, each being a cut edge of G.

For a k-galaxy G, where $k \geqslant 2$, every component of $G - V(I_G)$ is called a *tip*, and every node of a 1-galaxy is called a *tip*. In the case $k \geqslant 2$ a k-galaxy may consist of only a single tip (a graph with highest degree at most $k-1$), but must always contain at least one tip.

A hypomatchable graph has no node of degree 1 so a k-galaxy has no node of degree k. Furthermore, each component of I_G is a hypomatchable graph on at least 3 nodes. Galaxies generalize hypomatchable graphs because the 1-galaxies are exactly the hypomatchable graphs. Kaneko introduced the 2-galaxies under the name 'sun' [11]. See Figure 3 for some k-galaxies. The nodes of I_G are drawn as big dots, the edges of I_G as thick lines, and every tip is circled (for the 4-galaxy not all tips are circled for sake of visibility).

We will use the following fact at many places.

Lemma 9. [8] *A k-galaxy has no perfect k-piece packing.*

The following Edmonds–Gallai-type decomposition for the k-piece packings was proved in [10]. The classical Edmonds–Gallai theorem [5, 6, 7] first defines the node set D to consist of those nodes which can be missed by a maximum matching. In the k-piece packing problem we need a different formulation, and so Theorem 10 is not a direct generalization of the classical Edmonds–Gallai theorem. After the proof of Theorem 3 we try to explain the reason.

Theorem 10. [Edmonds–Gallai-type decomposition for k-piece packings] *For a graph G, let $U^k(G) = \{v \in V(G) :$ v is missed by a maximal k-piece packing of $G\}$, $D^k = \{v \in V(G) : \ |U^k(G-v)| < |U^k(G)|\}$, $A^k = \Gamma(D^k)$ and $C^k = V(G) \setminus (D^k \cup A^k)$. Then*

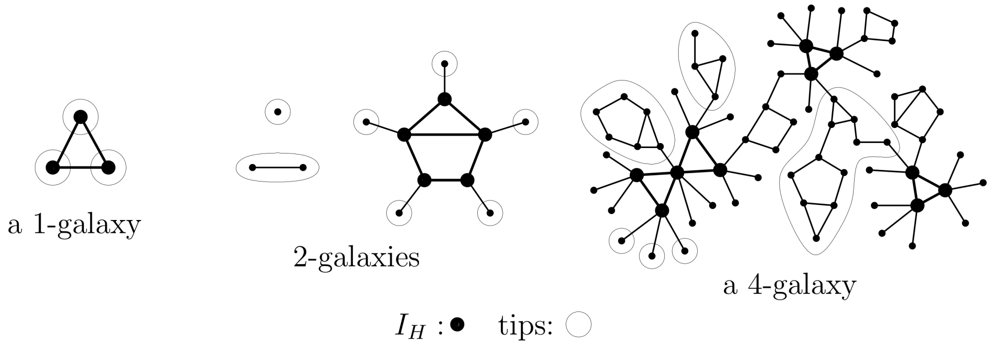


Figure 3: Galaxies

1. *every component of* $G[D^k]$ *is a* k*-galaxy,*
2. *for all* $\varnothing \neq A' \subseteq A^k$*, the number of the components of* $G[D^k]$ *that are adjacent to* A' *is at least* $k|A'| + 1$*, and*
3. $G[C^k]$ *has a perfect* k*-piece packing.*

In the graph packing terminologies, the node set A^k in the above theorem is called a *barrier*, and the k-galaxies are called *critical graphs* in the k-piece packing problem.

4 Proof of Theorem 3

In this section we make use of the established connection between j-restricted k-matchings and k-piece packings to derive Theorem 3 from Theorem 10.

4.1 (j, k)-galaxies and k-solar systems

In this subsection we define (j, k)-galaxies, introduce k-solar-systems, make observations on both of them, and review the rank of the transversal matroid of bipartite graphs.

Definition 11. For integers $0 \leqslant j < k$, a connected graph is a (j, k)-*galaxy* if it is a k-galaxy without a perfect j-restricted k-matching.

Lemma 12. *If* $k \geqslant 2$*,* $j = k - 1$ *and* M *is a* j*-restricted* k*-matching of a* k*-blossom* G*, then there exists a tip* T *of* G *such that* $V(T) \cap V(M) = \varnothing$*.*

Proof. Recall that every tip consists of only one node, and let T_G denote the set of these tip nodes. Suppose that every node in T_G is covered by M. Then also all edges in $E' = \{uv : u \in T_G, v \in V(I_G)\}$ are contained in M because $j \geqslant 1$. Note that every node $v \in V(I_G)$ has $k-1$ incident edges in E'. Let $M' = M[V(I_G)]$. Now $\deg_{M'}(v) \geqslant 1$ for all $v \in V(I_G)$, as otherwise the component of M containing v would only have $k-1$ edges, which is not a $(k-1)$-restricted k-matching. On the other hand, $\Delta(M') \leqslant 1$ as otherwise

$\Delta(M) > k$ would hold. Thus M' is a perfect matching of the hypomatchable graph I_G, which is impossible. □

The next theorem shows the connection between k-galaxies, k-blossoms and sub-j-graphs.

