Co-degrees resilience for perfect matchings in random hypergraphs

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Abstract

In this paper we prove an optimal co-degrees resilience property for the binomial k-uniform hypergraph model $H^k_{n,p}$ with respect to perfect matchings. That is, for a sufficiently large n which is divisible by k, and $p \geqslant C_k \log n/n$, we prove that with high probability every subgraph $H \subseteq H^k_{n,p}$ with minimum co-degree (meaning, the number of supersets every set of size $k-1$ is contained in) at least $(1/2 + o(1))np$ contains a perfect matching.

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1 Introduction

A perfect matching in a k-uniform hypergraph H is a collection of vertex-disjoint edges, covering every vertex of $V(H)$ exactly once. Clearly, a perfect matching in a k-uniform hypergraph cannot exist unless k divides n. From now on, we will always assume that this condition is met.

As opposed to graphs (that is, 2-uniform hypergraphs) where the problem of finding a perfect matching (if one exists) is relatively simple, the analogous problem in the hypergraph setting is known to be NP-hard (see [4]). Therefore, it is natural to investigate sufficient conditions for the existence of perfect matchings in hypergraphs.

A famous result by Dirac [2] asserts that every graph G on n vertices and with minimum degree $\delta(G) \geqslant n/2$ contains a Hamiltonian cycle (and therefore, by taking alternating edges along the cycle it also contains a perfect matching whenever n is even). Extending this result to hypergraphs provides us with some interesting cases, as one can

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study 'minimum degree' conditions for subsets of any size $1 \leqslant \ell < k$. That is, given a k-uniform hypergraph $H = (V, E)$ and a subset of vertices X, we define its *degree*

$$d(X) = |\{e \in E : X \subseteq e\}|.$$

Then, for every $1 \leqslant \ell < k$ we define

$$\delta_\ell(H) = \min\{d(X) : X \subseteq V(H), |X| = \ell\},$$

to be the *minimum ℓ-degree of H*. A natural question is: Given $1 \leqslant \ell < k$, what is the minimum $m_\ell(n)$ such that every k-uniform hypergraph on n vertices with $\delta_\ell(H) \geqslant m_\ell(n)$ contains a perfect matching?

The above question has attracted a lot of attention in the last few decades. For more details about previous work and open problems, we will refer the reader to surveys by Rödl and Ruciński [8] and Keevash [5]. In this paper we restrict our attention to the case where $\ell = k-1$. Following a long line of work in studying this property, which is expanded upon in the former survey, Kühn and Osthus proved in [6] that every k-uniform hypergraph with $\delta_{k-1} \geqslant n/2 + \sqrt{2n \log n}$ contains a perfect matching. This bound is optimal with an additive error term of $\sqrt{2n \log n}$. Note that one can view this result as follows: Start with a complete k-uniform hypergraph on n vertices (this clearly contains a perfect matching). Imagine that an adversary is allowed to delete 'many' edges in an arbitrary way, under the restriction that he/she cannot delete more than r edges that intersect on a subset of size at least $(k-1)$. What then, is the largest r for which the resulting hypergraph always contains a perfect matching? We refer to this value as the '$(k-1)$-local-resilience' of the hypergraph. The above mentioned result equivalently shows that such a hypergraph has '$(k-1)$-local-resilience' at least $n/2 - \sqrt{2n \log n}$.

Here we study a similar problem in the random hypergraph setting. Let $H^k_{n,p}$ be a random variable which outputs a k-uniform hypergraph on vertex set $[n]$ by including any k-subset $X \in \binom{[n]}{k}$ as an edge with probability p, independently. The existence of perfect matchings in a typical $H^k_{n,p}$ is a well studied problem with a very rich history. Unlike for random graphs where finding a 'threshold' for the existence of a perfect matching is quite simple, the problem of finding a 'threshold' function p for the existence of a perfect matching, with high probability, in the hypergraph setting is notoriously hard. After a few decades of study, in 2008 Johansson, Kahn and Vu [3] finally managed to determine the threshold. Among their results, one of particular note is that for $p \geqslant C \log n / n^{k-1}$, whp $H^k_{n,p}$ contains a perfect matching. On the other hand, it is quite simple to show that if $p \leqslant c \log n / n^{k-1}$ for some small constant c, then a typical $H^k_{n,p}$ contains isolated vertices and thus has no perfect matchings.

