On the sets of n points forming $n+1$ directions

Cédric Pilatte

Department of Mathematics
University of MONS (UMONS)
Place du Parc, 20
7000 Mons, Belgium

cedric.pilatte@umons.ac.be

Submitted: Nov 11, 2018; Accepted: Dec 22, 2019; Published: Jan 24, 2020

Abstract

Let S be a set of $n \geqslant 7$ points in the plane, no three of which are collinear. Suppose that S determines $n+1$ directions. That is to say, the segments whose endpoints are in S form $n+1$ distinct slopes. We prove that S is, up to an affine transformation, equal to n of the vertices of a regular $(n+1)$-gon. This result was conjectured in 1986 by R. E. Jamison.

Mathematics Subject Classifications: 52C10, 52C30, 52C35

1 Introduction

In 1970, inspired by a problem of Erdős, Scott [15] asked the following question, now known as *the slope problem*: what is the minimum number of directions determined by a set of n points in $\mathbb{R}^2$, not all on the same line? By the number of directions (or slopes) of a set S, we mean the size of the quotient set $\{PQ \mid P, Q \in S, P \neq Q\}/\!\sim$, where $\sim$ is the equivalence relation given by parallelism: $P_1Q_1 \sim P_2Q_2 \iff P_1Q_1 \parallel P_2Q_2$.

Scott conjectured that n points, not all collinear, determine at least $2\lfloor \frac{n}{2} \rfloor$ slopes. This bound can be achieved, for even n, by a regular n-gon; and for odd n, by a regular $(n-1)$-gon with its center. After some initial results of Burton and Purdy [2], this conjecture was proven by Ungar [16] in 1982, using techniques of Goodman and Pollack [5]. His beautiful proof is also exposed in the famous *Proofs from the Book* [1, Chapter 11]. Recently, Pach, Pinchasi and Sharir solved the tree-dimensional analogue of this problem, see[12, 13].

A lot of work has been done to determine the configurations where equality in Ungar's theorem is achieved. A *critical set* (respectively *near-critical set*) is a set of n non-collinear points forming $n-1$ slopes (respectively n slopes). Jamison and Hill described four infinite families and 102 sporadic critical configurations [6, 7, 10]. It is conjectured that this classification is accurate for $n \geqslant 49$. No classification is known in the near-critical case. See [8] for a survey of these questions, and other related ones.

In this paper, we suppose that no three points of S are collinear (we say that S is in general position). This situation was first investigated by Jamison [9], who proved that S must determine at least n slopes. As above, equality is possible with a regular n-gon. It is a well-known fact that affine transformations preserve parallelism. Therefore, the image of a regular n-gon under an affine transformation also determines exactly n slopes.[1] Jamison proved the converse, i.e. that the affinely regular polygons are the only configurations forming exactly n slopes.

A much more general statement is believed to be true: for some constant c_1, if a set of n points in general position forms $m = 2n - c_1$ slopes, then it is affinely equivalent to n of the vertices of a regular m-gon (see [9]). This would imply, in particular, that for every $c \geqslant 0$ and n sufficiently large, every simple configuration of n points determining $n + c$ slopes arises from an affinely regular $(n + c)$-gon, after deletion of c points. Jamison's result thus shows it for $c = 0$. Here, we will prove the case $c = 1$. The general conjecture is still open. In fact, for $c \geqslant 2$, it is not even known whether the points of S form a convex polygon.

Every affinely regular polygon is inscribed in an ellipse. Conics will play an important role in our proof. Another problem of Elekes [3] is the following: for all $m \geqslant 6$ and $C > 0$, there exists some $n_0(m, C)$ such that every set $S \subset \mathbb{R}^2$ with $|S| \geqslant n_0(m, C)$ forming at most $C|S|$ slopes contains m points on a (possibly degenerate) conic. It is still unsolved, even for $m = 6$.

2 Preliminary Remarks

Let S be a set of n points in the plane, in general position, that determines exactly $n+1$ slopes. If S had a point lying strictly inside its convex hull, there would be at least $n+2$ slopes, as was proved by Jamison [9, Theorem 7]. Therefore, we know that we can label the points of S as $A_1, \ldots, A_n$, such that $A_1A_2 \ldots A_n$ is a convex polygon.