Theorem 13. *For $j < k-1$, every (j,k)-galaxy is a sub-j-graph, and vice versa. For $j = k-1$, every (j,k)-galaxy is a k-blossom or a sub-j-graph, and vice versa.*

Proof. First, k-blossoms and sub-j-graphs are clearly k-galaxies. Now we prove that they have no perfect j-restricted k-matchings, and thus are (j,k)-galaxies for the given j,k. Trivially, this holds for sub-j-graphs for all $0 \leqslant j < k$. As for k-blossoms in the case $j = k-1$, for $k = 1$ observe that both $(0,1)$-galaxies and 1-blossoms are just hypomatchable graphs, and for $k \geqslant 2$ use Lemma 12.

Now we prove that every (j,k)-galaxy is either a k-blossom or a sub-j-graph for the given j,k's.

Let G be a (j,k)-galaxy. If $I_G = \varnothing$, that is $\Delta(G) < k$, then it is clear that G has no perfect j-restricted k-matching if and only if it is a sub-j-graph. So assume that $I_G \neq \varnothing$.

For the case of $j < k-1$, let $G' = (V(G), E(G) - E(I_G))$ be the subgraph of G without the edges of I_G. Clearly, $\Delta(G') = k-1$. Moreover, every component of G' has at least $k-1 > j$ edges because it contains a node from $V(I_G)$. Thus G' is a perfect j-restricted k-matching of G and so G cannot be a (j,k)-galaxy.

Let us consider the case of $j = k-1$. As we already observed, in the case of $k = 1$, both $(0,1)$-galaxies and 1-blossoms are just hypomatchable graphs, so let us assume that $k \geqslant 2$.

If I_G is connected, then we prove that every tip of G consists of a single node, and thus G is a k-blossom. Otherwise, suppose that some $v \in V(I_G)$ is connected to a tip of two or more nodes. Since I_G is hypomatchable, we let N' be a matching of I_G that covers all nodes of I_G but v. Then the subgraph $(V(G), E(G) - E(I_G) + E(N'))$ forms a perfect j-restricted k-matching, a contradiction to the definition of a (j,k)-galaxy.

If I_G has at least two components, we let T be a tip of G that is connected to at least two components of I_G. For every component C of I_G, let v_C be the unique node of C that is closest to T in G, and let N_C be a perfect matching of $C - v_C$. Let $E' = \bigcup\{E(N_C) : C$ is a component of $I_G\}$, and let $M = (V(G), E(G) - E(I_G) + E')$. We prove that M is a perfect j-restricted k-matching of G. As $M \cap E(I_G)$ has maximum degree 1, $\Delta(M) \leqslant k$, so it is enough to prove that $|E(C_M)| \geqslant k$ for every component C_M of M.

1. The component C_M containing T necessarily covers two nodes $u, v \in V(I_G)$ from two different components of I_G, thus C_M has at least $2(k-1) \geqslant k$ edges.

2. All other components C_M contain an edge $uv \in E'$ and the nodes u, v together are incident to $2(k-1)+1 > k$ edges of C_M. □

The maximal matchings of a hypomatchable graph G are exactly the perfect matchings of $G - v$ for the nodes $v \in V(G)$. The characterization of the maximal j-restricted k-matchings of a (j,k)-galaxy can be stated by means of the tips in Corollary 16.

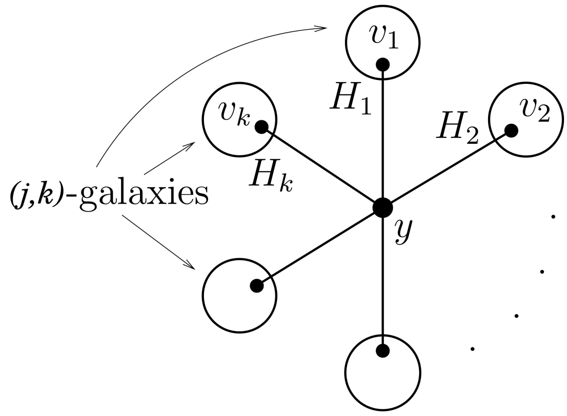


Figure 4: A k-solar system

Corollary 14. *If M is a j-restricted k-matching of a (j,k)-galaxy G, then there exists a tip T of G such that $V(M) \cap V(T) = \varnothing$.*

Proof. If G is a sub-j-graph, then M must be empty otherwise it would have a component with at most j edges. For k-blossoms, use the definition of hypomatchable graphs for $k = 1$ and apply Lemma 12 for $k \geqslant 2$. □

