Collapsibility of non-cover complexes of graphs

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Abstract

Let G be a graph on the vertex set V. A vertex subset W ⊆ V is a cover of G if V \ W is an independent set of G, and W is a non-cover of G if W is not a cover of G. The non-cover complex of G is a simplicial complex on V whose faces are non-covers of G. Then the non-cover complex of G is the combinatorial Alexander dual of the independence complex of G. Aharoni asked if the non-cover complex of a graph G without isolated vertices is (|V(G)| − iγ(G) − 1)-collapsible where iγ(G) denotes the independence domination number of G. Extending a result by the second author, who verified Aharoni’s question in the affirmative for chordal graphs, we prove that the answer to the question is yes for all graphs.

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1 Introduction

We consider only finite simple graphs. We use the common notation \([n]\) for \(\{1, \ldots, n\}\). Given a graph \(G\), let \(V(G)\) and \(E(G)\) denote the vertex set and edge set, respectively, of \(G\). An independent set of a graph is a subset of the vertices that induces no edge. A cover of \(G\) is a subset \(W\) of the vertices such that \(V(G) \setminus W\) is an independent set of \(G\); in other words, \(W\) contains an endpoint of every edge of \(G\). A subset of the vertices that is not a cover is called a non-cover.

The independence complex \(\mathcal{I}(G)\) of \(G\) is a simplicial complex defined as

\[
\mathcal{I}(G) := \{I \subseteq V(G) : I \text{ is an independent set of } G\},
\]

and the non-cover complex \(\mathcal{NC}(G)\) of \(G\), which is a simplicial complex defined as

\[
\mathcal{NC}(G) := \{W \subseteq V(G) : W \text{ is a non-cover of } G\}.
\]

These two simplicial complexes are highly related, in the sense that the non-cover complex \(\mathcal{NC}(G)\) is the (combinatorial) Alexander dual of \(\mathcal{I}(G)\), where the Alexander dual \(D(X)\) of a simplicial complex \(X\) on \(V\) is defined as

\[
D(X) := \{W \subseteq V : V \setminus W \notin X\}.
\]

Note that the non-cover complex of a graph with no edges is the void complex. If a graph with an isolated vertex \(v\) has an edge, then the non-cover complex is a cone with apex \(v\), and thus it is contractible. However, it is not easy to determine the non-cover complex of an arbitrary graph. Our main result connects the collapsibility of the non-cover complex and the independence domination number of the associated graph. We now introduce these two parameters.

For a graph \(G\) and \(A, D \subseteq V(G)\), if each \(v \in A\) has a neighbor in \(D\), then we say \(D\) dominates \(A\). We use \(\gamma(G; A)\) to denote the minimum size of a set that dominates \(A\). The independence domination number \(i\gamma(G)\) of \(G\) is defined as

\[
i\gamma(G) := \max\{\gamma(G; I) : I \in \mathcal{I}(G)\}.
\]

By convention, we let \(i\gamma(G) = \infty\) when \(G\) contains an isolated vertex.

For a finite simplicial complex \(X\), a face \(\sigma \in X\) is free if there is a unique facet of \(X\) containing \(\sigma\). An elementary \(d\)-collapse of \(X\) is the operation of deleting all faces containing a free face of size at most \(d\). We say \(X\) is \(d\)-collapsible if we can obtain the void complex from \(X\) by a finite sequence of elementary \(d\)-collapses. The notion of \(d\)-collapsibility of simplicial complexes was introduced in [16] and has been widely studied ever since [11, 12]. An easy observation is that an elementary \(d\)-collapse does not affect the (non-)vanishing property of homology groups of dimension at least \(d\). See also [7, 8] for applications regarding Helly-type theorems. In addition, the topological colorful Helly theorem [8] tells us that given a graph \(G\) with a \(d\)-collapsible non-cover complex, for every \(d + 1\) covers \(W_1, \ldots, W_{d+1}\) of \(G\), there is a cover \(W = \{w_{i_1}, \ldots, w_{i_k}\}\) of \(G\) such that
$1 \leq i_1 < \cdots < i_k \leq d + 1$ and $w_{i_j} \in W_{i_j}$ for each $j \in [k]$; the set $W$ is called a rainbow cover of $G$ for $W_1, \ldots, W_{d+1}$.