For every point $A_i \in S$, there are $n - 1$ segments, with distinct slopes, joining A_i to the other points of S. We will say that a slope is *forbidden* at A_i if it is not the slope of any segment A_iA_j, for $j \neq i$. Since S determines $n + 1$ slopes, *there are exactly two forbidden slopes* at each point of S.

We will denote by ∇A_iA_j the slope of the line A_iA_j. Thus, an equality like $\nabla A_{i_1}A_{i_2} = \nabla A_{i_3}A_{i_4}$ is equivalent to $A_{i_1}A_{i_2} \parallel A_{i_3}A_{i_4}$. Throughout our main proof, we will repeatedly make use of the next lemma. It will be particularly useful to prove that a slope is forbidden at a point or that two slopes are equal. As an obvious corollary, we have that $\nabla A_{i-1}A_{i+1}$ is forbidden at A_i for all $i \in \mathbb{Z}$. Throughout the paper, when we say "for all $i \in \mathbb{Z}$", we consider the indices modulo n, so that $A_{n+1} := A_1$, and so on.

[1] A polygon obtained as the image of a regular polygon by an affine transformation is sometimes called an *affine-regular* or *affinely regular* polygon.

Lemma 1. *Let $1 \leqslant i < j < k \leqslant n$. Exactly one of the following is true:*

- *the slope of A_iA_k is forbidden at A_j;*
- *$\exists p, i < p < k$ such that $A_iA_k \parallel A_jA_p$.*

Moreover, in the second case, $\nabla A_jA_p \notin \{\nabla A_iA_l \mid l \neq j, k\} \cup \{\nabla A_lA_j \mid l \neq i, k\}$.

Proof. This is almost immediate from the definition of a forbidden slope. In the second case, if $p \in \{1, \ldots, n\}$ were not between i and k, the segments A_iA_k and A_jA_p would intersect. Finally, if ∇A_jA_p were equal to some ∇A_iA_l, then $A_iA_k \parallel A_jA_p \parallel A_iA_l$, so A_i, A_k and A_l would be aligned, a contradiction. The same is true for the segments A_lA_j. □

We will also need the following result, which can be found in [14, Chapter 1].

Proposition 2. *Let $\mathcal{C}$ be a non-degenerate conic and O a point on $\mathcal{C}$. If P, Q are two points on $\mathcal{C}$, define $P + Q$ to be the unique point R on $\mathcal{C}$ such that $RO \parallel PQ$ (with the convention that XX is the tangent to $\mathcal{C}$ at X, for $X \in \mathcal{C}$). This addition turns $\mathcal{C}$ into an abelian group, of which O is the identity element.*

In particular, for P, Q, R, S four points on $\mathcal{C}$, we have $P + Q = R + S$ if and only if $PQ \parallel RS$. Lemma 3 will enable us to introduce conics in the proof, in order to use proposition 2.

Lemma 3. *Suppose $P_1, \ldots, P_6$ are points in the plane such that $P_1P_6 \parallel P_2P_5$, $P_2P_3 \parallel P_1P_4$ and $P_4P_5 \parallel P_3P_6$. Then $P_1, \ldots, P_6$ lie on a common conic.*

Proof. This follows immediately from Pascal's theorem applied to the hexagon $\mathcal{H} = P_1P_4P_5P_2P_3P_6$. Indeed, the intersections of the opposite sides of $\mathcal{H}$ are collinear on the line at infinity. □

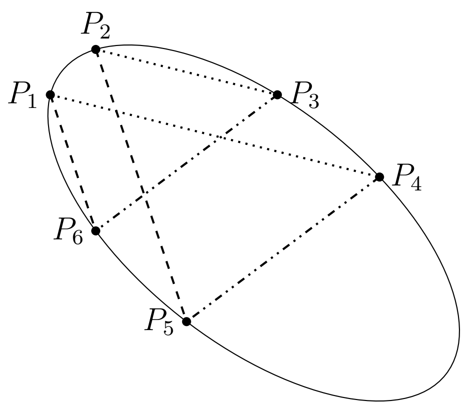

Figure 1: Illustration of lemma 3.

For the reader's convenience, we reproduce here a result of Korchmáros [11] (which is also discussed in [4]), that we will use twice in the proof.

