A statistics–respecting bijection between permutation matrices and descending plane partitions without special parts

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Abstract

We present a bijection between permutation matrices and descending plane partitions without special parts, which respects two of the statistics considered by Mills, Robbins and Rumsey, and an additional statistic considered by Behrend, Di Francesco and Zinn–Justin.

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1 Introduction

It is a well–known fact [11, 5, 4] that the enumeration

- of descending plane partitions with parts not exceeding n (let us denote the set of these objects by $\mathcal{D}_n$)

- and of alternating sign matrices of dimension n (let us denote the set of these objects by $\mathcal{A}_n$)

gives the same number:

$$|\mathcal{A}_n| = |\mathcal{D}_n| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1}$$

1.1 Search for a bijection

It appears to be quite difficult to find some "natural" bijection

$$\Phi : \mathcal{A}_n \to \mathcal{D}_n.$$

However, there are two additional informations which might help in the search for such bijection:

- There is a *quadruplet* (p, i, s, q) of statistics for both $\mathcal{A}_n$ and $\mathcal{D}_n$, i.e. there are functions

  $$\sigma_A : \mathcal{A}_n \to \mathbb{Z}^4 \text{ and } \sigma_D : \mathcal{D}_n \to \mathbb{Z}^4,$$

  such that for all $y := (p, i, s, q) \in \mathbb{Z}^4$ the preimages are *equinumerous*, i.e.,

  $$\left| \sigma_A^{-1}(y) \right| = \left| \sigma_D^{-1}(y) \right|$$

  (see [3, Theorem 1], the details are given in the next section).

- There are certain subsets of $\mathcal{A}_n$ and of $\mathcal{D}_n$, namely

  - alternating sign matrices with statistic $s = 0$ (let us denote this set by $\mathcal{A}_n^0$; it is, in fact, the set of $n \times n$ *permutation matrices*),
  - descending plane partitions with statistic $s = 0$ (let us denote this set by $\mathcal{D}_n^0$),

  which are *much simpler* to understand and for which it is, in fact, quite easy to give "natural" bijections (see below).

So an obvious approach would be to search for a bijection Φ which *respects* this quadruplet of statistics (p, i, s, q); in the sense that for all $A \in \mathcal{A}_n$ there should hold:

$$\sigma_A(A) = \sigma_D(\Phi(A)) \text{ for all } A \in \mathcal{A}_n. \tag{2}$$

Clearly, such bijection Φ restricted to the subset $\mathcal{A}_n^0 \subset \mathcal{A}_n$ would give a bijection

$$\Psi : \mathcal{A}_n^0 \to \mathcal{D}_n^0.$$

So if we find such "restricted" bijection Ψ which respects the triplet (p, i, q) of statistics in the sense of (2), then we might hope to "extend" it somehow to the desired "full" bijection Φ.

The purpose of this note is to present a simple bijection Ψ which indeed respects the triplet (p, i, q) of statistics in the sense of (2): The construction of Ψ relies on the representation of descending plane partitions as families of non–intersecting lattice paths and on a certain "visualization" of the statistic i (as number of certain entries) for matrices).

1.2 Other bijections

It should be noted that there are other bijections $\mathcal{A}_n^0 \to \mathcal{D}_n^0$: Maybe the simplest one was already mentioned by Lalonde [8, p.981], who referred to the inversion table of a permutation: This table (also called *inversion word*) gives a (well–known) unique encoding of permutations. The missing link from inversion words to $\mathcal{D}_n^0$ was explained by Striker [10, Lemma 5]. Striker uses monotone triangles as intermediate objects to establish the bijection between inversion words and $\mathcal{A}_n^0$, but this intermediate step is not necessary: Instead, we can employ directly the well–known encoding of permutations by inversion words. (We shall call these two mappings $\mathcal{A}_n^0 \to \mathcal{D}_n^0$ Striker's and Lalonde's bijection, respectively). Unfortunately, none of these two simple bijections respects the statistic q (see Figure 4). Ayyer [1] presented another (inductively constructed) bijection, which does not respect the statistic i (see [1, p. 1786]).

