On the volumes and affine types of trades

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Submitted: Dec 5, 2018; Accepted: Jan 10, 2020; Published: Jan 24, 2020

Abstract

A $[t]$-trade is a pair $T = (T_+, T_-)$ of disjoint collections of subsets (blocks) of a v-set V such that for every $0 \leqslant i \leqslant t$, any i-subset of V is included in the same number of blocks of T_+ and of T_-. It follows that $|T_+| = |T_-|$ and this common value is called the volume of T. If we restrict all the blocks to have the same size, we obtain the classical t-trades as a special case of $[t]$-trades. It is known that the minimum volume of a nonempty $[t]$-trade is 2^t. Simple $[t]$-trades (i.e., those with no repeated blocks) correspond to a Boolean function of degree at most $v - t - 1$. From the characterization of Kasami–Tokura of such functions with small number of ones, it is known that any simple $[t]$-trade of volume at most $2 \cdot 2^t$ belongs to one of two affine types, called Type (A) and Type (B) where Type (A) $[t]$-trades are known to exist. By considering the affine rank, we prove that $[t]$-trades of Type (B) do not exist. Further, we derive the spectrum of volumes of simple trades up to $2.5 \cdot 2^t$, extending the known result for volumes less than $2 \cdot 2^t$. We also give a characterization of "small" $[t]$-trades for $t = 1, 2$. Finally, an algorithm to produce $[t]$-trades for specified t, v is given. The result of the implementation of the algorithm for $t \leqslant 4$, $v \leqslant 7$ is reported.

Mathematics Subject Classifications: 05B05, 05B15, 97B05

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1 Introduction

Let v, k, t be positive integers such that $v > k > t$ and V be a v-set. Suppose that T_+ and T_- are two disjoint collections of k-subsets of V (called *blocks*) such that the occurrences of every t-subset of V in T_+ and T_- are the same. Then $T = (T_+, T_-)$ is called a t-(v, k) *trade* (or a t-*trade* when the role of v, k is not important). Basically, t-trades have been defined and utilized in connection with t-designs: if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two t-designs with the same parameters and the same ground set V, then $(\mathcal{D}_1 \setminus \mathcal{D}_2, \mathcal{D}_2 \setminus \mathcal{D}_1)$ is a t-trade. In this paper we consider $[t]$-*trades*, a generalization of t-trades, relaxed in the sense that the block size is not fixed. More precisely, a $[t]$-trade is a pair $T = (T_+, T_-)$ of disjoint collections of subsets of V such that for every $0 \leqslant i \leqslant t$, every i-subset of V is included in the same number of blocks of T_+ and of T_-. Note that any t-trade is also an i-trade for every $0 \leqslant i \leqslant t$, which means that any t-trade is a $[t]$-trade as well. On the other hand, $[t]$-trades can be naturally treated as trades of orthogonal arrays: given two orthogonal binary arrays A_1, A_2 with the same parameters and strength t, their difference pair $(A_1 \backslash A_2, A_2 \backslash A_1)$ is a $[t]$-trade (here, each array is treated as the set of its row-tuples).

For a $[t]$-trade $T = (T_+, T_-)$ we have $|T_+| = |T_-|$ and this common value is called the *volume* of T and denoted by $\mathrm{vol}(T)$. It is known that the smallest volume of a nonempty t-trade is 2^t which was determined independently in [5, 6] and [2]. For the volumes (of t-trades) between 2^t and $2 \cdot 2^t$, it was conjectured by Khosrovshahi and Malik [10, 14] and by Mahmoodian and Soltankhah [17, 13] (see also [4]) that any volume in this range is of the form $2^{t+1} - 2^i$ for some $i \in \{0, \dots, t-1\}$. This was known as "the gaps conjecture" which was proved recently in [11] for simple trades (for the trades with repeated blocked, the problem remains open). We note that the spectrum of volumes of t-trades and that of $[t]$-trades are the same [11] (i.e., a t-trade of volume m exists if and only if a $[t]$-trade of volume m exists). This is a key observation which allows one to translate problems related to the volumes of t-trades to the setting of $[t]$-trades; the strategy which was employed in settling the gaps conjecture for simple trades [11]. Further important problems in design theory can be described in terms of volumes of trades. For instance, the celebrated halving conjecture [3] can be considered as a partial case of the problem of determining the maximum volume of t-(v, k) trades (which is conjectured to be $\frac{1}{2}\binom{v}{k}$ whenever $\binom{v-i}{k-i}$ is even for all $i = 0, \dots, t$). This is one of the motivations to study $[t]$-trades as a new tool to attack problems in combinatorial design theory which can be described in terms of (volumes) of t-trades.

In this paper we further study $[t]$-trades and their volumes. As noted in [11], any simple (i.e., with no repeated blocks) $[t]$-trade corresponds to a Boolean function of degree at most $v - t - 1$ (where v is the number of arguments). From the characterization of such functions with small number of ones (given in [7]), it is observed that any simple $[t]$-trade of volume at most $2 \cdot 2^t$ belongs to one of the two affine types, called Type (A) and Type (B) (Type (A) $[t]$-trades are known to exist). Existence of $[t]$-trades of Type (B) was declared as an open problem in [11]. By considering the affine rank, we prove that $[t]$-trades of Type (B) do not exist. Also from our results on affine rank of trades, we derive the spectrum of volumes of trades up to $2.5 \cdot 2^t$ extending the gaps conjecture proved in

[11].

The paper is organized as follows. Section 2 contains main definitions. In Section 3 we prove some auxiliary statements. In Section 4, we consider the affine rank of simple $[t]$-trades. We utilize these considerations to prove the non-existence of simple $[t]$-trades of Type (B) as well as simple $[t]$-trades of volume $2^{t+1}+2^i$, $(t-1)/2 \leqslant i \leqslant t-4$. Based on this latter non-existential result and the construction of $[t]$-trades of volumes $2^{t+1}+2^{t-1}-2^i$, $0 \leqslant i \leqslant t-2$, and $2^{t+1}+2^{t-1}-3\cdot 2^i$, $0 \leqslant i \leqslant t-3$, in Section 5 we characterize the spectrum of volumes of simple $[t]$-trades up to the value $2.5\cdot 2^t$ exclusively. Section 6 is devoted to the characterization of [1]-trades of volume 3 and [2]-trades of volume 6. Section 7 contains the results of an exhaustive computer enumeration of the equivalence classes of trades for small t, and small foundations and volumes.

Finally, we note that our results are applicable to the classical t-trades. Indeed, on one hand, the t-trades are a special case of the $[t]$-trades; on the other hand, every $[t]$-trade can be mapped to a t-trade with a fixed block size by some affine transformation [11]. However, the characterization results for $[t]$-trades do not imply that the corresponding t-trades are also characterized up to isomorphism. Indeed, the class of equivalence transformations for $[t]$-trades is larger than that of t-trades (it contains shifts), and nonisomorphic t-trades could be equivalent as $[t]$-trades. As an example of the characterization of small t-trades, we mention the classification in [1, Table 3.4] of the Steiner 2-trades with block size 3, volume at most 9 and foundation size at most 11, where the additional "Steiner" property means that no pair of elements is included in more than one block of each leg of the trade.

2 Definitions

2.1 $[t]$-trades

Let t, v be positive integers with $t < v$. The subsets of $V = \{1,\ldots,v\}$ will be associated with their characteristic v-tuples, e.g., $\{2,3,6\} = (0,1,1,0,0,1,0) = 0110010$ for $v = 7$. The cardinality of a subset (the number of 1's in the corresponding tuple) will be referred to as its size. The set of all subsets of V is denoted by 2^V, which forms a group isomorphic to $\mathbb{Z}_2^v$, with the symmetric difference as the group operation. The symmetric difference corresponds to the bitwise modulo-2 addition of the characteristic v-tuples, and we will use $\oplus$ as the symbol for this operation. In many cases, we will omit this symbol, i.e., $XY := X \oplus Y$. For every $i \in V$, we denote $x_i := \{i\}$. Therefore, every $X = \{i_1, i_2, \ldots, i_w\} \in 2^V$ can be written as $X = x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_w} = x_{i_1}x_{i_2}\ldots x_{i_w}$.

By a $[t]$-*trade* we mean a pair $T = (T_+, T_-)$ of disjoint collections of 2^V such that for every $i \in [t]$, $[t] := \{0,\ldots,t\}$, every i-subset of V is included in the same number of elements of T_+ and of T_-. The sets T_+ and T_- are called the *legs* of T and the elements of T_+ and T_- are referred to as the *blocks* of T. A trade is called *simple* if it has no repeated blocks; in that case, T_+ and T_- can be considered as ordinary sets. The cardinality of a leg (which is, trivially, the same for both legs) is called the *volume* of T, denoted by $\mathrm{vol}(T)$. The *foundation* of T, denoted by $\mathrm{found}(T)$, is the set of all $\ell \in V$ such that ℓ

appears in some blocks of T. For any $\ell \in \mathrm{found}(T)$, the *replication* of ℓ is defined as

$$r_\ell := |\{B \in T_+ : \ell \in B\}| = |\{B \in T_- : \ell \in B\}|.$$

We use the same notation for the subsets $\alpha \subset \mathrm{found}(T)$ with $|\alpha| \leqslant t$:

$$r_\alpha := |\{B \in T_+ : \alpha \subseteq B\}| = |\{B \in T_- : \alpha \subseteq B\}|.$$

The trade of volume 0 is called *void*. A $[t]$-trade (T'_+, T'_-) is said to be a $[t]$-*subtrade* of a $[t]$-trade (T_+, T_-) if $T'_+ \subseteq T_+$ and $T'_- \subseteq T_-$. An element ℓ is said to be *essential* for a trade T if T has a block containing ℓ and a block not containing ℓ.

A trade can be treated as a $\mathbb{Z}$-valued function over 2^V, and written as

$$T = \sum_{X \in 2^V} \tau_X X, \tag{1}$$

where the positive coefficients τ_X equal the multiplicity of X in T_+, and the negative coefficients τ_X equal minus the multiplicity of X in T_-. In terms of such functions, (1), the definition of a $[t]$-trade can be rewritten as

$$\sum_{X \supseteq S} \tau_X = 0, \qquad \text{for every } S \in 2^V \text{ such that } |S| \leqslant t. \tag{2}$$

Below, we formally consider summation and multiplication of functions in form (1), using the rules of the group ring $\mathbb{Z}[(2^V, \oplus)]$. This language is convenient for the representation of the trades of small volumes.

A subset T of 2^V is said to be a $[t]$-*unitrade* if for every subset S of V with $|S| \leqslant t$, the number of blocks of T including S is an even number. A $[t]$-unitrade has necessarily an even number of blocks. If (T_+, T_-) is a simple $[t]$-trade, then clearly $T_+ \cup T_-$ is a $[t]$-unitrade. We extend the definition of volume, foundation and replication to include unitrades T by

$$\mathrm{vol}(T) := |T|/2, \quad r_\ell := |\{B \in T : \ell \in B\}|/2,$$

and similarly for subsets of $\mathrm{found}(T)$.

Remark 1. There is no reason to extend the concept of unitrade to multisets. Indeed, increasing or decreasing the multiplicity of any block by 2 does not change the $[t]$-unitrade property of the multiset. So, any multiset M is a (generalized) $[t]$-unitrade if and only if $\mathrm{odd}(M)$ is a $[t]$-unitrade, where $\mathrm{odd}(M)$ is the set of blocks with odd multiplicity in M. In particular, for any $[t]$-trade (T_+, T_-), the set $\mathrm{odd}(T_+ \uplus T_-)$ is a $[t]$-unitrade, where '$\uplus$' denotes union of multisets.

2.2 The binary vector space, Boolean functions and polynomials

The set 2^V with the addition operation $\oplus$ and the natural scalar multiplication by 0 and 1 is a v-dimensional vector space over the Galois field $\mathrm{GF}(2) = (\{0,1\}, \oplus)$. Every subset S of 2^V can be represented by the characteristic $\{0,1\}$-function over 2^V (such functions are known as *Boolean functions*), which, in turn, is uniquely represented as a polynomial of degree at most v in the vector coordinates $y_1, \ldots, y_v$ in the standard basis $x_1 = \{1\}, \ldots, x_v = \{v\}$, over GF(2). We will say that this polynomial is *associated* with the set S.