Lemma 15. *If T is a tip of a (j,k)-galaxy G, then $G - V(T)$ has a perfect j-restricted k-matching.*

Proof. If G is a sub-j-graph, then it is a tip itself, thus $G - T = \varnothing$ and so the statement is true. Otherwise G is a k-blossom and $T = \{t\}$ with $\deg_G(t) = 1$. Denote the neighbor of t by $v \in V(I_G)$. I_G has at least 3 nodes so we can choose a neighbor $w \in V(I_G)$ of v. Since I_G is hypomatchable, we let N' be a matching of I_G that covers all nodes of I_G but w. Then the subgraph $(V(G) - \{t\}, E(G) - E(I_G) + E(N') + wv)$ forms a perfect j-restricted k-matching of $G - t$. □

Corollary 16. *The maximal j-restricted k-matchings of a (j,k)-galaxy G are exactly the perfect j-restricted k-matchings of $G - V(T)$, where T is a tip of G.*

Proof. By Corollary 14 and Lemma 15. □

Definition 17. [10] A connected graph G is a *k-solar-system* (see Figure 4) if it has a node y, called *center*, such that $\deg_G(y) = k$ and $G - y$ has k components, each being a (j,k)-galaxy.

Lemma 18. *Every k-solar-system has a perfect j-restricted k-matching.*

Proof. In Lemma 3.10 of [10], it is proved that a k-solar system has a perfect k-piece packing. This is itself a perfect j-restricted k-matching. □

Lemma 18 is also easy to prove directly.

4.2 Tools from matroid theory

We will make use of transversal matroids of bipartite graphs. In this subsection we list the relevant results which we will use in later proofs. Let G be a graph and $A, D \subseteq V(G)$ disjoint node sets. Let $kA = \{z_1^v, \dots, z_k^v : v \in A\}$ and denote the set of components of $G[D]$ by $\mathcal{D}$. We denote by $K_{A,D}$ the bipartite graph $(kA, \mathcal{D}, \{z_i^v H : 1 \leqslant i \leqslant k, H \in \mathcal{D}$ is connected to node $v \in A\}$.

Definition 19. A is *k-matched into D by N* if N is a matching of $K_{A,D}$ such that $\deg_N(z_i^v) = 1$ for all $v \in A$, $1 \leqslant i \leqslant k$. A has *k-surplus in $K_{A,D}$* if for every component $H \in \mathcal{D}$, A can be k-matched into $D \setminus V(H)$.

Remark 20. By Kőnig's theorem [13], A has k-surplus in $K_{A,D}$ if and only if for every $\varnothing \neq A' \subseteq A$, A' is connected to at least $k|A'| + 1$ components of $G[D]$ in G.

The *transversal matroid* $\mathcal{T}_K$ of a bipartite graph $K = (U, V; E)$ is a matroid on V where a set $V' \subseteq V$ is independent if it can be covered by a matching of K. In a matroid, an element is a *bridge* if it is contained in every base. In terms of a transversal matroid $\mathcal{T}_K$, $v \in V$ is a bridge if it is covered by every maximum matching of K.

Theorem 21. [17] *The rank of $\mathcal{T}_K$ is $|V \setminus X| + |\Gamma_K(X)|$, where X is the set of the non-bridge elements of $\mathcal{T}_K$.*

For $A \subseteq V(G)$, let $D_A(G) = \bigcup\{V(H) : H$ is a (j,k)-galaxy component of $G - A\}$, and let $C_A(G) = V(G) \setminus (D_A \cup A)$. We sometimes use the notation D_A and C_A, respectively.

Definition 22. $A \subseteq V(G)$ has *k-surplus* if it has k-surplus in K_{A,D_A}. A is *perfect* if $G[C_A]$ has a perfect j-restricted k-matching.

Note the multiple meaning of perfectness in this paper. For packings (or subgraphs) *perfect* means spanning, while for node sets *perfect* is as defined in Definition 22.

4.3 The structure of maximal j-restricted k-matchings

Now we turn to establishing the canonical Edmonds–Gallai-type decomposition Theorem 3 for j-restricted k-matchings.

Theorem 23. *There exists a perfect node set $A \subseteq V(G)$ with k-surplus.*

Proof. Let us consider the decomposition $V(G) = D^k \dot\cup A^k \dot\cup C^k$ of Theorem 10. In this decomposition the components of $G[D^k]$ are k-galaxies, $G[C^k]$ has a perfect k-piece packing, and by Remark 20, A^k has k-surplus in K_{A^k,D^k}. Figure 5 is an illustration of this proof.

