The classification of homogeneous finite-dimensional permutation structures

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Abstract

We classify the homogeneous finite-dimensional permutation structures, i.e. homogeneous structures in a language of finitely many linear orders, giving a nearly complete answer to a question of Cameron, and confirming the classification conjectured by the first author. The primitive case was proven by the second author using model-theoretic methods, and those methods continue to appear here.

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1 Introduction

A countable relational structure is *homogeneous* if every finite partial automorphism extends to a total automorphism. This notion was introduced by Fraïssé to generalize the behavior of the rational order, which is the unique homogeneous linear order. (For the reader unfamiliar with amalgamation and Fraïssé limits, we refer to [4]*§2. For far more information, see [10].) Beginning with the case of partial orders [11], a program of classifying homogeneous structures in particular languages developed, which has included graphs [9], tournaments [8], directed graphs [5], and ongoing work on metrically homogeneous graphs [6].

Along this line, in [4] Cameron classified the homogeneous permutations, which he identified with homogeneous structures consisting of two linear orders. He then posed the problem of classifying, for each n, the homogeneous structures consisting of n linear orders, which we call *homogeneous n-dimensional permutation structures*.

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A construction producing many new imprimitive examples of such structures was introduced in [1]. The structures produced by a slight generalization of that construction, making use of the subquotient orders introduced in §2 in place of linear orders, were put forward as a conjecturally complete catalog in [2], which confirmed the case of 3 linear orders. Here, we confirm the completeness of that catalog as a whole.

Theorem 1. Γ *is a homogeneous finite-dimensional permutation structure iff there is a finite distributive lattice* Λ *such that* Γ *is interdefinable with an expansion of the generic* Λ*-ultrametric space by generic subquotient orders, such that every meet-irreducible of* Λ *is the bottom relation of some subquotient order.*

The primitive case, in which there is no $\varnothing$-definable equivalence relation, is foundational for proving the completeness of the catalog. The Primitivity Conjecture of [1] conjectured that, modulo the agreement of certain orders up to reversal, a primitive homogeneous finite-dimensional permutation structure is the Fraïssé limit of all finite n-dimensional permutation structures, for some n. In the case of 2 [4] and 3 [2] linear orders, the conjecture was proven by increasingly involved direct amalgamation arguments. A description of the ways linear orders can interact in certain ω-categorical structures, as well as of the closed sets $\varnothing$-definable in products of such structures, was given in [12], and as an application of these model-theoretic results, the Primitivity Conjecture was confirmed.

After reviewing the catalog and the relevant results of [12], our proof breaks into two sections. First, we examine the lattice of $\varnothing$-definable equivalence relations of a homogeneous finite-dimensional permutation structure, and in particular prove that each meet-irreducible element is convex with respect to some linear order in the language and that the reduct to the language of equivalence relations remains homogeneous. In the next section, we complete the classification by proving a finite-dimensional permutation structure may be presented in a language in which all the subquotient orders are generic.

Despite the fact that the catalog gives a simple description of *all* finite-dimensional permutation structures, it is difficult to determine the corresponding catalog for a *fixed number* of linear orders. This is because it is not known what lattices of $\varnothing$-definable equivalence relations can be realized with a given number of orders, nor is it true that one needs at most n orders to represent a structure with at most 2^n 2-types. For some discussion and results regarding these problems, see [3]*§4.4.

Problem 2. Given a lattice Λ, what is the minimal n such that Λ is isomorphic to the lattice of $\varnothing$-definable equivalence relations of some homogeneous n-dimensional permutation structure?

Given a homogeneous finite-dimensional permutation structure Γ presented in a language of equivalence relations and subquotient orders, what is the minimal n such that Γ is quantifier-free interdefinable with an n-dimensional permutation structure?

For the following proposition, see [3]*Corollary 4.4.3 for the upper bound and Corollary 39 for the lower bound.

Proposition 3. *Let Λ be a finite distributive lattice, Λ_0 the poset of meet-irreducibles of $\Lambda \setminus \{ \mathbb{0}, \mathbb{1} \}$, $\mathcal{L}$ a set of chains covering Λ_0, and ℓ the minimum size of any such $\mathcal{L}$. Let d_Λ be the minimum dimension of a homogeneous permutation structure with lattice of $\varnothing$-definable equivalence relations isomorphic to Λ. Then*

$$2\ell \leqslant d_\Lambda \leqslant |\mathcal{L}| + \sum_{L \in \mathcal{L}} \lceil \log_2(|L| + 1) \rceil .$$

However, neither bound describes the true behavior of d_Λ. To exceed the lower bound, let Λ be a chain. To beat the upper bound, the lattice consisting of the sum of the free boolean algebra on 2 atoms and a single point may be achieved using only 4 orders.

Although we use model-theoretic terminology throughout this paper, in the setting of a homogeneous structure M, many of these notions have equivalent presentations. In particular, an *(n)-type* is an orbit of the action of the automorphism group $Aut(M)$ on M^n. Equivalently (because of the homogeneity assumption), it is an isomorphism type of n labeled points. A subset $X \subset M^k$ is *definable* over $A \subset M$ (or A-definable) if the pointwise stabilizer of A in $Aut(M)$ fixes X setwise. In particular, $\varnothing$-definability is equivalent to $Aut(M)$-invariance. An element $c \in M$ is definable from a set A if the singleton $\{c\}$ is A-definable. If $E \subseteq M^2$ is a $\varnothing$-definable equivalence relation, then those notions carry through naturally to the quotient M/E.

2 The catalog

For proofs and further discussion of the results presented in this section, see [3]*Chap. 3.

Definition 4. Let M be a structure, equipped with an equivalence relation E and linear order $\leqslant$. Then we say that *E is $\leqslant$-convex*, or sometimes that *$\leqslant$ is E-convex*, if every E-class is convex with respect to $\leqslant$.

The construction of the structures in the catalog proceeds roughly as follows. One starts with a finite distributive lattice Λ and constructs the generic object with a lattice of $\varnothing$-definable equivalence relations isomorphic to Λ. This structure is then expanded by linear orders so that every $\varnothing$-definable equivalence relation is convex with respect to at least one $\varnothing$-definable order and the equivalence relations are then interdefinably replaced by additional linear orders.

However, we do not work directly with linear orders, but rather with certain partial orders which we call *subquotient orders*, which allow our expansion to be generic in a natural sense.

Definition 5. Let X be a structure, and $E \leqslant F$ equivalence relations on X. A *subquotient order from E to F* is a partial order on X/E in which two E-classes are comparable if and only if they lie in the same F-class (note, this pulls back to a partial order on X). Thus, this partial order provides a linear order of C/E for each $C \in X/F$. We call E the *bottom relation* and F the *top relation* of the subquotient order.

We say that a subquotient order $<$ from E to F is G-convex if E refines G and the projection to X/E of any G-class is $<$-convex.

