Collapsibility of non-cover complexes of graphs

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Abstract

Let G be a graph on the vertex set V. A vertex subset $W \subseteq V$ is a *cover* of G if $V \setminus W$ is an independent set of G, and W is a *non-cover* of G if W is not a cover of G. The *non-cover complex* of G is a simplicial complex on V whose faces are non-covers of G. Then the non-cover complex of G is the combinatorial Alexander dual of the independence complex of G. Aharoni asked if the non-cover complex of a graph G without isolated vertices is $(|V(G)| - i\gamma(G) - 1)$-collapsible where $i\gamma(G)$ denotes the independence domination number of G. Extending a result by the second author, who verified Aharoni's question in the affirmative for chordal graphs, we prove that the answer to the question is yes for all graphs.

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1 Introduction

We consider only finite simple graphs. We use the common notation $[n]$ for $\{1, \ldots, n\}$. Given a graph G, let $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively, of G. An *independent set* of a graph is a subset of the vertices that induces no edge. A *cover* of G is a subset W of the vertices such that $V(G) \setminus W$ is an independent set of G; in other words, W contains an endpoint of every edge of G. A subset of the vertices that is not a cover is called a *non-cover*.

The *independence complex* $\mathcal{I}(G)$ of G is a simplicial complex defined as

$$\mathcal{I}(G) := \{I \subseteq V(G) : I \text{ is an independent set of } G\},$$

and the *non-cover complex* $\mathcal{NC}(G)$ of G, which is a simplicial complex defined as

$$\mathcal{NC}(G) := \{W \subseteq V(G) : W \text{ is a non-cover of } G\}.$$

These two simplicial complexes are highly related, in the sense that the non-cover complex $\mathcal{NC}(G)$ is the *(combinatorial) Alexander dual* of $\mathcal{I}(G)$, where the Alexander dual $D(X)$ of a simplicial complex X on V is defined as

$$D(X) := \{W \subseteq V : V \setminus W \notin X\}.$$

Note that the non-cover complex of a graph with no edges is the void complex. If a graph with an isolated vertex v has an edge, then the non-cover complex is a cone with apex v, and thus it is contractible. However, it is not easy to determine the non-cover complex of an arbitrary graph. Our main result connects the collapsibility of the non-cover complex and the independence domination number of the associated graph. We now introduce these two parameters.

For a graph G and $A, D \subseteq V(G)$, if each $v \in A$ has a neighbor in D, then we say D *dominates* A. We use $\gamma(G; A)$ to denote the minimum size of a set that dominates A. The *independence domination number* $i\gamma(G)$ of G is defined as

$$i\gamma(G) := \max\{\gamma(G; I) : I \in \mathcal{I}(G)\}.$$

By convention, we let $i\gamma(G) = \infty$ when G contains an isolated vertex.

For a finite simplicial complex X, a face $\sigma \in X$ is *free* if there is a unique facet of X containing σ. An *elementary d-collapse* of X is the operation of deleting all faces containing a free face of size at most d. We say X is *d-collapsible* if we can obtain the void complex from X by a finite sequence of elementary d-collapses. The notion of d-collapsibility of simplicial complexes was introduced in [16] and has been widely studied ever since [11, 12]. An easy observation is that an elementary d-collapse does not affect the (non-)vanishing property of homology groups of dimension at least d. See also [7, 8] for applications regarding Helly-type theorems. In addition, the topological colorful Helly theorem [8] tells us that given a graph G with a d-collapsible non-cover complex, for every $d+1$ covers $W_1, \ldots, W_{d+1}$ of G, there is a cover $W = \{w_{i_1}, \ldots, w_{i_k}\}$ of G such that

$1 \leqslant i_1 < \cdots < i_k \leqslant d+1$ and $w_{i_j} \in W_{i_j}$ for each $j \in [k]$; the set W is called a *rainbow cover* of G for $W_1, \ldots, W_{d+1}$.