The set of all $\{0,1\}$-functions on 2^V represented by polynomials of degree at most m is denoted by $\mathrm{RM}(m,v)$ (in coding theory, this is known as the Reed–Muller code of order m).

3 Preliminary lemmas

In this section we establish some basic facts about $[t]$-trades which will be used in the rest of the paper. We start with a result which reveals the connection between $[t]$-trades and Reed–Muller codes. Consider $f(y_1, \ldots, y_6) = y_1y_2y_3 + y_1y_2y_4 \in \mathrm{RM}(3,6)$. The set of ones of f is

$$T = \{111000, 111001, 111010, 111011, 110100, 110101, 110110, 110111\}.$$

It is easily seen that T is indeed a [2]-unitrade. This is an example of the following general fact.

Lemma 2. *The subsets of 2^V associated with the polynomials from $\mathrm{RM}(m,v)$, $m < v$, are exactly the $[t]$-unitrades with $t = v - m - 1$.*

Proof. We divide the argument in three parts.

(i) Consider a monomial $f = y_{i_1} \cdots y_{i_\ell}$, and let T be the set of ones of f. Given a subset S of V, we count the number of the members of T 'including' S (in terms of tuples, having 1's in all positions from S). For a binary vector $\mathbf{a} = a_1 \ldots a_v$, we have $f(\mathbf{a}) = 1$ and $\mathbf{a}$ includes S if and only if $a_i = 1$ for all $i \in S \cup \{i_1, \ldots, i_\ell\}$. So the number of members of T including S is $2^{|V \setminus (S \cup \{i_1, \ldots, i_\ell\})|}$. This number is even if and only if $V \setminus (S \cup \{i_1, \ldots, i_\ell\})$ is nonempty.

(ii) In particular, if $l \leqslant m$, then $2^{|V \setminus (S \cup \{i_1, \ldots, i_\ell\})|}$ is even for every S of size $|S| \leqslant t = v - m - 1$. So, for every monomial of degree less than $v - t$, the associated set is a $[t]$-unitrade. This extends to every polynomial of degree less than $v - t$ (i.e., at most m), because any linear combination over GF(2) preserves the parity properties defining a $[t]$-unitrade.

(iii) On the other hand, if the degree s of a polynomial is $v - t$ or more, then it includes some monomial $y_{i_1} \cdots y_{i_s}$ with coefficient 1 and does not meet the definition of a $[t]$-unitrade with $S = V \backslash \{i_1, \ldots, i_s\}$, $|S| \leqslant t$. Indeed, by the 'only if' statement of (i), for this monomial, the set T of ones has odd number of elements including S; on the other hand, for every other monomial of degree at most s this number is even, by the 'if' statement of (i); hence, for the whole polynomial, it is odd. □

In view of Lemma 2, the next claim is just the well-known fact on Hamming distance of $\mathrm{RM}(m,v)$ (see, e.g., [12, Theorem 3 in 13.3]), which is easy to prove by induction on t.

Lemma 3. *If T is a nonempty $[t]$-unitrade, then $|T| \geqslant 2^{t+1}$, i.e., $\mathrm{vol}(T) \geqslant 2^t$.*

The same bound holds for $[t]$-trades. The following lemma gives the structure of $[t]$-trades with the minimum volume. A version of this result for t-trades is quite well-known, but it can be easily generalized to $[t]$-trades.

Lemma 4 ([2, 6])**.** *The minimum volume of a non-void $[t]$-trade is 2^t. Every $[t]$-trade of volume 2^t has the form*

$$X_0(X_1 - Y_1)(X_2 - Y_2) \cdots (X_{t+1} - Y_{t+1}),$$

where X_0, X_1, $\dots$, X_{t+1}, Y_1, $\dots$, Y_{t+1} are pairwise disjoint subsets of V and X_iY_i is nonempty for every $i = 1, \dots, t+1$.

For $Y \in 2^V$ and a function $T : 2^V \to \mathbb{Z}$, we call YT the Y*-shift*, or simply a *shift* of T.

Example 5. The function

$$x_1x_2x_3((1-x_1)(1-x_2) + (1-x_1x_2)(1-x_3)) = 1 - x_1x_2 - x_2x_3 - x_1x_3 + 2x_1x_2x_3.$$

is a [1]-trade of volume 3. The left part of the equation represents the trade as the sum of two simple [1]-trades of volume 2 shifted by $Y = \{1, 2, 3\}$.

Lemma 6 ([11])**.** *Any shift of a $[t]$-trade is also a $[t]$-trade.*

Given a trade T in the form (1) and an element $i \in V$, by the *i-projection*, or simply a *projection*, of T we mean the function T^i obtained from T by removing i from every block that contains i. Hence, $T^i = P + P'$, where $T = P + x_iP'$ and i does not occur in P and P'.

Note that after a projection, it is possible that two blocks cancel out each other, so the volume can be reduced. If the volume of T equals the volume of T^i, then we say that T is an *extension* of T^i. So, an extension of a $[t]$-trade T is a $[t]$-trade obtained from T by including a new element in some blocks of T.

Example 7. The following simple [1]-trade is an extension of the [1]-trade from Example 5:

$$1 - x_1x_2 - x_2x_3 + x_1x_2x_3 - x_1x_3\underline{x_4} + x_1x_2x_3\underline{x_4}.$$

The following four lemmas are straightforward from the definitions.

Lemma 8. *A projection of a $[t]$-trade is a $[t]$-trade.*

Lemma 9. *Let $T = P + x_iP'$ be a $[t]$-trade, where i does not occur in the blocks of P, P'. Then P, P', and x_iP' are $[t-1]$-trades.*

Lemma 10. *If (T_+, T_-) is a $[1]$-trade, then $\bigoplus_{X \in T_+ \cup T_-} X = \emptyset$.*

Lemma 11. *If P is a $[t-1]$-trade and the element i does not occur in its blocks, then $(1 - x_i)P$ is a $[t]$-trade.*

We say that a $[t]$-trade is *s-small* for some $s > 1$ if its volume is less than $s \cdot 2^t$. The 2-small trades will be referred to as *small.*

The following statement plays an important role in the computer-aided classification of small $[t]$-trades.

Corollary 12. *For each i from V, every $[t]$-trade T is decomposable to the sums*

$$
\begin{aligned}
T &= x_i T^i + (1 - x_i)P && (3)\\
&= T^i - (1 - x_i)P', && (4)
\end{aligned}
$$

where T^i is a $[t]$-trade, P and P' are $[t-1]$-trades, and the element i does not occur in T^i, P, P'. Moreover, if T is an s-small $[t]$-trade for some s, then T^i is an s-small $[t]$-trade and one of P, P' is an s-small $[t-1]$-trade.

Proof. If we present the $[t]$-trade in the form $T = P + x_i P'$ and define $T^i = P + P'$ to be the i-projection of T, then the first statement trivially follows from Lemmas 9 and 8. The volume of the projection is trivially not greater than the volume of the original trade; so, if T is s-small then so is T^i. Moreover, the volume of T is the sum of the volumes of P and P'; so, if it is less than $s \cdot 2^t$, then one of the summands is less than $s \cdot 2^{t-1}$, which means that the corresponding $[t-1]$-trade is s-small. □

As mentioned before, the minimum distance of $\mathrm{RM}(m, v)$ is $d = 2^{v-m}$. Kasami and Tokura [7] characterized codewords of $\mathrm{RM}(m, v)$ with weight at most $2d$. This result is the base of our characterization of $[t]$-trades with small volumes.

Lemma 13 ([7])**.** *Any Boolean function f from $\mathrm{RM}(m, v)$ of weight greater than 2^{v-m} and less than $2 \cdot 2^{v-m}$ can be reduced by an invertible affine transformation of its variables to one of the following forms:*

$$f(y_1, \ldots, y_v) = y_1 \cdots y_{m-\mu} \cdot (y_{m-\mu+1} \cdots y_m \oplus y_{m+1} \cdots y_{m+\mu}), \qquad \text{(A)}$$

$$f(y_1, \ldots, y_v) = y_1 \cdots y_{m-2} \cdot (y_{m-1} \cdot y_m \oplus y_{m+1} \cdot y_{m+2} \oplus \cdots \oplus y_{m+2\nu-3} \cdot y_{m+2\nu-2}), \qquad \text{(B)}$$

where $v \geqslant m + \mu$, $m \geqslant \mu \geqslant 2$, $v \geqslant m + 2\nu - 2$ and $\nu \geqslant 3$. Any Boolean function from $\mathrm{RM}(m, v)$ of minimum nonzero weight, 2^{v-m}, is the characteristic function of a $(v - m)$-dimensional affine subspace of 2^V.

Based on Lemma 2 and the Kasami–Tokura characterization, the gaps conjecture was proved in [11] in the more general setting of $[t]$-unitrades. For future reference, we state it as the following lemma.

Lemma 14. *If T is a nonempty $[t]$-unitrade with $\mathrm{vol}(T) < 2^{t+1}$, then*

$$\mathrm{vol}(T) \in \left\{ 2^t, \left(2 - \frac{1}{2}\right) 2^t, \ldots, \left(2 - \frac{1}{2^t}\right) 2^t \right\}.$$

In particular, the same holds for simple $[t]$-trades.

Lemma 15. *Every $(t+1)$-dimensional affine subspace of 2^V is a $[t]$-unitrade.*

Proof. Let A be a $(t+1)$-dimensional affine subspace of 2^V. Let $\{i_1,\dots,i_r\} \subset V$ with $r \leqslant t$. Consider the $(v-r)$-dimensional affine subspace $W = \{(y_1,\dots,y_v) : y_{i_1} = \cdots = y_{i_r} = 1\}$. Then $W \cap A$ is either empty or it is an affine subspace of 2^V of dimension at least $(v-r)+(t+1)-v \geqslant 1$ and so it has an even cardinality. Considering the vectors of A as subsets of V, this means that $\{i_1,\dots,i_r\}$ is contained in an even number of blocks of A. □

Lemma 16. *If T is a nonempty $[t]$-unitrade, then $\langle T\rangle \setminus T$ is also a $[t]$-unitrade, where $\langle T\rangle$ denotes the affine span of T.*

Proof. Let d be the dimension of $\langle T\rangle$. By Lemma 14, $|T| \geqslant 2^{t+1}$. Therefore, $d \geqslant t+1$, and hence by Lemma 15, $\langle T\rangle$ is a $[t]$-unitrade. It follows that $\langle T\rangle \setminus T$ is also a $[t]$-unitrade. □

Lemma 17. *Let $T = (T_+, T_-)$ be a $[t]$-trade. Let $\alpha, \beta \subset \mathrm{found}(T)$ with $\alpha \cap \beta = \emptyset$. Consider*

$$R^+ = \{B \in T_+ : \alpha \subset B,\, \beta \cap B = \emptyset\}, \qquad R^- = \{B \in T_- : \alpha \subset B,\, \beta \cap B = \emptyset\},$$

as multisets. Then (R_+, R_-) is a $(t - |\alpha| - |\beta|)$-trade.

Proof. The case $|\alpha|+|\beta| = 1$ is done by Lemma 9. The general case is proven by induction on $|\alpha| + |\beta|$. □

We denote the trade (R_+, R_-) of Lemma 17 by $T_{\alpha\overline{\beta}}$. In particular, we use the notation T_i for $\alpha = \{i\}$ and $\beta = \emptyset$ and $T_{\overline{j}}$ for $\alpha = \emptyset$ and $\beta = \{j\}$.

We call a $[t]$-trade T *reduced* if

$$r_i \leqslant \tfrac{1}{2}\mathrm{vol}(T), \qquad \text{for all } i \in \mathrm{found}(T).$$

Lemma 18. *Every $[t]$-trade can be transformed by some shifts into a reduced $[t]$-trade.*

Proof. Let T be a $[t]$-trade, and let I consist of all i's such that $r_i > \frac{1}{2}\mathrm{vol}(T)$. In $I \oplus T$, the I-shift of T, the replication of i is $\mathrm{vol}(T) - r_i < \frac{1}{2}\mathrm{vol}(T)$ for every $i \in I$ (the replications of elements in $V \setminus I$ remains the same). It follows that $I \oplus T$ is reduced. □

4 Affine rank of simple $[t]$-trades

Recall that by Lemma 2, unsigned simple $[t]$-trades with a foundation of size v can be regaded as codewords of the Reed–Muller code $\mathrm{RM}(v-t-1, v)$. As given in Lemma 13, the codewords of Reed–Muller codes with weights at most twice the minimum distance have been characterized in [7] and subsequently divided into Types (A) or (B). Accordingly, simple $[t]$-trades (and also $[t]$-unitrades) with volume at most 2^{t+1} can be categorized into Types (A) or (B). Krotov [11] considered this possible dichotomy and put forward the existence of $[t]$-trades of Type (B) as an open problem. In this section we establish some results about the affine rank of trades from which it follows that trades of Type (B) do not exist. In addition, the non-existence of simple $[t]$-trades with volumes $2^{t+1}+2^i$, $(t-1)/2 \leqslant i \leqslant t-4$ is also established.