Note a linear order is a subquotient order from $\mathbb{0}$ (equality) to $\mathbb{1}$ (the trivial relation). Starting with a linear order $\leqslant$ convex with respect to E and possibly additional equivalence relations, it can be interdefinably exchanged for its restriction within E-classes, a subquotient order from $\mathbb{0}$ to E, and the order it induces between E-classes, a subquotient order from E to $\mathbb{1}$. This process may then be iterated on the resulting subquotient orders until all convexity conditions are removed.

Further, instead of working directly in the language of equivalence relations, we find it convenient to work in the language of Λ-ultrametric spaces.

Definition 6. Let Λ be a lattice. A Λ-*ultrametric space* is a metric space where the metric takes values in Λ and the triangle inequality uses the join rather than addition.

The following proposition shows that Λ-ultrametric spaces are equivalent to structures equipped with a lattice of equivalence relations isomorphic to Λ, or to substructures of such structures. While the lattice of equivalence relations may collapse when passing to a substructure, such as a single point, Λ-ultrametric spaces have the benefit of keeping Λ fixed under passing to substructures.

Proposition 7 ([3]*Theorem 3.3.2)**.** *Fix a finite lattice Λ. Let $\mathcal{M}_\Lambda$ be the category of Λ-ultrametric spaces, with isometries as morphisms. Let $\mathcal{EQ}_\Lambda$ be the category of structures consisting of a set equipped with a family of not-necessarily-distinct equivalence relations $\{ E_\lambda \mid \lambda \in \Lambda \}$ satisfying the following conditions, with embeddings as morphisms.*

1. *$\{ E_\lambda \}$ forms a lattice.*

2. *The map $L\colon \lambda \mapsto E_\lambda$ is meet-preserving. In particular, if $\lambda_1 \leqslant \lambda_2$, then $E_{\lambda_1} \leqslant E_{\lambda_2}$.*

3. *$E_{\mathbb{0}}$ is equality and $E_{\mathbb{1}}$ is the trivial relation.*

Then $\mathcal{EQ}_\Lambda$ is isomorphic to $\mathcal{M}_\Lambda$. Furthermore, the functors of this isomorphism preserve homogeneity.

Given a system of equivalence relations as specified above, we get the corresponding Λ-ultrametric space by taking the same universe and defining $d(x,y) = \bigwedge \{ \lambda \in \Lambda \mid xE_\lambda y \}$. In the reverse direction, given a Λ-ultrametric space, we get the corresponding structure of equivalence relations by taking the same universe and defining $E_\lambda = \{ (x,y) \mid d(x,y) \leqslant \lambda \}$. As we wish to work in a finite relational language, we will usually consider our Λ-ultrametric spaces to be presented using a relation for each possible distance.

The next proposition explains the special status of distributive lattices.

Proposition 8 ([3]*Proposition 3.3.5, Corollary 5.2.6)**.** *Let Λ be a finite lattice. Then the class of all finite Λ-ultrametric spaces forms an amalgamation class if and only if Λ is distributive.*

The following theorem states that we may take the generic Λ-ultrametric space and expand it by the natural analogue for subquotient orders of generic linear orders.

Theorem 9 ([3]*Theorem 4.2.3). *Let Λ be a finite distributive lattice. Let $\mathcal{A}^*$ be the class of finite structures $(A, d, \{<_{E_i}\}_{i=1}^n)$ satisfying the following conditions:*

- *(A, d) is a Λ-ultrametric space;*
- *$<_{E_i}$ is a subquotient order with bottom relation E_i, for some meet-irreducible $E_i \in \Lambda$, and top relation $F_i \in \Lambda$.*

Then $\mathcal{A}^*$ is an amalgamation class.

Definition 10. Given a finite distributive lattice Λ, the *generic Λ-ultrametric space* Γ is the Fraïssé limit of all finite Λ-ultrametric spaces.

Suppose Γ^* is Γ equipped with some subquotient orders. We will say those subquotient orders are *generic* if Γ^* may be constructed as a Fraïssé limit of a class $\mathcal{A}^*$ as from Theorem 9.

Remark 11. The condition that the bottom relation of a generic subquotient order be meet-irreducible is analogous to the condition that for a Fraïssé class to be expandable by a generic linear order, it must have strong amalgamation. For if E is meet-irreducible, then our amalgamation procedure from Proposition 8 never forces the identification of E-classes. However, if $E = F \wedge F'$, then any E-class is the unique intersection of some F-class with some F'-class, so amalgamation may force the identification of E-classes.

Although the structures produced by Theorem 9 are presented in the language of equivalence relations and subquotient orders, we now give a sufficient condition for them to be representable in a language of linear orders.

Proposition 12 ([3]*Proposition 3.4.13). *Let $\mathcal{A}^*$ be a class as from Theorem 9, such that every meet-irreducible of Λ is the bottom relation of some subquotient order. Then the Fraïssé limit of $\mathcal{A}^*$ is interdefinable with a finite-dimensional permutation structure.*

Finally, the structures in our catalog are constructed as follows. Let Λ be a finite distributive lattice. Take the generic Λ-ultrametric space, and expand by generic subquotient orders with meet-irreducible bottom relation, such that every meet-irreducible of Λ is the bottom relation of at least one subquotient order.

3 Linear orders in ω-categorical structures

In this section, we review material from [12] leading to the proof of the Primitivity Conjecture, as well as introducing definitions and results that will be used later. For proofs and further discussion of the results presented in this section, see [12], particularly §3.

Notation 13. Throughout this section, we will assume that $(V; \leqslant, \cdots)$ is a $\varnothing$-definable substructure of an ω-categorical structure, equipped with a distinguished $\varnothing$-definable linear order $\leqslant$, and possibly other $\varnothing$-definable structure. Similarly for $(V_i; \leqslant_i, \cdots)$.

We first define the sorts of linear orders we will be concerned with, and the ways they can interact.

Definition 14. We say $(V; \leqslant, \cdots)$ is *weakly transitive* if it is dense and the set of realizations of any 1-type $p(x)$ over $\varnothing$ concentrating on V is dense in V.

We say $(V; \leqslant, \cdots)$ has *topological rank 1* if it does not admit any parameter-definable $\leqslant$-convex equivalence relation with infinitely many infinite classes.

Finally, $(V; \leqslant, \cdots)$ is *minimal* if it is weakly transitive and has topological rank 1.

Definition 15. By a *cut* in a dense order $(V, \leqslant)$, we mean an initial segment of it which is neither empty nor the whole of V and has no last element. We denote by $\overline{V}$ the set of parameter-definable cuts of V.

Definition 16. We say two $\varnothing$-definable weakly transitive orders $(V_0; \leqslant_0, \cdots)$ and $(V_1; \leqslant_1, \cdots)$ are *intertwined* if there is a $\varnothing$-definable non-decreasing map $f \colon V_0 \to \overline{V_1}$.