In this note we determine the '$(k-1)$-local-resilience' of a typical $H^k_{n,p}$. Note that if $p = o(\log n / n)$ then whp there exists a $(k-1)$-set of vertices which is not contained in any edge and therefore, for the study of $(k-1)$-resilience, it is natural to restrict our attention to $p \geqslant C \log n / n$ (which is significantly above the threshold for a perfect matching as obtained in [3]). The following theorem gives a complete solution to this problem for this range of p.

Theorem 1. *Let $k \in \mathbb{N}$, let $\varepsilon > 0$, and let $C := C(k, \varepsilon)$ be a sufficiently large constant. Then, for all $p \geqslant \frac{C \log n}{n}$, whp a hypergraph $H^k_{n,p}$ is such that the following holds: Every spanning subhypergraph $H \subseteq H^k_{n,p}$ with $\delta_{k-1}(H) \geqslant (1/2+\varepsilon)np$ contains a perfect matching.*

Next, we show that the above theorem is asymptotically tight.

Theorem 2. *Let $k \in \mathbb{N}$, let $\varepsilon > 0$, and let $C := C(k, \varepsilon)$ be a sufficiently large constant. Then, for all $p \geqslant \frac{C \log n}{n}$, any hypergraph $H^k_{n,p}$ is such that the following holds: Whp there exists $H \subseteq H^k_{n,p}$ with $\delta_{k-1}(H) \geqslant (1/2 - \varepsilon)np$ that does not contain a perfect matching.*

Sketch. This proof is based on an idea of Kühn and Osthus outlined in [6]. Fix a partition of $V(H) = V_1 \cup V_2$ into two sets of size roughly $n/2$, where $|V_1|$ is odd. Now, expose all the edges of $H^k_{n,p}$ and let H be the subhypergraph obtained by deleting all the hyperedges that intersect V_1 on an odd number of vertices. Clearly, H cannot have a perfect matching, as every edge covers an even number of vertices in V_1 and $|V_1|$ is odd. Now, we demonstrate that every $(k-1)$-subset of vertices still has at least $(1/2 - \varepsilon)np$ neighbors in H. Indeed, given any $(k-1)$ subset X, we distinguish between two cases:

1. $|X \cap V_1|$ is even – as we clearly kept all the edges of the form $X \cup \{v\}$, $v \in V_2$, and since $|V_2| \approx n/2$, by a standard application of Chernoff's bounds, X is contained in at least $(1/2 - \varepsilon)np$ many such edges as required.

2. $|X \cap V_1|$ is odd – as we clearly kept all the edges of the form $X \cup \{v\}$, $v \in V_1$, and since $|V_1| \approx n/2$, a similar reasoning as in 1. gives the desired.

All in all, whp the resulting subhypergraph has $\delta_{k-1}(H) \geqslant (1/2 - \varepsilon)np$ and does not contain a perfect matching. □

2 Notation

For the sake of brevity, we present the following, commonly used notation:

Given a graph G and $X \subseteq V(G)$, let $N(X) = \cup_{x \in X} N(x)$. For two subsets $X, Y \subseteq V(G)$ we define $E(X, Y)$ to be the set of all edges $xy \in E(G)$ with $x \in X$ and $y \in Y$, and set $e_G(X, Y) := |E(X, Y)|$.

