On q–covering designs

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Abstract

A q–covering design $\mathbb{C}_q(n,k,r)$, $k \geqslant r$, is a collection $\mathcal{X}$ of $(k-1)$–spaces of $\mathrm{PG}(n-1,q)$ such that every $(r-1)$–space of $\mathrm{PG}(n-1,q)$ is contained in at least one element of $\mathcal{X}$. Let $\mathcal{C}_q(n,k,r)$ denote the minimum number of $(k-1)$–spaces in a q–covering design $\mathbb{C}_q(n,k,r)$. In this paper improved upper bounds on $\mathcal{C}_q(2n,3,2)$, $n \geqslant 4$, $\mathcal{C}_q(3n+8,4,2)$, $n \geqslant 0$, and $\mathcal{C}_q(2n,4,3)$, $n \geqslant 4$, are presented. The results are achieved by constructing the related q–covering designs.

Mathematics Subject Classifications: 51E20, 05B40, 05B25, 51A05

1 Introduction

Let q be any prime power, let $\mathrm{GF}(q)$ be the finite field with q elements and let $\mathrm{PG}(n-1,q)$ be the $(n-1)$–dimensional projective space over $\mathrm{GF}(q)$. We will use the term k–space to denote a subspace of $\mathrm{PG}(n-1,q)$ of projective dimension k. Let $t \leqslant s$. A *blocking set* $\mathbb{B}$ is a set of $(t-1)$–spaces of $\mathrm{PG}(n-1,q)$ such that every $(s-1)$–space of $\mathrm{PG}(n-1,q)$ contains at least one element of $\mathbb{B}$. In the last fifty years the general problem of determining the smallest cardinality of a blocking set $\mathbb{B}$ has been studied by several authors (see [17, 4] and references therein) and in very few cases has been completely solved [5, 2, 3, 9, 18].

A blocking set $\mathbb{B}$ can be seen as a q–analog of a well known combinatorial design, called *Turán design*, see [11], [10]. Indeed, a blocking set $\mathbb{B}$ is also called a *q–Turán design* $\mathbb{T}_q(n,t,s)$. The dual structure of a q–Turán design $\mathbb{T}_q(n,t,s)$ is called *q–covering design* and it is denoted with $\mathbb{C}_q(n,n-t,n-s)$. In other words, a q–covering design $\mathbb{C}_q(n,k,r)$ is a collection $\mathcal{X}$ of $(k-1)$–spaces of $\mathrm{PG}(n-1,q)$ such that every $(r-1)$–space of $\mathrm{PG}(n-1,q)$ is contained in at least one element of $\mathcal{X}$. Let $\mathcal{C}_q(n,k,r)$ denote

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the minimum number of $(k-1)$–spaces in a q–covering design $\mathbb{C}_q(n,k,r)$. Lower and upper bounds on $\mathcal{C}_q(n,k,r)$ were considered in [11], [10]. Lower bounds are obtained by providing q–analogs of classical results which have been proved in the context of covering designs and Turán designs; upper bounds are obtained by explicit constructions of the related q–covering designs. A q–covering design $\mathbb{C}_q(n,k,r)$ which cover every $(r-1)$–space exactly once is called *q–Steiner system*. If $r=1$, a q–Steiner system $\mathbb{C}_q(n,k,r)$ is also known as $(k-1)$*–spread* of $\mathrm{PG}(n-1,q)$; spreads have been widely investigated in finite geometry and it is known that a $(k-1)$–spread of $\mathrm{PG}(n-1,q)$ exists if and only if k divides n, see [20].

The concept of q–covering design is of interest not only in projective geometry and design theory, but also in coding theory. Indeed, in recent years there has been an increasing interest in q–covering designs due to their connections with constant–dimension codes. An $(n,M,2\delta;k)_q$ constant–dimension subspace code (CDC) is a set $\mathcal{S}$ of $(k-1)$–spaces of $\mathrm{PG}(n-1,q)$ such that $|\mathcal{S}|=M$ and every $(k-\delta)$–space of $\mathrm{PG}(n-1,q)$ is contained in at most one member of $\mathcal{S}$ or, equivalently, any two distinct codewords of $\mathcal{S}$ intersect in at most a $(k-\delta-1)$–space. Subspace codes of largest possible size are said to be *optimal*. Therefore, a q–Steiner system is an optimal constant–dimension code (so far, apart from spreads, there is only one known example of q–Steiner system, i.e., the 2–covering design $\mathbb{C}_2(13,3,2)$ of smallest possible size [6]). Observe that, as shown in the inspiring article by Koetter and Kschischang [16], constant–dimension codes can be used for error–correction in random linear network coding theory.

In this paper we discuss bounds on q–covering designs. In Section 3, based on the q–covering design $\mathbb{C}_q(6,3,2)$ constructed in [7], an improved upper bound on $\mathcal{C}_q(2n,3,2)$, $n\geqslant 4$, is presented. In the last two sections, starting from a lifted MRD–code, improvements on the upper bounds of $\mathcal{C}_q(3n+8,4,2)$, $n\geqslant 0$, and $\mathcal{C}_q(2n,4,3)$, $n\geqslant 4$, are obtained. In particular, first q–covering designs $\mathbb{C}_q(8,4,r)$, $r=2,3$, of $\mathrm{PG}(7,q)$ are constructed. Then, by induction, q–covering designs $\mathbb{C}_q(3n+8,4,2)$, $n\geqslant 0$, and $\mathbb{C}_q(2n,4,3)$, $n\geqslant 4$, are presented.

In the sequel we will use the following notation $\theta_{n,q} := {\begin{bmatrix} n+1 \\ 1 \end{bmatrix}}_q = q^n+\ldots+q+1$.

2 Preliminaries

A *conic* of $\mathrm{PG}(2,q)$ is the set of points of $\mathrm{PG}(2,q)$ satisfying a quadratic equation: $a_{11}X_1^2+a_{22}X_2^2+a_{33}X_3^2+a_{12}X_1X_2+a_{13}X_1X_3+a_{23}X_2X_3=0$. There exist four kinds of conics in $\mathrm{PG}(2,q)$, three of which are degenerate (splitting into lines, which could be in the plane $\mathrm{PG}(2,q^2)$) and one of which is non–degenerate, see [13].