Let $\mathcal{H}^k = \{H : H$ is a component of $G[D^k]\}$, $\mathcal{H}' = \{H \in \mathcal{H}^k : H$ is a (j,k)-galaxy$\}$, and $D' = \bigcup\{V(H) : H \in \mathcal{H}'\}$. Let $\mathcal{T}^k$ be the transversal matroid on $\mathcal{H}^k$ in the bipartite graph K_{A^k,D^k}, and $\mathcal{T}' = \mathcal{T}^k|\mathcal{H}'$.

Applying Theorem 21 to $\mathcal{T}'$ we get that $r_{\mathcal{T}'} = |\mathcal{H}' \setminus \mathcal{D}| + k|\Gamma_G(D)| = |\mathcal{H}' \setminus \mathcal{D}| + k|A|$, where $\mathcal{D}$ consists of the non-bridge elements of $\mathcal{T}'$, $D = \bigcup\{V(H) : H \in \mathcal{D}\}$, and $A = \Gamma_G(D)$. Define $\mathcal{M} = \mathcal{T}'|\mathcal{D}$. That $\mathcal{M}$ has no bridge means that every element $H \in \mathcal{D}$

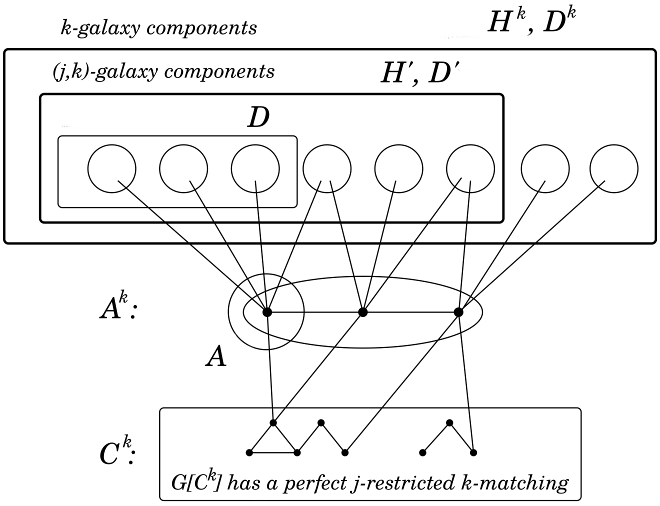


Figure 5: Creating the canonical decomposition for j-restricted k-matchings from the decomposition for k-piece packings ($k = 2$)

is missed by some base of $\mathcal{M}$, that is, A can be k-matched into $\mathcal{D} \setminus \{H\}$. Thus A has k-surplus in $K_{A,D}$.

We prove that A is perfect. By Theorem 10, $G[C^k]$ has a perfect k-piece packing, so it also has a perfect j-restricted k-matching M_1. We show that $G[(A^k \setminus A) \cup (D^k \setminus D)]$ has a perfect j-restricted k-matching, too. Take a base $\mathcal{B}$ of $\mathcal{T}'|\mathcal{D}$, extend it to a base $\mathcal{B}^k$ of $\mathcal{T}^k$, and take the matching N^k in K_{A^k,D^k} defining $\mathcal{B}^k$. Recall that $|\mathcal{B}| = k|A|$ and $|\mathcal{B}^k| = k|A^k|$, thus N^k matches kA to $\mathcal{B}$ in $K_{A,D}$ and $k(A^k \setminus A)$ to $\mathcal{B}^k \setminus \mathcal{B}$ in K_{A^k,D^k}.

Using Lemma 18, N^k gives rise to a perfect j-restricted k-matching M_2 in the subgraph induced by

$$(A^k \setminus A) \cup \bigcup\{V(H) :\ H \in \mathcal{H}^k \setminus \mathcal{D} \text{ is covered by } N^k\}.$$

As the components in $\mathcal{H}' \setminus \mathcal{D}$ are bridges in $\mathcal{T}'$, all these components are covered by N^k. Now consider a component $H \in \mathcal{H}^k \setminus \mathcal{H}'$ not covered by N^k. By definition, H has a perfect j-restricted k-matching M_H. Thus $M_1 \cup M_2 \cup \bigcup\{M_H : H \in \mathcal{H}^k \setminus \mathcal{H}' \text{ not covered by } N^k\}$ is a perfect j-restricted k-matching of $G - (A \cup D)$. As the components in $\mathcal{D}$ are (j,k)-galaxies, $C_A = V(G) \setminus (A \cup D)$, and so A is perfect. Moreover, $D_A = D$, and so A has k-surplus. □

Definition 24. Let $A_G = A$ as defined in the proof of Theorem 23, $D_G = D_A$ and $C_G = C_A$. The decomposition $V(G) = D_G \dot\cup A_G \dot\cup C_G$ is called the *canonical decomposition of* G for j-restricted k-matchings.

Now we investigate the structure of maximal j-restricted k-matchings of G.