The collapsibility of non-cover complexes of graphs is related to the topological connectivity of independence complexes. For a simplicial complex $X$, let $\eta(X)$ be the maximum integer $k$ such that $\tilde{H}_j(X) = 0$ for all $-1 \leq j \leq k - 2$. (We use $\tilde{H}_i(X)$ to denote the $i$th reduced homology group of $X$ over $\mathbb{Q}$.) Here, $\tilde{H}_{-1}(X) = 0$ if and only if $X$ is non-empty. In [2, 3] (see also [13, 14]), it was shown that large independence domination numbers of graphs give high connectivity of the independence complexes of graphs, in particular, Theorem 1. Research in this direction was motivated by a topological version of Hall’s marriage theorem [2].

**Theorem 1** ([2, 3]). For every graph $G$, $\eta(I(G)) \geq i\gamma(G)$.

As a consequence of Theorem 1 and the Alexander duality theorem\footnote{Alexander duality theorem ([6,15]) Let $X$ be a simplicial complex on the vertex set $V$. If $V \notin X$, then for all $-1 \leq i \leq |V| - 2$, $\tilde{H}_i(D(X)) \cong \tilde{H}_{|V|-i-3}(X)$.} (see [6,15]) we obtain that for every graph $G$ with at least one edge, the reduced homology group of the non-cover complex of $G$ satisfies

$$\tilde{H}_i(\mathcal{NC}(G)) = 0 \text{ for all } i \geq |V(G)| - i\gamma(G) - 1.$$ (1)

Aharoni [1] asked the following question:

**Question 2** ([1]). If $G$ is a graph with no isolated vertices, then is it true that the non-cover complex of $G$ is $(|V(G)| - i\gamma(G) - 1)$-collapsible?

The verification of Question 2 for all graphs implies not only the property in (1), but also the stronger property that for every $W \subseteq V(G)$, the reduced homology group of the subcomplex $\mathcal{NC}(G)[W]$ induced by $W$ satisfies

$$\tilde{H}_i(\mathcal{NC}(G)[W]) = 0 \text{ for all } i \geq |V(G)| - i\gamma(G) - 1.$$ (1)

In [10], the second author of this paper verified Question 2 for chordal graphs. We extend this result by resolving Question 2 completely in the affirmative.

**Theorem 3.** For a graph $G$ without isolated vertices, the non-cover complex of $G$ is $(|V(G)| - i\gamma(G) - 1)$-collapsible.

The main tool for our proof of Theorem 3 is minimal exclusion sequences [12] (see also [11]), which we review in section 2 along with the proof of Theorem 3. We end the paper by providing some remarks in section 3.
2 Proof

2.1 Minimal exclusion sequences

In this subsection, we review a result in [12], which will play a key role in the proof.

For a simplicial complex $X$ on the vertex set $[n]$, take a linear ordering $\prec_F: \sigma_1, \ldots, \sigma_m$ of the facets of $X$. Given a face $\sigma$ of $X$, we define the minimal exclusion sequence $\text{mes}_{\prec_F}(\sigma)$ as follows. Let $i$ denote the smallest index such that $\sigma \subseteq \sigma_i$. If $i = 1$, then $\text{mes}_{\prec_F}(\sigma)$ is the null sequence. If $i \geq 2$, then $\text{mes}_{\prec_F}(\sigma) = (v_1, \ldots, v_{i-1})$ is a finite sequence of length $i - 1$ such that $v_1 = \min(\sigma \setminus \sigma_1)$ and for each $k \in \{2, \ldots, i - 1\}$,

$$v_k = \begin{cases} \min\{v_1, \ldots, v_{k-1}\} \cap (\sigma \setminus \sigma_k) & \text{if } \{v_1, \ldots, v_{k-1}\} \cap (\sigma \setminus \sigma_k) \neq \emptyset, \\ \min(\sigma \setminus \sigma_k) & \text{otherwise}. \end{cases}$$

Let $M_{\prec_F}(\sigma)$ denote the set of vertices appearing in $\text{mes}_{\prec_F}(\sigma)$, and define

$$d_{\prec_F}(X) := \max_{\sigma \in X} |M_{\prec_F}(\sigma)|.$$ 

The following was proved in [12] (see also [11]).