For a k-uniform hypergraph H on vertex set $V(H)$, and for two subsets $X, Y \subseteq V(H)$ we define

$$d(X, Y) = |\{e \in E(H) : X \subseteq e \text{ and } e \setminus X \subseteq Y\}|.$$

Given any k-partite, k-uniform hypergraph with parts $V(H) = V_1 \cup \ldots \cup V_k$ of the same size m we consider all V_i to be disjoint copies of the integers 1 to m, without loss of generality.

Finally, for every random variable X, we let $M(X)$ be its *median*.

3 Outline

In this section we give a brief outline of our argument. Consider a typical $H^k_{n,p}$, and let $H \subseteq H^k_{n,p}$ with $\delta_{k-1}(H) \geqslant (\frac{1}{2}+\varepsilon)np$. In order to show that H contains a perfect matching, we first show that some auxiliary bipartite graph B contains a perfect matching. Then, we show that every perfect matching in B can be translated into a perfect matching in H.

To this end, we first find a partition $V(H) = V_1 \cup \cdots \cup V_k$, with all V_i's having the exact same size $m = \frac{n}{k}$, such that the following property holds: For every subset $X \in \binom{[n]}{k-1}$ and for every $1 \leqslant i \leqslant k$ we have

$$d_H(X, V_i) \in (1 \pm \varepsilon) \cdot \frac{d_H(X)}{k}.$$

Then, we let H' be the k-partite, k-uniform subhypergraph induced by this partition of $V(H)$.

Now, given some set of permutations $\pi = \{\pi_1, \pi_2, \cdots, \pi_{k-1}\}$, $\pi_i = [m] \to V_i$, we can construct a bipartite graph $B_\pi(H')$ as follows:

The parts of $B_\pi(H')$ are V_k and

$$X_\pi = \{\{\pi_1(i), \pi_2(i), \ldots, \pi_{k-1}(i)\} \mid 1 \leqslant i \leqslant m\}.$$

The edges of $B_\pi(H')$ consist of all pairs $xv \in X_\pi \times V_k$, for which $x \cup \{v\} \in E(H')$.

A moment's thought now reveals that a perfect matching in any such $B_\pi(H')$ corresponds to a perfect matching in H', which itself corresponds to a perfect matching in H. Therefore, the main part of the proof consists of showing that, with high probability, there exists a π such that $B_\pi(H')$ contains a perfect matching.

4 Tools and Preliminary Results

In this section we present some tools to be used in the proof of our main result.

4.1 Chernoff's inequalities

First, we need the following well-known bound on the upper and lower tails of the binomial distribution, outlined by Chernoff (see Appendix A in [1]).

Lemma 3 (Chernoff's inequality). *Let $X \sim Bin(n,p)$ and let $\mathbb{E}(X) = \mu$. Then*

- $\mathbb{P}(X < (1-a)\mu) < e^{-a^2\mu/2}$ *for every* $a > 0$*;*
- $\mathbb{P}(X > (1+a)\mu) < e^{-a^2\mu/3}$ *for every* $0 < a < 3/2$.

Remark 4. These bounds also hold when X is hypergeometrically distributed with mean μ.

In addition, we will make use of the following simple bound.

Lemma 5. *Let $X \sim Bin(m,q)$. Then, for all k we have*

$$\Pr[X \geqslant k] \leqslant \left(\frac{emq}{k}\right)^k.$$

Proof. Note that

$$\Pr[X \geqslant k] \leqslant \binom{m}{k} q^k \leqslant \left(\frac{emq}{k}\right)^k$$

as desired. □

4.2 Talagrand's type inequality

Our main concentration tool is the following theorem from McDiarmid [7].

Theorem 6. *Given a set S of size m, we let $Sym(S)$ denote the set of all $m!$ permutations of S. Let $\{B_1, \dots, B_k\}$ be a family of finite non-empty sets, and let $\Omega = \prod_i Sym(B_i)$. Let $\pi = \{\pi_1, \dots, \pi_k\}$ be a family of independent permutations, such that for i, $\pi_i \in Sym(B_i)$ is chosen uniformly at random.*

Let c and r be constants, and suppose that the nonnegative real-valued function h on Ω satisfies the following conditions for each $\pi \in \Omega$.