Lemma 4. *Let $P_1, \ldots, P_n$ be distinct points on a non-degenerate conic. Suppose that, for all $j \in \mathbb{Z}$, $P_{j+1}P_{j+2} \parallel P_jP_{j+3}$. Then, P is affinely equivalent to a regular n-gon.*

3 Main Theorem

In this section, we prove the following theorem, using the results from section 2.

Theorem 5. *Any set S of $n \geqslant 7$ points in the plane, in general position, that determines exactly $n+1$ slopes, is affinely equivalent to n of the vertices of a regular $(n+1)$-gon.*

Proof. We use the notations of section 2: $S = \{A_1, A_2, \ldots, A_n\}$ where $A_1A_2\ldots A_n$ is a convex polygon. We will split the proof into two cases. In the first case, we suppose that, for every $i \in \mathbb{Z}$, $A_{i+1}A_{i+2} \parallel A_iA_{i+3}$. If this fails for some i, we can assume that this i is 1.

Case 1 For every $i \in \mathbb{Z}$, $A_{i+1}A_{i+2} \parallel A_iA_{i+3}$.

We will distinguish subcases according to which segments are parallel to A_iA_{i+5}. As we will see, none of the subcases are actually possible.

Case 1.1 For all $i \in \mathbb{Z}$, $A_iA_{i+5} \parallel A_{i+1}A_{i+4}$.

Let $A_{k+1}, \ldots, A_{k+6}$ be any six consecutive points of S. We have $A_{k+1}A_{k+6} \parallel A_{k+2}A_{k+5}$, $A_{k+2}A_{k+3} \parallel A_{k+1}A_{k+4}$ and $A_{k+4}A_{k+5} \parallel A_{k+3}A_{k+6}$ from our two assumptions. Thus, lemma 3 implies that the six points lie on a common conic. As this is true for any six consecutive points, and since five points in general position determine a unique conic, all the A_i's lie on the same conic. Together with the fact that $\forall i$, $A_{i+1}A_{i+2} \parallel A_iA_{i+3}$, this implies that $A_1A_2\ldots A_n$ is affinely equivalent to a regular n-gon, by lemma 4. Therefore, S determines exactly n directions, which is a contradiction.

Case 1.2 For some $i \in \mathbb{Z}$, we have $A_iA_{i+5} \parallel A_{i+2}A_{i+4}$.

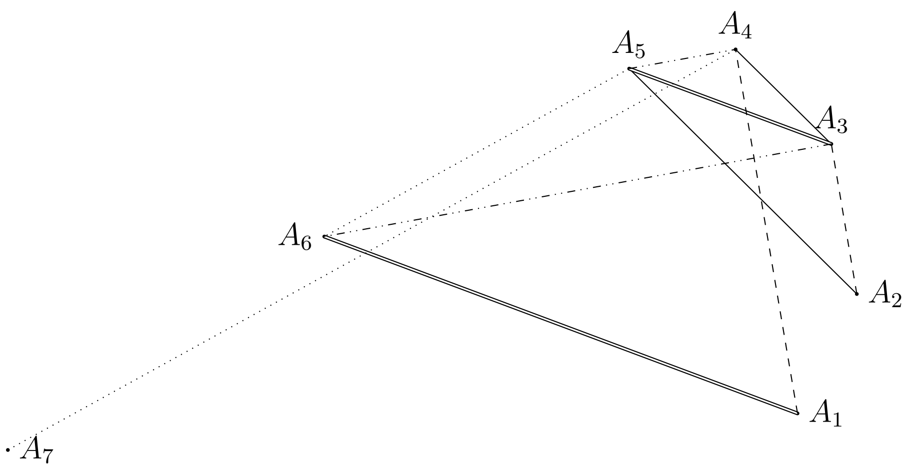

Figure 2: Case 1.2.