1.3 Organization of this note

This note is organized as follows:

- Section 2 contains basic definitions and background information,

- Section 3 presents a "visualization" of inversions in alternating sign matrices used for our bijection,

- Section 4 presents a representation of descending plane partitions as families of non–intersecting lattice paths (in the "obvious" way: for the expert it will suffice to look at Figure 2),

- Section 5 presents a bijection $\Psi : \mathcal{A}_n^0 \to \mathcal{D}_n^0$ which respects the triplet (p, i, q) of statistics.

2 Background information

For reader's convenience, we recall some background information.

2.1 Descending plane partitions

Here is the definition of descending plane partitions as given by Mills, Robbins and Rumsey [9, Definitions 2–4]:

Definition 1 (descending plane partition). A *descending plane partition* is an array $\pi = (a_{i,j})$, $1 \leqslant i \leqslant j < \infty$, of *positive integers*

$$\pi = \begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & & \cdots & & \cdots & a_{1,\mu_1} \\ & a_{2,2} & a_{2,3} & & \cdots & \cdots & a_{2,\mu_2} \\ & & & \cdots & \\ & & & \cdots & \\ & & a_{k,k} & \cdots & a_{k,\mu_k} \end{matrix}$$

such that

1. rows are *weakly decreasing*, i.e., $a_{i,j} \geqslant a_{i,j+1}$ for all $i = 1, \ldots, k$ and $i \leqslant j < \mu_i$,

2. columns are *strictly decreasing*, i.e., $a_{i,j} > a_{i+1,j}$ for all $i = 1, \ldots, k-1$ and $i < j \leqslant \mu_{i+1}$,

3. $a_{i,i} > \mu_i - i + 1$ for all $i = 1, \ldots, k$,

4. $a_{i,i} \leqslant \mu_{i-1} - i + 2$ for all $i = 2, \ldots, k$.

Clearly, conditions 3 and 4 imply

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant k.$$

The *parts* of a descending plane partition are the *numbers* (with repetitions) that appear in the array. The *empty* array, which we denote by $\varnothing$, is explicitly allowed.

A descending plane partition π where no part is greater than n (i.e., π has at most $n - 1$ rows) is said to have dimension n. (So a descending plane partition of dimension n may also be viewed as a descending plane partition of dimension k, for all $k > n$.)

By the *length* of row i in descending plane partition π we define the number of parts it contains (i.e., $\mu_i - i + 1$). So we may rephrase conditions 3 and 4 as follows:

3′. The first part of row i is greater than the length of row i for $i = 1, \ldots, k$,

4′. The first part of row i is less or equal than the length of the preceding row $i - 1$ for $i = 2, \ldots, k$.

A part $a_{i,j}$ in a descending plane partition is called *special* if it does not exceed the number of parts to its left (in its row i), i.e., if

$$a_{i,j} \leqslant j - i.$$

Example 2. A typical example is the array

$$
\begin{array}{ccccc}
6 & 6 & 6 & 4 & \underline{2} \\
  & 5 & 3 & \underline{2} & \underline{1} \\
  &   & 2 &   &   \\
\end{array}
$$

with 3 rows and 10 parts (written in descending order)

$$6, 6, 6, 5, 4, 3, 2, \underline{2}, \underline{2}, \underline{1},$$

three of which are special parts (indicated as underlined numbers; note that the 2 in the last row is *not* a special part):

$$\underline{2}, \underline{2}, \underline{1}.$$

(This is the example D_0 considered by Lalonde [7, Fig. 1].)

From now on, we shall use the shorthand *DPP* for descending plane partitions.

2.2 Alternating sign matrices

Here is the definition of alternating sign matrices as given by Mills, Robbins and Rumsey, see [9, Definition 1]:

Definition 3 (alternating sign matrix)**.** An *alternating sign matrix* of dimension n is an $n \times n$ square matrix which satisfies

- all entries are 1, -1 or 0,

- every row and column has sum 1,

- in every row and column the nonzero entries alternate in sign.

Suppose that $A = (A_{i,j})_1^n$ is an alternating sign matrix of dimension n. Then the number of *inversions* in A is defined to be [9, p. 344]

$$\sum_{\substack{1 \leqslant i < k \leqslant n \\ 1 \leqslant l < j \leqslant n}} A_{i,j} A_{k,l}. \tag{3}$$

Example 4. The following matrix is an example of an alternating sign matrix of dimension 5:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0. \end{bmatrix}$$

From now on, we shall use the shorthand *ASM* for alternating sign matrices.