1. *Swapping any two elements in any π_i can change the value of h by at most $2c$.*
2. *If $h(\pi) = s$, there exists a set $\pi_{proof} \subseteq \pi$ of size at most rs, such that $h(\pi') \geqslant s$ for any $\pi' \in \Omega$ where $\pi' \supseteq \pi_{proof}$.*

Then for each $t \geqslant 0$ we have

$$\Pr[h \leqslant M(h(\pi)) - t] \leqslant 2\exp\left(-\frac{t^2}{16rc^2M}\right).$$

4.3 Hall's theorem

It is convenient for us to work with the following equivalent version of Hall's theorem (the proof is an easy exercise).

Theorem 7. *Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = n$. Then, G contains a perfect matching if and only if the following holds:*

1. *For all $X \subseteq A$ of size $|X| \leqslant n/2$ we have $|N(X)| \geqslant |X|$, and*
2. *For all $Y \subseteq B$ of size $|Y| \leqslant n/2$ we have $|N(Y)| \geqslant |Y|$.*

4.4 Properties of random hypergraphs

In this section we collect some properties that a typical $H^k_{n,p}$ satisfies. First, we show that all the $(k-1)$-degrees are 'more or less' the same.

Lemma 8. *Let $\varepsilon > 0$ and let $k \geqslant 2$ be any integer. Then, whp we have*

$$(1-\varepsilon)np \leqslant \delta_{k-1}(H^k_{n,p}) \leqslant \Delta_{k-1}(H^k_{n,p}) \leqslant (1+\varepsilon)np,$$

provided that $p = \omega(\log n/n)$.

Proof. Let us fix some $X \in \binom{[n]}{k-1}$. Observe that $d(X) \sim Bin(n-k+1,p)$, and therefore

$$\mu := \mathbb{E}[d(X)] = (n-k+1)p.$$

Hence, by Chernoff's inequalities we obtain that

$$\Pr[d(X) \notin (1 \pm \varepsilon)\mu] \leqslant 2e^{-\frac{\varepsilon^2 \mu}{3}} = o(1/n^k).$$

All in all, by taking a union bound over all sets $\binom{[n]}{k-1}$, we conclude that

$$\Pr[\exists X \in \binom{[n]}{k-1} \text{ s.t. } d(X) \notin (1 \pm \varepsilon)\mu] = o(1).$$

This completes the proof. □

In the proof of our main result we will convert the problem of finding a perfect matching in H into the problem of finding a perfect matching in some auxiliary bipartite graph. In order to do so, we wish to partition our hypergraph $H \subseteq H^k_{n,p}$ into k equal parts satisfying some 'degree assumptions', and then to define our auxiliary bipartite graph based on such a partition. In the following lemma we show that, given a k-uniform hypergraph H with 'relatively large' $(k-1)$-degree, a random partition of its vertices into equally sized parts satisfies these assumptions.

Lemma 9. *For every $\varepsilon > 0$ there exists $C := C(\varepsilon)$ for which the following holds. Let H be a k-uniform hypergraph on n vertices, where n is sufficiently large. Suppose that $\delta_{k-1}(H) \geqslant C \log n$ and that n is divisible by k. Then, there exists a partition $V(H) = V_1 \cup \ldots \cup V_k$ into sets of the exact same size satisfying the following property: For every subset $X \in \binom{[n]}{k-1}$ and for every $1 \leqslant i \leqslant k$ we have*

$$d_H(X, V_i) \in (1 \pm \varepsilon) \cdot \frac{d_H(X)}{k}.$$

Proof. Let H be a a k-uniform hypergraph on n vertices, where n is sufficiently large. Consider the random partition $V(H) = V_1 \cup \ldots \cup V_k$ into sets of the exact same size. For some fixed X and i, observe that $d_H(X, V_i)$ is hypergeometrically distributed with an expected value of $\frac{d_H(X)}{k}$. Therefore, we can use Lemma 3 to determine that

$$\Pr[d_H(X, V_i) > (1+\varepsilon) \cdot \frac{d_H(X)}{k}] \leqslant e^{-\varepsilon^2 \frac{d_H(X)}{k}/3} \leqslant e^{-k \log n} = n^{-k},$$

where the last inequality holds for a large enough C.