Say $i = 1$, meaning $A_1A_6 \parallel A_3A_5$. By lemma 1 applied three times, we see that ∇A_2A_6 is forbidden at A_3, A_4 and A_5 (here, we have used that $A_2A_6 \nparallel A_3A_5$ and, for A_4, that $A_3A_4 \parallel A_2A_5$ and $A_4A_5 \parallel A_3A_6$). For $l = 3, 4, 5$, we know that ∇A_2A_6 and $\nabla A_{l-1}A_{l+1}$ are exactly the two forbidden slopes at A_l. Therefore, ∇A_1A_5 is not forbidden at A_4, hence, by lemma 1 again, we conclude that $A_1A_5 \parallel A_2A_4$. Similarly, ∇A_3A_7 is not forbidden at A_4, so $A_3A_7 \parallel A_4A_6$. As the slope of A_2A_7 is not forbidden at A_5, we conclude $A_2A_7 \parallel A_4A_5 (\parallel A_3A_6)$. We have $A_3A_4 \parallel A_2A_5$, $A_3A_6 \parallel A_2A_7$ and we just showed that $A_2A_7 \parallel A_3A_6$. By lemma 3, $A_2, A_3, \ldots, A_7$ lie on a common conic.

We will equip this conic with the group structure descibed in lemma 2, with A_7 the zero element. We will write $A_7 = 0$ and $A_6 = x$. Then, $A_5A_6 \parallel A_4A_7$, $A_4A_5 \parallel A_3A_6$ and $A_4A_6 \parallel A_3A_7$ together imply $A_5 = 2x$, $A_4 = 3x$ and $A_3 = 4x$. Also, $A_3A_4 \parallel A_2A_5$ gives $A_2 = 5x$. Let B be the point on the conic with $B = 6x$. We thus have $A_2A_3 \parallel BA_4$ and $A_2A_4 \parallel BA_5$. However, there can only be one point P with $A_2A_3 \parallel PA_4$ and $A_2A_4 \parallel PA_5$. As A_1 is such a point, $A_1 = B = 6x$. This contradicts $A_1A_6 \parallel A_3A_5$, as $A_1 + A_6 = 6x + x \neq 4x + 2x = A_3 + A_5$.

Case 1.3 For some $i \in \mathbb{Z}$, we have $A_iA_{i+5} \parallel A_{i+1}A_{i+3}$.

This is exactly the previous case after having relabelled every A_i as A_{n+1-i}.

Case 1.4 The previous cases do not apply.

If none of the previous cases is possible, there must be some i, say $i = 1$, for which A_1A_6 is not parallel to any of A_2A_5, A_3A_5 and A_2A_4. Then, ∇A_1A_6 is forbidden at A_2, A_3, A_4 and A_5. Once again, we deduce that the forbidden slopes at A_l, $2 \leqslant l \leqslant 5$, are ∇A_1A_6 and $\nabla A_{l-1}A_{l+1}$. We use lemma 1 to find $A_2A_6 \parallel A_3A_5$ (applied with $A_k = A_4$) and $A_1A_5 \parallel A_2A_4$ ($A_k = A_2$).

Let $\mathcal{C}$ be the conic passing through $A_1, A_2, \ldots, A_5$. We use lemma 2 to define a group structure on $\mathcal{C}$, with $A_1 = 0$. Let $A_2 = x$ and $A_3 = y$. From $A_2A_3 \parallel A_1A_4$ and $A_3A_4 \parallel A_2A_5$, we have $A_4 = x + y$ and $A_5 = 2y$. But $A_2A_4 \parallel A_1A_5$ implies $y = 2x$, so $A_i = (i-1)x$ for $1 \leqslant i \leqslant 5$. We use the same argument as before. Let $B = 5x$, then $A_4A_5 \parallel A_3B$ and $A_3A_5 \parallel A_2B$, so $B = A_6 = 5x$. We deduce $A_1A_6 \parallel A_2A_5$, a contradiction.

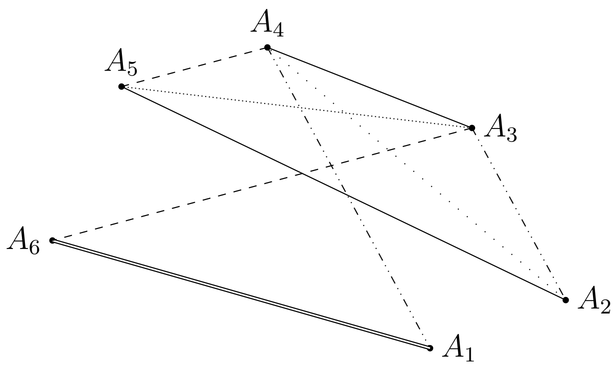

Figure 3: Case 1.4.

Case 2 We have $A_2A_3 \nparallel A_1A_4$ (without loss of generality).