If $(V_0; \leqslant_0, \cdots)$ and $(V_1; \leqslant_1, \cdots)$ are minimal, we say they are *independent* if V_0 is intertwined with neither V_1 nor its reverse.

The definition of independence in [12] is on the face of it stronger, however Lemma 3.19 of that paper states that for minimal orders, independence is equivalent to the definition that we give here. The stronger property will be useful for us though, and we record it in the following lemma.

Lemma 17 (follows from [12]*Lemma 3.19)**.** *Let $(V_0; \leqslant_0, \cdots)$,$(V_1; \leqslant_1, \cdots)$ be minimal independent linear orders. Let X_0, X_1 be infinite A-definable subsets of V_0, V_1 respectively, transitive over A. Then X_0, X_1 are independent over A.*

The following proposition is a special case of Proposition 3.23 in [12].

Proposition 18. *Let $(M; \leqslant_1, \ldots, \leqslant_n, \cdots)$ be ω-categorical, transitive, with each $\leqslant_i$ a linear order of topological rank 1. Assume that no two distinct orders are intertwined. Then the reduct of M to the language $L_0 = \{ \leqslant_1, \ldots, \leqslant_n \}$ is completely determined up to isomorphism by whether, $\leqslant_i, \leqslant_j$ are equal, reverse of each other, or independent, for any $i, j \leqslant n$.*

To apply this proposition, we need to know that a primitive finite-dimensional permutation structure has topological rank 1.

Definition 19. We say an ω-categorical structure is *binary* if it eliminates quantifiers in a finite binary relational language.

Definition 20. We say $(V; \leqslant, \cdots)$ is *topologically primitive* if it does not admit any proper $\varnothing$-definable $\leqslant$-convex equivalence relation besides equality.

Lemma 21 ([12]*Lemma 7.3)**.** *Let $(M; \leqslant, \cdots)$ be a binary structure which is topologically primitive. Then $(M; \leqslant, \cdots)$ has topological rank 1.*

The proof of Lemma 21 uses the following result, which is a special case of [12]*Lemma 7.1.

Lemma 22. *Let M be a binary structure. Then we cannot find a sequence $(F_k)_{k<\omega}$ of uniformly parameter-definable equivalence relations and a sequence $M \supset C_0 \supset C_1 \supset \cdots$ such that each C_k is an F_k-class which splits into infinitely many F_{k+1}-classes.*

From those results, one obtains the following theorem confirming the Primitivity Conjecture.

Theorem 23 ([12]*Theorem 7.4)**.** *Let $(\Gamma; \leqslant_1, \ldots, \leqslant_n)$ be a primitive homogeneous finite-dimensional permutation structure such that no two orders are equal or opposite of each other. Then Γ is the Fraïssé limit of all finite sets equipped with n orders.*

Finally, we close with several results that will also be used later. The first two propositions describe the closed sets $\varnothing$-definable in a minimal linear order and then in a product of pairwise independent linear orders.

Proposition 24 ([12]*Proposition 3.11)**.** *Let $(V; \leqslant, \cdots)$ be a minimal definable linear order. Let $p(x_0, \ldots, x_{n-1})$ be a type in V^n such that $p \vdash x_0 < x_1 < \ldots < x_{n-1}$. Then given open $\leqslant$-intervals $I_0 < \cdots < I_{n-1}$ of V, we can find $a_i \in I_i$ such that $(a_0, \ldots, a_{n-1}) \vDash p$.*

Lemma 25 ([12]*Lemma 3.1)**.** *Let $(V, \leqslant, \ldots)$ be infinite and transitive. Then $\leqslant$ is dense, and for any $a \in V$, $\operatorname{acl}(a) = \{ a \}$.*

Proposition 26 ([12]*Proposition 3.21)**.** *Choose pairwise independent minimal orders. $(V_0; \leqslant_0, \cdots), \ldots, (V_{n-1}; \leqslant_{n-1}, \cdots)$. Then any $\varnothing$-definable closed set $X \subseteq V_0^{k_0} \times \cdots \times V_{n-1}^{k_{n-1}}$ is a finite union of products of the form $D_0 \times \cdots \times D_{n-1}$, where each D_i is a $\varnothing$-definable closed subset of $V_i^{k_i}$.*

Proposition 27 ([12]*Proposition 6.1)**.** *Assume that M is NIP and binary. Let $X, Y \subset M$ be $\varnothing$-definable, and let $p(x, y)$ be the complete type of a thorn-independent pair from $X \times Y$. Let $(V; \leqslant, \cdots)$ be a $\varnothing$-definable minimal linear order. and let $f\colon p(X \times Y) \to V$ be a $\varnothing$-definable function. Then for any $(a, b) \vDash p$, $f(a, b) \in \operatorname{dcl}(a) \cup \operatorname{dcl}(b)$.*

In particular, for any $A \subset V$, $\operatorname{dcl}(A) \cap V = A$.

Remark 28. A definition of thorn-independence in our setting may be found in [12]*§2.3. However, we will only need the following three facts:

1. We can always find a thorn-independent pair in a product of $\varnothing$-definable sets.

2. If (a, b) is a thorn-independent pair and E is a $\varnothing$-definable equivalence relation with infinitely many classes, then a and b are in different E-classes.

3. If (a, b) is a thorn-independent pair, $a' \in \operatorname{dcl}(a)$ and $b' \in \operatorname{dcl}(b)$, then (a', b') is a thorn-independent pair.

For applications of Proposition 27, note that a homogeneous finite-dimensional permutation structure is NIP, as it has quantifier elimination, NIP is preserved by boolean combinations, and "$x \leqslant y$" is NIP.

4 The lattice of $\varnothing$-definable equivalence relations

In this section, we investigate the $\varnothing$-definable equivalence relations of a homogeneous finite-dimensional permutation structure. The main result for the first subsection, Lemma 36, is that each meet-irreducible element of the lattice is convex with respect to some linear order in the language, and that of the next subsection, Proposition 45, is that the reduct to the language of $\varnothing$-definable equivalence relations is generic.

Some lemmas will be proven in a more general setting, so we introduce the following definition.

Definition 29. A structure M is *order-like* if for any complete type $p(x, y)$ in two variables over $\varnothing$, we have $p(x, y) \wedge p(y, z) \to p(x, z)$.

Note that a homogeneous finite-dimensional permutation structure is order-like. Conversely, we do not know if every transitive, order-like, binary NIP structure is (bi-definable with) a finite-dimensional permutation structure.

Notation 30. For this section, $\leqslant$ will denote a linear order, as will $\leqslant_i$.

4.1 Convexity

We first establish an analogue of Lemma 21 when working in the quotient of a binary structure.