By applying a union bound over all possible X's and i's, we obtain that the probability of having such a set and an index i is at most

$$\binom{n}{k-1} k n^{-k} = o(1).$$

Similarly, we obtain that

$$\Pr\left[\exists X \text{ and } i : d_H(X, V_i) < (1-\varepsilon)\cdot \frac{d_H(X)}{k}\right] = o(1).$$

This completes the proof. □

Definition 10. Let $\varepsilon > 0$, $p \in (0,1]$, and $m \in \mathbb{N}$. A bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = m$ is called (ε, p)-*pseudorandom* if it satisfies the following properties:

1. $\delta(G) \geqslant (1/2 + \varepsilon)mp$,
2. for every $X \subseteq A$ and $Y \subseteq B$ with $|X| - 1 = |Y| \leqslant m/10$ we have $e_G(X, Y) \leqslant mp|X|/2$,
3. for every $X \subseteq A$ and $Y \subseteq B$ with $m/10 \leqslant |X| - 1 = |Y| \leqslant m/2$ we have $e_G(X, Y) \leqslant (1/2 + \varepsilon/2)mp|X|$

Definition 11. Let H' be a k-partite, k-uniform hypergraph with parts $V(H') = V_1 \cup \ldots \cup V_k$ of the same size m. Given a set of permutations $\pi = \{\pi_1, \pi_2, \ldots \pi_{k-1}\}$, $\pi_i : [m] \to V_i$, we construct an auxiliary bipartite graph, $B_\pi := B_\pi(H')$, as follows:

Let $X_\pi = \{\{\pi_1(i), \pi_2(i), \ldots, \pi_{k-1}(i)\}; 1 \leqslant i \leqslant m\}$ and V_k be the parts of B_π. For every pair xv with $x \in X_\pi$ and $v \in V_k$, we let $xv \in E(B_\pi)$ iff $x \cup \{v\} \in E(H')$.

Remark 12. Note that every edge in a given $B_\pi(H')$ with parts $x \in X_\pi$ and $v \in V_k$ corresponds to an edge $\pi_1(i) \cup \pi_2(i) \ldots \pi_{k-1}(i) \cup \{v\}$ in H' for some $1 \leqslant i \leqslant m$. Therefore, if $B_\pi(H')$ contains a perfect matching, clearly H' contains a perfect matching as well. Having established this fact, our main goal is to show that there exists a π for which B_π contains a perfect matching.

We now wish to demonstrate that given a 'proper' k-partite, k-uniform hypergraph H', a randomly chosen π results in a $B_\pi(H')$ with a sufficiently large minimum degree. As will be seen soon, the 'problematic' random variables that we need to control are $d_{B_\pi}(v)$, where $v \in V_k$. In order to prove that these variables concentrate about their expectation, we will use Theorem 6.

For the sake of simplicity in the following lemma, we define this notation: Suppose that H' is a k-partite, k-uniform hypergraph with parts $V(H') = V_1 \cup \ldots \cup V_k$. Let $W_i := V_1 \times \ldots V_{i-1} \times V_{i+1} \times \ldots \times V_k$. For every $X \in W_i$ (note that $|X| = k-1$) define

$$\delta^*_{k-1}(H') := \min\{d(X, V_i) : X \in W_i, \text{ and } 1 \leqslant i \leqslant k\}.$$