2.3 The Mills–Robbins–Rumsey conjecture

Here is the Conjecture of Mills, Robbins and Rumsey [9, Conjecture 3], slightly rephrased to fit our exposition:

Theorem 5. *Suppose that n, p, i, s are nonnegative integers, $0 \leqslant p \leqslant n-1$. Let $\mathcal{A}_n(p, i, s)$ be the set of ASMs such that*

1. *the size of the matrix is $n \times n$ (i.e., its dimension is n),*

2. *the number of 0's to the left of the 1 in the first row is p,*

3. *the number of -1's in the matrix is s,*

4. *the number of inversions in the matrix is $i + s$.*

On the other hand, let $\mathcal{D}_n(p, i, s)$ be the set of DPPs such that

1. *no part exceeds n (i.e., the dimension of the DPP is n),*

2. there are exactly p parts equal to n,

3. there are exactly s special parts,

4. there are a total of $i + s$ parts.

Then $\mathcal{A}_n (p, i, s)$ and $\mathcal{D}_n (p, i, s)$ have the same cardinality — the sets $\mathcal{A}_n$ and $\mathcal{D}_n$ are equidistributed with respect to the triplet of statistics (p, i, s).

This conjecture was proved by Behrend, Di Francesco and Zinn–Justin [2, Theorem 1].

2.4 The fourth statistic given by Behrend, Di Francesco and Zinn–Justin

In [3], Behrend, Di Francesco and Zinn–Justin presented a fourth statistic q for matrices and DPPs and showed that matrices and DPPs are equidistributed with respect to the *quadruplet of statistics* (p, i, s, q) [3, Theorem 1]: This statistic q is

- for n–dimensional ASMs equal to the number of 0's to the right of the 1 in the last row,

- for n–dimensional DPPs equal to the number of parts $n - 1$ plus the number of rows of length $n - 1$.

2.5 Permutation matrices and inversions

Let $\sigma \in \mathfrak{S}_n$ be a permutation of the first n natural numbers $\{1, 2, \dots, n\}$.

2.5.1 Inversions of a permutation

Recall that an *inversion* of σ is a pair (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$. For the number $\mathbf{inv}(\sigma)$ of *all* inversions of σ we have $0 \leqslant \mathbf{inv}(\sigma) \leqslant \frac{n(n-1)}{2}$.

We may assign to σ its *inversion word* $(a_1, a_2, \dots, a_{n-1})$, where a_k is the number of inversions (i, j) with $\sigma(j) = k$, $k = 1, 2, \dots, n - 1$. Clearly we have $0 \leqslant a_k \leqslant n - k$ and $a_1 + a_2 + \cdots + a_{n-1} = \mathbf{inv}(\sigma)$.

Considering the *one–line notation*

$$[\sigma(1), \sigma(2), \dots, \sigma(n)],$$

of σ, the inversion word's k–th entry a_k is simply the number of elements *to the left* of k (in the one–line notation) which are *greater* than k, and it is easy to see that *every* word $(b_1, b_2, \dots, b_{n-1})$ with $0 \leqslant b_k \leqslant n - k$ determines a *unique* permutation: Inversion words are, in this sense, just another "encoding" for permutations.

2.5.2 Permutation matrices

A permutation $\sigma \in \mathfrak{S}_n$ can be represented by an $n \times n$–matrix M with entries

$$M_{i,j} = \delta_{i,\sigma(j)}$$

(where $\delta_{x,y}$ denotes Kronecker's delta: $\delta_{x,y} = 1$ if $x = y$, $\delta_{x,y} = 0$ if $x \neq y$; this is the transpose of the usual permutation matrix). We call this matrix the *permutation matrix* of σ: Clearly, it contains precisely one entry 1 in every row and column.

Example 6. Let $n = 6$ and $\sigma \in \mathfrak{S}_6$ be the permutation with one–line notation

$$\sigma = [6, 3, 1, 4, 2, 5].$$

The corresponding permutation matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the corresponding inversion word is

$$(2, 3, 1, 1, 1).$$

Note that every *permutation matrix* is an ASM (which does not contain entries -1), and that the definition of inversions (3) for ASMs is a generalization of the number of inversions of a permutation (see also Section 3.2).