We denote the affine rank (the dimension of the affine span) of a subset S of the vector space 2^V by $\operatorname{afrk}(S)$. If $T=(T_+,T_-)$ is a simple $[t]$-trade, by $\operatorname{afrk}(T)$ we mean $\operatorname{afrk}(T_+ \cup T_-)$.

We first show how the types of $[t]$-trades can be distinguished by means of their affine rank.

Proposition 19. *Let T be a simple $[t]$-trade with $\operatorname{vol}(T)=2^{t+1}-2^i$ for $i \in \{0,1,\dots,t-1\}$.*

(i) *If T is of Type (A), then $\operatorname{afrk}(T)=2t+2-i$.*

(ii) *If T is of Type (B), then $(t-1)/2 \leqslant i \leqslant t-2$ and $\operatorname{afrk}(T)=t+3$.*

In particular, if either $\operatorname{afrk}(T) \geqslant t+4$, $i=t-1$ or $i<(t-1)/2$, then T is of Type (A).

Proof. Let T' denote the corresponding $[t]$-unitrade with T. Note that an invertible affine transformation of the variables does not change the affine rank and the cardinality of the set of ones of the polynomials given in Lemma 13. So we may assume that T' is the set of ones of such polynomials.

(i) Considering the associated polynomial of T' given by Lemma 13 (A), it is seen that T' is the symmetric difference of two intersecting affine subspaces of dimension $t+1$. If the dimension of the intersection is i, $0 \leqslant i < t$, then the cardinality of T' is $2^{t+2}-2^{i+1}$ and its affine rank is $2t+2-i$.

(ii) T' is the set of ones of the polynomial given by Lemma 13 (B). By a counting argument, we have

$$\begin{aligned}
|T'| &= 2^{v-m-2\nu+2} \sum_{j \text{ odd}} \binom{\nu}{j} 3^{\nu-j} \\
&= 2^{v-m-2\nu+2} \cdot \frac{1}{2}\big((3+1)^\nu-(3-1)^\nu\big) \\
&= 2^{t+2}-2^{t+2-\nu} \qquad (\text{as } t=v-m-1).
\end{aligned}$$

We have $\nu \geqslant 3$ and $v \geqslant m-2+2\nu$, so $3 \leqslant \nu \leqslant (t+3)/2$. As $|T'| = 2\operatorname{vol}(T) = 2^{t+2}-2^{i+1}$, it follows that $i=t+1-\nu$ and thus $(t-1)/2 \leqslant i \leqslant t-2$.

A unitrade of Type (B) is an intersection of an affine subspace of dimension $t+3$ and the set of ones of a quadratic function. So $\operatorname{afrk}(T') \leqslant t+3$. If $\operatorname{afrk}(T') \leqslant t+2$, then by Lemma 16, $\langle T' \rangle \setminus T'$ is $[t]$-unitrade with volume 2^i for some $0 \leqslant i \leqslant t-1$ which is a contradiction to Lemma 14. It follows that $\operatorname{afrk}(T')=t+3$. □

From Lemma 13 it is clear that $[t]$-unitrades of Type (B) (and so with affine rank $t+3$) do exist. However, we manage to prove that this is not the case for $[t]$-trades. It follows that unitrades of Type (B) are not 'splittable.' This means that, although an unsigned $[t]$-trade gives a $[t]$-unitrades, but this is not reversible in general.

Lemma 20. *Let $t \geqslant 3$ and T be a simple $[t]$-trade such that for all $i \in \mathrm{found}(T)$, $r_i = 2^{t-1}$. If $\mathrm{vol}(T) > 2^t$, then $\mathrm{afrk}(T) \geqslant t+4$.*

Proof. Suppose that $\mathrm{vol}(T) > 2^t$. So by Lemma 14, $\mathrm{vol}(T) \geqslant 1.5{\cdot}2^t$. For any $i \in \mathrm{found}(T)$, T_i is a $[t-1]$-trade of volume r_i. Choose $i, j \in \mathrm{found}(T)$ so that $r_{ij} \notin \{0, 2^{t-1}\}$. Then $r_{ij} = 2^{t-2}$. As T_{ij} is a $[t-2]$-trade of minimum volume and $t \geqslant 3$, there exists some $k \in \mathrm{found}(T)$ with $r_{ijk} = 2^{t-3}$. It turns out that $r_{ik}, r_{jk} \notin \{0, 2^{t-1}\}$ and so $r_{ik} = r_{jk} = 2^{t-2}$. Then

$$\begin{aligned}\mathrm{vol}(T_{\overline{ijk}}) &= \mathrm{vol}(T) - \mathrm{vol}(T_i) - \mathrm{vol}(T_j) - \mathrm{vol}(T_k) + \mathrm{vol}(T_{ij}) + \mathrm{vol}(T_{ik}) + \mathrm{vol}(T_{jk}) - \mathrm{vol}(T_{ijk}) \\ &\geqslant 1.5 \cdot 2^t - 3 \cdot 2^{t-1} + 3 \cdot 2^{t-2} - 2^{t-3} = 1.25 \cdot 2^{t-1} > 2^{t-1}.\end{aligned}$$

It follows that $T_{\overline{ijk}}$ has affine rank at least $t+1$. On the other hand, as $\mathrm{vol}(T_{i\overline{jk}}) = \mathrm{vol}(T_{j\overline{ik}}) = \mathrm{vol}(T_{k\overline{ij}}) = 2^{t-3} \neq 0$, there are three more affinely independent vectors in T each containing exactly one of i, j or k. This means that the affine rank of T is at least $t+4$. □

Lemma 21. *Let T be a $[t]$-unitrade.*

(i) *If $\mathrm{vol}(T) = 2^{t+1} \pm 2^i$, $(t-1)/2 \leqslant i \leqslant t-1$, and $\mathrm{afrk}(T) = t+3$, then the associated polynomial corresponding to T can be obtained from*

$$f(y_1, \ldots, y_v) = y_1 \cdots y_{m-2} \cdot (y_{m-1} \cdot y_m \oplus y_{m+1} \cdot y_{m+2} \oplus \cdots \oplus y_{m+2\nu-3} \cdot y_{m+2\nu-2} \oplus a) \quad (5)$$

by an invertible affine transformation of variables, where $m = v-t-1$, $\nu = t+1-i$, $a = 1$ if $\mathrm{vol}(T) = 2^{t+1} + 2^i$ and $a = 0$ if $\mathrm{vol}(T) = 2^{t+1} - 2^i$.

(ii) *If $2^{t+1} < \mathrm{vol}(T) < 2^{t+1} + 2^{t-3}$, then $\mathrm{vol}(T) = 2^{t+1} + 2^i$, for some i, $(t-1)/2 \leqslant i \leqslant t-4$, the associated polynomial to T is of the form (5) with $a = 1$, and $\mathrm{afrk}(T) = t+3$.*

Proof. (i) If $\mathrm{afrk}(T) = t+3$, then there is an invertible affine variable transformation that sends T to a $[t]$-unitrade T' whose affine span is defined by the equations

$$y_1 = 1, \ \ldots, \ y_{m-2} = 1 \quad (m = v-t-1). \quad (6)$$

It follows from (6) that the polynomial associated to T' has the form

$$g(y_1, \ldots, y_v) = y_1 \cdots y_{m-2} \cdot h(y_{m-1}, \ldots, y_v).$$

By Lemma 2, g has degree at most m, and hence h has degree at most 2. The polynomial h, as a polynomial in the $t+3$ variables $y_{m-1}, \ldots, y_v$, has $2\mathrm{vol}(T)$ ones, which is either $2^{t+2} - 2^{i+1}$ or $2^{t+2} + 2^{i+1}$. By the results of [16], h is affinely equivalent to

$$y_{m-1} \cdot y_m \oplus y_{m+1} \cdot y_{m+2} \oplus \cdots \oplus y_{m+2\nu-3} \cdot y_{m+2\nu-2} \oplus a$$

with $a = 0$ or $a = 1$, respectively. Therefore, g is affinely equivalent to f in (5).

(ii) is straightforward from Lemma 2 and the characterization of the codewords of $\mathrm{RM}(m, v)$ of weight smaller than $2.5 \cdot 2^{m-v}$ [8]. □

Example 22. The set $T = \{00111, 10011, 01011, 11001, 11100, 11010\}$ is a [1]-unitrade of volume $2^2 - 2^0$. Consider the linear transformation f between two 4-dimensional linear spaces that maps $00000 \to 00000$, $10100 \to 00010$, $01100 \to 00001$, $11110 \to 01111$, $11011 \to 01011$, $11101 \to 00111$. Note that $\{10100, 01100, 11110, 11011\}$ and $\{00010, 00001, 01111, 01011\}$ are linearly independent sets and $11101 = 10100 + 01100 + 11110 + 11011 \to 00111 = 00010 + 00001 + 01111 + 01011$. Now $\mathbf{x} \mapsto f(\mathbf{x} + 00111) + 11100$ is the invertible affine transformation that maps $00111 \to 11100$, $10011 \to 11110$, $01011 \to 11101$, $11001 \to 10011$, $11100 \to 10111$, $11010 \to 11011$. So T is mapped onto the [1]-unitrade $\{11100, 11110, 11101, 10011, 10111, 11011\}$ which is the set of ones of the polynomial $y_1 \cdot (y_2 \cdot y_3 \oplus y_4 \cdot y_5)$ of type (5).

Lemma 23. *Let T be a $[t]$-unitrade with* $\mathrm{afrk}(T) = t + 3$ *and* $\mathrm{vol}(T) = 2^{t+1} \pm 2^i$ *where* $t/2 \leqslant i \leqslant t - 1$*. Then* $r_j = \mathrm{vol}(T)/2$ *for some* $j \in \mathrm{found}(T)$*.*

Proof. By Lemma 21, the associated polynomial corresponding to T can be obtained from (5) by an invertible affine transformation of variables. We have

$$m + 2\nu - 2 = v - 1 + t - 2i \leqslant v - 1.$$

So y_v is a free variable of f, which implies that $r_v = \mathrm{vol}(T)/2$. In fact the set of ones of f is of the form $S \times \{0,1\}$ for some $S \subset 2^{[v-1]}$ with $|S| = \mathrm{vol}(T)$. Let $\mathbf{y} \mapsto \mathbf{y}M + \mathbf{b}$ be the invertible affine transformation which gives the associated polynomial of T. Hence T is the set of ones of $g(\mathbf{y}) = f(\mathbf{y}M + \mathbf{b})$, i.e.,

$$T = \{(\mathbf{y} - \mathbf{b})M^{-1} : \mathbf{y} \in S \times \{0,1\}\} = \{\mathbf{x}M^{-1} : \mathbf{x} \in S' \times \{0,1\}\},$$

for some S' with $|S'| = |S|$. The last row of M^{-1} should be nonzero. So we may assume that the j-th column of M^{-1}, say $\mathbf{a}^\top$ has its last component equal to 1. Then we have either $\mathbf{x}\mathbf{a}^\top = 1$ for all $\mathbf{x} \in S' \times \{0\}$ or $\mathbf{x}\mathbf{a}^\top = 1$ for all $\mathbf{x} \in S' \times \{1\}$. This means that $r_j = |S'|/2 = \mathrm{vol}(T)/2$. ☐

Lemma 24. *For any simple $[t]$-trade T, there exists a simple $[t]$-trade T' with*

$$|\mathrm{found}(T')| = \mathrm{afrk}(T') = \mathrm{afrk}(T)$$

and $\mathrm{vol}(T') = \mathrm{vol}(T)$*.*

Proof. Denote by A the affine span of T, and by A_i, the i-projection of A. If $|A_i| < |A|$ for all $i \in \mathrm{found}(T)$, then $A = 2^{\mathrm{found}(T)}$, and the statement trivially holds with $T' = T$. Otherwise, $|A_i| = |A|$ for some $i \in \mathrm{found}(T)$, and the i-projecting acts bijectively on A. It follows that the i-projection of T has the same volume and affine rank as T, but smaller foundation. Repeating this operation $|\mathrm{found}(T)| - \mathrm{afrk}(T)$ times, we find a required T'. ☐

Lemma 25. *The simple [2]-trades of foundation size* 5 *and volume* 6, 8, 10 *satisfy the following properties.*

(i) *In any simple* $[2]$*-trade with volume* 6 *and foundation size* 5*, the number of elements with odd replication (the only possible odd value is* 3*) is odd.*

(ii) *In any simple* $[2]$*-trade with volume* 8 *and foundation size* 5*, the number of elements with odd replication (*3 *or* 5*) is even.*

(iii) *In any simple* $[2]$*-trade with volume* 10 *and foundation size* 5*, the number of elements with odd replication (the only possible odd value is* 5*) is odd.*

The proof of Lemma 25 is by computation and will be addressed in Section 7. The sharpening claims in the parenthesis can be easily shown theoretically, but we will not use them in the further discussion.