A *regulus* is the set of lines intersecting three skew (disjoint) lines and has size $q+1$. The *hyperbolic quadric* $\mathcal{Q}^+(3,q)$, is the set of points of $\mathrm{PG}(3,q)$ which satisfy the equation $X_1X_2+X_3X_4=0$. The hyperbolic quadric $\mathcal{Q}^+(3,q)$ consists of $(q+1)^2$ points and $2(q+1)$ lines that are the union of two reguli. Through a point of $\mathcal{Q}^+(3,q)$ there pass two lines belonging to different reguli.

A 1–spread is also called *line–spread*. Recall that a line–spread of $\mathrm{PG}(3,q)$ is a set $\mathcal{S}$ of q^2+1 lines of $\mathrm{PG}(3,q)$ with the property that each point of $\mathrm{PG}(3,q)$ is incident with exactly one element of $\mathcal{S}$. A 1–*parallelism* of $\mathrm{PG}(3,q)$ is a collection $\mathcal{P}$ of q^2+q+1 line–spreads such that each line of $\mathrm{PG}(3,q)$ is contained in exactly one line–spread of $\mathcal{P}$. In [1] the author proved that there exist 1–parallelisms in $\mathrm{PG}(3,q)$.

The *Klein quadric* $\mathcal{Q}^+(5,q)$, is the set of points of $\mathrm{PG}(5,q)$ which satisfy the equation $X_1X_2+X_3X_4+X_5X_6=0$. The Klein quadric contains $(q^2+1)(q^2+q+1)$ points of and two families each consisting of q^3+q^2+q+1 planes called *Latin planes* and *Greek planes*. Two distinct planes in the same family share exactly one point, whereas planes lying in distinct families are either disjoint or meet in a line. A line of $\mathrm{PG}(5,q)$ not contained in $\mathcal{Q}^+(5,q)$ is either *external*, or *tangent*, or *secant* to $\mathcal{Q}^+(5,q)$, according as it contains $0,1$ or 2 points of $\mathcal{Q}^+(5,q)$. A hyperplane of $\mathrm{PG}(5,q)$ contains either q^3+2q^2+q+1 or q^3+q^2+q+1 points of $\mathcal{Q}^+(5,q)$. In the former case the hyperplane is called *tangent*, contains the $2(q+1)$ planes of $\mathcal{Q}^+(5,q)$ through one of its points, say R, and meets $\mathcal{Q}^+(5,q)$ in a cone having as vertex the point R and as base a hyperbolic quadric $\mathcal{Q}^+(3,q)$. In the latter case the hyperplane is called *secant* and contains no plane of $\mathcal{Q}^+(5,q)$. The stabilizer of $\mathcal{Q}^+(5,q)$ in $\mathrm{PGL}(6,q)$, say G, contains a subgroup isomorphic to $\mathrm{PGL}(4,q)$. Also, the stabilizer in G of a plane g of $\mathcal{Q}^+(5,q)$ contains a subgroup H isomorphic to $\mathrm{PGL}(3,q)$ acting in its natural representation on the points and lines of g. For more details see [14, Chapter 1]. A *Singer cyclic subgroup* of $\mathrm{PGL}(k,q)$ is a cyclic group acting regularly on points and hyperplanes of a projective space $\mathrm{PG}(k-1,q)$.

2.1 Lifting an MRD–code

The set $\mathcal{M}_{n\times m}(q)$, $n\leqslant m$, of $n\times m$ matrices over the finite field $\mathrm{GF}(q)$ forms a metric space with respect to the *rank distance* defined by $d_r(A,B)=rank(A-B)$. The maximum size of a code of minimum distance δ, with $1\leqslant\delta\leqslant n$, in $(\mathcal{M}_{n\times m}(q),d_r)$ is $q^{m(n-\delta+1)}$. A code $\mathcal{A}\subset\mathcal{M}_{n\times m}(q)$ attaining this bound is said to be a $(n\times m,\delta)_q$ *maximum rank distance code* (or *MRD–code* in short). A rank distance code $\mathcal{A}$ is called $\mathrm{GF}(q)$*–linear* if $\mathcal{A}$ is a subspace of $\mathcal{M}_{n\times m}(q)$ considered as a vector space over $\mathrm{GF}(q)$. Linear MRD–codes exist for all possible parameters [8, 12, 19, 21].

We recall the so–called *lifting process* for a matrix $A\in\mathcal{M}_{n\times m}(q)$, see [22]. Let I_n be the $n\times n$ identity matrix. The rows of the $n\times n+m$ matrix $(I_n|A)$ can be viewed as coordinates of points in general position of an $(n-1)$–space of $\mathrm{PG}(n+m-1,q)$. This subspace is denoted by $L(A)$. Hence the matrix A can be "lifted" to the $(n-1)$–space $L(A)$.

Here and in the sequel we denote by U_i the point of the ambient projective space represented by the vector having 1 in i–th position and 0 elsewhere; furthermore we denote by Σ the $(m-1)$–space of $\mathrm{PG}(n+m-1,q)$ containing $U_{n+1},\dots,U_{n+m}$. Note that if $A\in\mathcal{A}$, then $L(A)$ is disjoint from Σ. The following results are well known, see for instance [10, Theorem 12].

Proposition 1.

i) *If $\mathcal{A}$ is a $(3 \times m, 2)_q$ MRD–code, $m \geqslant 3$, then $\mathcal{X} = \{L(A) \mid A \in \mathcal{A}\}$ is a set of q^{2m} planes of $\mathrm{PG}(m+2, q)$ such that every line of $\mathrm{PG}(m+2, q)$ disjoint from Σ is contained in exactly one element of $\mathcal{X}$.*

ii) *If $\mathcal{A}$ is a $(4 \times m, 3)_q$ MRD–code, $m \geqslant 4$, then $\mathcal{X} = \{L(A) \mid A \in \mathcal{A}\}$ is a set of q^{2m} solids of $\mathrm{PG}(m+3, q)$ such that every line of $\mathrm{PG}(m+3, q)$ disjoint from Σ is contained in exactly one element of $\mathcal{X}$.*

iii) *If $\mathcal{A}$ is a $(4 \times m, 2)_q$ MRD–code, $m \geqslant 4$, then $\mathcal{X} = \{L(A) \mid A \in \mathcal{A}\}$ is a set of q^{3m} solids of $\mathrm{PG}(m+3, q)$ such that every plane of $\mathrm{PG}(m+3, q)$ disjoint from Σ is contained in exactly one element of $\mathcal{X}$.*

In [10, Theorem 15, Theorem 17], the author showed that it is possible to obtain a 2–covering design $\mathbb{C}_2(n+k-1, k, 2)$ or $\mathbb{C}_2(n+2, 4, 3)$ starting form a 2–covering design $\mathbb{C}_2(n, k, 2)$ or $\mathbb{C}_2(n, 4, 3)$, respectively. These results can be easily generalized for any q.