3 Inversions in ASMs

3.1 Orientation of cells in ASMs

Definition 7. By a *cell* in an ASM A we simply mean the *position* (i, j) (at row i and column j). If $A_{i,j} = 0$, we call this a *zero–cell*, otherwise a *non–zero cell*: So, non–zero cells can be either 1–cells or (-1)–cells.

Observe that for every *zero–cell* (i, j) in an ASM there must be

- to the left or to the right (or both) of (i, j) a *closest non–zero cell* in row i, precisely one of which must be a 1–cell: If this 1–cell lies to the left of (i, j), then call this cell *left–oriented*, otherwise *right-oriented*,

- above or below (or both) (i, j) a *closest non–zero* cell in column j, precisely one of which must be a 1–cell: If this 1–cell lies below (i, j), then call this cell *down–oriented*, otherwise *up-oriented*.

A zero–cell (i, j) is called

- *rd–cell*, if it is *right–oriented* and *down–oriented*.

- *lu–cell*, if it is *left–oriented* and *up–oriented*.

Remark: rd–cells and lu–cells correspond to two of the four vertex states in the *six vertex model* which are replaced by zeroes by the well–known bijection between square ice and alternating sign matrices (the remaining two of the altogether six possible vertex states are replaced by $+1$ and -1, see [6, Figure 5]).

3.2 Inversions of matrices

Observe that the *rd–cells* in the *permutation matrix* $A = \left(\delta_{i,\sigma(j)}\right)$ of some permutation $\sigma \in \mathfrak{S}_n$ are in one–to–one correspondence with the *inversions* of σ: (i, j) is a rd–cell if

- entry 1 in row i is in column $y = \sigma^{-1}(i) > j$ (to the right of column j),

- and entry 1 in column j is in row $\sigma(j) > i$ (below row i).

This is equivalent to

$$j < y \text{ and } \sigma(j) > (i = \sigma(y)),$$

i.e., (j, y) is an *inversion* of the permutation σ.

More generally, observe that the fourfold sum (3), which defines the number of inversions of an ASM, may be rewritten as follows

$$\sum_{(i,l)=(1,1)}^{(n,n)} \sum_{(k,j)=(i+1,l+1)}^{(n,n)} A_{i,j} A_{k,l}.$$

Now observe that the inner sum is simply the product

$$\left(\sum_{k=i+1}^{n} A_{k,l}\right)\left(\sum_{j=l+1}^{n} A_{i,j}\right).$$

A moment's thought shows that this product is equal to

$$[(i, l) \text{ is a } (-1)\text{–cell or an rd–cell}].$$

Here, we used *Iverson's bracket*:

$$[\text{assertion}] := \begin{cases} 1 & \text{if the assertion is true,} \\ 0 & \text{otherwise.} \end{cases}$$

So the number of inversions of an ASM A (according to (3)) is equal to the number of rd–cells of A *plus* the number of (-1)–cells of A.

The left picture shows the two paths constructed for rd–cell $(1,1)$: The first cell of crossing of these paths is lu–cell $(4,6)$, so $(1,1)$ is mapped to $(4,6)$.

The right picture shows the two paths constructed for rd–cell $(3,2)$: The first cell of crossing of these paths is lu–cell $(6,5)$, so $(3,2)$ is mapped to $(6,5)$. Observe that these paths also *meet* (but do not *cross*) at the -1–cell $(5,4)$.

Figure 1: The bijection mapping rd–cells to lu–cells.

3.3 Quadruplet of statistics, reformulated

These considerations immediately lead to the following reformulation of the quadruplet of statistics (p,i,s,q):

Definition of the statistic f:		
f	for $D \in \mathcal{D}_n$:	for $A \in \mathcal{A}_n$:
p	# (parts equal to n)	# (rd–cells in first row)
i	# (non–special parts)	# (rd–cells)
s	# (special parts)	# ((-1)–entries)
q	# (parts $(n-1)$) + # (rows of length $(n-1)$)	# (lu–cells in last row)

3.4 rd–cells and lu–cells are equinumerous

Observe that for every ASM A the number of rd–cells equals the number of lu–cells. We show this by a bijection (see Figure 1): For some rd–cell (i,j), we construct two paths,

- both starting at (i,j),

- both proceeding only horizontally to the right or vertically downwards,

- and both changing horizontal/vertical direction of movement whenever they encounter a non–zero cell.