**Theorem 4** ([12]). If $\prec_F$ is a linear ordering of the facets of $X$, then $X$ is $d_{\prec_F}(X)$-collapsible.

2.2 Proof of Theorem 3

Let $G$ be a graph without isolated vertices. For simplicity, assume $V(G) = [n]$ and denote $\overline{S} := [n] \setminus S$ for $S \subseteq [n]$. Let $I$ be an independent set of $G$ such that $\gamma(G; I) = \gamma(G)$. Let $|I| = i$. We may assume that $I$ is a maximal independent set and $I := [n] \setminus [n - i]$.

Note that every facet of $\mathcal{NC}(G)$ is the complement of an edge of $G$. We define a linear ordering $\prec_F$ of the facets of $\mathcal{NC}(G)$ as follows. For two edges $a_1b_1$ and $a_2b_2$ where $a_i < b_i$ for $i \in [2]$, let $\prec_{\mathcal{AL}}$ be the anti-lexicographic ordering of $<$, that is, $a_1b_1 <_{\mathcal{AL}} a_2b_2$ if either (i) $b_1 < b_2$ or (ii) $b_1 = b_2$ and $a_1 < a_2$. For two distinct facets $\sigma$ and $\tau$ of $\mathcal{NC}(G)$, we denote $\sigma \prec_F \tau$ if $\overline{\sigma} <_{\mathcal{AL}} \overline{\tau}$.

**Claim 5.** For $\sigma, \sigma' \in \mathcal{NC}(G)$, if $\overline{\sigma} \cap \overline{\tau} = \overline{\sigma'} \cap \overline{\tau}$ and $G[\overline{\sigma} \cap \overline{\tau}]$ contains an edge, then $\text{mes}_{\prec_F}(\sigma) = \text{mes}_{\prec_F}(\sigma')$.

**Proof.** Let $j$ be the length of $\text{mes}_{\prec_F}(\sigma)$. Note that an edge between $I$ and $\overline{\tau}$ comes after all the edges of $G[\overline{\tau}]$ in the linear ordering $<_{\mathcal{AL}}$. Since $G[\overline{\sigma} \cap \overline{\tau}]$ has an edge, for the $(j + 1)$th facet $\sigma_{j + 1}$, $\sigma_{j + 1}$ is an edge such that $\overline{\sigma_{j + 1}} \subseteq \overline{\tau}$. By the definition of $\prec_F$, it also follows that for every $k \in [j + 1]$, the $k$th facet $\sigma_k$ satisfies $\overline{\sigma_k} \subseteq \overline{\tau}$. Clearly, $\sigma \cap \overline{\tau} = \sigma' \cap \overline{\tau}$. Thus, we have

$$\overline{\sigma_k} \cap \sigma = \overline{\sigma_k} \cap \sigma' \cap \overline{\tau} = \overline{\sigma_k} \cap \sigma' \cap \overline{\tau} = \overline{\sigma_k} \cap \sigma'.$$

Thus the length of $\text{mes}_{\prec_F}(\sigma')$ is also $j$ and for every $k \in [j]$, the $k$th entry of $\text{mes}_{\prec_F}(\sigma)$ is equal to that of $\text{mes}_{\prec_F}(\sigma')$. \qed
Claim 6. For every $S \subseteq \overline{T}$,

$$|S| - |N(S) \cap I| \geq i\gamma(G) - |I|,$$

where $N(S) = \{v \in V(G) : uv \in E(G) \text{ for some } u \in S\}$.