Definition 25. For $A \subseteq V(G)$, let $W_A(G)$ or simply $W_A = \bigcup\{T_H : T_H \text{ is the node set of a tip of a } (j,k)\text{-galaxy component } H \text{ of } G[D_A]\}$.

Recall the definition of $U(G)$ in Theorem 3.

Lemma 26. *Let $A \subseteq V(G)$ be perfect and k-matchable into D_A. Then a subgraph M of G is a maximal j-restricted k-matching of G if and only if*

1. *exactly $k|A|$ components of $G[D_A]$ are entered by M and these components are completely covered by M,*

2. *if H is a component of $G[D_A]$ not entered by M, then there exists a tip T of H such that $M[H]$ is a perfect j-restricted k-matching of $H - V(T)$, and*

3. *$M[C_A]$ is a perfect j-restricted k-matching of $G[C_A]$.*

It holds that $U(G) \subseteq W_A$. Moreover, if A has k-surplus then $U(G) = W_A$.

Proof. Assume that M is a j-restricted k-matching satisfying the properties 1, 2 and 3, and M' is a j-restricted k-matching with $V(M) \subsetneq V(M')$. By Lemma 14, M' must enter more than $k|A|$ components of $G[D_A]$, which is not possible. Thus M is maximal.

Now let M be a maximal j-restricted k-matching of G. We construct a j-restricted k-matching M' for which $V(M) \subseteq V(M')$ holds, and if M fails any of the properties 1, 2 and 3, then even $V(M) \subsetneq V(M')$. This is clearly enough to prove.

Let $\mathcal{H} = \{H :\ H \text{ is a component of } G[D_A]\}$, and let $\mathcal{T}$ be the transversal matroid on $\mathcal{H}$ in the bipartite graph K_{A,D_A}. Let $\mathcal{H}_M = \{H \in \mathcal{H} :\ H \text{ is entered by } M\}$, and take a base $\mathcal{B} \supseteq \mathcal{H}_M$ of $\mathcal{T}$. A can be k-matched into D_A, so $r_{\mathcal{T}} = k|A|$ and thus $|\mathcal{B}| = k|A|$. By Lemma 14, one can choose a tip T_H missed by M in each component $H \in \mathcal{H} \setminus \mathcal{B}$. Similarly as in the proof of Theorem 23, we construct a j-restricted k-matching M' of G missing exactly these tips. First, take a matching N in K_{A,D_A} that defines the base $\mathcal{B}$. In a component $H \in \mathcal{H} \setminus \mathcal{B}$, take a perfect j-restricted k-matching M_H of $H - V(T_H)$. The union of these j-restricted k-matchings is M_1. Using Lemma 18, N gives rise to a perfect j-restricted k-matching M_2 in the subgraph induced by

$$A \cup \bigcup \{V(H) :\ H \in \mathcal{B}\}.$$

Finally, take a perfect j-restricted k-matching M_3 of $G[C_A]$. Now $M' = M_1 \cup M_2 \cup M_3$ is a j-restricted k-matching with $V(M') = V(G) \setminus \bigcup \{V(T_H) : H \in \mathcal{H} \setminus \mathcal{B}\} \supseteq V(M)$.

Trivially, $|\mathcal{H}_M| \leqslant k|A|$. In fact, $|\mathcal{H}_M| = k|A|$ holds because otherwise the matching N would enter strictly more components of $G[D_A]$ than M, resulting in $V(M) \subsetneq V(M')$, a contradiction. The properties 1 and 2 are straightforward by the maximality of M and by Corollary 16. For the property 3, observe that M has no edge joining A to C_A because otherwise $|\mathcal{H}_M| < k|A|$ would hold.

It clearly follows that $U(G) \subseteq W_A$. We show that $U(G) = W_A$ if A has k-surplus. Let T be an arbitrary tip in a (j,k)-galaxy component H_0 of $G[D_A]$. A has k-surplus, so $\mathcal{T}$ has a base $\mathcal{B} \subseteq \mathcal{H} \setminus H_0$. Now choose a tip T_H in each component $H \in \mathcal{H} \setminus (H_0 \cup \mathcal{B})$. Similarly as above, one can construct a maximal j-restricted k-matching M of G missing exactly these tips, including T. □

4.4 Uniqueness of the canonical decomposition

In the matching case, that is, when $k = 1$, it holds that $W_A = D_A$, thus Lemma 26 itself characterizes D_A in the canonical decomposition. In the general case, only $W_A \subseteq D_A$ holds, so we have to go one step further in order to characterize D_A in Theorem 28. First we need the following lemma.

Lemma 27. *If G is a (j,k)-galaxy and $v \in V(G)$, then every component of $G - v$ is either a (j,k)-galaxy or has a perfect j-restricted k-matching. Moreover, with*

$$W^v = \{u : \text{ } u \text{ is in a tip in a } (j,k)\text{-galaxy component } H \text{ of } G - v\}$$

and $W_G = \{u : \text{ } u \text{ is in a tip of } G\}$, we have $W^v \subsetneq W_G$.