Lemma 31. *Let $(M; \leqslant, \cdots)$ be ω-categorical and binary. Let E be the coarsest proper $\varnothing$-definable $\leqslant$-convex equivalence relation. Then $(M/E; \leqslant, \cdots)$ has topological rank 1.*

Proof. The proof is an adaptation of that of [12]*Lemma 7.3. Assume that there is a proper parameter-definable $\leqslant$-convex equivalence relation F on $(M/E; \leqslant, \cdots)$. Let F be defined over $\bar{a}$ and write $F = F_{\bar{a}}$. Let $R(x, y)$ be the relation on M/E which holds of a pair (c, d) if for every $\bar{b}$ having the same type as $\bar{a}$ over $\varnothing$, there are finitely many $F_{\bar{b}}$-equivalence classes between c and d. Then R is $\varnothing$-definable and is a $\leqslant$-convex equivalence relation. By the maximality assumption on E, R is equality. Then for any $F_{\bar{a}}$-class C, there is $\bar{b} \equiv \bar{a}$ such that C splits into infinitely many $F_{\bar{b}}$-classes. Let $F_0 = F$, $C_0 = C$, $F_1 = F_{\bar{b}}$, and let C_1 be any F_1-class inside C_0. We can iterate the construction to obtain a sequence $(F_k)_{k<\omega}$ of equivalence relations on M/E and a decreasing sequence $(C_k)_{k<\omega}$ such that each C_k is an F_k-class that splits into infinitely many F_{k+1}-classes. This entire situation lifts to M and contradicts Lemma 22. □

Lemma 32. *Let $(M; \leqslant, \cdots)$ be ω-categorical order-like, transitive, and binary. Let E be the coarsest proper convex $\varnothing$-definable equivalence relation. Then given $a \in M$, for any a-definable cut c of $(M, \leqslant)$, we have* $\inf(a/E) \leqslant c \leqslant \sup(a/E)$.

Proof. We write $a \ll b$ for $a/E < b/E$, equivalently $a < b$ and $a/E \neq b/E$. If c is a cut, then $a \ll c$ means that c is greater than the supremum of the E-class of a.

Assume that there is a cut c definable from a, with $a \ll c$. Let $c(a)$ be the minimal such cut. Consider the cut $c_* := \sup\{c(a') : a'Ea\}$. Note that c_* depends only on the E-class of

a, so we can write $c_* = f_*(a/E)$ for some function f_*. Assume c_* is not $+\infty$ and let $g_*(a/E)$ be the image of $f_*(a/E)$ in the quotient M/E. Then g_* is a function from $(M/E, \leqslant)$ to its Dedekind completion with $x < g_*(x)$. By Lemma 31, M/E has topological rank 1. Therefore by Proposition 24, the graph of g_* must be dense in $\{(x, y) : x < y\}$. We can then find $b \in M$ such that $a \ll b \ll c(a) \ll c(b)$. Then as $c(b)$ is the minimal cut definable from b above b/E, there is $d \in M$ such that $\mathrm{tp}(a, b) = \mathrm{tp}(b, d)$ and $a \ll b \ll c(a) \ll d \ll c(b)$. So $\mathrm{tp}(a, d) \neq \mathrm{tp}(a, b)$ and this is a contradiction to M being order-like.

If c_* is $+\infty$, then we can also find b as above, just by definition of c_*. $\square$

Corollary 33. *Let $(M; \leqslant, \cdots)$ be ω-categorical order-like, transitive, and binary. Let E be the coarsest proper $\leqslant$-convex $\varnothing$-definable equivalence relation. Let F be a $\varnothing$-definable equivalence relation not refining E. Then no F-class defines a cut in $(M; \leqslant, \cdots)$.*

In particular:

1. *Every F-class intersects a dense set of E-classes.*
2. *Suppose F is the coarsest proper $\leqslant'$-convex $\varnothing$-definable equivalence relation, for some $\varnothing$-definable order $\leqslant'$. Then $(M/E; \leqslant, \cdots)$ and $(M/F; \leqslant', \cdots)$ are independent.*

Proof. Let C be an F-class, and suppose that a cut in $(M; \leqslant, \cdots)$ is definable from C. Then that cut is definable from any $a \in C$. Let $a_1, a_2 \in C$ lie in distinct E-classes. By Lemma 32, the only cut of $(M/E, \leqslant)$ definable from a_i is that of a_i/E. As these cuts are distinct, C can define neither.

For (1), let C be an F-class, and let $\sim$ be the $\leqslant$-convex equivalence relation on M/E defined by:

$$x \sim y \iff \text{the interval } [x, y] \text{ lies in the complement of C.}$$

As M/E has topological rank 1 by Lemma 31, $\sim$ has finitely many classes. There must be multiple $\sim$-classes, but then their endpoints would be C-definable cuts.

For (2), note that an intertwining map would require each F-class to define a cut in $(M/E; \leqslant, \cdots)$, which we ruled out above. $\square$

Definition 34. Let $(\Gamma, \leqslant_1, \ldots, \leqslant_n)$ be homogeneous, and E a $\varnothing$-definable equivalence relation. We say E is *convex* if it is $\leqslant_i$-convex for some i.

Lemma 35. *Let $(\Gamma, \leqslant_1, \ldots, \leqslant_n)$ be homogeneous. Then any maximal $\varnothing$-definable equivalence relation F is convex with respect to at least two linear orders in the language.*

Proof. For each $i \leqslant n$, let E_i denote the maximal $\leqslant_i$-convex $\varnothing$-definable equivalence relation, let $V_i = (\Gamma/E_i; \leqslant_i, \cdots)$ be the structure induced on the quotient, and let $W_1, \ldots, W_k$ be representatives of the V_i's up to $\varnothing$-definable monotonic bijection. Then by Corollary 33(2) and Lemma 31, the W_i's are pairwise independent topological rank 1 ordered sets.

First, suppose F is not convex. Then by Corollary 33(1), each F-class projects densely on each W_i. For any F-class C, we may expand the language by a unary predicate naming C. Each resulting W_i is still minimal, as C can define no cuts by Corollary 33. Let

$C^* \subset \prod W_i$, where each element of C^* is the tuple of projections onto each W_i of an element of C. By Proposition 26, C^* is dense in $\prod W_i$ equipped with the product topology, i.e. if a non-empty open $\leqslant_i$-interval is chosen for each W_i, then there is some $c \in C^*$ lying in all the chosen intervals. By quantifier elimination $\bigwedge_i x <_i y$ implies a complete type on (x, y). However, by the density of C^* for any F-class C, this type is consistent both with $F(x, y)$ and $\neg F(x, y)$.