The collapsibility of non-cover complexes of graphs is related to the topological connectivity of independence complexes. For a simplicial complex X, let $\eta(X)$ be the maximum integer k such that $\tilde{H}_j(X) = 0$ for all $-1 \leqslant j \leqslant k-2$. (We use $\tilde{H}_i(X)$ to denote the ith reduced homology group of X over $\mathbb{Q}$.) Here, $\tilde{H}_{-1}(X) = 0$ if and only if X is non-empty. In [2, 3] (see also [13, 14]), it was shown that large independence domination numbers of graphs gives high connectivity of the independence complexes of graphs, in particular, Theorem 1. Research in this direction was motivated by a topological version of Hall's marriage theorem [2].

Theorem 1 ([2, 3])**.** *For every graph G, $\eta(\mathcal{I}(G)) \geqslant i\gamma(G)$.*

As a consequence of Theorem 1 and the Alexander duality theorem[1] (see [6, 15]) we obtain that for every graph G with at least one edge, the reduced homology group of the non-cover complex of G satisfies

$$\tilde{H}_i(\mathcal{NC}(G)) = 0 \text{ for all } i \geqslant |V(G)| - i\gamma(G) - 1. \tag{1}$$

Aharoni [1] asked the following question:

Question 2 ([1])**.** If G is a graph with no isolated vertices, then is it true that the non-cover complex of G is $(|V(G)| - i\gamma(G) - 1)$-collapsible?

The verification of Question 2 for all graphs implies not only the property in (1), but also the stronger property that for every $W \subseteq V(G)$, the reduced homology group of the subcomplex $\mathcal{NC}(G)[W]$ induced by W satisfies

$$\tilde{H}_i(\mathcal{NC}(G)[W]) = 0 \text{ for all } i \geqslant |V(G)| - i\gamma(G) - 1.$$

In [10], the second author of this paper verified Question 2 for chordal graphs. We extend this result by resolving Question 2 completely in the affirmative.

Theorem 3. *For a graph G without isolated vertices, the non-cover complex of G is $(|V(G)| - i\gamma(G) - 1)$-collapsible.*

The main tool for our proof of Theorem 3 is minimal exclusion sequences [12] (see also [11]), which we review in section 2 along with the proof of Theorem 3. We end the paper by providing some remarks in section 3.

[1] **Alexander duality theorem**([6,15]) Let X be a simplicial complex on the vertex set V. If $V \notin X$, then for all $-1 \leqslant i \leqslant |V| - 2$, $\tilde{H}_i(D(X)) \cong \tilde{H}_{|V|-i-3}(X)$.

2 Proof

2.1 Minimal exclusion sequences

In this subsection, we review a result in [12], which will play a key role in the proof.

For a simplicial complex X on the vertex set $[n]$, take a linear ordering $\prec_F$: $\sigma_1, \dots, \sigma_m$ of the facets of X. Given a face σ of X, we define the *minimal exclusion sequence* $\mathrm{mes}_{\prec_F}(\sigma)$ as follows. Let i denote the smallest index such that $\sigma \subseteq \sigma_i$. If $i = 1$, then $\mathrm{mes}_{\prec_F}(\sigma)$ is the null sequence. If $i \geqslant 2$, then $\mathrm{mes}_{\prec_F}(\sigma) = (v_1, \dots, v_{i-1})$ is a finite sequence of length $i - 1$ such that $v_1 = \min(\sigma \setminus \sigma_1)$ and for each $k \in \{2, \dots, i-1\}$,

$$v_k = \begin{cases} \min(\{v_1, \dots, v_{k-1}\} \cap (\sigma \setminus \sigma_k)) & \text{if } \{v_1, \dots, v_{k-1}\} \cap (\sigma \setminus \sigma_k) \neq \emptyset, \\ \min(\sigma \setminus \sigma_k) & \text{otherwise.} \end{cases}$$

Let $M_{\prec_F}(\sigma)$ denote the set of vertices appearing in $\mathrm{mes}_{\prec_F}(\sigma)$, and define

$$d_{\prec_F}(X) := \max_{\sigma \in X} |M_{\prec_F}(\sigma)|.$$

The following was proved in [12] (see also [11]).