Theorem 2. *If there exists a q–covering design $\mathbb{C}_q(n, k, 2)$, $n \geqslant 6$, say $\mathbb{S}_n$, and a hyperplane Λ_n of $\mathrm{PG}(n-1, q)$ such that there are x_n $(k-1)$–spaces of $\mathbb{S}_n$ not contained in Λ_n and y_n $(k-1)$–spaces of $\mathbb{S}_n$ contained in Λ_n, then there exists a q–covering design $\mathbb{C}_q(n+k-1, k, 2)$, say $\mathbb{S}_{n+k-1}$, such that $|\mathbb{S}_{n+k-1}| = q^{2(n-1)} + \frac{q^k-1}{q-1} x_n + y_n$.*

Moreover there exists an $(n+k-3)$–space of $\mathrm{PG}(n+k-2, q)$, say Λ_{n+k-1}, such that there are $x_{n+k-1} = q^{2n-2} + q^{k-1} x_n$ $(k-1)$–spaces of $\mathbb{S}_{n+k-1}$ not contained in Λ_{n+k-1} and $y_{n+k-1} = \frac{q^{k-1}-1}{q-1} x_n + y_n$ $(k-1)$–spaces of $\mathbb{S}_{n+k-1}$ contained in Λ_{n+k-1}.

Proof. In $\mathrm{PG}(n+k-2, q)$, let Λ_n be the $(n-2)$–space $\langle U_{k+1}, \ldots, U_{n+k-1} \rangle$. Let $\mathcal{A}$ be a $(k \times (n-1), k-1)_q$ MRD–code and let $\mathcal{U} = \{L(A) \mid A \in \mathcal{A}\}$ be the set of $q^{2(n-1)}$ $(k-1)$–spaces of $\mathrm{PG}(n+k-2, q)$ obtained by lifting the matrices of $\mathcal{A}$. Let Π be the $(k-1)$–space $\langle U_1, \ldots, U_k \rangle$. Thus Π is disjoint from Λ_n. Let us fix a point $\bar{P}$ of Π. From the hypothesis there is a q–covering design $\mathbb{C}_q(n, k, 2)$ of $\langle \Lambda_n, \bar{P} \rangle$, say $\mathbb{S}_n$, such that $|\mathbb{S}_n| = x_n + y_n$ and y_n is the number of $(k-1)$–spaces of $\mathbb{S}_n$ contained in Λ_n.

Let $M \in \mathrm{GL}(k, q)$ such that the projectivities of $\mathrm{PGL}(k, q)$ induced by the matrices M^i, $1 \leqslant i \leqslant q^k - 1$, form a Singer cyclic group of $\mathrm{PGL}(k, q)$. Then the projectivities of $\mathrm{PGL}(n+k-1, q)$ associated with the matrices

$$\left(\begin{array}{c|c} M^{i(q-1)} & 0 \\ \hline 0 & I_{n-1} \end{array} \right), 1 \leqslant i \leqslant q^k - 1,$$

give rise to a subgroup C of $\mathrm{PGL}(n+k-1, q)$ having order $(q^k-1)/(q-1)$. In particular, the group C fixes pointwise Λ_n and permutes the points of Π in a single orbit. Hence, if $g, g' \in C$, $g \neq g'$, then $\mathbb{S}_n^g \cap \mathbb{S}_n^{g'}$ consists of the y_n members of $\mathbb{S}_n^g$ contained in Λ_n.

Let $\mathcal{V} = \bigcup_{g \in C} {\mathbb{S}_n}^g$. Observe that $\mathcal{U} \cup \mathcal{V}$ is a q–covering design $\mathbb{C}_q(n+k-1, k, 2)$. Indeed, from Proposition 1, every line of $\mathrm{PG}(n+k-2, q)$ disjoint from Λ_n is contained in exactly one element of $\mathcal{U}$. On the other hand, if r is a line of $\mathrm{PG}(n+k-2, q)$ meeting

Λ_n in at least a point, then r is contained in $\langle \Lambda_n, \bar{P}^g \rangle$, for some $g \in C$, and r is contained in at least an element of $\mathbb{S}_n^g$. Hence $\mathcal{U} \cup \mathcal{V}$ is a q–covering design $\mathbb{C}_q(n+k-1, k, 2)$. Note that $|\mathcal{U} \cup \mathcal{V}| = q^{2(n-1)} + \frac{q^k-1}{q-1} x_n + y_n$.

Let σ be a $(k-2)$–space of Π and let Λ_{n+k-1} be the hyperplane $\langle \Lambda_n, \sigma \rangle$ of $\mathrm{PG}(n+k-2, q)$. Since every $(k-1)$–space of $\mathcal{U}$ is disjoint from Λ_n, we have that no member of $\mathcal{U}$ is contained in Λ_{n+k-1}. The elements of $\mathcal{V}$ not contained in Λ_{n+k-1} are $(k-1)$–spaces of $\langle \Lambda_n, P \rangle$, for some point $P \in \Pi \setminus \sigma$, not contained in Λ_n. Hence there are

$$q^{2n-2} + q^{k-1} x_n$$

$(k-1)$–spaces of $\mathcal{U} \cup \mathcal{V}$ not contained in Λ_{n+k-1}. Finally note that the members of $\mathcal{U} \cup \mathcal{V}$ contained in Λ_{n+k-1} are $(k-1)$–spaces of $\langle \Lambda_n, P \rangle$, for some point $P \in \sigma$. Hence there are

$$\frac{q^{k-1}-1}{q-1} x_n + y_n$$

$(k-1)$–spaces of $\mathcal{U} \cup \mathcal{V}$ contained in Λ_{n+k-1}. □

Theorem 3. *Let $\mathbb{S}_n$ be a q–covering design $\mathbb{C}_q(2n, 4, 3)$, $n \geqslant 4$, such that there is a $(2n-3)$–space of $\mathrm{PG}(2n-1, q)$, say Λ_n, containing precisely α_n elements of $\mathbb{S}_n$ and every hyperplane of $\mathrm{PG}(2n-1, q)$ through Λ_n contains β_n members of $\mathbb{S}_n$. Then there exists a q–covering design $\mathbb{C}_q(2n+2, 4, 3)$, say $\mathbb{S}_{n+1}$, where*

$$|\mathbb{S}_{n+1}| = q^{6(n-1)} + (q^2+1)(q^2+q+1)|\mathbb{S}_n| - q(q+1)^2(q^2+1)\beta_n + q^3(q^2+q+1)\alpha_n.$$