Now suppose F is only $\leqslant_j$-convex. Then we carry out the same argument, omitting W_j. Again, each F-class is dense in the product $\prod_{i \neq j} W_i$. Let $C_1 <_j C_2$ be two F-classes. By density, we can find $a, a' \in C_1$ and $b, b' \in C_2$ such that $\bigwedge_{i \neq j} a <_i b$ and $\bigwedge_{i \neq j} b' <_i a'$. It follows, both $x <_j y \wedge \bigwedge_{i \neq j} x <_i y$ and $y <_j x \wedge \bigwedge_{i \neq j} x <_i y$ are consistent with $\neg F(x, y)$. By quantifier elimination, those formulas must imply $\neg F(x, y)$. However, by density of C_1, we can find $(c, d) \in C_1^2$ satisfying one of those formulas. This is a contradiction. □

Lemma 36. *Let $(\Gamma; \leqslant_1, \ldots, \leqslant_n)$ be homogeneous, with lattice of $\varnothing$-definable equivalence relations Λ. Then any meet-irreducible $E \in \Lambda$ is convex with respect to at least two linear orders in the language.*

Proof. Let $E \in \Lambda$ be meet-irreducible, and E^+ the cover of E. Fix an E^+-class C^+, and $C \subset C^+$ an E-class. By Lemma 35, there are distinct $i, j \leqslant n$ such that C is $\leqslant_i$-convex and $\leqslant_j$-convex in C^+.

If E^+ is both $\leqslant_i$-convex and $\leqslant_j$-convex, we are finished, so assume neither E^+ nor E is $\leqslant_i$-convex. Let $\overline{C^+}$ be the $\leqslant_i$-convex closure of C^+. The structure $(\overline{C^+}; \leqslant_1, \ldots, \leqslant_n)$ is homogeneous. Let G be the maximal $\varnothing$-definable equivalence relation that is $\leqslant_i$-convex in $\overline{C^+}$, so G is also $\leqslant_i$-convex in Γ. Then E^+ does not refine G, since otherwise C^+/G would equal $\overline{C^+}$; thus E also does not refine G, since we cannot have $E = G$ as E is not $\leqslant_i$-convex. By Corollary 33, the projections of both C and C^+ are dense in $(\overline{C^+}/G; \leqslant_i)$. As C is $\leqslant_i$-convex in C^+, these projections must be equal. As $(\overline{C^+}/G; \leqslant_i)$ is without endpoints, applying $\leqslant_i$-convexity again gives $C = C^+$, which is a contradiction. □

Corollary 37. *Let $(\Gamma; \leqslant_1, \ldots, \leqslant_n)$ be homogeneous, then any $\varnothing$-definable equivalence relation is an intersection of convex $\varnothing$-definable equivalence relations.*

Remark 38. By the proof of [3]*Lemma 3.4.10, any intersection of convex equivalence relations is convex for some $\varnothing$-definable order (not necessarily one of $\leqslant_1, \ldots, \leqslant_n$).

Corollary 39. *Let Λ be a finite distributive lattice, Λ_0 the poset of meet-irreducibles of $\Lambda \setminus \{ \mathbb{0}, \mathbb{1} \}$, and ℓ the minimum number of chains needed to cover Λ_0. Let d_Λ be the minimum dimension of a homogeneous permutation structure with lattice of $\varnothing$-definable equivalence relations isomorphic to Λ. Then $2\ell \leqslant d_\Lambda$.*

Proof. By Lemma 36, every element of Λ_0 must be convex for at least two linear orders in the language. However, if $E, F \in \Lambda$ are incomparable, then they cannot be convex with respect to the same order. □

4.2 Distributivity and genericity

We now prove distributivity of the lattice of $\varnothing$-definable equivalence relations and genericity of the reduct to the language of $\varnothing$-definable equivalence relations. Distributivity is proven first, although by Proposition 8 it follows from genericity, as we will use Proposition 8 to prove genericity.

Definition 40. Two equivalence relations E and F are *cross-cutting* if every E-class intersects every F-class.

We now prove that two $\varnothing$-definable equivalence relations are cross-cutting if their join is $\mathbb{1}$. Note this would be immediate if we already knew the genericity of the reduct to the language of $\varnothing$-definable equivalence relations.

Lemma 41. *Let $(M; E, F, \cdots)$ be ω-categorical, transitive, and order-like, where E and F are $\varnothing$-definable equivalence relations. Let $a, a', b \in M$ such that aEa' and aFb, then there is $b' \in M$ with bEb' and $a'Fb'$.*

Proof. Let e_1 be the E-class of a and e_2 the E-class of b. Take e_3 so that $\mathrm{tp}(b, e_3) = \mathrm{tp}(a', e_2)$. Next take $c \in e_2$ such that $\mathrm{tp}(c, e_3) = \mathrm{tp}(a, e_2)$ (this is possible as $\mathrm{tp}(e_1, e_2) = \mathrm{tp}(e_2, e_3)$). Finally let $d \in e_3$ be such that $\mathrm{tp}(a, c) = \mathrm{tp}(c, d)$, so, since M is order-like, $\mathrm{tp}(a, d) = \mathrm{tp}(a, c)$. Let d' be such that $\mathrm{tp}(a, d, d') = \mathrm{tp}(a, c, b)$. Then we have aFb and dEd'. So $d' \in e_3$ and bFd'. Since $\mathrm{tp}(b, e_3) = \mathrm{tp}(a', e_2)$, there is $b' \in e_2$ such that $a'Fb'$. $\square$

Corollary 42. *Let $(M; E, F, \cdots)$ be ω-categorical, transitive, and order-like, with lattice of $\varnothing$-definable equivalence relations Λ. Let $E, F \in \Lambda$ such that $E \vee F = \mathbb{1}$. Then E and F are cross-cutting.*

Proof. We first show that Lemma 41 implies that given two E-classes e_1 and e_2, either e_1 and e_2 intersect the same F-classes, or they intersect disjoint sets of F-classes. Let f be an F-class such that e_1 and e_2 intersect f. Let $a \in e_1 \cap f$ and $b \in e_2 \cap f$. Given any $a' \in e_1$, let $f' = a'/F$. Then by Lemma 41, we may find some $b' \in e_2 \cap f'$.

Now, let G be the equivalence relation on M which holds for (a, b) if the E-class of a and the E-class of b intersect the same F-classes. Then G is definable and is coarser than both E and F, and in fact $G = E \vee F$. As $E \vee F = \mathbb{1}$, E and F must be cross-cutting. $\square$

Lemma 43. *Let $(\Gamma; \leqslant_1, \ldots, \leqslant_n)$ be homogeneous. Let E, F, G be $\varnothing$-definable equivalence relations such that E is maximal, $F \vee G = \mathbb{1}$, and neither F nor G refines E. Then $F \wedge G$ does not refine E.*

Proof. Assume that $F \wedge G$ refines E. Pick any F-class C_F, let C_G be a thorn-independent G-class, and let $p(x, y)$ be the type of (C_F, C_G). By Corollary 42, we may find a such that $a/F = C_F$ and $a/G = C_G$. We then have a function $f \colon p(\Gamma/F, \Gamma/G) \to \Gamma/E$, given by $f(x/F, x/G) = x/E$. This is well-defined as $F \wedge G \leqslant E$. By Proposition 27, $a/E \in \mathrm{dcl}(a/F) \cup \mathrm{dcl}(a/G)$. Assume say $a/E \in \mathrm{dcl}(a/F)$, then the F-class of a is included in an E-class (by transitivity of Γ), so F refines E. $\square$

Proposition 44. *Let* $(\Gamma; \leqslant_1, \ldots, \leqslant_n)$ *be homogeneous, with lattice of* $\varnothing$*-definable equivalence relations* Λ*. Then* Λ *is distributive.*

Proof. We must prove the lattices M_3 and N_5 (see Figure 1) do not appear as sublattices of Λ.