Lemma 26. *Let* $t = 2, 3$*, and* T *be a simple* $[t]$*-trade with* $1.5 \cdot 2^t < \mathrm{vol}(T) < 2.5 \cdot 2^t$ *and* $\mathrm{vol}(T) \neq 2 \cdot 2^t$*. Then the affine rank of* T *is at least* $t + 4$*.*

Proof. As shifts do not change the volume and affine rank of trades, in view of Lemma 18, we may assume that T is a reduced simple $[t]$-trade.

First let $t = 2$. Then $\mathrm{vol}(T) = 7$ or 9. If $\mathrm{vol}(T) = 7$, then by Proposition 19, it has affine rank 6 (this is even true for [2]-unitrades of volume 7). Let $\mathrm{vol}(T) = 9$. We have $\mathrm{afrk}(T) \geqslant \lceil \log_2(2\mathrm{vol}(T)) \rceil \geqslant 5$. If $\mathrm{afrk}(T) = 5$, by Lemma 16, there exists a [2]-unitrade with affine rank 5 and $2^5 - 18 = 14$ blocks, which cannot exist as just shown. It follows that $\mathrm{afrk}(T) \geqslant 6$.

Now assume that $t = 3$. We have $\mathrm{vol}(T) \in \{14, 15, 17, 18, 19\}$. If $\mathrm{vol}(T) = 15$, we are done by Proposition 19. Let $\mathrm{vol}(T)$ is 17 (respectively, 19). We have $\mathrm{afrk}(T) \geqslant \lceil \log_2(2\mathrm{vol}(T)) \rceil \geqslant 6$. If $\mathrm{afrk}(T) = 6$, then by Lemma 16, there exists a [3]-unitrade with affine rank 6 and cardinality 13 (respectively 15) which is impossible by Lemma 14 (by the above argument). So $\mathrm{afrk}(T) \geqslant 7$. It remains to prove the assertion for volumes 14 and 18.

Suppose $\mathrm{vol}(T) = 14$. For a contradiction, let $\mathrm{afrk}(T) = 6$. By Lemma 24, we may assume that $|\mathrm{found}(T)| = 6$. Applying Lemma 14 to T_i we obtain $r_i \in \{4, 6, 7\}$ for all $i \in \mathrm{found}(T)$. If $r_i = 7$ for some $i \in \mathrm{found}(T)$, then T_i is a [2]-trade of volume 7 and has affine rank at least 6 by Proposition 19. Hence $\mathrm{afrk}(T) \geqslant 7$, a contradiction. Hence for all $i \in \mathrm{found}(T)$, $r_i = 4$ or 6. If for all $i \in \mathrm{found}(T)$, $r_i = 4$, then we are done by Lemma 20. So assume that $r_i = 6$ for some $i \in \mathrm{found}(T)$. Here T_i is a [2]-trade of volume 6 and $|\mathrm{found}(T_i)| = 5$ ($|\mathrm{found}(T_i)|$ cannot be smaller than 5 as $\mathrm{afrk}(T_i) = 5$). Note that $\mathrm{vol}(T_{\bar{i}}) = 8$. Also $|\mathrm{found}(T_{\bar{i}})| = 5$, because $T_{\bar{i}j}$ is a [1]-trade and so $\mathrm{afrk}(T_{\bar{i}j}) \geqslant 4$, it follows that $\mathrm{afrk}(T_{\bar{i}}) \geqslant 5$. On the other hand, $\mathrm{afrk}(T_{\bar{i}}) \leqslant \mathrm{afrk}(T) - 1 = 5$. Our aim is to obtain a contradiction by considering the replications of elements in both T_i and $T_{\bar{i}}$. In view of Lemma 25 applied to T_i, the number of $j \in \mathrm{found}(T)$ with $r_{ij} = 3$ must be odd. We further claim that $r_{ij} = 3$ if and only if $r_{\bar{i}j} = 3$. The claim follows from the fact that if either $r_{ij} = 3$ or $r_{\bar{i}j} = 3$, then $r_j = 6$; since otherwise, $r_j = 4$, and then T_{ij} or $T_{\bar{i}j}$ would be a [1]-trade of volume 1, a contradiction. Also there are no $k \in \mathrm{found}(T)$ with $r_{\bar{i}k} = 5$; since otherwise r_k is necessarily 6, and so $T_{\bar{i}\bar{k}}$ would be a [1]-trade of volume 1, again a contradiction. The above argument shows that the number of elements with an odd replication in T_i is the same as the number of elements with an odd replication in $T_{\bar{i}}$. However, by Lemma 25, the former is an odd number and the latter is an even number, again a contradiction.

Finally, suppose that $\mathrm{vol}(T) = 18$ and $\mathrm{afrk}(T) = 6$. By Lemma 24, we may assume that $|\mathrm{found}(T)| = 6$. By Lemma 14 and since T is reduced we have $r_i \in \{4, 6, 7, 8, 9\}$ for all $i \in \mathrm{found}(T)$. If $r_i \in \{7, 9\}$, for some $i \in \mathrm{found}(T)$, then T_i is a [2]-trade of volume 7 or 9 and consequently $\mathrm{afrk}(T_i) \geqslant 6$ as we just showed. Hence $\mathrm{afrk}(T) \geqslant 7$, a contradiction. So $r_i \in \{4, 6, 8\}$ for all $i \in \mathrm{found}(T)$.

We claim that $r_k = 8$ for some $k \in \mathrm{found}(T)$. Otherwise, $r_i \in \{4, 6\}$ for all $i \in \mathrm{found}(T)$. If for all i, $r_i = 4$, then by Lemma 20 we have that $\mathrm{afrk}(T) \geqslant 7$. If $r_i = 6$, for some $i \in \mathrm{found}(T)$, then by Lemma 23 applied to T_i, we obtain that $r_{ij} = 3$ for some $j \in \mathrm{found}(T)$. It turns out that $r_j = 6$. Thus $T_{\overline{ij}}$ has 18 blocks; so $\mathrm{afrk}(T_{\overline{ij}}) \geqslant 5$. It follows that $\mathrm{afrk}(T) \geqslant 7$, a contradiction. Hence, the claim follows.

Therefore, we assume that $r_k = 8$ and so $\mathrm{vol}(T_{\bar{k}}) = 10$. Also $\mathrm{afrk}(T_{\bar{k}}) = |\mathrm{found}(T_{\bar{k}})| = 5$. For every $i \in \mathrm{found}(T)$, r_i is even (4, 6, or 8); hence, the volumes of T_{ki} and $T_{\bar{k}i}$ have the same parity. It follows that the number of elements with an odd replication in T_k is the same as the number of elements with an odd replication in $T_{\bar{k}}$. However, the former is an odd number by Lemma 25(ii) and the latter is an even number by Lemma 25(iii), a contradiction. □

Now, we are ready to prove the main result of this section.

Theorem 27. *If T is a simple $[t]$-trade with $1.5 \cdot 2^t < \mathrm{vol}(T) < 2.5 \cdot 2^t$ and $\mathrm{vol}(T) \neq 2^{t+1}$, then the affine rank of T is at least $t + 4$.*

Proof. We proceed by induction on t. For $t = 1$, there is no trade satisfying the assumptions, and $t = 2, 3$ has been settled in Lemma 26. Hence we assume that $t \geqslant 4$.

Since shifts do not change the affine rank of trades, we may assume that T is reduced. As T is reduced, $r_i \leqslant \mathrm{vol}(T)/2 < 2.5 \cdot 2^{t-1}$ for all $i \in \mathrm{found}(T)$. If there exists some $i \in \mathrm{found}(T)$ with $r_i \neq 2^t$ and $r_i > 1.5 \cdot 2^{t-1}$, then T_i is a simple $[t-1]$-trade with $\mathrm{vol}(T_i) \neq 2^t$ and $1.5 \cdot 2^{t-1} < \mathrm{vol}(T_i) < 2.5 \cdot 2^{t-1}$. So by the induction hypothesis, $\mathrm{afrk}(T_i) \geqslant t + 3$. Therefore, $\mathrm{afrk}(T) \geqslant t + 4$, and we are done. Hence we can assume that

$$\text{for all } i \in \mathrm{found}(T), \text{ either } r_i = 2^t \text{ or } r_i \leqslant 1.5 \cdot 2^{t-1}. \tag{7}$$

So it suffices to consider the following two cases.

Case 1. There exist some $i \in \mathrm{found}(T)$ with $r_i = 2^t$.

As we assumed that T is reduced, $\mathrm{vol}(T) \geqslant 2r_i = 2^{t+1}$, so $2 \cdot 2^{t-1} < \mathrm{vol}(T) - r_i < 3 \cdot 2^{t-1}$. If further, $\mathrm{vol}(T_{\bar{i}}) = \mathrm{vol}(T) - r_i < 2.5 \cdot 2^{t-1}$, then by the induction hypothesis, $\mathrm{afrk}(T_{\bar{i}}) \geqslant t + 3$, and so we are done. Therefore, we assume that $\mathrm{afrk}(T_{\bar{i}}) = t + 2$ and $2.5 \cdot 2^{t-1} \leqslant \mathrm{vol}(T_{\bar{i}}) < 3 \cdot 2^{t-1}$. Then by Lemma 16, there exists a $[t-1]$-unitrade T' with $2^{t-1} < \mathrm{vol}(T') = 2^{t+1} - \mathrm{vol}(T_{\bar{i}}) \leqslant 1.5 \cdot 2^{t-1}$. By Lemma 14, $\mathrm{vol}(T') = 1.5 \cdot 2^{t-1}$ and so $\mathrm{vol}(T_{\bar{i}}) = 2.5 \cdot 2^{t-1}$ implying that $\mathrm{vol}(T) = 4.5 \cdot 2^{t-1}$. If $\mathrm{afrk}(T) \leqslant t+3$, then by Lemma 23, $r_j = 4.5 \cdot 2^{t-2}$ for some j, which is impossible in view of (7). So $\mathrm{afrk}(T) \geqslant t + 4$ and we are done.

Case 2. For all $i \in \text{found}(T)$, $r_i \leqslant 1.5 \cdot 2^{t-1}$.