Proof. If G is a sub-j-graph, then all components of $G - v$ are sub-j-graphs and thus are (j,k)-galaxies. Moreover, $W^v = V(G) \setminus \{v\} \subsetneq V(G) = W_G$.

Let $j = k - 1$ and G be a k-blossom. For $k = 1$, the statement follows from the definition of hypomatchable graphs. For $k \geqslant 2$, there are two cases to consider:

1. v is a tip. Let $u \in V(I_G)$ be the neighbor of v, and $x \in V(I_G)$ some neighbor of u. Take a perfect matching N of $I_G - x$, and let $N' = N \cup \{ux\}$. Clearly, $\deg_{N'}(y) = 1$ for all $y \in V(I_G) \setminus \{u\}$ and $\deg_{N'}(u) = 2$. Now consider the graph $J = (V(G), E(G) - E(I_G) + E(N')) - v$. Now every component of J has maximum degree k, so it is a perfect j-restricted k-matching of $G - v$. Clearly, $W^v = \varnothing \subsetneq W_G$.

2. $v \in V(I_G)$. Denote by W' the tips connected to v in G. In $G - v$, the tips in W' become singletons, and thus (j,k)-galaxies. Now take a perfect matching N of $I_G - v$. Consider the graph $J = (V(G), E(G) - E(I_G) + E(N')) - (W' + v)$. Now every component of J has maximum degree k, so it is a perfect j-restricted k-matching of $G - (W' + v)$. It follows that $W^v \subsetneq W_G$. □

The uniqueness of the canonical decomposition will follow from the next theorem.

Theorem 28. *Every graph G has a unique perfect node set $A \subseteq V(G)$ with k-surplus. For this node set A, it holds that*

$$D_A = \{v : \text{ } U(G - v) \subsetneq U(G)\} = \{v : \text{ } |U(G - v)| < |U(G)|\}.$$

Proof. Let $A \subseteq V(G)$ be perfect with k-surplus. By Lemma 26, we know that $U(G) = W_A$. Now we investigate the canonical decomposition of $G - v$ for a node $v \in V(G)$ in the following three cases:

1. $v \in C_A$. Denote the graph $G[C_A] - v$ by G'. Observe that in $G - v$ the set $A'' = A \cup A_{G'}$ is perfect with k-surplus. Thus by Lemma 26, $U(G - v) = W_{A''} \supseteq W_A = U(G)$.

2. $v \in A$. In the graph $G - v$ the set $A \setminus \{v\}$ is perfect with k-surplus, so by Lemma 26, $U(G - v) = W_{A \setminus \{v\}} = W_A = U(G)$.

3. $v \in V(H)$ for a (j,k)-galaxy component H of $G[D_A]$. By Lemma 27, $\varnothing$ is perfect and has k-surplus in the graph $H - v$. Let $D' = \{V(K) : K$ is a (j,k)-galaxy component of $H - v\}$ and $C' = \{V(K) : K$ is a component of $H - v$ with a perfect j-restricted k-matching$\}$. Furthermore, let $D'' = (D_A \backslash V(H)) \cup D'$ and $C'' = C_A \cup C'$. Lemma 27 implies that $W_A(G - v) \subsetneq W_A(G)$. In the graph $G - v$, the set A is perfect because $G[C'']$ has a perfect j-restricted k-matching. Moreover, A can be k-matched into D'' in $G - v$ because A has k-surplus in G. So by Lemma 26 we have $U(G - v) \subseteq W_A(G - v) \subsetneq W_A(G) = U(G)$.

We have proved that, if $A \subseteq V(G)$ is perfect with k-surplus, then

$$D_A = \{v : U(G - v) \subsetneq U(G)\} = \{v : |U(G - v)| < |U(G)|\}.$$

As here the right hand side does not depend on A, the set D_A is unique across the perfect node sets $A \subseteq V(G)$ with k-surplus and thus equals D_G. Finally we show that the uniqueness of D_A implies the uniqueness of A. By definition, $\Gamma(D_A) \subseteq A$. On the other hand, the k-surplus of A implies that $A \subseteq \Gamma(D_A)$. Thus $A = A_G$. □

At this point the proof of Theorem 3 is straightforward using the results of this section.

Proof of Theorem 3. By Theorem 28, $D = D_G$, and thus $A = A_G$ and $C = C_G$. Now the property 1 holds by definition. A_G is perfect with k-surplus, which is just tantamount to the properties 2 and 3. The property 4 follows from Lemma 26. □

We try to give an explanation why in Theorems 3 and 10 the canonical set D is defined in an unusual way. In the classical Edmonds–Gallai decomposition theorem for matchings [5, 6, 7]

1. D is defined as the set of nodes which are missed by a maximum matching of G.

(Maximal would also be possible here.) An alternative, rarely used definition would be that

2. D is the set of nodes $v \in V(G)$ for which $\mathrm{def}(G - v) < \mathrm{def}(G)$, where the deficiency def is defined as $\mathrm{def}(G) = \max\{c(G[D]) - |\Gamma(D)| : D \subseteq V(G), G[D]$ consists of hypomatchable components$\}$.