Theorem 4 ([12])**.** *If $\prec_F$ is a linear ordering of the facets of X, then X is $d_{\prec_F}(X)$-collapsible.*

2.2 Proof of Theorem 3

Let G be a graph without isolated vertices. For simplicity, assume $V(G) = [n]$ and denote $\overline{S} := [n] \setminus S$ for $S \subseteq [n]$. Let I be an independent set of G such that $\gamma(G; I) = i\gamma(G)$. Let $|I| = i$. We may assume that I is a maximal independent set and $I := [n] \setminus [n-i]$.

Note that every facet of $\mathcal{NC}(G)$ is the complement of an edge of G. We define a linear ordering $\prec_F$ of the facets of $\mathcal{NC}(G)$ as follows. For two edges a_1b_1 and a_2b_2 where $a_i < b_i$ for $i \in [2]$, let $<_{AL}$ be the anti-lexicographic ordering of $<$, that is, $a_1b_1 <_{AL} a_2b_2$ if either (i) $b_1 < b_2$ or (ii) $b_1 = b_2$ and $a_1 < a_2$. For two distinct facets σ and τ of $\mathcal{NC}(G)$, we denote $\sigma \prec_F \tau$ if $\overline{\sigma} <_{AL} \overline{\tau}$.

Claim 5. *For $\sigma, \sigma' \in \mathcal{NC}(G)$, if $\overline{\sigma} \cap \overline{I} = \overline{\sigma'} \cap \overline{I}$ and $G[\overline{\sigma} \cap \overline{I}]$ contains an edge, then* $\mathrm{mes}_{\prec_F}(\sigma) = \mathrm{mes}_{\prec_F}(\sigma')$.

Proof. Let j be the length of $\mathrm{mes}_{\prec_F}(\sigma)$. Note that an edge between I and $\overline{I}$ comes after all the edges of $G[\overline{I}]$ in the linear ordering $<_{AL}$. Since $G[\overline{\sigma} \cap \overline{I}]$ has an edge, for the $(j+1)$th facet σ_{j+1}, $\overline{\sigma_{j+1}}$ is an edge such that $\overline{\sigma_{j+1}} \subseteq \overline{I}$. By the definition of $\prec_F$, it also follows that for every $k \in [j+1]$, the kth facet σ_k satisfies $\overline{\sigma_k} \subseteq \overline{I}$. Clearly, $\sigma \cap \overline{I} = \sigma' \cap \overline{I}$. Thus, we have

$$\overline{\sigma_k} \cap \sigma = \overline{\sigma_k} \cap \sigma \cap \overline{I} = \overline{\sigma_k} \cap \sigma' \cap \overline{I} = \overline{\sigma_k} \cap \sigma'.$$

Thus the length of $\mathrm{mes}_{\prec_F}(\sigma')$ is also j and for every $k \in [j]$, the kth entry of $\mathrm{mes}_{\prec_F}(\sigma)$ is equal to that of $\mathrm{mes}_{\prec_F}(\sigma')$. □

Claim 6. *For every $S \subseteq \overline{I}$,*

$$|S| - |N(S) \cap I| \geqslant i\gamma(G) - |I|,$$

where $N(S) = \{v \in V(G) : uv \in E(G) \text{ for some } u \in S\}$.