Applying Lemma 14 to T_i we obtain that $r_i = 2^{t-1}$ or $1.5 \cdot 2^{t-1}$ for all $i \in \text{found}(T)$. If for all $i \in \text{found}(T)$, $r_i = 2^{t-1}$, then we are done by Lemma 20. So assume that $r_i = 1.5 \cdot 2^{t-1}$ for some $i \in \text{found}(T)$. It follows that

$$1.5 \cdot 2^{t-1} < \text{vol}(T_{\bar{i}}) = \text{vol}(T) - r_i < 3.5 \cdot 2^{t-1}, \quad \text{vol}(T_{\bar{i}}) \neq 2.5 \cdot 2^{t-1}.$$

Note that we also have $\text{vol}(T_{\bar{i}}) \neq 2 \cdot 2^{t-1}$ (since otherwise $\text{vol}(T) = 3.5 \cdot 2^{t-1}$ and so Lemma 23 implies the existence of some $j \in \text{found}(T)$ with $r_j = 3.5 \cdot 2^{t-2}$, hence a contradiction). Therefore, if $\text{vol}(T_{\bar{i}}) < 2.5 \cdot 2^{t-1}$, then $T_{\bar{i}}$ satisfies the induction hypothesis, and so $\text{afrk}(T_{\bar{i}}) \geqslant t+3$ implying that $\text{afrk}(T) \geqslant t+4$. Now suppose that $\text{vol}(T_{\bar{i}}) > 2.5 \cdot 2^{t-1}$. Then $\text{afrk}(T_{\bar{i}}) \geqslant \lceil \log_2(2\text{vol}(T_{\bar{i}})) \rceil = t+2$. If $\text{afrk}(T_{\bar{i}}) = t+2$, then by Lemma 16, $T' = \langle T_{\bar{i}} \rangle \setminus T_{\bar{i}}$ is a $[t-1]$-unitrade with $\text{vol}(T') < 1.5 \cdot 2^{t-1}$. So by Lemma 14, $\text{vol}(T') = 2^{t-1}$ which in turn implies that $\text{vol}(T) = 4.5 \cdot 2^{t-1}$. Now Lemma 23 implies the existence of some $j \in \text{found}(T)$ with $r_j = 4.5 \cdot 2^{t-2}$, a contradiction. So $\text{afrk}(T_{\bar{i}}) \geqslant t+3$ and thus $\text{afrk}(T) \geqslant t+4$. □

Now, by Theorem 27 and Proposition 19 we have the following corollary which answers an open problem of [11].

Corollary 28. *There do not exist simple* $[t]$*-trades* T *with* $2^t < \text{vol}(T) < 2^{t+1}$ *of Type* (B).

The following corollary will be used in the next section.

Corollary 29. *There do not exist simple* $[t]$*-trades* T *with* $\text{vol}(T) = 2^{t+1} + 2^i$ *for* $(t-1)/2 \leqslant i \leqslant t-4$.

Proof. Suppose for a contradiction that T is a simple $[t]$-trade with $\text{vol}(T) = 2^{t+1} + 2^i$, $(t-1)/2 \leqslant i \leqslant t-4$. By Theorem 27, $\text{afrk}(T) \geqslant t+4$. On the other hand, let T' be the unitrade associated with T. By Lemma 21(ii), $\text{afrk}(T) = \text{afrk}(T') = t+3$, a contradiction. □

5 Spectrum of volumes of simple $[t]$-trades between $2 \cdot 2^t$ and $2.5 \cdot 2^t$

Based on the characterization of codewords of Reed-Muller code with weights within the range 2 and 2.5 times the minimum distance by Kasami *et al.* [8], the following was obtained in [11].

Theorem 30. *If the volume of a* $[t]$*-trade is between* $2 \cdot 2^t$ *and* $2.5 \cdot 2^t$*, then it has one of the following forms:*

(i) $2^{t+1} + 2^i$ *for* $i = \lceil (t-1)/2 \rceil, \ldots, t-2$;

(ii) $2^{t+1} + 2^{t-1} - 2^i$ *for* $i = 0, \ldots, t-2$;

(iii) $2^{t+1} + 2^{t-1} - 3 \cdot 2^i$ *for* $i = 0, \ldots, t-3$.

In Corollary 29, we showed that $[t]$-trades with volumes of the form (i) do not exist (except for $i = t-2$ and $t-3$ which can be represented in the form (ii) and (iii), respectively). In this section, we show by construction that they do exist with volumes of the forms (ii) and (iii). So the spectrum of volumes of $[t]$-trades in the range $2 \cdot 2^t$ and $2.5 \cdot 2^t$ is completely determined. For the construction, we employ the following observation of [11].

Lemma 31. *Assume that (T_+, T_-) and (T'_+, T'_-) are two different simple $[t]$-trades such that $T_+ \cap T'_+ = T_- \cap T'_- = \emptyset$. Then $\big((T_+ \cup T'_+) \setminus (T_- \cup T'_-), (T_- \cup T'_-) \setminus (T_+ \cup T'_+)\big)$ is a simple $[t]$-trade.*

Theorem 32. *There exist simple $[t]$-trades of volumes:*

(i) $2^{t+1} + 2^{t-1} - 2^i$ *for* $i = 0, \dots, t-2$;

(ii) $2^{t+1} + 2^{t-1} - 3 \cdot 2^i$ *for* $i = 0, \dots, t-3$.

Proof. (i) Let

$$T_1 := \langle \{1\}, \dots, \{t+1\} \rangle\,,$$
$$T_2 := \langle \{1\}, \dots, \{t-1\}, \{t+2\}, \{t+3\} \rangle.$$

Define T_1^+ (T_1^-) to be the set of vectors of T_1 with an odd (even) weight and T_2^+ (T_2^-) to be the set of vectors of T_2 with an even (odd) weight. We have $T_1^+ \cap T_2^+ = T_1^- \cap T_2^- = \emptyset$. So $T_3 = (T_3^+, T_3^-)$ with

$$T_3^+ := (T_1^+ \cup T_2^+) \setminus (T_1^- \cup T_2^-), \quad T_3^- := (T_1^- \cup T_2^-) \setminus (T_1^+ \cup T_2^+)$$

is a $[t]$-trade of volume $|T_1 \oplus T_2|/2$ where $\oplus$ denotes symmetric difference. Now let

$$T_4 := \langle \{1\}, \dots, \{i\}, \{t\}, \{t+4\}, \dots, \{2t-i+3\} \rangle,$$

with T_4^+ (T_4^-) being the set of vectors of T_4 with an even (odd) weight. We have $T_3^+ \cap T_4^+ = \emptyset$. To see this, let $B \in T_3^+ \cap T_4^+$. As $T_3^+ \subseteq T_1^+ \cup T_2^+$, we have $B \in T_4^+ \cap (T_1^+ \cup T_2^+)$. The blocks of both T_4^+, T_2^+ have even weights while those of T_1^+ have odd weights. It follows that $B \in T_4^+ \cap T_2^+ \subseteq \langle \{1\}, \dots, \{i\} \rangle$. So $B \in \langle \{1\}, \dots, \{i\} \rangle$ with an even weight and so $B \in T_1^-$ which implies that $B \notin T_3^+$, a contradiction. Analogously we have $T_3^- \cap T_4^- = \emptyset$. So $T_5 := (T_5^+, T_5^-)$ with

$$T_5^+ := (T_3^+ \cup T_4^+) \setminus (T_3^- \cup T_4^-), \quad T_5^- := (T_3^- \cup T_4^-) \setminus (T_3^+ \cup T_4^+) \tag{8}$$

is a $[t]$-trade similarly. For its volume we have

$$\begin{aligned} 2\mathrm{vol}(T_5) &= |T_1 \oplus T_2 \oplus T_4| \\ &= |T_1| + |T_2| + |T_4| - 2|T_1 \cap T_2| - 2|T_1 \cap T_4| - 2|T_2 \cap T_4| + 4|T_1 \cap T_2 \cap T_4| \\ &= 2^{t+1} + 2^{t+1} + 2^{t+1} - 2 \cdot 2^{t-1} - 2 \cdot 2^{i+1} - 2 \cdot 2^i + 4 \cdot 2^i \\ &= 2(2^{t+1} + 2^{t-1} - 2^i), \end{aligned}$$

as required.

(ii) Let $T_j = (T_j^+, T_j^-)$ for $j = 1, 2, 3$ be as in the case (i) and

$$T_4 := \langle \{1\}, \dots, \{i\}, \{t\}, \{t+1\}, \{t+4\}, \dots, \{2t-i+2\} \rangle,$$

with T_4^+ (T_4^-) to be the set vectors of T_4 of even (odd) weight. Here we have $T_3^+ \cap T_4^+ = T_3^- \cap T_4^- = \emptyset$. We define $T_5 := (T_5^+, T_5^-)$ similar to (8). So it is a $[t]$-trade with

$$\begin{aligned} 2\mathrm{vol}(T_5) &= |T_1 \oplus T_2 \oplus T_4| \\ &= |T_1| + |T_2| + |T_4| - 2|T_1 \cap T_2| - 2|T_1 \cap T_4| - 2|T_2 \cap T_4| + 4|T_1 \cap T_2 \cap T_4| \\ &= 2^{t+1} + 2^{t+1} + 2^{t+1} - 2 \cdot 2^{t-1} - 2 \cdot 2^{i+2} - 2 \cdot 2^i + 4 \cdot 2^i \\ &= 2(2^{t+1} + 2^{t-1} - 3 \cdot 2^i), \end{aligned}$$

as desired. □

From Corollary 29, Theorems 30 and 32, we have the following.

Corollary 33. *The spectrum of volumes of $[t]$-trades T with $2 \cdot 2^t < \mathrm{vol}(T) < 2.5 \cdot 2^t$ is*

$$\{2^{t+1} + 2^{t-1} - 2^i : i = 0, \dots, t-2\} \cup \{2^{t+1} + 2^{t-1} - 3 \cdot 2^i : i = 0, \dots, t-3\}.$$

6 Characterization of small $[t]$-trades for $t = 1, 2$

We say that two trades are *equivalent* if one is obtained from the other by some permutation of the elements of V, some shifts, and, optionally, the swap of the two components T_+, T_- of the trade. In this section we characterize [1]-trades of volume 3 and [2]-trades of volume 6 up to equivalence.

6.1 [1]-trades of volume 3

By the definition, a small [1]-trade has volume smaller than 4. Lemma 4 describes the [1]-trades of minimum nonzero volume 2; the remaining value is considered in the following simple theorem.

Theorem 34. *Every [1]-trade of volume 3 is a shift of $(\{Y_1, Y_2, Y_3\}, \{Z_1, Z_2, Z_3\})$, where Y_1, Y_2, Y_3 are mutually disjoint, Z_1, Z_2, Z_3 are also mutually disjoint, $Y_1Y_2Y_3 = Z_1Z_2Z_3$, and $Y_i \neq Z_j$ for every $i, j \in \{1, 2, 3\}$.*

Proof. Let (T_+, T_-) be a [1]-trade, then every element i occurs in the same number of blocks from T_+ and from T_-. If this number is 2 or 3, then we consider the x_i-shift, for which it is 1 or 0. Making this for all elements, we get a [1]-trade satisfying the conditions from the conclusion of the theorem. □

6.2 [2]-trades of volume 6

In the following four propositions, we define four types of [2]-trades of volume 6. The main result of this section states that every [2]-trade of volume 6 is of one of these four types.

Proposition 35. *Assume that a [2]-trade $T = (T_+, T_-)$ of volume 6 is represented as*

$$T = (1 - XY_1)(1 - XY_2)(1 - XY_3) - (1 - XZ_1)(1 - XZ_2)(1 - XZ_3)$$

where X, Y_1, Y_2, Y_3 are mutually disjoint sets, X, Z_1, Z_2, Z_3 are also mutually disjoint sets, Y_1, Y_2, Y_3, Z_1, Z_2, Z_3 are mutually different nonempty sets, and $Y_1Y_2Y_3 = Z_1Z_2Z_3$ (we note that X can be empty and a relation of type $Z_i = Y_jY_k$ is possible). Then, every extension T' of T has the same form, up to a shift.

Proof. We have

$$T_+ = \{XZ_1, XZ_2, XZ_3, Y_1Y_2, Y_2Y_3, Y_1Y_3\}, \quad T_- = \{XY_1, XY_2, XY_3, Z_1Z_2, Z_2Z_3, Z_1Z_3\}.$$

By Lemma 9 and the definition, an extension (T'_+, T'_-) has the form $T'_+ = S_+ \uplus x_sQ_+$, $T'_- = S_- \uplus x_sQ_-$, where $T_+ = S_+ \uplus Q_+$, $T_- = S_- \uplus Q_-$, and $S = (S_+, S_-)$, $Q = (Q_+, Q_-)$ are [1]-trades. (Note that the multiset union $\uplus$ is essential here, as some blocks can have multiplicity 2; e.g., if $XZ_1 = Y_2Y_3$.) W.l.o.g., we may assume that $\mathrm{vol}(Q) \leqslant 3$ (otherwise, we consider the x_s-shift). If it is 0, the statement holds trivially; 1 is not possible by Lemma 4. So it suffices to consider the following two cases.

Case 1. $\mathrm{vol}(Q) = 2$.