Without loss of generality, we can also suppose that the point A_4 is closer to the line A_2A_3 than is A_1. In this situation, the line parallel to A_2A_3 passing through A_4 intersects the segment $[A_1A_2]$ in its relative interior, and the line parallel to A_2A_3 passing through A_1 does not intersect the segment $[A_3A_4]$.

From $A_2A_3 \nparallel A_1A_4$, we deduce that the forbidden slopes at A_2 and A_3 are ∇A_1A_3, ∇A_1A_4 and $\nabla A_2A_4, \nabla A_1A_4$, respectively. Thus, $A_1A_2 \parallel A_nA_3$ and $A_2A_5 \parallel A_3A_4$. We now show that A_2A_3 is forbidden at A_4. Suppose, for some k, that $A_2A_3 \parallel A_4A_k$. Then, k has to be between 5 and n, so $A_1A_2A_3A_4A_k$ must be a convex polygon, with $A_2A_3 \parallel A_4A_k$. We can see that this contradicts the fact that A_4 is closer than A_1 to the line A_2A_3.

Case 2.1 $A_{n-1}A_2 \parallel A_nA_1$.

We want to show that this case is impossible. From lemma 1, we find $A_{n-1}A_3 \parallel A_nA_2$. When we apply this lemma again with the slope of A_nA_4, we find that A_nA_4 is parallel to A_1A_3, because A_2A_3 is forbidden at A_4. In the same way, we get $A_{n-1}A_4 \parallel A_nA_3$.

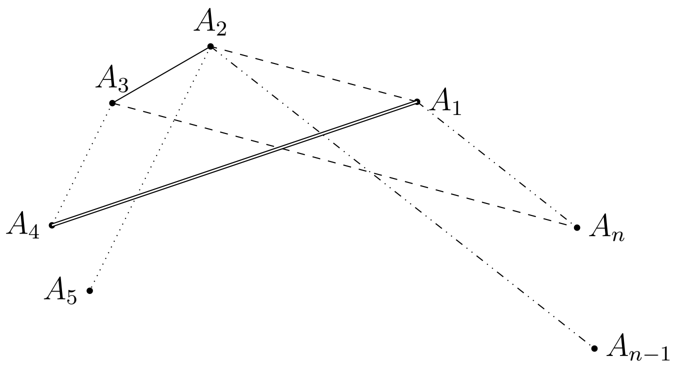

Figure 4: Case 2.1.

Let $\mathcal{C}$ be the conic passing through A_3, A_2, A_1, A_n and A_{n-1}. Again, we use proposition 2, setting $A_{n-1} = 0$. Let $A_n = x$ and $A_2 = y$. From $A_{n-1}A_3 \parallel A_nA_2$ we deduce $A_3 = x+y$, and from $A_{n-1}A_4 \parallel A_nA_3$ we get $A_4 = y+2x$. Let $B = 2x$. Then $A_nA_4 \parallel BA_3$ and $A_nA_3 \parallel BA_2$. This means that B belongs to the line parallel to A_nA_4 through A_3 and to the line parallel to A_nA_3 through A_2. So $B = A_1$, i.e. $A_1 = 2x$. On the one hand, the relation $A_{n-1}A_2 \parallel A_nA_1$ gives $0+y = x+2x$. On the other hand, $A_2A_3 \nparallel A_1A_4$ yields $y + (y + x) \neq 2x + (y + 2x)$. This is a contradiction.

Case 2.2 $A_{n-1}A_2 \nparallel A_nA_1$.

This is the last case of the proof, and the only case that produces valid configurations of points. As $A_{n-1}A_2 \nparallel A_nA_1$, $\nabla A_{n-1}A_2$ is forbidden at A_1. With ∇A_0A_2, those are the two forbidden slopes at A_1. Therefore, none of ∇A_2A_i, $3 \leqslant i \leqslant n-2$ is forbidden at A_1. So, every ∇A_2A_i, $3 \leqslant i \leqslant n-2$, corresponds to a unique ∇A_1A_j for some j.