Lemma 13. *Let $0 < \alpha < 1/2$ and let $m \in \mathbb{N}$ be sufficiently large. Let H' be a k-partite, k-uniform hypergraph with parts $V(H') = V_1 \cup \ldots \cup V_k$ of the same size m. Suppose that $\delta^*_{k-1}(H') \geqslant 200/\alpha^2$. Let B_π be the auxiliary-bipartite graph formed from the set of permutations $\pi := \{\pi_1, id_2, \ldots, id_{k-1}\}$, where π_1 is a random permutation of V_1 and each id_j is the identity permutation of V_j. Let $\mu_v = \mathbb{E}[d_{B_\pi}(v)]$. Then, for every $v \in V_k$ we have*

$$M_v = M(d_{B_\pi}(v)) \in (1 \pm \alpha)\mu_v.$$

Remark 14. The above lemma enables us to use μ_v instead of M_v in Theorem 6 when it is applied to $d_{B_\pi}(v)$.

Proof. Consider the B_π, formed from the set of permutations $\pi := \{\pi_1, id_2, \dots, id_{k-1}\}$, where π_1 is a random permutation of V_1 and each id_j is the identity permutation of V_j. Let v be some element in V_k. For each $1 \leqslant i \leqslant m$, let $A_i := \{id_2(i), id_3(i) \dots, id_{k-1}(i)\}$, and let $d_i(v)$ be the number of extensions of $\{v\} \cup A_i$ into V_1 (that is, the number of edges $e \in E(H')$ for which $\{v\} \cup A_i \subseteq e$). Moreover, let $d_v = \sum_i d_i(v)$, and for each i define a indicator random variable $\mathbb{1}_i$, where $\mathbb{1}_i = 1$ if $\{\pi_1(i)\} \cup A_i \cup \{v\} \in E(H')$. Observe that $d_{B_\pi}(v) = \sum \mathbb{1}_i$.

Our plan is to compute $\mu_v := \mathbb{E}[d_{B_\pi}(v)]$ and $\sigma^2 = Var(d_{B_\pi}(v))$ and to show that $\sigma^2 \leqslant \alpha^2\mu_v^2/100$. The desired result will then be easily obtained as follows: First, note that by Chebyshev's inequality we have

$$\mathbb{P}[|d_{B_\pi}(v) - \mu_v| \geqslant \alpha\mu_v] \leqslant \frac{\sigma^2}{\alpha^2\mu_v^2} \leqslant 1/100.$$

Since with probability at least 99/100 we have that $d_{B_\pi}(v) \in (1 \pm \alpha)\mu_v$, we conclude that the median also lies in this interval.

It remains to compute μ_v and σ^2. Since $\mathbb{P}[\mathbb{1}_i = 1] = \frac{d_i(v)}{m}$, by linearity of expectation we obtain

$$\mu_v = \sum_{i=1}^{m} \mathbb{E}[\mathbb{1}_i] = \sum_{i=1}^{m} \frac{d_i(v)}{m} = \frac{d_v}{m}.$$

To compute the variance, note that

$$\begin{aligned}
Var\left(d_{B_\pi}(v)\right) &= Var\left(\sum_{i=1}^{m} \mathbb{1}_i\right) = \sum_{i=1}^{m} Var\left(\mathbb{1}_i\right) + 2\sum_{i<j} Cov(\mathbb{1}_i, \mathbb{1}_j) \\
&\leqslant \mu_v + 2\sum_{i<j} \left(\mathbb{E}[\mathbb{1}_i\mathbb{1}_j] - \mathbb{E}[\mathbb{1}_i]\mathbb{E}[\mathbb{1}_j]\right) \\
&\leqslant \mu_v + 2\sum_{i<j} \left(\frac{d_i(v)d_j(v)}{m(m-1)} - \frac{d_i(v)d_j(v)}{m^2}\right) = \mu_v + 2\sum_{i<j} \left(\frac{d_i(v)d_j(v)}{m^2(m-1)}\right) \\
&\leqslant \mu_v + 2\sum_{i=1}^{m}\sum_{j=1}^{m} \left(\frac{d_i(v)d_j(v)}{m^2(m-1)}\right) \leqslant \mu_v + 2\sum_{i=1}^{m} \left(\frac{d_i(v)d_v}{m^2(m-1)}\right) \\
&= \mu_v + \frac{2d_v^2}{m^2(m-1)} = \mu_v + \frac{2\mu_v^2}{m-1}.
\end{aligned}$$