It is not difficult to see that Q cannot be a subtrade of T. Indeed, if $Q_+ = \{Y_1Y_2, Y_1Y_3\}$ (similarly, $\{Y_1Y_2, Y_2Y_3\}$ or $\{Y_1Y_3, Y_2Y_3\}$), then every element of Y_1 occurs twice in the blocks of Q_+. The same should be true for Q_-; so, either Q_- contains XY_1, or $Q_- = \{Z_iZ_j, Z_iZ_k\}$. In the first case, utilizing the definition of a [1]-trade, we see that the second block of Q_- is $XY_1Y_2Y_3$, which is not a block from T_-, a contradiction. In the second case, taking into account that $Y_1Y_2Y_3 = Z_iZ_jZ_k$, we conclude that $Y_1 = Z_i$, which does not fit the hypothesis of the proposition.

If $Q_+ = \{XZ_1, XZ_2\}$ (similarly, $\{XZ_1, XZ_3\}$ or $\{XZ_2, XZ_3\}$), then the elements of Z_3 do not occur in the blocks of Q_+. The same should be true for Q_-. So, Q_- does not contain Z_1Z_3 or Z_2Z_3. If it contains Z_1Z_2, then the second block is X, which is not from T_-, again a contradiction. Therefore, $Q_+ = \{XY_i, XY_j\}$ and w.l.o.g., $Q_+ = \{XY_1, XY_2\}$. But this leads to $Z_1Z_2 = Y_1Y_2$, and from $Z_1Z_2Z_3 = Y_1Y_2Y_3$ we find that $Z_3 = Y_3$, which contradicts the hypothesis of the proposition.

If $Q_+ = \{XZ_1, Y_1Y_2\}$ (similarly, every remaining case), then we can assume that $Q_- = \{XY_i, Z_jZ_k\}$ (the other cases are shown above). From $XZ_1Y_1Y_2 = XY_iZ_jZ_k$ we see that $Q_- = \{XY_3, Z_2Z_3\}$. We now see that every element occurs exactly twice in blocks of $Q_+ \cup Q_-$. By the definition of a [1]-trade, every element occurs exactly once in blocks of Q_+ (similarly, Q_-). But this means that $Z_1 = Y_3$, a contradiction.

Case 2. $\mathrm{vol}(Q) = 3$ (and so $\mathrm{vol}(S) = 3$).

Either Q_+, or S_+ contains XZ_i and XZ_j for some different i and j. W.l.o.g. we can assume that Q_+ contains XZ_1, XZ_2. Consider the following two subcases.

(2a) $Q_+ = \{XZ_1, XZ_2, XZ_3\}$. All elements of $Z_1Z_2Z_3$ occur exactly once in the blocks of Q_+ and, hence, in the blocks of Q_-. So, Q_- cannot have two blocks from Z_1Z_2, Z_1Z_3, Z_2Z_3 and must have at least two blocks from XY_1, XY_2, XY_3. The third block of Q_- is uniquely determined and $Q_- = \{XY_1, XY_2, XY_3\}$. We see that the claim of the proposition holds with

$$T' = (1 - X'Y_1)(1 - X'Y_2)(1 - X'Y_3) - (1 - X'Z_1)(1 - X'Z_2)(1 - X'Z_3), \qquad X' = x_s X.$$

(2b) W.l.o.g., let $Q_+ = \{XZ_1, XZ_2, Y_1Y_2\}$. We can assume that Q_- contains XY_i, Z_1Z_j for some $i \in \{1, 3\}$, $j \in \{2, 3\}$ (the other possibilities are similar or considered in the subcase (2a). Then the third element of Q_- is $W = XZ_1 \oplus XZ_2 \oplus Y_1Y_2 \oplus XY_i \oplus Z_1Z_j = XZ_2Z_jY_1Y_2Y_i$.

If $j = 2$, then W can only be XY_2, in which case

$$T' = (1 - XY_1)(1 - XY_2)(x_s - XY_3) - (1 - XZ_1)(1 - XZ_2)(x_s - XZ_3). \tag{9}$$

Then, the x_s-shift of T has the required form.

If $j = 3$ and $i = 3$, then $W = XZ_1$, which is not a block of T_-.

If $j = 3$ and $i = 1$, we have $Q_- = \{XY_1, Z_1Z_3, W\}$, where $W = XZ_2Z_3Y_2$ should be a block of T_-. Clearly, $W \neq XY_2$ and $\neq Z_2Z_3$; also $W \neq XY_1$ (as $Z_2Z_3 \neq Y_1Y_2$ by the proposition hypothesis) and, similarly, $W \neq XY_3$. If $W = Z_1Z_2$, then $XY_2 = Z_1Z_3$, which is possible, but then $Q_- = \{XY_1, Z_1Z_3 = XY_2, Z_1Z_2\}$ corresponds to (9), considered above. Finally, if $W = Z_1Z_3$, then we have $Z_1Z_2 = XY_2$, which means that $XZ_3 = Y_1Y_2$ and leads to the subcase (2a). ☐

Proposition 36. *Assume that a* [2]*-trade* $T = (T_+, T_-)$ *of volume* 6 *is represented as*

$$T = (1 - Y_1)(1 - Y_2)(1 - Y_3) - (1 - Z_1)(1 - Z_2)(1 - Z_3)$$

where Y_1*,* Y_2*,* Y_3 *are mutually disjoint nonempty sets, and likewise* Z_1*,* Z_2*,* Z_3 *are mutually disjoint nonempty sets,* Y_1*,* Y_2*,* Y_3*,* Z_1*,* Z_2*,* Z_3 *are mutually different nonempty sets, and* $Y_1Y_2 = Z_1Z_2$*. Then every extension* T' *of* T *has the same form, up to a shift.*

Proof. We have

$$T_+ = \{Z_1, Z_2, Y_2Y_3, Y_1Y_3, Z_3, Z_1Z_2Z_3\}, \quad T_- = \{Y_1, Y_2, Y_3, Y_1Y_2Y_3, Z_2Z_3, Z_1Z_3\}.$$

Repeating the arguments of the previous proof, we conclude that we have to check all possibilities for a [1]-subtrade $Q = (Q_+, Q_-)$ of volume 2 or 3.

Denote

$$\boldsymbol{X} := \{Z_1, Z_2, \underline{Y_1}, \underline{Y_2}\}, \quad \boldsymbol{Y} := \{Y_2Y_3, Y_1Y_3, \underline{Y_3}, \underline{Y_1Y_2Y_3}\}, \quad \boldsymbol{Z} := \{Z_3, Z_1Z_2Z_3, \underline{Z_2Z_3}, \underline{Z_1Z_3}\}$$

(the underlined blocks are from T_-, the other are from T_+). We first note the following fact.

(*) *The sets Q_+ and Q_- have the same number of elements from each of $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$.*

Indeed, since Y_3 and Z_3 are different, we have $Y_3\backslash Z_3 \neq \emptyset$ or $Z_3\backslash Y_3 \neq \emptyset$. Assume w.l.o.g. that $Z_3\backslash Y_3$ is not empty; i.e., it contains some element x_i. By Lemma 9, $Q_+ \cap \boldsymbol{Z}$ and $Q_- \cap \boldsymbol{Z}$ are the legs of a [0]-trade; hence, the cardinalities of this intersection are equal. Next, consider an element x_j from Y_3. If $x_j \notin Z_3$, then, similar to the argument above, we obtain that $|Q_+\cap\boldsymbol{Y}| = |Q_-\cap\boldsymbol{Y}|$. If $x_j \in Z_3$, then we have $|Q_+\cap(\boldsymbol{Y}\cup\boldsymbol{Z})| = |Q_-\cap(\boldsymbol{Y}\cup\boldsymbol{Z})|$. In any case, the whole statement of (*) follows.

Case 1. $\mathrm{vol}(Q) = 2$.

Assume that Q_+ has one block from $\boldsymbol{X}$, say X, and one block from $\boldsymbol{Y}$, say Y. Then, from (*), Q_+ also has one block from $\boldsymbol{X}$, say X', and one block from $\boldsymbol{Y}$, say Y'. We have $XX' = Z_iY_j$ and $YY' = Y_k$ for some $i, j, k \in \{1, 2\}$. In any case, $XX'YY' = Z_l$ for some $l \in \{1, 2\}$, which contradicts Lemma 10. So, Q_+ cannot have one block from $\boldsymbol{X}$ and one from $\boldsymbol{Y}$. Similarly, Q_+ cannot have one block from $\boldsymbol{X}$ and one from $\boldsymbol{Z}$, or one block from $\boldsymbol{Y}$ and one from $\boldsymbol{Z}$. The remaining possibilities satisfy the statement of the proposition:

(a) $Q_+ = \{Z_1, Z_2\}$, $Q_- = \{Y_1, Y_2\}$; then the extension of T is

$$T' = x_s((1 - Y_1)(1 - Y_2)(1 - x_sY_3) - (1 - Z_1)(1 - Z_2)(1 - x_sZ_3)).$$

(b) $Q_+ = \{Y_2Y_3, Y_1Y_3\}$, $Q_- = \{Y_3, Y_1Y_2Y_3\}$; then the extension of T is

$$T' = (1 - Y_1)(1 - Y_2)(1 - x_sY_3) - (1 - Z_1)(1 - Z_2)(1 - Z_3).$$

(c) $Q_+ = \{Z_3, Z_1Z_2Z_3\}$, $Q_- = \{Z_2Z_3, Z_1Z_3\}$; then the extension of T is

$$T' = (1 - Y_1)(1 - Y_2)(1 - Y_3) - (1 - Z_1)(1 - Z_2)(1 - x_sZ_3).$$

Case 2. $\mathrm{vol}(Q) = 3$

Q_+ cannot intersect one of $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{Z}$ in two blocks, otherwise it contains a [1]-subtrade of volume 2 ((a), (b), or (c)) and the difference would be a [1]-trade of volume 1. So, $Q_+ = \{X, Y, Z\}$ and $Q_- = \{X', Y', Z'\}$ for some X, Y, Z, X', Y', Z' from $\boldsymbol{X}\cap T_+$, $\boldsymbol{Y}\cap T_+$, $\boldsymbol{Z} \cap T_+$, $\boldsymbol{X} \cap T_-$, $\boldsymbol{Y} \cap T_-$, $\boldsymbol{Z} \cap T_-$, respectively. We have $YY' = Y_i$ and $ZZ' = Z_j$, where $i, j \in \{1, 2\}$. Assume w.l.o.g. that $YY' = Y_1$ and $ZZ' = Z_1$. It follows from Lemma 10 that $XX' = Y_1Z_1$. With these assumptions, X', Y', and Z' are uniquely determined by X, Y, and Z. It remains to consider the eight possibilities to choose X, Y, and Z ($X \in \{Z_1, Z_2\}$, $Y \in \{Y_1Y_3, Y_2Y_3\}$, $Z \in \{Z_3, Z_1Z_2Z_3\}$). The following two possibilities are in agree with the proposition statement:

(d) $Q_+ = \{Z_1, Y_1Y_3, Z_3\}$, $Q_- = \{Y_1, Y_3, Z_1Z_3\}$; then the extension of T is

$$T' = (1 - x_sY_1)(1 - Y_2)(1 - Y_3) - (1 - x_sZ_1)(1 - Z_2)(1 - Z_3).$$

(e) $Q_+ = \{Z_2, Y_2Y_3, Z_1Z_2Z_3\}$, $Q_- = \{Y_2, Y_1Y_2Y_3, Z_2Z_3\}$; then the extension of T is

$$T' = x_s((1 - x_sY_1)(1 - Y_2)(1 - Y_3) - (1 - x_sZ_1)(1 - Z_2)(1 - Z_3)).$$

Consider the six other possibilities to choose X, Y, Z from $\boldsymbol{X} \cap T_+$, $\boldsymbol{Y} \cap T_+$, $\boldsymbol{Z} \cap T_+$. For example, let $Q_+ = \{Z_2, Y_2Y_3, Z_3\}$ (the other five cases are similar); so, $Q_- = \{Y_2, Y_1Y_2Y_3, Z_1Z_3\}$. Subtracting (Q_-, Q_+) from the [1]-trade (e) above, we get

$$(\{Z_1Z_2Z_3, Z_1Z_3\}, \{Z_2Z_3, Z_3\}),$$

which is not a [1]-trade (compare with (c) above). Hence, (Q_-, Q_+) is not a [1]-trade either.