Proof. Since G has no isolated vertex, for each $v \in I \setminus (N(S) \cap I)$, we can take a neighbor $u_v \in \overline{I} \setminus S$ of v. Let $T = \{u_v : v \in I \setminus (N(S) \cap I)\}$. Note that $|T| \leqslant |I| - |N(S) \cap I|$ and $S \cup T$ dominates I. Thus we obtain

$$|S| + |I| - |N(S) \cap I| \geqslant |S| + |T| \geqslant i\gamma(G). \qquad \square$$

By Theorem 4, it is sufficient to show that

$$|M_{\prec_F}(\sigma)| \leqslant |V(G)| - i\gamma(G) - 1 \quad \text{for every } \sigma \in \mathcal{NC}(G). \tag{2}$$

For a face $\sigma \in \mathcal{NC}(G)$, let $\beta(\sigma) = |N(\overline{\sigma} \cap \overline{I}) \cap \overline{\sigma} \cap I|$. Suppose that $\beta(\sigma) = 0$. Then $G[\overline{\sigma} \cap \overline{I}]$ must have an edge. Consider $\sigma' = \sigma \cap \overline{I}$. Then $\overline{\sigma} \cap \overline{I} = \overline{\sigma'} \cap \overline{I}$. By Claim 5, $\text{mes}_{\prec_F}(\sigma) = \text{mes}_{\prec_F}(\sigma')$ and therefore, $M_{\prec_F}(\sigma) = M_{\prec_F}(\sigma')$. On the other hand, we know $\beta(\sigma') \geqslant 1$ by the definition of σ'. Thus, it is sufficient to check (2) under the assumption $\beta(\sigma) \geqslant 1$.

We claim that for $v \in \sigma \cap I$, if $v \in M_{\prec_F}(\sigma)$, then v is a neighbor of some vertex in $\overline{\sigma} \cap \overline{I}$. Let k be the first index such that the kth entry of $\text{mes}_{\prec_F}(\sigma)$ is v. Then $v \in \sigma \setminus \sigma_k$, which means that v is in the edge $\overline{\sigma_k}$. Let $\overline{\sigma_k} = wv$ for some vertex $w \in \overline{I}$. Since $w < v$ and v is the kth entry of $\text{mes}_{\prec_F}(\sigma)$, we obtain $w \notin \sigma$. Thus v is a neighbor of $w \in \overline{\sigma} \cap \overline{I}$.

Thus,

$$\begin{aligned}
|M_{\prec_F}(\sigma)| &\leqslant |\sigma \cap \overline{I}| + |N(\overline{\sigma} \cap \overline{I}) \cap (\sigma \cap I)| \\
&= |\overline{I}| - |\overline{\sigma} \cap \overline{I}| + |N(\overline{\sigma} \cap \overline{I}) \cap I| - \beta(\sigma) \\
&\leqslant |\overline{I}| - i\gamma(G) + |I| - \beta(\sigma) \\
&= |V(G)| - i\gamma(G) - \beta(\sigma),
\end{aligned}$$

where the last inequality holds by applying Claim 6 to the set $\overline{\sigma} \cap \overline{I}$. As we assumed that $\beta(\sigma) \geqslant 1$, (2) follows, and this concludes the proof of Theorem 3.

3 Concluding remarks

For a graph G and $A, W \subseteq V(G)$, if each $w \in A$ has a neighbor in W or $w \in W$, then we say W *weakly dominates* A. We use $\gamma_w(G; A)$ to denote the minimum size of a set that weakly dominates A. The *weak independence domination number* $i\gamma_w(G)$ of G is defined as

$$i\gamma_w(G) := \max\{\gamma_w(G; I) : I \text{ is an independent set of } G\}.$$

The following is a straightforward application of Theorem 3.

Corollary 7. *For a graph G, the non-cover complex of G is $(|V(G)| - i\gamma_w(G) - 1)$-collapsible.*

Proof. If G has no isolated vertex, then $i\gamma_w(G) = i\gamma(G)$ and we are done by Theorem 3. Assume G has k isolated vertices for some integer $k \geqslant 1$. Let W be the set of isolated vertices of G, and let G' be the graph obtained from G by removing all vertices in W.