Moreover there exists a $(2n-1)$–space of $\mathrm{PG}(2n+1, q)$, say Λ_{n+1}, containing $\alpha_{n+1} = |\mathbb{S}_n|$ elements of $\mathbb{S}_{n+1}$ and such that every hyperplane of $\mathrm{PG}(2n+1, q)$ through Λ_{n+1} contains β_{n+1} members of $\mathbb{S}_{n+1}$, where

$$\beta_{n+1} = (q^2+q+1)|\mathbb{S}_n| - (q^3+q^2+q)\beta_n + q^3\alpha_n.$$

Proof. Let Λ_n be the $(2n-3)$–space of $\mathrm{PG}(2n+1, q)$ generated by $U_5, \dots, U_{2n+2}$, let $\mathcal{A}$ be a $(4 \times (2n-2), 2)$ MRD–code and let $\mathcal{U}$ be the set of $q^{6(n-1)}$ solids obtained by lifting the matrices of $\mathcal{A}$. Let Π be the solid $\langle U_1, U_2, U_3, U_4 \rangle$. Thus Π is disjoint from Λ_n. From the hypothesis there is a line ℓ of Π and a q–covering design $\mathbb{C}_q(2n, 4, 3)$, say $\mathbb{S}_n$, of $\langle \Lambda_n, \ell \rangle$ such that α_n elements of $\mathbb{S}_n$ are contained in Λ_n and every $2(n-1)$–space of $\langle \Lambda_n, \ell \rangle$ through Λ_n contains β_n members of $\mathbb{S}_n$. Let $\bar{\mathcal{W}}$ be the set of $|\mathbb{S}_n| - \alpha_n$ solids of $\mathbb{S}_n$ not contained in Λ_n and let $\mathcal{Z}$ denote the α_n solids of $\mathbb{S}_n$ contained in Λ_n. For a point P of ℓ, there are β_n solids of $\mathbb{S}_n$ contained in $\langle \Lambda_n, P \rangle$, among which α_n are contained in Λ_n. Let $\bar{\mathcal{V}}$ be the set of solids of $\mathbb{S}_n$ not contained in none of the $2(n-1)$–spaces of $\langle \Lambda_n, \ell \rangle$ through Λ_n. Then $\bar{\mathcal{V}}$ consists of $|\mathbb{S}_n| - \beta_n - q(\beta_n - \alpha_n)$ solids and every plane of $\langle \Lambda_n, \ell \rangle$ intersecting Λ_n in one point is contained in at least one element of $\bar{\mathcal{V}}$. Note that $\bar{\mathcal{V}} \subset \bar{\mathcal{W}}$.

For a line ℓ' of Π, let $M_{\ell'} \in \mathrm{GL}(4, q)$ such that the projectivity of $\mathrm{PGL}(4, q)$ induced by the matrix $M_{\ell'}$, maps the line ℓ to the line ℓ'. Hence the projectivity $g_{\ell'}$ of $\mathrm{PGL}(2n+2, q)$ associated with the matrix

$$\left(\begin{array}{c|c} M_{\ell'} & 0 \\ \hline 0 & I_{2n-2} \end{array} \right),$$

sends $\mathbb{S}_n$ to a q–covering design $\mathbb{C}_q(2n, 4, 3)$ of $\langle \Lambda_n, \ell' \rangle$. Varying r among the lines of Π, we obtain a set G of $(q^2+1)(q^2+q+1)$ projectivities g_r of $\mathrm{PGL}(2n+2, q)$ and each of them fixes pointwise Λ_n. If r, r' are two distinct lines of Π, then $\langle \Lambda_n, r \rangle \cap \langle \Lambda_n, r' \rangle$ is at most a $2(n-1)$–space containing Λ_n; hence $|\bar{\mathcal{V}}^{g_r}| = |\bar{\mathcal{V}}^{g_{r'}}|$ and $|\bar{\mathcal{V}}^{g_r} \cap \bar{\mathcal{V}}^{g_{r'}}| = 0$. Let $\mathcal{S}$ be a line–spread of Π such that $\ell \in \mathcal{S}$. We have that if r, r' are two distinct lines of $\mathcal{S}$, then $|\bar{\mathcal{W}}^{g_r}| = |\bar{\mathcal{W}}^{g_{r'}}|$ and $|\bar{\mathcal{W}}^{g_r} \cap \bar{\mathcal{W}}^{g_{r'}}| = 0$. Denote by $\mathcal{V}$ the following set of solids:

$$\bigcup_{g_r \in G, r \notin \mathcal{S}} \bar{\mathcal{V}}^{g_r}$$

and by $\mathcal{W}$ the following set of solids:

$$\bigcup_{g_r \in G, r \in \mathcal{S}} \bar{\mathcal{W}}^{g_r}.$$

Let $\mathbb{S}_{n+1} = \mathcal{U} \cup \mathcal{V} \cup \mathcal{W} \cup \mathcal{Z}$. We claim that $\mathbb{S}_{n+1}$ is a q–covering design $\mathbb{C}_q(2n+2, 4, 3)$. Let π be a plane of $\mathrm{PG}(2n+1, q)$. If π is disjoint from Λ_n, then, from Proposition 1, there is a unique solid of $\mathcal{U}$ containing π. If π meets Λ_n in a point, then $\langle \Lambda_n, \pi \rangle$ is a $(2n-1)$–space meeting the solid Π in a line, say r. Then there is at least one solid of $\bar{\mathcal{V}}^{g_r}$ or of $\bar{\mathcal{W}}^{g_r}$ containing π, according as $r \notin \mathcal{S}$ or $r \in \mathcal{S}$, respectively. If π shares with Λ_n a line, then $\langle \Lambda_n, \pi \rangle$ is a $2(n-1)$–space meeting the solid Π in a point Q. Let ℓ' be the unique member of $\mathcal{S}$ containing Q; thus there is a solid of $\bar{\mathcal{W}}^{g_{\ell'}}$ containing π. Finally, if $\pi \subset \Lambda_n$, then there is at least a solid of $\bar{\mathcal{W}}^{g_\ell} \cup \mathcal{Z}$ containing π.