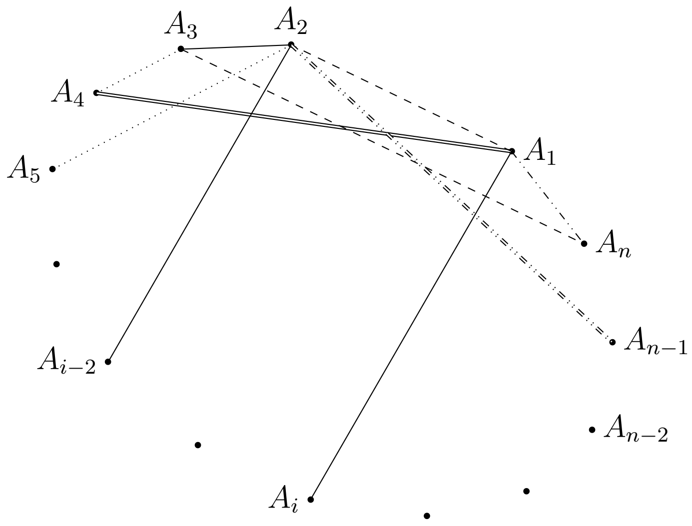

Figure 5: Case 2.2.

A simple but important observation is that, for all $3 \leqslant i_1, i_2 \leqslant n-2$ and $4 \leqslant j_1, j_2 \leqslant n$,

$$\begin{cases} A_2A_{i_1} \parallel A_1A_{j_1} \\ A_2A_{i_2} \parallel A_1A_{j_2} \end{cases} \implies \quad (i_1 < i_2 \iff j_1 < j_2)\,.$$

That is, the assignment f that maps every $3 \leqslant i \leqslant n-2$ to the unique $4 \leqslant j \leqslant n$ such that $A_2A_i \parallel A_1A_j$ must be strictly increasing. Moreover, it has to satisfy $f(3) \neq 4$ as we assumed $A_2A_3 \nparallel A_1A_4$. The unique possibility is then $f(i) = i+2$ for every i. We have proven that, for $5 \leqslant i \leqslant n$, $A_2A_{i-2} \parallel A_1A_i$.

Claim 2.2.1. For every $i \in \{5, \ldots, n\}$,

1. $A_{i-2}A_i$ and A_2A_{i-2} are the two forbidden slopes at A_{i-1}, and;
2. $\forall k \in \{3, \ldots, i-2\}$, $A_{i-1}A_k \parallel A_iA_{k-1}$.

Proof of claim. For $i = 5$, we have already proven those two statements. We will prove them for $i = j$, assuming it has already been proven for all $5 \leqslant i \leqslant j-1$.

1. We have to show that A_2A_{j-2} is forbidden at A_{j-1}. This is clear as $A_2A_{j-2} \parallel A_1A_j$ and there is no point of S between A_1 and A_2.
2. Since we know the forbidden slopes at A_{j-1}, we can use lemma 1 at the point A_{j-1} several times, with different slopes. First, ∇A_jA_{j-3} is not forbidden, so A_jA_{j-3} and $A_{j-1}A_{j-2}$ are parallel. Then ∇A_jA_{j-4} is not forbidden, and is distinct from $\nabla A_{j-1}A_{j-2} = \nabla A_jA_{j-3}$, so $A_jA_{j-4} \parallel A_{j-1}A_{j-3}$. We can continue this way, until we get $A_jA_2 \parallel A_{j-1}A_3$. This concludes the proof of the claim. □

In particular, for every $i \in \{6, \ldots, n-1\}$, we have $A_3A_i \parallel A_2A_{i+1}$, $A_5A_i \parallel A_4A_{i+1}$. As $A_3A_4 \parallel A_2A_5$, we can use lemma 1, which shows that A_2, A_3, A_4, A_5, A_i and A_{i+1} lie on a conic. As this is true for every $6 \leqslant i \leqslant n-1$, we know that the A_i's, for $2 \leqslant i \leqslant n$, all lie on a common conic (because there is a unique conic passing through five points in general position).

As we have done several times in this proof, we use the group structure on the conic given by parallelism. Choose A_2 to be the identity element, let $A_3 = x$. Solving

$$\begin{cases} A_3A_4 \parallel A_2A_5 \\ A_3A_6 \parallel A_4A_5 \\ A_3A_5 \parallel A_2A_6 \end{cases}$$

gives $A_4 = 2x$, $A_5 = 3x$ and $A_6 = 4x$. Then, a simple induction (using $A_{i-1}A_3 \parallel A_iA_2$) gives $A_i = (i-2)x$ for all $i \in \{2, \ldots, n\}$. Let B be the point on the conic with $B = -2x$. Then $A_2A_3 \parallel BA_5$ and $A_2A_4 \parallel BA_6$. However, we proved before that $A_2A_3 \parallel A_1A_5$ and $A_2A_4 \parallel A_1A_6$, so $A_1 = B = -2x$.