Proof. Since $G$ has no isolated vertex, for each $v \in I \setminus (N(S) \cap I)$, we can take a neighbor $u_v \in \overline{T} \setminus S$ of $v$. Let $T = \{u_v : v \in I \setminus (N(S) \cap I)\}$. Note that $|T| \leq |I| - |N(S) \cap I|$ and $S \cup T$ dominates $I$. Thus we obtain

$$|S| + |I| - |N(S) \cap I| \geq |S| + |T| \geq i\gamma(G).$$

By Theorem 4, it is sufficient to show that

$$|M_{<f}(\sigma)| \leq |V(G)| - i\gamma(G) - 1 \quad \text{for every } \sigma \in \mathcal{NC}(G). \quad (2)$$

For a face $\sigma \in \mathcal{NC}(G)$, let $\beta(\sigma) = |N(\overline{\sigma} \cap \overline{T}) \cap \sigma \cap I|$. Suppose that $\beta(\sigma) = 0$. Then $G[\overline{\sigma} \cap \overline{T}]$ must have an edge. Consider $\sigma' = \sigma \cap I$. Then $\overline{\sigma} \cap \overline{T} = \overline{\sigma'} \cap \overline{T}$. By Claim 5, $\mes_{<f}(\sigma) = \mes_{<f}(\sigma')$ and therefore, $M_{<f}(\sigma) = M_{<f}(\sigma')$. On the other hand, we know $\beta(\sigma') \geq 1$ by the definition of $\sigma'$. Thus, it is sufficient to check (2) under the assumption $\beta(\sigma) \geq 1$.

We claim that for $v \in \sigma \cap I$, if $v \in M_{<f}(\sigma)$, then $v$ is a neighbor of some vertex in $\overline{\sigma} \cap \overline{T}$. Let $k$ be the first index such that the $k$th entry of $\mes_{<f}(\sigma)$ is $v$. Then $v \in \sigma \setminus \sigma_k$, which means that $v$ is in the edge $\overline{\sigma_k}$. Let $\overline{\sigma_k} = wv$ for some vertex $w \in \overline{T}$. Since $w < v$ and $v$ is the $k$th entry of $\mes_{<f}(\sigma)$, we obtain $w \notin \sigma$. Thus $v$ is a neighbor of $w \in \overline{\sigma} \cap \overline{T}$.

Thus,

$$|M_{<f}(\sigma)| \leq |\sigma \cap \overline{T}| + |N(\overline{\sigma} \cap \overline{T}) \cap (\sigma \cap I)|$$

$$= |\overline{T}| - |\overline{\sigma} \cap \overline{T}| + |N(\overline{\sigma} \cap \overline{T}) \cap I| - \beta(\sigma)$$

$$\leq |\overline{T}| - i\gamma(G) + |I| - \beta(\sigma)$$

$$= |V(G)| - i\gamma(G) - \beta(\sigma),$$

where the last inequality holds by applying Claim 6 to the set $\overline{\sigma} \cap \overline{T}$. As we assumed that $\beta(\sigma) \geq 1$, (2) follows, and this concludes the proof of Theorem 3.

3 Concluding remarks

For a graph $G$ and $A, W \subseteq V(G)$, if each $w \in A$ has a neighbor in $W$ or $w \in W$, then we say $W$ weakly dominates $A$. We use $\gamma_w(G; A)$ to denote the minimum size of a set that weakly dominates $A$. The weak independence domination number $i\gamma_w(G)$ of $G$ is defined as

$$i\gamma_w(G) := \max\{\gamma_w(G; I) : I \text{ is an independent set of } G\}.$$

The following is a straightforward application of Theorem 3.
Corollary 7. For a graph $G$, the non-cover complex of $G$ is $(|V(G)| - i\gamma_{w}(G) - 1)$-collapsible.

Proof. If $G$ has no isolated vertex, then $i\gamma_{w}(G) = i\gamma(G)$ and we are done by Theorem 3. Assume $G$ has $k$ isolated vertices for some integer $k \geq 1$. Let $W$ be the set of isolated vertices of $G$, and let $G'$ be the graph obtained from $G$ by removing all vertices in $W$.