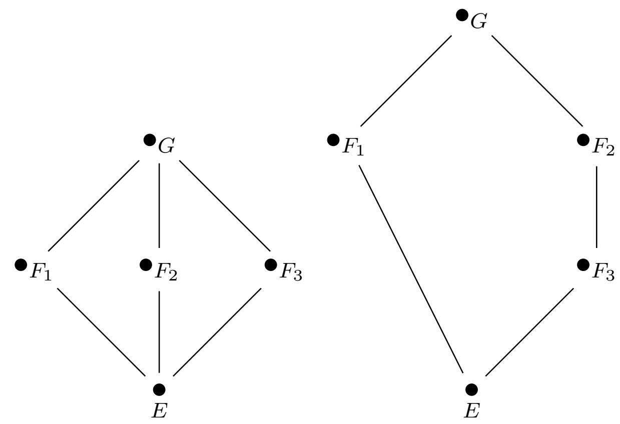

Figure 1: M_3 and N_5

Suppose M_3 appears as a sublattice. Let E be the minimum element, G the maximum element, and F_1, F_2, F_3 the non-trivial elements. We pick a G-class and work within it, so we may assume $G = \mathbb{1}$. Let $F' \geqslant F_1$ be maximal below G. Then neither F_2 nor F_3 refines F', so by Lemma 43 neither does $F_2 \wedge F_3$. But this is a contradiction.

Now suppose N_5 appears as a sublattice. Let E be the minimum element, G the maximum element, and F_1, F_2, F_3 the non-trivial elements, with $F_2 > F_3$. We pick a G-class and work within it, so we may assume $G = \mathbb{1}$. Then F_1 and F_3 are cross-cutting by Corollary 42, as $F_1 \vee F_3 = \mathbb{1}$. As there are infinitely many F_3-classes in each F_2-class (see [3]*Lemma 5.2.2), we cannot have that $F_1 \wedge F_3 = F_1 \wedge F_2$. $\square$

Proposition 45. *Let* $(\Gamma, \leqslant_1, \ldots, \leqslant_n)$ *be homogeneous. Then the reduct of* Γ *to the language of* $\varnothing$*-definable equivalence relations is generic.*

Proof. Let Λ be the lattice of $\varnothing$-definable equivalence relations of Γ. Let $A \subset \Gamma$ be finite. We must realize any maximal quantifier-free 1-type $p(x/A)$, in the language of equivalence relations, that is consistent with the Λ-triangle inequality. (For the equivalence between being a Fraïssé limit and satisfying such a 1-point extension property, see [7]*§7.1.) We proceed by induction on the height of Λ. The statement is trivial if Λ has 2 elements.

Let $a \in A$ such that $p \vdash xFa$ for some maximal $F \in \Lambda$, and let $C = a/F$. We now wish to inductively continue inside C, but might not have $A \subset C$. For every $E_i \in \Lambda$ such that F and E_i are incomparable and every $a_i \in A$ such that $p \vdash d(x, a_i) = E_i$, we will find some $a_i' \in \Gamma$ such that $a_i' \in C \cap a_i/E_i$. Let A' be A with each a_i replaced by a_i'. We then create a new type $p'(x/A')$ by choosing distances $d_i = d(a_i', x) \leqslant E_i$ such that p' is still consistent with the Λ-triangle inequality. By induction, there will be some element realizing p', which will then also realize p.

For each E_i incomparable to F, we have that E_i and F are cross-cutting by Corollary 42. Thus for each a_i such that $p \vdash d(x, a_i) = E_i$, we have that $C \cap a_i/E_i \neq \varnothing$. We may pick any element of this intersection to be a_i'.

We view the assignment of the distances d_i as an amalgamation problem. The base is A, the first factor is $A \cup \{ x \}$, and the second factor is $A \cup A'$. As Λ is distributive by Proposition 44, we can apply Proposition 8 to complete the amalgamation diagram while respecting the Λ-triangle inequality. This forces $d_i \leqslant d(a_i', a_i) \vee d(a_i, x) = E_i$. □

5 Classification

All homogeneous finite-dimensional permutation structures are assumed to be presented in a language of equivalence relations and subquotient orders.

It is not immediate that the quotient of a homogeneous finite-dimensional permutation structure by a $\varnothing$-definable equivalence relation is again homogeneous. However, the next few results establish the Primitivity Conjecture (or actually something slightly stronger) when working in the quotient.

Lemma 46. *Let Γ be a homogeneous permutation structure with lattice of $\varnothing$-definable equivalence relations Λ. Let $E, F \in \Lambda$ with $E < F$. Let C be an F-class, and let $\leqslant_1, \ldots, \leqslant_n$ be $\varnothing$-definable topologically primitive linear orders on C/E. Then $\leqslant_1, \ldots, \leqslant_n$ are generic, modulo the agreement of certain orders up to reversal.*

Proof. By Lemma 31, C/E has topological rank 1 with respect to each $\leqslant_i$. By Proposition 18, it suffices to show that no order is intertwined with another, or its reverse.

Suppose $\leqslant_i$ is intertwined with $\leqslant_j$ via some intertwining map f. Then, given any $a \in C/E$, f produces an a-definable cut in $\leqslant_j$. If $\leqslant_i, \leqslant_j$ are not equal, there must be some $a \in C/E$ for which $f(a) \neq a$. But this contradicts Lemma 32. □

Definition 47. Let Γ be a homogeneous finite-dimensional permutation structure, $<$ a subquotient order on Γ, and E a $\varnothing$-definable equivalence relation. Then the *restriction of $<$ to E*, when defined, is the subquotient order $<\restriction_E$ with top relation E given by $x <\restriction_E y$ iff $(xEy) \wedge (x < y)$.

Lemma 48. *Let Γ be a homogeneous finite-dimensional permutation structure, with lattice of $\varnothing$-definable equivalence relations Λ. Let $<_i, <_j$ be subquotient orders with bottom relation $E \in \Lambda$ and top relations $F_i, F_j \in \Lambda$, respectively, with $F_i \leqslant F_j$. Assume $<_i, <_j$ are convex with respect to no $\varnothing$-definable equivalence relations between E and F_i.*

Let $G \in \Lambda$ with $E < G < F_i$. If $<_i\restriction_G = <_j\restriction_G$, then $<_i = <_j\restriction_{F_i}$.