To summarize, we know that all the n points of S are on a conic, $A_i = (i-2)x$ for $i \in \{2, \ldots, n\}$ and $A_1 = -2x$. We use the group structure one last time: $A_3A_n \parallel A_1A_2$ implies $x + (n-2)x = -2x + 0$, so $(n+1)x = 0$. Therefore, the subgroup generated by $A_3 = x$ is a finite cyclic group of order $n+1$:

$$\langle A_3 \rangle = \Big\{ \underbrace{A_2}_{0}, \underbrace{A_3}_{x}, \underbrace{A_4}_{2x}, \ldots, \underbrace{A_{n-1}}_{(n-3)x}, \underbrace{A_n}_{(n-2)x}, \underbrace{A_1}_{(n-1)x}, -x \Big\}.$$

To finish the proof, we use the more convenient notations $P_j := jx$ for $0 \leqslant j \leqslant n$ (so that every A_i is a P_j). If the indices are considered modulo $n+1$, we have, for all $j \in \mathbb{Z}$, $P_{j+1}P_{j+2} \parallel P_jP_{j+3}$, because $(j+1)x + (j+2)x = jx + (j+3)x$. By lemma 4, $P_0P_1P_2 \ldots P_n$ is, up to an affine transformation, a regular $(n+1)$-gon. □

Acknowledgements

I would like to thank Christian Michaux, without whom this work would not have been possible.

References

[1] Martin Aigner and Günter Ziegler. *Proofs from the Book*, volume 274. Springer, 2010.

[2] G. R. Burton and G. B. Purdy. The directions determined by n points in the plane. *Journal of the London Mathematical Society*, 2(1):109–114, 1979.

[3] György Elekes. On linear combinatorics III. Few directions and distorted lattices. *Combinatorica*, 19(1):43–53, 1999.

[4] J. Chris Fisher and Robert E. Jamison. Properties of affinely regular polygons. *Geometriae Dedicata*, 69(3):241–259, 1998.

[5] Jacob E. Goodman and Richard Pollack. A combinatorial perspective on some problems in geometry. *Congressus Numerantium*, 32(0981):383–394, 1981.

[6] Robert E. Jamison. Planar configurations which determine few slopes. *Geometriae Dedicata*, 16(1):17–34, 1984.

[7] Robert E. Jamison. Structure of slope-critical configurations. *Geometriae Dedicata*, 16(3):249–277, 1984.

[8] Robert E. Jamison. A survey of the slope problem. *Annals of the New York Academy of Sciences*, 440(1):34–51, 1985.

[9] Robert E. Jamison. Few slopes without collinearity. *Discrete Mathematics*, 60:199–206, 1986.

[10] Robert E. Jamison and Dale Hill. A catalogue of sporadic slope-critical configurations. *Congressus Numerantium*, 40:101–125, 1983.

[11] G. Korchmáros. Poligoni affin-regolari dei piani di galois d'ordine dispari. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.(8)*, 56(5):690–697, 1974.

[12] János Pach, Rom Pinchasi, and Micha Sharir. On the number of directions determined by a three-dimensional points set. *Journal of Combinatorial Theory, Series A*, 108(1):1–16, 2004.

[13] János Pach, Rom Pinchasi, and Micha Sharir. Solution of scott's problem on the number of directions determined by a point set in 3-space. *Discrete & Computational Geometry*, 38(2):399–441, 2007.

[14] Viktor V. Prasolov and Yuri P. Solovyev. *Elliptic functions and elliptic integrals*, volume 170. American Mathematical Soc., 1997.

[15] Paul R. Scott. On the sets of directions determined by n points. *The American Mathematical Monthly*, 77(5):502–505, 1970.

[16] Peter Ungar. $2n$ noncollinear points determine at least $2n$ directions. *Journal of Combinatorial Theory, Series A*, 33(3):343–347, 1982.