By construction it follows that

$$\begin{aligned} |\mathbb{S}_{n+1}| &= q^{6(n-1)} + (q^2+1)(q^2+q)\left(|\mathbb{S}_n| - \beta_n - q(\beta_n - \alpha_n)\right) + (q^2+1)(|\mathbb{S}_n| - \alpha_n) + \alpha_n \\ &= q^{6(n-1)} + (q^2+1)(q^2+q+1)|\mathbb{S}_n| - q(q+1)^2(q^2+1)\beta_n + q^3(q^2+q+1)\alpha_n. \end{aligned}$$

In order to complete the proof, set $\Lambda_{n+1} = \langle \Lambda_n, \ell \rangle$. The number of solids of $\mathbb{S}_{n+1}$ that are contained in Λ_{n+1} coincides with $|\mathbb{S}_n|$. Hence $\alpha_{n+1} = |\mathbb{S}_n|$. A hyperplane $\mathcal{H}$ of $\mathrm{PG}(2n+1, q)$ through Λ_{n+1} meets Π in a plane, say σ, where $\ell \subset \sigma$. Since the unique line of $\mathcal{S}$ contained in σ is ℓ, we have that the solids of $\mathbb{S}_{n+1}$ contained in $\mathcal{H}$ are either the solids of $\bar{\mathcal{W}}^{g_\ell}$ or the solids contained in

$$\bigcup_{r \textit{ line of } \sigma,\, r \neq \ell} \bar{\mathcal{V}}^{g_r},$$

or the image under $g_r \in G$ of the $\beta_n - \alpha_n$ solids of $\mathbb{S}_n$ contained in $\langle \Lambda_n, P \rangle$, where $P \in \ell$, $r \in \mathcal{S}$, $r \neq \ell$, and $P^{g_r} \in \sigma$.

Therefore

$$\begin{aligned} \beta_{n+1} &= (q^2+q)\left(|\mathbb{S}_n| - \beta_n - q(\beta_n - \alpha_n)\right) + q^2(\beta_n - \alpha_n) + |\mathbb{S}_n| \\ &= (q^2+q+1)|\mathbb{S}_n| - q(q^2+q+1)\beta_n + q^3\alpha_n. \end{aligned}$$ □

3 On $\mathcal{C}_q(2n, 3, 2)$

In this section we provide an upper bound on $\mathcal{C}_q(2n, 3, 2)$, $n \geqslant 4$. In [7], a constructive upper bound on $\mathcal{C}_q(6, 3, 2)$ has been given. In what follows we recall the construction and some of the properties of this q–covering design.

Construction 4. Let g be a Greek plane of $\mathcal{Q}^+(5, q)$. From [7, Lemma 2.2], there exists a set $\mathcal{X}$ of $q^6 - q^3$ planes disjoint from g and meeting $\mathcal{Q}^+(5, q)$ in a non–degenerate conic that, together with the set $\mathcal{Y}$ of $q^3 + q^2 + q$ Greek planes of $\mathcal{Q}^+(5, q)$ distinct from g, cover every line ℓ of $\mathrm{PG}(5, q)$ that is either disjoint from g or contained in $\mathcal{Q}^+(5, q) \setminus g$.

Let ℓ be a line of g. Through the line ℓ there pass $q-1$ planes meeting $\mathcal{Q}^+(5, q)$ exactly in ℓ and a unique Latin plane π. Varying the line ℓ over the plane g and considering the planes meeting $\mathcal{Q}^+(5, q)$ exactly in ℓ, we get a family $\mathcal{Z}$ of consisting of $(q-1)(q^2+q+1) = q^3 - 1$ planes. From [7, Lemma 2.3], every line that is tangent to $\mathcal{Q}^+(5, q)$ at a point of g is contained in exactly a plane of $\mathcal{Z}$.

Let P be a point of ℓ. Through the point P there pass q lines of π and q lines of g distinct from ℓ and contained in $\mathcal{Q}^+(5, q)$. Let S be the set of q^2 planes generated by a line of π through P distinct from ℓ and a line of g through P distinct from ℓ. Let C be a Singer cyclic group of the group $H \simeq \mathrm{PGL}(3, q)$. Here H is a subgroup of G stabilizing the plane g. Let $\mathcal{T}$ be the orbit of the set S under C. Then $\mathcal{T}$ consists of $q^2(q^2+q+1)$ planes and each of these planes has $2q + 1$ points in common with $\mathcal{Q}^+(5, q)$ on two intersecting lines of $\mathcal{Q}^+(5, q)$. From [7, Lemma 2.4], every line that is secant to $\mathcal{Q}^+(5, q)$ and has a point on g is contained in exactly one plane of $\mathcal{T}$.

Theorem 5 ([7, Theorem 2.5])**.** *The set $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z} \cup \mathcal{T}$ is a q–covering design $\mathbb{C}_q(6, 3, 2)$ of size $q^6 + q^4 + 2q^3 + 2q^2 + q - 1$.*

We will need the following result.

Theorem 6. *There exists a hyperplane Γ of $\mathrm{PG}(5, q)$ such that $q^3 + 2q^2 + q - 1$ elements of $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z} \cup \mathcal{T}$ are contained in Γ.*

Proof. Let Γ be a hyperplane of $\mathrm{PG}(5, q)$ containing g. Then Γ is a tangent hyperplane and contains the planes of $\mathcal{Q}^+(5, q)$ through a point R of g. In particular, there are q planes of $\mathcal{Y}$ contained in Γ. First of all observe that no plane of $\mathcal{X}$ is contained in Γ. Indeed, by way of contradiction, assume that a plane of $\mathcal{X}$ is contained in Γ. Then such a plane would meet g in at least a point, contradicting the fact that every plane of $\mathcal{X}$ is disjoint from g. A plane of $\mathcal{Z}$ that is contained in Γ has to contain the point R. On the other hand, the $q - 1$ planes of $\mathcal{Z}$, passing through a line of g which is incident with R, are contained in Γ. Hence there are $(q + 1)(q - 1) = q^2 - 1$ planes of $\mathcal{Z}$ contained in Γ. If π is a Latin plane contained in Γ, then $\pi \cap g$ is a line, say ℓ. By construction there is a point $P \in \ell$ such that the set $\mathcal{T}$ contains q^2 planes meeting π in a line through P and g in a line through P. Note that these q^2 planes of $\mathcal{T}$ are contained in Γ. It follows that there are $q^2(q + 1)$ planes of $\mathcal{T}$ contained in Γ. The result follows. □