One of these paths starts horizontally (to the right), the other starts vertically (downwards): The pictures in Figure 1 illustrate this simple idea. Now observe that the paths starting horizontally must *necessarily* end vertically, and vice versa. Hence the paths *must* eventually *cross* at some cell, and any such cell of crossing must *necessarily* be a lu–cell (the paths could meet but not cross at some (-1)–cell, see the right picture in Figure 1): Map (i,j) to the *first* cell of crossing (k,l) thus obtained. By symmetry (reflection at the second diagonal of A) it is immediately clear that this construction gives a *bijection*.

Consider the following two DPPs of dimension 6:

$$
\begin{array}{ccccc}
6 & 6 & 6 & 4 & \underline{\color{red}2} \\
  & 5 & 3 & \underline{\color{red}2} & \underline{\color{red}1} \\
  &   & 2 &   &
\end{array}
\qquad\qquad
\begin{array}{ccccc}
6 & 6 & 6 & 5 & 0 \\
  & 4 & 3 & \underline{\color{red}1} & \\
  &   & 2 &   &
\end{array}
$$

The pictures below show the non–intersecting lattice paths corresponding to the above DPPs:

Note that the the right DPP has a zero–padded first row, and that the special parts of these DPPs correspond to the horizontal steps (at heights > 0) in the "special range" below the main diagonal $y = x$ (indicated by the gray triangle).

Figure 2: Representation of DPPs as lattice paths.

4 Representation of DPPs as lattice paths

If some row i in a DPP $\pi = (a_{i,j})$ is *shorter* than $a_{i,i} - 1$, i.e.,

$$
\delta = (a_{i,i} - 1) - (\mu_i - i + 1) > 0,
$$

then we pad this row with δ trailing zeroes (so the *length* of some row is the number of *non–zero* parts in that row.)

Now we employ the well–known encoding of (shifted) tableaux as non–intersecting lattice paths; i.e., we encode a DPP $\pi = (a_{i,j})$ of dimension n with r rows as an r–tuple of non–intersecting lattice paths in the lattice $\mathbb{Z}^2$. For reader's convenience, we shall describe the details of this encoding below, but the idea can easily be obtained by looking at the illustrative example in Figure 2.

These lattice paths shall only use horizontal steps to the right or vertical steps downwards, i.e., steps leading from lattice point (x, y) to lattice point $(x + 1, y)$ or to lattice point $(x, y - 1)$.

The starting points of these r lattice paths are the points

$$
S_i := (0, a_{i,i}),
$$

i.e., the lattice path corresponding to row i starts on the vertical axis at height equal to the first part of row i.

The ending points of these lattice paths are the points

$$E_i := (a_{i,i} - 1, 0),$$

i.e., the lattice path corresponding to row i ends on the horizontal axis and consists of $a_{i,i} - 1$ horizontal steps at heights corresponding to the parts of row i; including padded zeroes (if any), which correspond to steps at height 0.

It is easy to see that the set $\mathcal{D}_n$ of n–dimensional DPPs is in bijection with the set of non–intersecting lattice paths as defined above, with no starting point higher than n. (But *note* that in this representation, the number of horizontal steps of a path is equal to the length of the corresponding row *plus* the number of horizontal steps at height *zero*.)

4.1 Bijection between inversion words and DPPs without special parts

If we have some ("incomplete") lattice path $\mathfrak{p}$ starting in $(0, h)$ and ending in (x, y), where $x \leqslant h - 1$ and $y \geqslant 0$, then by the *completion* of $\mathfrak{p}$ we mean the path obtained by appending to $\mathfrak{p}$

- y vertical steps (downwards)

- and $h - 1 - x$ horizontal steps (at height 0, to the right)

(i.e., the completion of $\mathfrak{p}$ ends in $(h - 1, 0)$).

Striker's bijection [10, Lemma 5] between inversion words and DPPs without special parts corresponds to the following construction.