Proof. Suppose not. Let C be an F_i-class. By Lemma 46, $<_i$ and $<_j$ are independent on C/E. Let $a \in C/E$ and $A = a/G$. Let $b \in C/E \cap (A \setminus \{ a \})$, $p = \mathrm{tp}(b/a)$, and $P \subset C/E$ the realizations of p. Then P is infinite, as $\mathrm{acl}(a) = \{ a \}$ by Lemma 25, and so $<_i$ and $<_j$ are independent on P by Lemma 17. However, this is a contradiction as P lies within a single G-class, so $<_i = <_j$ on P. □

Corollary 49. *Let Γ be a homogeneous finite-dimensional permutation structure, with lattice of $\varnothing$-definable equivalence relations Λ. Let E be meet-irreducible in Λ and E^+ its unique cover. Let C be an E^+-class, and consider C/E equipped with the restriction to E^+ of every subquotient order with bottom relation E. If none of the original subquotient orders are equal up to reversal to any restriction of another, then C/E is generic.*

Proof. By Lemma 46, C/E is generic modulo the agreement of certain orders up to reversal. By Lemma 48, as none of the original subquotient orders were equal up to reversal, none of the restricted subquotient orders are either. $\square$

The structures in the catalog have subquotient orders with only meet-irreducible bottom relations. The next few results show we may ignore the possibility of subquotient orders with meet-reducible bottom relations.

Lemma 50. *Let Γ be a homogeneous finite-dimensional permutation structure with lattice of $\varnothing$-definable equivalence relations Λ. Let $<$ be a subquotient order from E to F, convex with respect to no intermediate $\varnothing$-definable equivalence relation. Let $E = G_1 \wedge G_2$. Then $F \not\geqslant G_1 \vee G_2$.*

Proof. Suppose $F \geqslant G_1 \vee G_2$. Define Γ' to be Γ/E restricted to a single $G_1 \vee G_2$-class. Then $<$ is a transitive linear order on Γ', and by Lemma 31 has topological rank 1, and thus is minimal. Choose a thorn-independent pair $a_1, a_2 \in \Gamma'$. By Remark 28(2), $d(a_1, a_2) = G_1 \vee G_2$. Also, by Remark 28(3), $(a_1/G_1, a_2/G_2)$ is a thorn-independent pair. But $a_1/G_1 \cap a_2/G_2 \in \Gamma'$ is neither in $\mathrm{dcl}(a_1/G_1)$ nor in $\mathrm{dcl}(a_2/G_2)$, contradicting Proposition 27. $\square$

Lemma 51. *Let Γ be a homogeneous finite-dimensional permutation structure, E a $\varnothing$-definable equivalence relation, and $C_1, C_2 \subset \Gamma$ two E-classes. Then C_1 remains homogeneous after expanding the language by a unary predicate naming C_2.*

Proof. Let $A \subset C_1$ be finite. It suffices to find some $c \in C_2$ such that c has the same type over every $a \in A$, as we may then always extend any finite partial isomorphism of C_1 to one fixing c.

Let $F \geqslant E$ be maximal such that C_1, C_2 lie in distinct F-classes. Note F must be meet-irreducible, so let F^+ be its cover. Let C be the F^+-class of C_1, which is also the F^+-class of C_2 and we now work in C. So we may assume $F^+ = \mathbb{1}$. Let C_2' be the F-class of C_2.

We now move to the language of linear orders. If $\leqslant_i$ is an order for which there is $\leqslant_i$-convex G with $E \leqslant G \leqslant F$, then any $x \in C_2'$ has the same $\leqslant_i$-type over every $a \in A$. Enumerate the remaining orders—those for which there is no such G—as $\leqslant_1, \dots, \leqslant_m$. For each $i \leqslant m$, let $E_i \in \Lambda$ be maximal $\leqslant_i$-convex below F^+ and let $V_i = (C/E_i; \leqslant_i)$. By Corollary 33, C_2' projects densely onto each V_i.

We now work inside C_2'. Let $F_i \in \Lambda$ be the maximal $\leqslant_i$-convex relation below F. Then E does not refine F_i, as we ignored orders where that would be the case. Now let $W_i = (C_2'/F_i; \leqslant_i)$, and $X_1, \dots, X_k$ representatives of the W_i's up to monotonic bijection. Let $C_2^* \subset \prod X_i$, where each element of C_2^* is the tuple of projections onto each X_i of an element of C_2. Then the same argument as in Lemma 35 gives that C_2^* is dense in the product $\prod_i X_i$ equipped with the product topology. Namely, by Corollary 33(1), C_2 is dense in each X_i. As each X_i is minimal, Proposition 26 gives density in $\prod_i X_i$. By the last sentence of the previous paragraph, each W_i contains a point $\leqslant_i$-greater than all of A. Thus we may find some $c \in C_2$ that is $\leqslant_i$-greater than all of A for each i. □

For the next lemma, note that if $<'$ is a subquotient order in a generic Λ-ultrametric space from F to $F \vee G$, then $<' \restriction_G$ is a subquotient order from $F \wedge G$ to G (see [3]*Lemma 3.4.7).

Lemma 52. *Let Γ be a homogeneous finite-dimensional permutation structure with lattice of $\varnothing$-definable equivalence relations Λ. Let $E \in \Lambda$ be meet-reducible, and $<$ a subquotient order from E to G, convex with respect to no intermediate $\varnothing$-definable equivalence relations. Then there exists some $F > E$ and subquotient order $<'$ from F to $F \vee G$ such that $< \,=\, <' \restriction_G$.*

Furthermore, $<$ and $<'$ are interdefinable.

Proof. By Lemma 50, G cannot be above two covers of E, so we can find $F \in \Lambda$ such that $E = F \wedge G$. Suppose the first part of the lemma is false for this F. Then there exist F-classes C_1, C_2 and $x_i, y_i \in C_i$ such that $x_1 < x_2$ and $y_1 > y_2$. In particular, $x_1 G x_2$ and $y_1 G y_2$, so x_1, y_1, and C_2 determine x_2/E and y_2/E. We wish to produce an automorphism of C_1 sending $(x_1/E, x_2/E)$ to $(y_1/E, y_2/E)$, which will yield a contradiction. It suffices to produce an automorphism sending x_1 to y_1 and leaving C_2 invariant. By Lemma 51, there is such an automorphism, and we are finished.