Starting from the q–covering design $\mathbb{C}_q(6,3,2)$ of Theorem 5, Theorem 2 can be used recursively to obtain a q–covering design $\mathbb{C}_q(2n,3,2)$, $n \geqslant 4$, of size

$$q^2\theta_{2n-4,q^2} + q^{2n-3} - 1 + \sum_{i=2}^{n-1} \left(\theta_{4i-5,q} - \theta_{2i-4,q}\right).$$

In particular there is a hyperplane Γ of $\mathrm{PG}(2n-1,q)$ such that there are

$$q^{2n-3} + \sum_{j=0}^{n-2} q^{2(n+j-1)}$$

planes of $\mathbb{C}_q(2n,3,2)$ not contained in Γ and

$$(q+1)\left(\sum_{i=2}^{n-1}\left(q^{2i-3} + \sum_{j=0}^{i-2} q^{2(i+j-1)}\right)\right) - 1$$

planes of $\mathbb{C}_q(2n,3,2)$ contained in Γ.

Theorem 7. *If $n \geqslant 3$, then*

$$\mathcal{C}_q(2n,3,2) \leqslant q^2\theta_{2n-4,q^2} + q^{2n-3} - 1 + \sum_{i=2}^{n-1} \left(\theta_{4i-5,q} - \theta_{2i-4,q}\right).$$

4 On $\mathcal{C}_q(3n+8,4,2)$

In this section we provide an upper bound on $\mathcal{C}_q(3n+8,4,2)$, $n \geqslant 0$. We first deal with the case $n = 0$.

Construction 8. Let $\mathcal{A}$ be a $(4\times 4, 3)_q$ MRD–code and let $\mathcal{X} = \{L(A) \mid A \in \mathcal{A}\}$ be the set of q^8 solids of $\mathrm{PG}(7,q)$ obtained by lifting the matrices of $\mathcal{A}$. Let Σ' be the solid of $\mathrm{PG}(7,q)$ containing U_1, U_2, U_3, U_4. Then Σ' is disjoint from Σ. Let $\mathcal{S} = \{\ell_i \mid 1 \leqslant i \leqslant q^2+1\}$ be a line–spread of Σ, let $\mathcal{S}' = \{\ell'_i \mid 1 \leqslant i \leqslant q^2+1\}$ be a line–spread of Σ' and let $\mu : \ell'_i \in \mathcal{S}' \longmapsto \ell_i \in \mathcal{S}$ be a bijection. Let Γ_i denote the 5–space containing Σ and ℓ'_i, $1 \leqslant i \leqslant q^2+1$. If γ is a plane of Σ, then there are q^2+q solids of Γ_i meeting Σ exactly in γ. Let $\mathcal{Y}_i$ be the set of $q(q+1)^2$ solids of Γ_i (distinct from Σ) meeting Σ in a plane containing $\mu(\ell'_i)$. Let $\mathcal{Y} = \bigcup_{i=1}^{q^2+1} \mathcal{Y}_i$. Then $\mathcal{Y}$ consists of $q(q+1)^2(q^2+1)$ solids.

Theorem 9. *The set $\mathcal{X} \cup \mathcal{Y}$ is a q–covering design $\mathbb{C}_q(8,4,2)$ of size $q^8 + q(q+1)^2(q^2+1)$.*

Proof. Let r be a line of $\mathrm{PG}(7,q)$. If r is disjoint from Σ, then from Proposition 1, we have that r is contained in exactly one element of $\mathcal{X}$. If r meets Σ in one point, say P, then let Λ be the 4–space $\langle \Sigma, r\rangle$, let ℓ_j be the unique line of $\mathcal{S}$ containing P, let P' be the point $\Sigma' \cap \Lambda$ and let ℓ'_k be the unique line of $\mathcal{S}'$ containing P'. If $j = k$, then $P \in \ell_k$ and r is contained in the $q+1$ solids $\langle \alpha, r\rangle$ of $\mathcal{Y}$, where α is a plane of Σ containing ℓ_k. If $j \neq k$, then $P \notin \ell_k$. Let β be the plane of Σ containing ℓ_k and P. Then r is contained in $\langle \beta, r\rangle$, where $\langle \beta, r\rangle$ is a solid of $\mathcal{Y}$. Finally let r be a line of Σ, then r is contained in $q(q+1)^2$ solids of $\mathcal{Y}$. □

Remark 10. Let $\mathcal{L}$ be a Desarguesian line–spread of $\mathrm{PG}(7,q)$. There are $(q^4+1)(q^4+q^2+1)$ solids of $\mathrm{PG}(7,q)$ containing exactly q^2+1 lines of $\mathcal{L}$. If $\mathcal{Z}$ denotes the set of these solids, then it is not difficult to see that every line of $\mathrm{PG}(7,q)$ is contained in at least a solid of $\mathcal{Z}$. In [17, p. 221], K. Metsch posed the following question: "Is $(q^4+1)(q^4+q^2+1)$ the smallest cardinality of a set of 3–spaces of $\mathrm{PG}(7,q)$ that cover every line?" Theorem 9 provides a negative answer to this question.

Remark 11. When $q=2$, in the proof of [10, Theorem 13], the existence of a 2–covering design $\mathbb{C}_2(8,4,2)$ of size 346 has been shown.

Proposition 12. *There exists a hyperplane* Γ *of* $\mathrm{PG}(7,q)$ *such that precisely* $q(q+1)(2q+1)$ *members of* $\mathcal{X}\cup\mathcal{Y}$ *are contained in* Γ.