If the inversion word $w = (a_1, a_2, \ldots, a_{n-1})$ equals $(0, 0, \ldots, 0)$, then simply assign to w the *empty* DPP (with no lattice path at all). Otherwise we start with the empty DPP, to which the construction will successively add lattice paths. Let k be the smallest index for which $a_k > 0$. *View w as* (the encoding of) a lattice path $\mathfrak{p}$ which starts at $(0, n - k + 1)$ and ends at $(a_1 + a_2 + \cdots + a_{n-1}, 0)$, containing a_j horizontal steps at height $n - j + 1$ for $j = k, k + 1, \ldots, n - 1$ (see Figure 3). As long as there is an intersection $P = (h, h)$ of $\mathfrak{p}$ with the boundary $y = x$ of the "special region", *cut* $\mathfrak{p}$ at P in two parts $\mathfrak{p}_1$ (containing $\mathfrak{p}$'s starting point) and $\mathfrak{p}_2$ (containing $\mathfrak{p}$'s ending point), remove any *vertical* steps at the beginning of $\mathfrak{p}_2$, and

- *add* the *completion* of $\mathfrak{p}_1$ to the (representation of the) DPP under construction;

- if $\mathfrak{p}_2$ contains horizontal steps, the first of which starts at some point $Q = (h, h')$ where $h' \leqslant h$, then *shift* $\mathfrak{p}_2$ by h units to the left (so that it starts at $(0, h')$) and set $\mathfrak{p}$ equal to this shifted part; else stop the construction.

If the construction has not yet stopped, then $\mathfrak{p}$ is a lattice path which does *not intersect* the boundary $y = x$: Simply add the completion of $\mathfrak{p}$ to the DPP and stop the construction.

See Figure 3 for an illustration of this construction: It is easy to see that it gives always a DPP without special parts and is, in fact, a bijection.

Consider the inversion word $w = (2, 1, 3, 1, 1)$ and "view" it as the encoding $(6, 6, 5, 4, 4, 4, 3, 2)$ of the lattice path $\mathfrak{p}$ shown as thick gray line: The construction "cut at the boundary $y = x$, complete and shift" applied to this path gives the three non–intersecting lattice paths shown as thin black lines; these three paths correspond DPP $\begin{smallmatrix} 6 & 6 & 5 & 4 \\ & 4 & 4 & 3 \\ & & 2 & \end{smallmatrix}$ with three rows
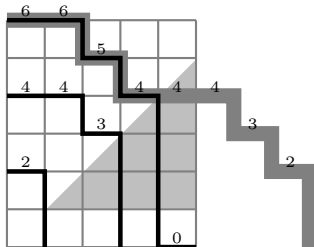

Figure 3: Illustration of Striker's bijection.

4.2 Striker's and Lalonde's bijection do not respect statistic q

Now it is easy to show by a simple example that neither Lalonde's nor Striker's bijection respect statistic q: Look at Figure 4, where we indicated

- the 1–cells by symbol $\oplus$,

- the (-1)–cells by symbol $\ominus$,

- the rd–cells by small equilateral right–angled triangles, where the sides of the right angle point to the right and downwards,

- and the lu–cells by small equilateral right–angled triangles, where the sides of the right angle point to the left and upwards.

5 The bijection between DPPs without special parts and permutation matrices

5.1 The statistic q for DPPs without special parts

Observe that if a DPP $\pi \in \mathcal{D}_n$ *without* special parts has a row of length $n - 1$, this row

- must start with part n

- and must not contain parts smaller than $n - 1$

(this fact is immediately seen from the lattice path representation of π).

Consider the 4–dimensional DPP $\pi = \begin{smallmatrix} 4 & 4 & 4 \\ & 3 & 2 \end{smallmatrix}$: $q(\pi) = 2$, since π has one part 3 and one row of length 3.

The lattice path representation of π is shown to the left, the inversion word (needed for both Lalonde's and Striker's bijection) corresponding to π is $(3, 1, 1)$, and the *monotone triangle* (needed for Striker's "intermediate" bijective step [10]) corresponding to π is shown on the top.