Therefore, under the assumption that $YY' = Y_1$ and $ZZ' = Z_1$, in only two subcases, (d) and (e), we have trades. The other cases ($YY' = Y_1$ and $ZZ' = Z_2$, $YY' = Y_2$ and $ZZ' = Z_1$, $YY' = Y_2$ and $ZZ' = Z_2$) are similar. □

Proposition 37. *Let a* [2]*-trade* $T = (T_+, T_-)$ *of volume* 6 *be represented as*

$$T = (1 - Y_1)(1 - Y_2)(1 - Y_3) - (1 - Z_1)(1 - Z_2)(1 - Y_1Y_2Y_3)$$

where Y_1*,* Y_2*,* Y_3*,* Z_1*,* Z_2 *are mutually disjoint nonempty sets. Then, every extension* T' *of* T *has the same form, up to a shift.*

Proof. We have

$$T_+ = \{Y_1Y_2, Y_1Y_3, Y_2Y_3, Z_1, Z_2, Y_1Y_2Y_3Z_1Z_2\},$$
$$T_- = \{Y_1, Y_2, Y_3, Z_1Z_2, Z_1Y_1Y_2Y_3, Z_2Y_1Y_2Y_3\}.$$

Repeating the arguments of the proofs of Propositions 35 and 36, we need to check all possibilities for a [1]-subtrade $Q = (Q_+, Q_-)$ of volume 2 or 3.

Denote

$$\boldsymbol{Y} := \{\underline{Y_1}, \underline{Y_2}, \underline{Y_3}, Y_1Y_2, Y_1Y_3, Y_2Y_3\},$$
$$\boldsymbol{Z} := \{Z_1, \underline{Z_1Z_2}, \underline{Z_1Y_1Y_2Y_3}, Z_1Z_2Y_1Y_2Y_3\},$$
$$\boldsymbol{Z}' := \{Z_2, \underline{Z_1Z_2}, \underline{Z_2Y_1Y_2Y_3}, Z_1Z_2Y_1Y_2Y_3\}.$$

Similarly to the claim (*) in the proof of Proposition 36, we have

(*) Q_+ *and* Q_- *have the same number of elements from each of* $\boldsymbol{Z}$ *and* $\boldsymbol{Z}'$*.*

Now, assume that Q is a [1]-subtrade of volume 2 or 3. Consider the following four cases, which exhaust all possibilities.

Case 1. $|Q_+ \cap \boldsymbol{Z}| = 2$ or $|Q_+ \cap \boldsymbol{Z}'| = 2$.

Without loss of generality assume $|Q_+ \cap \boldsymbol{Z}| = 2$. Necessarily we have $|Q_- \cap \boldsymbol{Z}| = 2$, and so $Q_+ \supseteq \{Z_1, Z_1Z_2Y_1Y_2Y_3\}$, $Q_- \supseteq \{Z_1Z_2, Z_1Y_1Y_2Y_3\}$. We see that

$$(\{Z_1, Z_1Z_2Y_1Y_2Y_3\}, \{Z_1Z_2, Z_1Y_1Y_2Y_3\})$$

is a [1]-trade, and we cannot add one more element to each leg keeping the [1]-trade property. So, $\mathrm{vol}(Q) = 2$ and

$$T' = (1 - Y_1)(1 - Y_2)(1 - Y_3) - (1 - x_sZ_1)(1 - Z_2)(1 - Y_1Y_2Y_3).$$

Case 2. $|Q_+ \cap \boldsymbol{Z}| = |Q_+ \cap \boldsymbol{Z}'| = 0$.

In this case we have $Q_+ \subseteq \{Y_1Y_2, Y_1Y_3, Y_2Y_3\}$ and $Q_- \subseteq \{Y_1, Y_2, Y_3\}$. The leg Q_+ has two intersecting blocks, but the blocks of Q_- are mutually disjoint; we have an obvious contradiction with the definition of a [1]-trade.

Case 3. $|Q_+ \cap \boldsymbol{Z}| = 1$ and $|Q_+ \cap \boldsymbol{Z}'| = 0$ (similarly, $|Q_+ \cap \boldsymbol{Z}| = 0$ and $|Q_+ \cap \boldsymbol{Z}'| = 1$).

From (*) we have that $Z_1 \in Q_+$, $Z_1Y_1Y_2Y_3 \in Q_-$, and every other block of Q_+ or Q_- belongs to $\boldsymbol{Y}$. Since all elements of $Y_1Y_2Y_3$ occur in Q_-, at least two of Y_1Y_2, Y_1Y_3, Y_2Y_3 belong to Q_+ (in particular, the volume of Q is 3, not 2). W.l.o.g. assume $Q+ = \{Z_1, Y_1Y_2, Y_1Y_3\}$. We see that the elements of Y_1 occurs twice in Q_+; hence, Q_- contains Y_1. By Lemma 10, the third block in Q_- is $Z_1 \oplus Y_1Y_2 \oplus Y_1Y_3 \oplus Z_1Y_1Y_2Y_3 \oplus Y_1$, i.e., $\emptyset$. Since $\emptyset \notin T_-$, hence we reach at a contradiction.

Case 4. $|Q_+ \cap \boldsymbol{Z}| = |Q_+ \cap \boldsymbol{Z}'| = 1$.

Consider the following subcases.

(4a) $Z_1Z_2Y_1Y_2Y_3 \in Q_+$, $Z_1Z_2 \in Q_-$, the other blocks are from $\boldsymbol{Y}$. Since all elements of $Y_1Y_2Y_3$ occur in Q_+, Q_- must contain each of Y_1, Y_2, Y_3, which is impossible since $|Q_-| \leqslant 3$.

(4b) $Z_1, Z_2 \in Q_+$, $Z_1Y_1Y_2Y_3, Z_2Y_1Y_2Y_3 \in Q_-$, the other blocks are from $\boldsymbol{Y}$. Since all elements of $Y_1Y_2Y_3$ occur in Q_- twice, Q_+ must contain each of Y_1Y_2, Y_1Y_3, Y_2Y_3, which is impossible as $|Q_+| \leqslant 3$.

(4c) $Z_1Z_2Y_1Y_2Y_3 \in Q_+$, $Z_1Y_1Y_2Y_3, Z_2Y_1Y_2Y_3 \in Q_-$, the other blocks are from $\boldsymbol{Y}$. Since $Z_1Z_2Y_1Y_2Y_3 \oplus Z_1Y_1Y_2Y_3 \oplus Z_2Y_1Y_2Y_3 = Y_1Y_2Y_3 \notin T_+$, from Lemma 10 we observe that the [1]-trade (Q_+, Q_-) cannot have volume 2. So, Q_+ has two elements from $\boldsymbol{Y}$, say Y_iY_j and Y_iY_k. By Lemma 10 we find $Y_i \in Q_-$, and so

$$T' = (1 - x_sY_i)(1 - Y_j)(1 - Y_k) - (1 - Z_1)(1 - Z_2)(1 - x_sY_1Y_2Y_3).$$

(4d) $Z_1, Z_2 \in Q_+$, $Z_1Z_2 \in Q_-$, the other blocks are from $\boldsymbol{Y}$. Similarly to the subcase (4c), we have

$$T' = x_s(1 - x_sY_i)(1 - Y_j)(1 - Y_k) - x_s(1 - Z_1)(1 - Z_2)(1 - x_sY_1Y_2Y_3).$$

□

Proposition 38. *Assume that*

$$T = (\{Y_1, Y_2, Y_3, XZ_1, XZ_2, XZ_3\}, \{Z_1, Z_2, Z_3, XY_1, XY_2, XY_3\}),$$

where X, Y_1, Y_2, Y_3 are mutually disjoint, X, Z_1, Z_2, Z_3 are mutually disjoint, $Y_1Y_2Y_3 = Z_1Z_2Z_3$, $Y_i \neq Z_j$ for every $i, j \in \{1, 2, 3\}$ and $X \neq \emptyset$ Then, every extension of T has the same form, up to a shift.

Proposition 38 is a partial case of the following more general fact.

Proposition 39. *Assume that*

$$T = (1 - X)\bar{\sigma}$$

where $\bar{\sigma}$ is a $[t-1]$-trade of volume less than 2^t (i.e., small) and X is a nonempty set, disjoint from the blocks of $\bar{\sigma}$ (so, T is a small $[t]$-trade). Let T' be an extension of T. Then

$$T' = (1 - x_sX)\bar{\sigma}, \qquad T' = (x_s - X)\bar{\sigma}, \tag{10}$$

or

$$T' = (1 - X)\bar{\sigma}', \tag{11}$$

where $\bar{\sigma}'$ is an extension of $\bar{\sigma}$.

Proof. We have $T' = x_s\bar{\kappa} + (T - \bar{\kappa})$, where $\bar{\kappa}$ is a $[t-1]$-subtrade of T. W.l.o.g., we can assume that $\bar{\kappa}$ is small. Let $\bar{\kappa}^p$ be the projection of $\bar{\kappa}$ in X. Then $\bar{\kappa}^p$ is a small $[t-1]$-trade, whose blocks are blocks of $\bar{\sigma}$. Let us prove the following claim:

(*) *If $\bar{\kappa}^p$ is not void, then all blocks of the $[t-1]$-trade $\bar{\kappa}^p + \bar{\sigma}$ have even multiplicity.*

Denote by a and b the number of different blocks of $\bar{\sigma}$ of odd and even multiplicity, respectively. The volume of $\bar{\sigma}$ is at least $(a+2b)/2$; since $\bar{\sigma}$ is a small $[t-1]$-trade, we have

$$(a + 2b)/2 < 2^t. \tag{12}$$

Denote by a' and b' the number of blocks of $\bar{\kappa}^p$ of odd multiplicity whose multiplicity in $\bar{\sigma}$ is odd and even, respectively. So, the number of blocks of odd multiplicity in $\bar{\kappa}^p + \bar{\sigma}$ is $a - a' + b'$.

Next, since $\bar{\kappa}^p$ is a small non-void $[t-1]$-trade, by Lemma 3 we have

$$a' + b' \geqslant 2^t \tag{13}$$

Now, using (12), (13), and the trivial fact that $b' \leqslant b$, for the number $a - a' + b'$ of odd-multiplicity blocks of $\bar{\kappa}^p + \bar{\sigma}$ we have

$$a - a' + b' = a + 2b' - a' - b' \leqslant (a + 2b) - (a' + b') < 2 \cdot 2^t - 2^t = 2^t.$$

By Lemma 3, this number is 0. Hence (*) follows.

If $\bar{\kappa}^p$ is void, we have (11). By (*), it remains to consider the case when all blocks of the $[t-1]$-trade $\bar{\kappa}^p + \bar{\sigma}$ have even multiplicity.

(**) *$(\bar{\kappa}^p + \bar{\sigma})/2$ is a $[t-1]$-subtrade of $\bar{\sigma}$.* Equivalently, every block of $(\bar{\kappa}^p + \bar{\sigma})/2$ has the same sign in $(\bar{\kappa}^p + \bar{\sigma})/2$ as in $\bar{\sigma}$ and at most the same multiplicity. Indeed, by the definition of $\bar{\kappa}^p$, the coefficient α at each of its blocks satisfies $|\alpha| \leqslant |\beta|$. It follows that $0 \leqslant \left|\frac{\alpha+\beta}{2}\right| \leqslant |\beta|$ and $\frac{\alpha+\beta}{2}$ and β are of the same sign. So (**) follows.

Since $\bar{\sigma}$ is a small $[t-1]$-trade, it does not have proper subtrades, and $(\bar{\kappa}^p + \bar{\sigma})/2$ is either zero or $\bar{\sigma}$. In the first case, $\bar{\kappa}^p = -\bar{\sigma}$, and $\bar{\kappa} = -X\bar{\sigma}$. In the second case, $\bar{\kappa}^p = \bar{\kappa} = \bar{\sigma}$. Therefore, in every case, we obtain that T' has on the forms given in (10). □

Theorem 40. *Every* $[2]$*-trade of volume* 5 *or* 6 *have one of the forms described in Propositions* 35–38*.*

In particular, Theorem 40 implies that there are no $[2]$-trades of volume 5, which is a known fact [6].

Proof. We proceed by induction on the number of the elements involved in the blocks of a trade. If this number is zero, then the statement is trivial (there are no non-void trades), which gives the induction base. Let us consider a $[2]$-trade T of volume 5 or 6. If it has a projection of volume 5 or 6, then by the inductive hypothesis the statement of the theorem holds for this projection. Hence, it is true for T, by Propositions 35–38.