Both variants fail for j-restricted k-matchings (and also for k-piece packings). Definition 1. fails because a non-tip node in a (j,k)-galaxy component in $G[D]$ is covered by every maximum j-restricted k-matching by Theorem 26. Definition 2. fails because the analogue of the deficiency, $\mathrm{def}_{j,k}(G) = \max\{c(G[D]) - k|\Gamma(D)| : D \subseteq V(G), G[D]$ consists of (j,k)-galaxy components$\}$ may even increase. Indeed, for $k = 3$ and G a triangle with two pendant edges at all three nodes (a 3-blossom) we have $D = V(G)$ and $\mathrm{def}_{j,k}(G) = 1$, however, $\mathrm{def}_{j,k}(G - v) = 2$ for every non-tip node $v \in V(G)$. That is why we need to use the tips in the galaxies in $G[D]$, and define D in Theorem 3 via $U(G)$.

5 Matroidality and maximum weight packings

Definition 29. We say that the $\mathcal{F}$-packing problem is **matroidal** if for all graphs G those node sets $X \subseteq V(G)$ which can be covered by an $\mathcal{F}$-packing of G form a matroid.

Loebl and Poljak [15] express their belief that for graph sets $\mathcal{F}$ with $K_2 \in \mathcal{F}$ the $\mathcal{F}$-packing problem is polynomial if and only if it is matroidal. This question is still open. That the condition $K_2 \in \mathcal{F}$ is indeed required was shown in [8], where it was proved that the k-piece packing problem for $k \geqslant 2$ is polynomial but not matroidal. This applies to the j-restricted k-matching problem for $0 \leqslant j < k$ as well, which is polynomial by [14], but not matroidal for $j > 0$, as shown by Theorem 4.

Proof of Theorem 4. Lemma 26 implies that the following considerations hold. Let

$$\pi = \{\{v\} : v \notin W_A\} \cup \{V(T) : T \text{ is a tip of a } (j,k)\text{-galaxy component of } G[D]\}.$$

To create matroid $\mathcal{P}$, we make use of matroid $\mathcal{M}$ in the proof of Theorem 23. First, for each component H of $G[D]$, replace H in $\mathcal{M}$ with

$$\pi_H = \{V(T) : T \text{ is a tip of } H\} \subseteq \pi,$$

such that the elements of π_H are in series with each other. Second, add as a direct sum the elements $\{v\}$ as bridges for $v \notin W_A$. The resulting matroid is $\mathcal{P}$. □

Let $\text{def}(G) = c(G[D]) - k|A|$. The co-rank of $\mathcal{M}$ is $\text{def}(G)$ thus the co-rank of $\mathcal{P}$ is $\text{def}(G)$, too. Note that by Lemma 26 for each maximal j-restricted k-matching M of G, every node set of π is either fully covered or fully missed by M and the number of the fully missed node sets is $\text{def}(G)$. In the case $j = 0$, a tip has exactly one node so π is the partition into singletons. A special case is the classical matching problem for $j = 0$, $k = 1$. For $j > 0$, a tip has at most j nodes so the node sets of π are of size at most j.

Because the ground set of the matroid $\mathcal{P}$ is a partition into different size sets, in the j-restricted k-matching problem a *maximal* packing is not necessarily *maximum*, as it is the case in the known polynomial packing problems with $K_2 \in \mathcal{F}$.

Theorem 6 on the characterization of the maximum weight j-restricted k-matchings was first proved in [14]. It can be deduced from the properties of matroid $\mathcal{P}$ as follows.

Proof of Theorem 6. Let us take the maximum weight bases of $\mathcal{P}$ with the weight function $X \mapsto \sum\{w(v) : v \in X\}$ for $X \in \pi$. Now the maximum weight bases of $\mathcal{P}$ correspond to the maximum weight j-restricted k-matchings. So one can apply the greedy algorithm to find the maximum weight j-restricted k-matchings, which yields the formula in the statement. □

Clearly, A_1 in Theorem 6 can be chosen to be the barrier A in the canonical decomposition. In the case $k = 1$ we get the Berge-theorem on maximum matchings [1].