Both Striker's and Lalonde's bijection map π to the ASM shown to the right, with statistic q equal to 3 (the number of lu–cells, indicated by equilateral right–angled triangles, in the last row is 3). However, a bijection respecting the quadruplet of statistics should map π to the ASM shown at the bottom, which is the *only* ASM of dimension 4 with quadruplet of statistics $(3, 5, 0, 2)$.
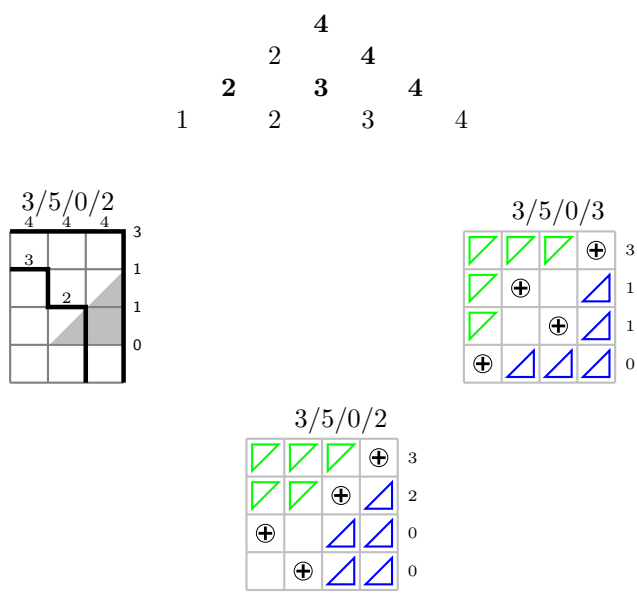

Figure 4: Neither Lalonde's nor Striker's bijection respect statistic q.

Stated otherwise: The somewhat complicated condition "existence of a path of length $n-1$" can be simply expressed as

$$\# \,(\text{parts equal to } n) + \# \,(\text{parts equal to } n-1) \geqslant n-1. \qquad (4)$$

in the case of DPPs without special parts.

5.2 The bijective construction

Given some *permutation matrix* A (i.e., some ASM without entries -1) of dimension n, set $k=1$, $A_1 = A$, and repeat the following step $n-1$ times:

- Note down the number a_k of rd–cells in the first row of A_k,

- delete the first row and the column containing the 1 (i.e., column $a_k + 1$),

- rotate the matrix by 180°,

- let A_{k+1} be the $(n-k) \times (n-k)$–matrix thus obtained, and increase k by 1.

(See Figure 5 for an illustration of this construction.)

Theorem 8. *The sequence of $(n-1)$ numbers $(a_1, a_2, \ldots, a_{n-1})$ obtained by the construction described above has the following properties:*

- $0 \leqslant a_i \leqslant n-i$ *for* $i = 1, 2, \ldots, n-1$, *i.e.,* $(a_1, a_2, \ldots, a_{n-1})$ *is an* inversion word,

- $a_1 = p\,(A)$,

- $a_2 = q\,(A) - [q\,(A) + p\,(A) \geqslant n]$,

- $a_1 + a_2 + \cdots + a_{n-1} = i\,(A)$.

Moreover, this construction gives a bijection *between permutation matrices and inversion words.*

Proof. It is clear that $0 \leqslant a_k \leqslant n-k$, since a_i is (by construction) the number of rd–cells in the first row of an $(n-k+1) \times (n-k+1)$ permutation matrix, and this number cannot exceed $n-k$.

By construction, we also have $a_1 = p\,(A)$.

Of the $q\,(A)$ lu–cells in the last row of A, precisely *one* will be deleted in the first step of our construction *if and only if* $q\,(A) + p\,(A) \geqslant n$: By the rotation at the end of the first step, precisely these "surviving" lu–cells will turn up as the rd–cells in the first row of the matrix at the beginning of the second step, whose number will give a_2.

Finally, if there are a_k rd–cells in the first row of the $(n+1-k) \times (n+1-k)$–matrix A_k at the beginning of step k, then the column $a_k + 1$ of A_k contains precisely a_k lu–cells, since the submatrix given by rows 2 to $n+1-k$ and columns 1 to a_k must contain *precisely* a_k 1–cells. So in every step, the number of deleted rd–cells *equals* the number of deleted lu–cells, and after $n-1$ steps,

- *all* of the rd–cells and lu–cells of A (recall that their number is *twice* the number of rd–cells) have been deleted,

- and *half* of these cells have been noted down during the construction; their number is equal to $a_1 + \cdots + a_{n-1}$.