Proof. Let Γ be a hyperplane of $\mathrm{PG}(7,q)$ containing Σ. Then no element of $\mathcal{X}$ is contained in Γ, otherwise such a solid would meet Σ, contradicting the fact that every solid in $\mathcal{X}$ is disjoint from Σ. The hyperplane Γ intersects Σ' in a plane σ. The plane σ contains exactly one line of $\mathcal{S}'$, say ℓ'_k. Hence the $q(q+1)^2$ solids of $\mathcal{Y}$ meeting Σ in a plane through the line $\mu(\ell'_k)=\ell_k$ are contained in Γ. Let $\ell'_j\in\mathcal{S}'$, with $j\neq k$, then $\ell'_j\cap\sigma$ is a point, say R. In this case the $q+1$ solids generated by R and a plane of Σ through $\mu(\ell'_j)=\ell_j$ is contained in Γ. Since the elements of $\mathcal{Y}$ are those contained in the 5–space $\langle\Sigma,\ell'_i\rangle$, where $\ell'_i\in\mathcal{S}'$, and meeting Σ in a plane through ℓ_i, the proof is complete. □

As before, by using Theorem 2, the q–covering design of Theorem 9 can be used recursively to obtain a q–covering design $\mathbb{C}_q(3n+8,4,2)$, $n\geqslant 1$, of size

$$q^{3n+5}\theta_{n+1,q^3}+\sum_{i=0}^{n-1}(\theta_{6i+10,q}-\theta_{3i+4,q})+\sum_{i=0}^{n}\left(q^{3i+2}(2q^2-1)\right)+q(q+1)(2q+1).$$

In particular, there exists a hyperplane Γ of $\mathrm{PG}(3n+7,q)$ such that there are

$$q^{3n+2}(2q^2-1)+\sum_{j=0}^{n+1}q^{3(n+j)+5}$$

solids of $\mathbb{C}_q(3n+8,4,2)$ not contained in Γ and

$$(q^2+q+1)\left(\sum_{i=0}^{n-1}\left(q^{3i+2}(2q^2-1)+\sum_{j=0}^{i+1}q^{3(i+j)+5}\right)\right)+q(q+1)(2q+1)$$

solids of $\mathbb{C}_q(3n+8,4,2)$ contained in Γ.

Theorem 13. *If* $n\geqslant 0$, *then*

$$\mathcal{C}_q(3n+8,4,2)\leqslant q^{3n+5}\theta_{n+1,q^3}+\sum_{i=0}^{n-1}(\theta_{6i+10,q}-\theta_{3i+4,q})+\sum_{i=0}^{n}\left(q^{3i+2}(2q^2-1)\right)+q(q+1)(2q+1).$$

5 On $\mathcal{C}_q(2n, 4, 3)$

The main goal of this section is to give an upper bound on $\mathcal{C}_q(2n, 4, 3)$, $n \geqslant 4$. We begin by providing a construction in the case $n = 4$.

Construction 14. Let $\mathcal{A}$ be a $(4 \times 4, 2)_q$ MRD–code and let $\mathcal{X} = \{L(A) \mid A \in \mathcal{A}\}$ be the set of q^{12} solids of $\mathrm{PG}(7, q)$ obtained by lifting the matrices of $\mathcal{A}$. Let Σ' be the solid of $\mathrm{PG}(7, q)$ containing U_1, U_2, U_3, U_4. Then Σ' is disjoint from Σ. Let $\mathcal{P} = \{\mathcal{S}_i \mid 1 \leqslant i \leqslant q^2 + q + 1\}$ be a 1–parallelism of Σ, let $\mathcal{P}' = \{\mathcal{S}'_i \mid 1 \leqslant i \leqslant q^2 + q + 1\}$ be a 1–parallelism of Σ' and let $\mu : \mathcal{S}'_i \in \mathcal{P}' \longmapsto \mathcal{S}_i \in \mathcal{P}_i$ be a bijection. For a line ℓ' of Σ', let $\Gamma_{\ell'}$ denote the 5–space containing Σ and ℓ'. Since $\mathcal{P}'$ is a 1–parallelism of Σ', there exists a unique j, with $1 \leqslant j \leqslant q^2 + q + 1$, such that $\ell' \in \mathcal{S}'_j$. Note that $\mu(\mathcal{S}'_j) = \mathcal{S}_j$ is a line–spread of Σ. Let ℓ be a line of $\mathcal{S}_j$ and let $\mathcal{Y}_\ell$ be the set of q^4 solids of $\Gamma_{\ell'}$ (distinct from Σ) meeting Σ exactly in ℓ. Let $\mathcal{Z}_{\ell'} = \bigcup_{\ell \in \mathcal{S}_j} \mathcal{Y}_\ell$. Then $\mathcal{Z}_{\ell'}$ consists of $q^4(q^2 + 1)$ solids. Varying ℓ' among the lines of Σ', we get a set

$$\mathcal{Z} = \bigcup_{\ell' \textit{ line of } \Sigma'} \mathcal{Z}_{\ell'}$$

consisting of $q^4(q^2 + 1)^2(q^2 + q + 1)$ solids.

Theorem 15. *The set $\mathcal{X} \cup \mathcal{Z} \cup \{\Sigma\}$ is a q–covering design $\mathbb{C}_q(8, 4, 3)$ of size $q^{12} + q^4(q^2 + 1)^2(q^2 + q + 1) + 1$.*

Proof. Let π be a plane of $\mathrm{PG}(7, q)$. If π is disjoint from Σ, then, from Proposition 1, we have that π is contained in exactly one element of $\mathcal{X}$. If π meets Σ in one point, say P, then let Λ be the 5–space $\langle \Sigma, \pi \rangle$ and let ℓ' be the line of Σ' obtained by intersecting Σ' with Λ. Note that $\Lambda = \Gamma_{\ell'}$. Let $\mathcal{S}'_j$ be the unique line–spread of $\mathcal{P}'$ containing ℓ'. Then there exists a unique line ℓ of $\mathcal{S}_j = \mu(\mathcal{S}'_j)$ such that $P \in \ell$ and π is contained in $\langle \pi, \ell \rangle$, that is a solid of $\mathcal{Z}$. If π meets Σ in a line, say r, then let $\mathcal{S}_k$ be the unique line–spread of $\mathcal{P}$ containing r and let Λ be the 4–space $\langle \Sigma, \pi \rangle$. Then $\Lambda \cap \Sigma'$ is a point, which belongs to a unique line, say r', of the line–spread $\mu^{-1}(\mathcal{S}_k) = \mathcal{S}'_k$ of $\mathcal{P}'$. Since there are q^2 solids of $\Gamma_{r'}$ meeting Σ exactly in r and containing π, we have that in this case π is contained in q^2 members of $\mathcal{Z}$. Finally if π is a plane of Σ, then π is contained in Σ. □

Remark 16. Note that, as regard as the case $q = 2$, in the proof of [10, Theorem 16] the author exhibited a 2–covering design $\mathbb{C}_2(8, 4, 3)$ of size 6897.