We remark that our approach provides an alternative polynomial time algorithm to find a maximum weight j-restricted k-matching. First, the Edmonds–Gallai-type decomposition $V(G) = D^k \dot\cup A^k \dot\cup C^k$ for the k-piece packing problem can be determined in polynomial time [8, 10]. Theorem 28 and the construction in Theorem 23 shows that the canonical decomposition $V(G) = D\dot{\cup}A\dot{\cup}C$ for the j-restricted k-matching problem can be determined in polynomial time as well. With the greedy algorithm in the proof of Theorem 6 above these provide a polynomial time algorithm to find a maximum weight j-restricted k-matching for $0 \leqslant j < k$. As a counterpart, [14] proved that this problem is NP-complete for $j \geqslant k$.

One can construct the maximum weight packings in other ways as well. In [14] minimum cost flows are applied in the polynomial time alternating tree algorithm, while in [8] a direct argument is given for the k-piece packing problem.

6 Conclusions

An important relation between k-piece packings and j-restricted k-matchings is that every k-piece is a j-restricted k-matching for every $0 \leqslant j < k$. Using this connection, in this paper we gave an alternative proof to Theorem 6 of Li [14], and we proved two new results on the j-restricted k-matching problem. Theorem 3 is an Edmonds–Gallai-type decomposition, and Theorem 4 is a characterization of the maximal j-restricted k-matchings.

We may consider a generalization of j-restricted k-matchings inspired by the (l, u)-piece packings defined in [8], where l and u are assumed to be constant functions on the nodes satisfying $0 \leqslant l \leqslant u$. This generalization is called *j-restricted (l, u)-piece packing*, where l, j and u are constants such that $0 \leqslant l \leqslant j < u$. A connected graph G is called a *j-restricted (l, u)-piece* if $l \leqslant \Delta(G) \leqslant u$ and $|E(G)| > j$. It is easy to see that a j-restricted k-matching ($0 \leqslant j < k$) is just a j-restricted $(0, k)$-piece packing. One can probably establish an Edmonds–Gallai-type decomposition for j-restricted (l, u)-piece packings with $0 \leqslant l \leqslant j < u$.

Another question is, given a graph and not necessarily non-negative node weights, whether one can solve the maximum node weight j-restricted k-matching problem ($1 \leqslant j < k$) in polynomial time. The same question is open for k-piece packings ($k \geqslant 2$).

References

[1] Claude Berge. Sur le couplage maximum d'un graphe. *C.R. Acad. Sci. Paris Ser. I Math.*, 247:258–259, 1958.

[2] Gérard Cornuéjols and David Hartvigsen. An extension of matching theory. *Journal of Combinatorial Theory, Series B*, 40(3):285–296, 1986.

[3] Gerard Cornuéjols, David Hartvigsen, and William R. Pulleyblank. Packing subgraphs in a graph. *Operations Research Letters*, 1(4):139–143, 1982.

[4] Gérard Cornuéjols and William R. Pulleyblank. Critical graphs, matchings and tours or a hierarchy of relaxations for the travelling salesman problem. *Combinatorica*, 3(1):35–52, 1983.

[5] Jack Edmonds. Paths, trees, and flowers. *Canadian Journal of Mathematics*, 17(3):449–467, 1965.

[6] Tibor Gallai. Kritische Graphen. II. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 8:373–395 (1964), 1963.

[7] Tibor Gallai. Maximale Systeme unabhängiger Kanten. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:401–413 (1965), 1964.

[8] David Hartvigsen, Pavol Hell, and Jácint Szabó. The k-piece packing problem. *Journal of Graph Theory*, 52(4):267–293, 2006.

[9] Pavol Hell and David G. Kirkpatrick. Packings by cliques and by finite families of graphs. *Discrete Mathematics*, 49(1):45–59, 1984.

[10] Marek Janata, Martin Loebl, and Jácint Szabó. The Edmonds-Gallai decomposition for the k-piece packing problem. *The Electronic Journal of Combinatorics*, 12(1):#R8, 2005.

[11] Atsushi Kaneko. A necessary and sufficient condition for the existence of a path factor every component of which is a path of length at least two. *Journal of Combinatorial Theory, Series B*, 88(2):195–218, 2003.

[12] Mikio Kano, Gyula Y. Katona, and Zoltán Király. Packing paths of length at least two. *Discrete mathematics*, 283(1):129–135, 2004.

[13] Dénes Kőnig. Gráfok és mátrixok (graphs and matrices). *Matematikai és Fizikai Lapok*, 38:116–119, 1931.

[14] Yanjun Li. Nonnegative node weight j-restricted k-matching problems. *Mathematics of Operations Research*, 39(3):930–948, 2014.

[15] Martin Loebl and Svatopluk Poljak. On matroids induced by packing subgraphs. *Journal of Combinatorial Theory, Series B*, 44(3):338–354, 1988.

[16] László Lovász and Michael D. Plummer. *Matching theory*, volume 367. American Mathematical Soc., 2009.

[17] R. Rado. A theorem on independence relations. *Quart. J. Math., Oxford Ser.*, 13:83–89, 1942.