Recall that $\mathcal{NC}(G)$ is a cone with apex $v$ if $v$ is an isolated vertex of $G$. Thus $\mathcal{NC}(G)$ is $d$-collapsible if and only if the subcomplex of $\mathcal{NC}(G)$ induced by $V(G)\setminus\{v\}$ is $d$-collapsible. Moreover, since the subcomplex of $\mathcal{NC}(G)$ induced by $V(G)\setminus W$ is equal to $\mathcal{NC}(G')$, it follows that $\mathcal{NC}(G)$ is $d$-collapsible if and only if $\mathcal{NC}(G')$ is $d$-collapsible. Thus, it is sufficient to show $\mathcal{NC}(G')$ is $(|V(G)| - i\gamma_{w}(G) - 1)$-collapsible. By Theorem 3, $\mathcal{NC}(G')$ is $(|V(G')| - i\gamma(G') - 1)$-collapsible. Since $|V(G')| = |V(G)| - k$ and $i\gamma_{w}(G) = i\gamma(G') + k$, we obtain $|V(G')| - i\gamma(G') - 1 = |V(G)| - i\gamma_{w}(G) - 1$. \qed

We finish the section by stating a direct consequence of the topological colorful Helly theorem [8] from our main result.

**Corollary 8.** Let $G$ be a graph on $n$ vertices and let $W_{1},\ldots,W_{n-i\gamma(G)} \subseteq V(G)$. Assume that every set $A \subseteq V(G)$ satisfying the following two conditions is a cover of $G$:

(i) $A \cap W_{i} \neq \emptyset$ for $i \in [n - i\gamma(G)]$.

(ii) $W_{j} \subseteq A$ for some $j \in [n - i\gamma(G)]$.

Then there is a cover $W$ of $G$ where $W = \{w_{i_{1}},\ldots,w_{i_{k}}\}$ with $1 \leq i_{1} < \cdots < i_{k} \leq n - i\gamma(G)$ and $w_{i_{j}} \in W_{i_{j}}$ for each $j \in [k]$.

Dao and Schweig [4] showed a weaker version of Theorem 3 concerning a topological property known as “Lerayness” via an algebraic approach. Let us briefly introduce their result. For a simplicial complex $X$, we say $X$ is $d$-Leray if $\tilde{H}_{i}(Y) = 0$ for all induced subcomplexes $Y$ of $X$ and all integers $i \geq d$. Wegner showed that $d$-collapsibility implies $d$-Lerayness [16], yet the converse is not always true [12]. Hochster [5] proved the relation between the Leray number$^{2}$ and the Castelnuovo-Mumford regularity of the Stanley-Reisner ideal of a simplicial complex. From this relationship and the result in [4], it was shown that for a graph $G$, the non-cover complex $\mathcal{NC}(G)$ is $(|V(G)| - i\gamma(G) - 1)$-Leray.

There is an active line of research in this direction, see [9,17] for more details. By applying the topological colorful Helly theorem of the Lerayness version, we obtain the following:

**Corollary 9.** Let $G$ be a graph on $n$ vertices. For every $n - i\gamma(G)$ covers $W_{1},\ldots,W_{n-i\gamma(G)}$ of $G$, there is a cover $W$ of $G$ where $W = \{w_{i_{1}},\ldots,w_{i_{k}}\}$ with $1 \leq i_{1} < \cdots < i_{k} \leq n - i\gamma(G)$ and $w_{i_{j}} \in W_{i_{j}}$ for each $j \in [k]$.

Note that Corollary 9 is weaker than Corollary 8, since if we have $n - i\gamma(G)$ covers for a graph $G$, then a set $A \subseteq V(G)$ satisfying (ii) is a cover of $G$. As mentioned in the introduction, the set $W$ in Corollary 8 and 9 is also known as a rainbow cover of $G$ for $W_{1},\ldots,W_{n-i\gamma(G)}$. The following example demonstrates that Corollaries 8 and 9 are tight.

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$^{2}$For a simplicial complex $X$, the Leray number of $X$ is the minimum integer $k$ such that $X$ is $k$-Leray.
**Example 10.** Let $C_{3k}$ be a cycle of length $3k$ for an integer $k \geq 2$. It is easy to verify $i\gamma(C_{3k}) = k$ and so $|V(C_{3k})| - i\gamma(C_{3k}) = 2k$. Consider $M \subseteq V(C_{3k})$ that induces a matching of size $k$, so that $M$ is a cover of $C_{3k}$. Let $W_i = M$ for all $i \in [2k-1]$. It is again easy to verify that there is no rainbow cover with respect to $W_1, \ldots, W_{2k-1}$.

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**References**