If T has a void projection, then it has the form $T = (1 - x_i)\bar{\sigma}$, where $\bar{\sigma}$ is a $[1]$-trade of volume 3. In this case, the statement is straightforward from Theorem 34.

It remains to consider the case when all projections have volume 4. For a given i, the i-projection has the form

$$(1-X)(1-Y)(1-Z) = 1 - X - Y - Z + XY + XZ + YZ - XYZ,$$

up to a shift. Then

$$T = \alpha_{000} - \alpha_{100}X - \alpha_{010}Y - \alpha_{001}Z + \alpha_{110}XY + \alpha_{101}XZ + \alpha_{011}YZ - \alpha_{111}XYZ \pm (1-x_i)V \pm (1-x_i)W,$$

where $\alpha_{000}, \alpha_{100}, \alpha_{010}, \alpha_{001}, \alpha_{110}, \alpha_{101}, \alpha_{011}, \alpha_{111} \in \{1, x_i\}$ and V, W are some blocks with $i \notin V, W$. The number of blocks of T with (or without) element i is at least 2 and at most 10; taking into account Lemma 9, it is 4, 6, or 8. So, the number p_i of coefficients $\alpha_{...}$ equal to x_i is 2, 4, or 6. W.l.o.g. (up to the x_i-shift) we may assume that it is $p_i = 2$ or 4. The case of $p_i = 2$, up to a shift and renaming X, Y, and Z, is exhausted by the Cases 1-3 below.

Case 1. $\alpha_{000} = \alpha_{100} = x_i$, the other coefficients are 1:

$$T = x_i - x_iX - Y - Z + XY + XZ + YZ - XYZ + (1-x_i)V - (1-x_i)W.$$

Considering the [1]-subtrade $x_i - x_iX - x_iV + x_iW$, we see that X and V are disjoint and $W = XV$. We find that the case falls under the conditions of Proposition 38, with $Y_1 := x_i$, $Y_2 := V$, $Y_3 := YZ$, $Z_1 := Y$, $Z_2 := Z$, $Z_3 := x_iXV$, and $X = X$.

Case 2. $\alpha_{000} = \alpha_{110} = x_i$, the other coefficients are 1:

$$T = x_i - X - Y - Z + x_iXY + XZ + YZ - XYZ + (1-x_i)V + (1-x_i)W.$$

Considering the [1]-subtrade $x_i + x_iXY - x_iV - x_iW$, we see that V and W are disjoint and $VW = XY$. We find that the case falls under the conditions of Proposition 36, with $Y_1 := X$, $Y_2 := Y$, $Y_3 := Z$, $Z_1 := V$, $Z_2 := W$, $Z_3 := x_i$.

Case 3. $\alpha_{000} = \alpha_{111} = x_i$, the other coefficients are 1:

$$T = x_i - X - Y - Z + XY + XZ + YZ - x_iXYZ + (1-x_i)V - (1-x_i)W.$$

Similar to Case 1, XYZ and V are disjoint and $W = XYZ \oplus V$. The case falls under the conditions of Proposition 37, with $Y_1 := X$, $Y_2 := Y$, $Y_3 := Z$, $Z_1 := V$, $Z_2 := x_i$.

Case 4. $p_i = 4$.

We can assume that $p_j = 4$ for any element j involved in the trade T (otherwise we will be in one of the Cases 1-3); so,

(*) *for every essential element j, in the decomposition $T = P + x_jP'$ the volume of the* [1]*-trades P and P' is* 3.

In particular,

(**) *V and W consist of elements of XYZ*, as any other element contradicts (*).

(***) $VW = XYZ$ (indeed, from (*) we see that every element j from XYZ belongs to exactly one of V, W).

We consider two subcases.

(4a) Firstly, assume that one of X, Y, Z, say X, has two different elements j and k. Since the j-projection of T has volume 4, we find that V (and hence W) differs from one of 1, X, Y, Z, XY, XZ, YZ, XYZ in only one element j. Up to a shift, we assume that $V = x_j$. The same can be said about k; so, $X = x_j x_k$. Now, neither Y nor Z can have more than one element (otherwise, there are projections of volume 6).

Let, w.l.o.g., $\alpha_{000} = x_i$. The [1]-subtrade of T consisting of all blocks containing x_i has six blocks, three of which we know: x_i, x_iV, and x_iW. The other three blocks must sum up to $x_i \oplus x_iV \oplus x_iW = x_iXYZ$; so, they are either x_iX, x_iY, x_iZ, or x_iXY, x_iYZ, x_iXZ. The last case is impossible because the four blocks x_i, x_iXY, x_iYZ, x_iXZ have the same sign. We conclude that

$$T = x_i - x_iX - x_iY - x_iZ + XY + XZ + YZ - XYZ - V + x_iV - W + x_iW,$$

where $X = x_jx_k$, $V = x_j$, $W = x_kYZ$, which has the j-projection of volume 6, contradicting our assumption.

(4b) The remaining subcase is $|X| = |Y| = |Z| = 1$. Each of V, W is one of 1, X, Y, Z, XY, XZ, YZ, XYZ. It is not difficult to conclude that, up to a shift,

$$T = 2 - X - Y - Z - x_i + XYZ + XYx_i + XZx_i + YZx_i - 2XYZx_i,$$

which is the case of Proposition 35. □

7 Computational results

In this section we present an algorithm to construct $[t]$-trades with a given foundation of size v. We implement this algorithm and enumerate all small $[t]$-trades for $t \leqslant 4$.

7.1 Algorithm

Corollary 12 allows to compute all possible s-small $[t]$-trades T with a foundation of size v if we know all s-small $[t]$-trades T' and s-small $[t-1]$-trades T'' with foundations of size $v-1$. This gives the possibility to classify, for a given s, all s-small $[t]$-trades of small foundation recursively, starting from $t = 0$. The following algorithm describes the recursive step.

0 Set $\mathcal{T} := \emptyset$.

1 For all s-small $[t]$-trades T' and all s-small $[t-1]$-trades T'' do Steps 1.1–1.2.

  1.1 Add $T' - (1 - x_v)T''$ to $\mathcal{T}$.

  1.2 If $T' - T''$ is not small, then add $x_vT' + (1 - x_v)T''$ to $\mathcal{T}$.

At the end, $\mathcal{T}$ will be the set of all s-small $[t]$-trades. Indeed, for every such trade T, consider the representation $T = P + x_vP'$, where $v \notin \mathrm{found}(P), \mathrm{found}(P')$. If P' is s-small, then T is added at Step 1.1 with $T' = P + P'$ and $T'' = P'$. If P' is not s-small, then P is s-small, and T is added at Step 1.2 with $T' = P + P'$ and $T'' = P$.

From $\mathcal{T}$, we can choose a complete collection of nonequivalent s-small $[t]$-trades (to be exact, representatives of all equivalence classes). The graph isomorphism routine [15] is employed to deal with the equivalence rejection. See [9] for general technique of representing subsets of 2^V by graphs, for checking the equivalence. If we do not need the list of all trades, we can check equivalence at Steps 1.1 and 1.2, and collect only nonequivalent representatives. In this case, there is an obvious improvement: it is sufficient to consider either only nonequivalent $[t]$-trades T', or only nonequivalent $[t-1]$-trades T''. However, the second component, T'' or T', must be chosen from all different trades with corresponding parameters, and this approach does not allow to make all steps of the recursion by considering only nonequivalent representatives.

7.2 Validity of computational results

The correctness of the computer classification can be partially verified by the following double-counting arguments (see [9, Ch. 10]). Denote by $\mathrm{Aut}(T)$ the full automorphism group of a trade T, which consists of all equivalence transformations that send T to itself (recall that an equivalence transformation consists of a shift, a permutation of the elements of V, and, optionally, the swap of the components T_+, T_- of the trade $T = (T_+, T_-)$). The number of all different s-small $[t]$-trades with foundation contained in V can be calculated as

$$\sum |\mathrm{Aut}(2^V)|/|\mathrm{Aut}(T)|, \tag{14}$$

where the summation is over all nonequivalent representatives and $\mathrm{Aut}(2^V)$ is the group of all equivalence transformations with $|\mathrm{Aut}(2^V)| = 2 \cdot 2^v \cdot v!$. On the other hand, this number can be found as the total number of solutions found by the algorithm (if T' or T'' runs over nonequivalent representatives, then every solution is counted with the factor $2^v(v-1)!/|\mathrm{Aut}(T')|$ or $2^v(v-1)!/|\mathrm{Aut}(T'')|$, respectively). Coinciding this number with (14) means that the probability of errors of different kinds is very-very small.

7.3 Results: Construction of small $[t]$-trades with $t \leqslant 4$ and $|\mathrm{found}| \leqslant 7$

The tables below show the number of $[t]$-trades in 2^V, for given $|V|$ and given volume. The first number in a cell indicates the number of equivalence classes of all $[t]$-trades. The second number (in parentheses) indicates the number of equivalence classes of non-degenerate $[t]$-trades. The third, the number of equivalence classes of all simple $[t]$-trades. The fourth, the number of equivalence classes of non-degenerate simple $[t]$-trades. Note that the row "$v = \ldots$" reflects the numbers for trades in 2^V with $|V| = v$, but the foundation size of the trades can be smaller; so, the same trades are necessarily counted in the next row, together with the trades of foundation size $v+1$.

$t = 1$:

vol.	0	2	3
$v \leqslant 1$	1	0	0
$v = 2$	1	1(1) 1(1)	0
$v = 3$	1	2(1) 2(1)	1(1) 0(0)
$v = 4$	1	4(1) 4(1)	5(4) 3(3)
$v = 5$	1	6(1) 6(1)	17(8) 13(7)
$v = 6$	1	9(1) 9(1)	51(12) 44(11)
$v = 7$	1	12(1) 12(1)	126(14) 115(13)

$t = 2$:

vol.	0	4	6	7
$v \leqslant 2$	1	0	0	0
$v = 3$	1	1(1) 1(1)	0	0
$v = 4$	1	2(1) 2(1)	2(2) 0(0)	0
$v = 5$	1	4(1) 4(1)	12(9) 7(7)	7(7) 0(0)
$v = 6$	1	7(1) 7(1)	43(17) 32(15)	88(63) 52(52)
$v = 7$	1	11(1) 11(1)	130(24) 109(22)	515(161) 391(148)

$t = 3$:

vol.	0	8	12	14	15
$v \leqslant 3$	1	0	0	0	0
$v = 4$	1	1(1) 1(1)	0	0	0
$v = 5$	1	2(1) 2(1)	2(2) 0(0)	0	1(1) 0(0)
$v = 6$	1	4(1) 4(1)	15(11) 9(9)	14(14) 0(0)	7(6) 0(0)
$v = 7$	1	7(1) 7(1)	56(20) 41(18)	165(110) 89(89)	74(51) 0(0)

$t = 4$:

vol.	0	16	24	28	30	31
$v \leqslant 4$	1	0	0	0	0	0
$v = 5$	1	1(1) 1(1)	0	0	0	0
$v = 6$	1	2(1) 2(1)	2(2) 0(0)	0	2(2) 0(0)	0
$v = 7$	1	4(1) 4(1)	15(11) 9(9)	17(17) 0(0)	15(12) 0(0)	0

7.4 Proof of Lemma 25

For $t = 2$, we can further implement our algorithm to construct all $[t]$-trades T with $2 \cdot 2^t \leqslant \mathrm{vol}(T) \leqslant 3 \cdot 2^t$ and $|\mathrm{found}(T)| = 5$. In particular, Lemma 25 is derived. The enumeration of these trades is given in the table below.

vol.	8	9	10	11	12
$v \leqslant 2$	0	0	0	0	0
$v = 3$	1(1) 0(0)	0	0	0	1(1) 0(0)
$v = 4$	7(6) 2(2)	2(2) 0(0)	3(3) 0(0)	0	18(17) 0(0)
$v = 5$	94(80) 39(36)	85(82) 0(0)	479(471) 20(20)	771(771) 0(0)	3195(3154) 26(26)

Acknowledgments

The research of the first author was in part supported by a grant from IPM (No. 98050211). The research of the second and third authors was partially supported by IPM. The research of the fourth author was supported by the Program of fundamental scientific researches of the SB RAS No.I.5.1. (project No. 0314-2019-0016).

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