It is easy to see that the mapping from permutation matrices to inversion words thus obtained is *injective*, whence it is a bijection. $\qquad\square$

Now we simply employ Striker's bijection [10] between inversions words and DPPs without special parts: This bijection maps an inversion word $(a_1, a_2, \ldots, a_{n-1})$ to a DPP π of dimension n without special parts and with precisely a_i parts $(n + 1 - i)$, whence we obtain

- $i(\pi) = a_1 + a_2 + \cdots + a_{n-1}$,

- $p(\pi) = a_1$,

- $q(\pi) = a_2 + [a_2 + a_1 \geqslant n - 1]$.

(the last assertion is due to the simple characterization (4)).

Theorem 9. *The bijection $\Psi : \mathcal{A}_n^0 \to \mathcal{D}_n^0$ established by the combination of Theorem 8 and Striker's bijection respects the triplet of statistics (p, i, q).*

Proof. The assertion is obvious for i and p, so it remains to show this for q: Letting $\epsilon := [q(A) + a_1 \geqslant n]$, we have $a_2 = q(A) - \epsilon$ and thus obtain

$$q(\pi) = q(A) - \epsilon + [(q(A) - \epsilon) + a_1 \geqslant n - 1].$$

We have to consider two cases:

- $\epsilon = 1 \iff q(A) + a_1 \geqslant n$: This implies $[q(A) - \epsilon + a_1 \geqslant n - 1] = \epsilon$.

- $\epsilon = 0 \iff q(A) + a_1 < n$: Observe that $q(A) + a_1 = q(A) + p(A) = n - 1$ is *not possible* for a *permutation matrix A*, since this would imply that the entries 1 in the first and last row appear in the *same* column. So we must have $q(A) + a_1 < n - 1$ in this case, whence we again obtain $[q(A) - \epsilon + a_1 \geqslant n - 1] = \epsilon$.

So in both cases, we have $q(\pi) = q(A)$. $\qquad\square$

5.3 An open question

Our construction was successful because we have the "translation" (4) of the (somewhat complicated) condition

DPP π of dimension n has a row of length $n - 1$

for DPPs *without* special parts to an obvious (and quite simple) corresponding condition for matrices *without* entries -1.

In the search for a "natural" bijection between ASMs and DPPs, it might be helpful to identify such "corresponding condition" for *general* matrices.

Starting with the 7×7 permutation matrix corresponding to the permutation $\sigma = (3, 2, 5, 7, 1, 6, 4)$, the pictures show the 6 steps of the bijective construction: The rows and columns to be deleted are indicated by thick gray lines. The inversion word thus obtained is $(4, 2, 1, 1, 0, 1)$.
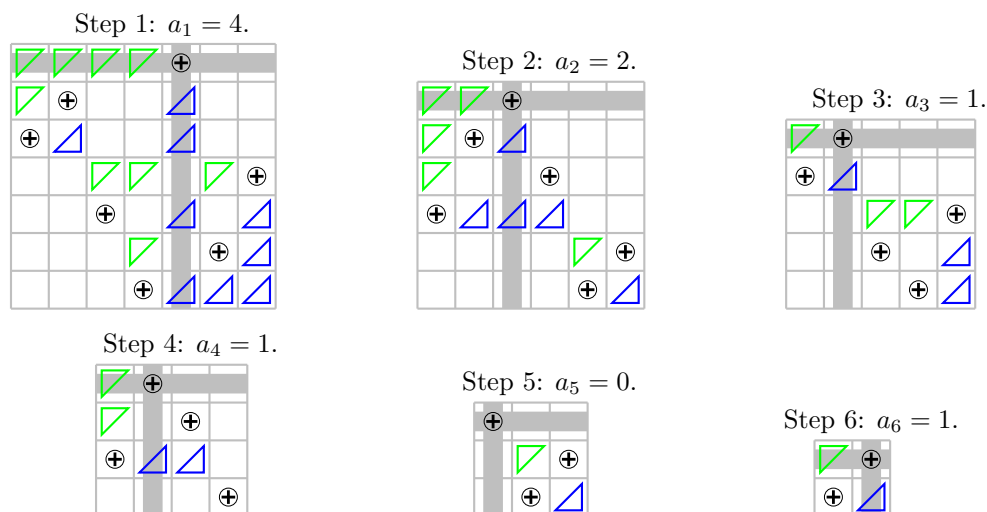

Figure 5: Illustration of the steps of the bijective construction.

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