For the last part, first note any projection of $<'$ is $\varnothing$-definable from $<'$. For the other direction, we have that F and G-classes within the same $(F \vee G)$-class are cross-cutting by Corollary 42, so we may define $x <' y \iff \exists z((xFz) \wedge (z < y))$. □

Corollary 53. *Let Γ be a homogeneous finite-dimensional permutation structure with lattice of $\varnothing$-definable equivalence relations Λ. Let $<$ be a subquotient order with a meet-reducible bottom relation, convex with respect to no intermediate $\varnothing$-definable equivalence relations. Then $<$ is interdefinable with some subquotient order $<'$ with a meet-irreducible bottom relation.*

Proof. Starting with $<$, iteratively apply Lemma 52 until a subquotient order with meet-irreducible bottom relation is produced. This must eventually happen, as at each step the bottom relation moves upward in Λ, and the maximal elements of Λ are meet-irreducible. □

Finally, we establish that the subquotient orders are generic by proving a suitable 1-point extension property.

Lemma 54. *Let Γ be a homogeneous finite-dimensional permutation structure. Suppose every subquotient order of Γ has a meet-irreducible bottom relation, and that no one is equal up to reversal to the restriction of another. Then the subquotient orders of Γ are generic.*

Proof. We assume that no subquotient orders are convex with respect to any intermediate $\varnothing$-definable equivalence relation. Let Λ be the lattice of $\varnothing$-definable equivalence relations of Γ. Let $A \subset \Gamma$ be finite, and $p(x)$ a complete quantifier-free 1-type over A in the language of equivalence relations and subquotient orders, such that its restriction to the language of equivalence relations and any single subquotient order is consistent. To satisfy Definition 10, it suffices to show $p(x)$ is realized. (For the equivalence between being a Fraïssé limit and satisfying such a 1-point extension property, see [7]*§7.1.) Our plan will be to produce a consistent type $q(x)$ solely in the language of $\varnothing$-definable equivalence relations such that $q(x) \implies p(x)$. As the reduct to the language of $\varnothing$-definable equivalence relations is generic, we may then realize $q(x)$.

We will produce a finite sequence of types $p_0(x), \dots, p_\ell(x) = q(x)$ such that the reduct of each to the language of $\varnothing$-definable equivalence relations is consistent, $p_i(x) \implies p_{i-1}(x)$, and $p_0(x) \implies p(x)$. Let $p_0(x)$ be $p(x)$, but removing any condition $x <_{i,j} a$ or $x >_{i,j} a$ for any $a \in A$ and any subquotient order $<_{i,j}$ with bottom relation E_i such that there is some $b \in A$ such that $p \vdash xE_ib$. By the consistency conditions on p, $p_0(x) \implies p(x)$.

Let $E_1, \dots, E_\ell \in \Lambda$, ordered such that if $i < j$ then $E_i \not< E_j$, be the meet-irreducibles of Λ such that there is some subquotient order $<_{i,j}$ with bottom relation E_i and some $a_i \in A$ such that $p_0 \vdash a_i <_{i,j} x$ or $p_0 \vdash x <_{i,j} a_i$. Given p_{k-1}, we will produce p_k by removing any condition $x <_{k,j} a$ or $x >_{k,j} a$ for any $a \in A$ and any subquotient order $<_{k,j}$ with bottom relation E_k, then finding a suitable $b_k \in \Gamma$ and adding xE_kb_k.

Suppose we have found p_{k-1}. Enumerate the subquotient orders with bottom relation E_k as $<_i$ for $i \leqslant n$, let F_i be the top relation of $<_i$, and let E_k^+ be the cover of E_k. As the reduct of Γ to the language of equivalence relations is generic, the corresponding reduct of $p_{k-1}(x)$ is realized. Pick a realization, and let $C_{E_k^+}$ be its E_k^+-class. As $C_{E_k^+}$ is homogeneous, $C_{E_k^+}/E_k$ is generic by Corollary 49.

For each $i \leqslant n$, $p_{k-1}(x)$ restricts x/E_k to a $<_i$-interval J_i with endpoints in $A \cup \{\pm\infty\}$. It also restricts x to lie in a given F_i-class C_{F_i}, with C_{F_i} containing the endpoints of J_i. By the consistency of the reduct of $p_{k-1}(x)$ to the language of $\varnothing$-definable equivalence relations, we have $C_{F_i} = C_{E_k^+}/F_i$.

Let $I_i = C_{E_k^+}/E_k \cap J_i$, so I_i is a $<_i$-interval in $C_{E_k^+}/E_k$. Working within C_{F_i}/E_k, since $<_i$ is not convex with respect to any intermediate $\varnothing$-definable equivalence relations, we may apply Corollary 33(1) (where E and F there are E_k and E_k^+ here, respectively) to get that $C_{E_k^+}/E_k$ is $<_i$-dense in C_{F_i}/E_k, and so $I_i \neq \varnothing$. By genericity of $C_{E_k^+}/E_k$, $\bigcap I_i \neq \varnothing$. Let $y \in \bigcap I_i$ be an E_k-class not among the finitely many $p_{k-1}(x)$ specifies are to be avoided. Then we may take $b_k \in \Gamma$ to be any element of y.

We now show the reduct of $p_k(x)$ to the language of $\varnothing$-definable equivalence relations is consistent. Suppose the reduct of $p_{k-1}(x)$ forces x to be in some G_i-class C_{G_i}, for some $G_i \in \Lambda$. Then for each such i, $G_i \not< E_k$ by assumption, so $\bigwedge G_i \not< E_k$ since E_k is meet-irreducible and Λ is distributive, so $((\bigwedge G_i) \vee E_k) \geqslant E_k^+$. As $C_{E_k^+}$ is the E_k^+-class of some realization of p_{k-1}, we have that the $((\bigwedge G_i) \vee E_k)$-class of $C_{E_k^+}$ equals the $((\bigwedge G_i) \vee E_k)$-class of $\bigcap C_{G_i}$. Then by Corollary 42, $\bigcap C_{G_i}$ meets any E_k-class in $C_{E_k^+}$, and so meets b_k/E_k. $\square$

Theorem 55. *Let Γ be a homogeneous finite-dimensional permutation structure. Then there is a finite distributive lattice Λ such that Γ is interdefinable with an expansion of the generic Λ-ultrametric space by generic subquotient orders, such that every meet-irreducible of Λ is the bottom relation of some subquotient order.*

Proof. Let Λ be the lattice of $\varnothing$-definable equivalence relations of Γ. Then Λ is distributive by Proposition 44, and the reduct of Γ to the language of $\varnothing$-definable equivalence relations gives the generic Λ-ultrametric space by Proposition 45.

We may assume Γ is presented in a language such that no subquotient order is equal up to reversal to the restriction of another, nor convex with respect to any intermediate $\varnothing$-definable equivalence relation. By Corollary 53, we may also assume every subquotient order has a meet-irreducible bottom relation. By Lemma 36, every meet-irreducible is the bottom relation of some subquotient order. By Lemma 54, all subquotient orders are generic. $\square$

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