Recall that $\mathcal{NC}(G)$ is a cone with apex v if v is an isolated vertex of G. Thus $\mathcal{NC}(G)$ is d-collapsible if and only if the subcomplex of $\mathcal{NC}(G)$ induced by $V(G)\setminus\{v\}$ is d-collapsible. Moreover, since the subcomplex of $\mathcal{NC}(G)$ induced by $V(G) \setminus W$ is equal to $\mathcal{NC}(G')$, it follows that $\mathcal{NC}(G)$ is d-collapsible if and only if $\mathcal{NC}(G')$ is d-collapsible. Thus, it is sufficient to show $\mathcal{NC}(G')$ is $(|V(G)| - i\gamma_w(G) - 1)$-collapsible. By Theorem 3, $\mathcal{NC}(G')$ is $(|V(G')| - i\gamma(G') - 1)$-collapsible. Since $|V(G')| = |V(G)| - k$ and $i\gamma_w(G) = i\gamma(G') + k$, we obtain $|V(G')| - i\gamma(G') - 1 = |V(G)| - i\gamma_w(G) - 1$. □

We finish the section by stating a direct consequence of the topological colorful Helly theorem [8] from our main result.

Corollary 8. *Let G be a graph on n vertices and let $W_1, \ldots, W_{n-i\gamma(G)} \subseteq V(G)$. Assume that every set $A \subseteq V(G)$ satisfying the following two conditions is a cover of G:*

(i) $A \cap W_i \neq \emptyset$ for $i \in [n - i\gamma(G)]$.

(ii) $W_j \subseteq A$ for some $j \in [n - i\gamma(G)]$.

Then there is a cover W of G where $W = \{w_{i_1}, \ldots, w_{i_k}\}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n-i\gamma(G)$ and $w_{i_j} \in W_{i_j}$ for each $j \in [k]$.

Dao and Schweig [4] showed a weaker version of Theorem 3 concerning a topological property known as "Lerayness" via an algebraic approach. Let us briefly introduce their result. For a simplicial complex X, we say X is *d-Leray* if $\tilde{H}_i(Y) = 0$ for all induced subcomplexes Y of X and all integers $i \geqslant d$. Wegner showed that d-collapsiblity implies d-Lerayness [16], yet the converse is not always true [12]. Hochster [5] proved the relation between the Leray number[2] and the Castelnuovo-Mumford regularity of the Stanley-Reisner ideal of a simplicial complex. From this relationship and the result in [4], it was shown that for a graph G, the non-cover complex $\mathcal{NC}(G)$ is $(|V(G)| - i\gamma(G) - 1)$-Leray. There is an active line of research in this direction, see [9,17] for more details. By applying the topological colorful Helly theorem of the Lerayness version, we obtain the following:

Corollary 9. *Let G be a graph on n vertices. For every $n-i\gamma(G)$ covers $W_1, \ldots, W_{n-i\gamma(G)}$ of G, there is a cover W of G where $W = \{w_{i_1}, \ldots, w_{i_k}\}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n-i\gamma(G)$ and $w_{i_j} \in W_{i_j}$ for each $j \in [k]$.*

Note that Corollary 9 is weaker than Corollary 8, since if we have $n - i\gamma(G)$ covers for a graph G, then a set $A \subseteq V(G)$ satisfying (ii) is a cover of G. As mentioned in the introduction, the set W in Corollary 8 and 9 is also known as a *rainbow cover* of G for $W_1, \ldots, W_{n-i\gamma(G)}$. The following example demonstrates that Corollaries 8 and 9 are tight.

[2] For a simplicial complex X, the *Leray number* of X is the minimum integer k such that X is k-Leray.

Example 10. Let C_{3k} be a cycle of length $3k$ for an integer $k \geqslant 2$. It is easy to verify $i\gamma(C_{3k}) = k$ and so $|V(C_{3k})| - i\gamma(C_{3k}) = 2k$. Consider $M \subseteq V(C_{3k})$ that induces a matching of size k, so that M is a cover of C_{3k}. Let $W_i = M$ for all $i \in [2k-1]$. It is again easy to verify that there is no rainbow cover with respect to $W_1, \ldots, W_{2k-1}$.

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