Proposition 17. *There exists a 5–space Λ of $\mathrm{PG}(7, q)$ containing exactly $q^4(q^2 + 1) + 1$ members of $\mathcal{X} \cup \mathcal{Z} \cup \{\Sigma\}$. Moreover every hyperplane of $\mathrm{PG}(7, q)$ through Λ contains precisely $q^4(q^2 + 1)(q^2 + q + 1) + 1$ solids of $\mathcal{X} \cup \mathcal{Z} \cup \{\Sigma\}$.*

Proof. Let Λ be a 5–space containing Σ. Then Λ meets Σ' in a line, say r, and $\Lambda = \langle \Sigma, r \rangle$. The line r belongs to a unique line–spread $\mathcal{S}'_i$ of the 1–parallelism $\mathcal{P}'$ of Σ'. Then $\mu(\mathcal{S}'_i) = \mathcal{S}_i$ is a line–spread belonging to the 1–parallelism $\mathcal{P}$ of Σ. The $q^4(q^2 + 1)$ solids of $\mathcal{Z}$ lying in $\langle \Sigma, r \rangle$ meet Σ in a line of $\mathcal{S}_i$ and are contained in Λ. Let s be a line of Σ' such that $s \neq r$. In this case none of the $q^4(q^2 + 1)$ solids of $\mathcal{Z}$ lying in $\langle \Sigma, s \rangle$ is contained in Λ. Indeed, assume by contradiction that there is a solid Δ contained in $\Lambda \cap \langle \Sigma, s\rangle$, then $\Delta \subset \langle \Sigma, s \cap r\rangle$ and hence $\Delta \cap \Sigma$ is a plane of Σ, contradicing the fact that every solid of $\mathcal{Z}$ meets Σ in a line. On the other hand, no solid of $\mathcal{X}$ is contained in Λ, otherwise such a solid would meet Σ not trivially. Finally note that Σ is a solid of Λ.

Let Γ be a hyperplane of $\mathrm{PG}(7, q)$ through Λ. Then $\Gamma \cap \Sigma'$ is a plane, say σ, containing the line r. Repeating the previous argument for every line of the plane σ, it turns out that there are $q^4(q^2 + 1)(q^2 + q + 1)$ solids of $\mathcal{Z}$ in Γ, as required. □

Let $\mathbb{S}_4$ denotes $\mathcal{X} \cup \mathcal{Z} \cup \{\Sigma\}$. As in the previous sections, $\mathbb{S}_4$ can be used as a basis for a recursive construction of a q–covering designs $\mathbb{C}_q(2n, 4, 3)$, $n \geqslant 5$.

Theorem 18.

$$\mathcal{C}_q(8, 4, 3) \leqslant q^{12} + q^4(q^2 + 1)^2(q^2 + q + 1) + 1$$

$$\mathcal{C}_q(10, 4, 3) \leqslant q^{18} + q^4(q^2 + 1)(q^2 + q + 1)(q^8 + q^6 + q^4 + q^3 + q^2 + 1) + 1$$

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References

[1] A. Beutelspacher. On parallelisms in finite projective spaces. *Geom. Dedicata*, 3:35–40, 1974.

[2] A. Beutelspacher. On t–covers in finite projective spaces. *J. Geom.*, 12(1):10–16, 1979.

[3] A. Beutelspacher and J. Ueberberg. A characteristic property of geometric t–spreads in finite projective spaces. *European J. Combin.*, 12(4):277–281, 1991.

[4] A. Blokhuis, P. Sziklai and T. Szőnyi. *Blocking sets in projective spaces.* In *Current Research Topics in Galois Geometry*, Nova Science Publishers, Inc., pp. 63–86, 2014.

[5] R. C. Bose and R. C. Burton. A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the MacDonald codes. *J. Combinatorial Theory*, 1:96–104, 1966.

[6] M. Braun, T. Etzion, P. R. J. Östergård, A. Vardy, and A. Wassermann. Existence of q–analogs of Steiner systems. *Forum Math. Pi*, 4 e7:14 pp., 2016.

[7] A. Cossidente and F. Pavese. A note on q–covering designs in $\mathrm{PG}(5, q)$. *J. Combin. Des.*, 24:383–388, 2016.

[8] P. Delsarte. Bilinear forms over a finite field, with applications to coding theory. *J. Combin. Theory Ser. A*, 25:226–241, 1978.

[9] J. Eisfeld and K. Metsch. Blocking s–dimensional subspaces by lines in $\mathrm{PG}(2s, q)$. *Combinatorica*, 17(2):151–162, 1997.

[10] T. Etzion. Covering of subspaces by subspaces. *Des. Codes Cryptogr.*, 72:405–421, 2014.

[11] T. Etzion and A. Vardy. On q–Analogs for Steiner Systems and Covering Designs. *Adv. Math. Commun.*, 5:161–176, 2011.

[12] E. M. Gabidulin. Theory of codes with maximum rank distance. *Problems of Information Transmission*, 21:1–12, 1985.

[13] J. W. P. Hirschfeld. *Projective Geometries over Finite Fields.* Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1998.

[14] J. W. P. Hirschfeld. *Finite projective spaces of three dimensions.* Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1985.

[15] J. W. P. Hirschfeld and J. A. Thas. *General Galois Geometries.* Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1991.

[16] R. Koetter and F. R. Kschischang. Coding for errors and erasures in random network coding. *IEEE Trans. on Inform. Theory*, 54:3579–3591, 2008.

[17] K. Metsch. Blocking sets in projective spaces and polar spaces. *J. Geom.*, 76:216–232, 2003.

[18] K. Metsch. Blocking subspaces by lines in $\mathrm{PG}(n, q)$. *Combinatorica*, 24:459–486, 2004.

[19] R. M. Roth. Maximum–rank array codes and their application to crisscross error correction. *IEEE Trans. Inform. Theory*, 37:328–336, 1991.

[20] B. Segre. Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane. *Ann. Mat. Pura Appl.*, 64:1–76, 1964.

[21] J. Sheekey. MRD Codes: Constructions and Connections. In *Combinatorics and finite fields: Difference sets, polynomials, pseudorandomness and applications. Radon Series on Computational and Applied Mathematics*, K.-U. Schmidt and A. Winterhof (eds.), De Gruyter, 2019.

[22] D. Silva, F. R. Kschischang, and R. Koetter. A rank–metric approach to error control in random network coding. *IEEE Trans. Inform. Theory*, 54:3951–3967, 2008.