To complete the proof let us first observe that since m is sufficiently large we have $\frac{2\mu_v^2}{m-1} \leqslant \alpha^2\mu_v^2/200$. Second, note that since $\mu_v \geqslant 200/\alpha^2$ we have that $\mu_v \leqslant \alpha^2\mu_v^2/200$. Plugging these estimates into the last line of the above equation gives us the desired. □

Lemma 15. *For every $\varepsilon > 0$ there exists $C := C(\varepsilon)$ for which the following holds for sufficiently large $m \in \mathbb{N}$ and $p = C\log m/m$. Let H' be a k-partite, k-uniform hypergraph with parts $V(H') = V_1 \cup \ldots \cup V_k$ of the same size m. Suppose that $\delta^*_{k-1}(H') \geqslant (\frac{1}{2}+\varepsilon)mp$. Then there exists $\pi := \{\pi_1, \pi_2, \ldots, \pi_{k-1}\}$, $\pi_i : [m] \to V_i$, s.t. $\delta(B_\pi) \geqslant (\frac{1}{2}+\frac{\varepsilon}{2})mp$.*

Proof. Consider the B_π, formed from the set of permutations $\pi := \{\pi_1, id_2, \ldots, id_{k-1}\}$, where π_1 is random and id_j is the identity permutation for V_j. As $\delta^*_{k-1}(H') \geqslant (\frac{1}{2}+\varepsilon)mp$, it is guaranteed that for all $x \in X_\pi$ we have (deterministically) that $d_{B_\pi}(x) \geqslant (\frac{1}{2}+\varepsilon)mp$.

Consider some $v \in V_k$ and observe from the proof of Lemma 13, under the same notation, that $\mathbb{E}[d_{B_\pi}(v)] = \frac{d_v}{m} \geqslant (1/2+\varepsilon)mp$.

In order to complete the proof, we want to show that the $d_{B_\pi}(v)$'s are 'highly concentrated' using Theorem 6. To this end, let $h(\pi) = d_{B_\pi}(v)$ and note that swapping any two elements of π_1 can change h by at most 2. Moreover, note that if $h(\pi) \geqslant s$, then it is enough to specify only s elements of V. Therefore, $h(\pi)$ satisfies the conditions outlined by Talagrand's type inequality with $c = 1$ and $r = 1$.

Now, let $\alpha = \varepsilon/100$, and observe that by Lemma 13 we have that the median M of $d_{B_\pi}(v)$ lies in the interval $(1 \pm \alpha)\mathbb{E}[d_{B_\pi}(v)]$.

Therefore, we have

$$\Pr[h \leqslant (\frac{1}{2}+\varepsilon/2)mp] \leqslant \Pr[h \leqslant (1-\varepsilon/2)\mathbb{E}[d_{B_\pi}(v)]]$$

and the latter is at most

$$\Pr[h \leqslant (1-\varepsilon/2)(1+\alpha)M] \leqslant \Pr[h \leqslant (1-\varepsilon/4)M].$$

Now, by Theorem 6 we obtain that

$$\Pr[h \leqslant (1/2+\varepsilon/2)mp] \leqslant 2\exp\left(-\frac{(\varepsilon M/4)^2}{16M}\right).$$

Next, using (again) the fact that $M \in (1\pm\alpha)\mathbb{E}[d_{B_\pi}(v)]$ and that $\mathbb{E}[d_{B_\pi}(v)] = \Theta(mp) \geqslant C\log m$, we can upper bound the above right hand side by

$$2\exp\left(-\Theta(mp)\right) \leqslant n^{-2}.$$

Finally, in order to complete the proof, we take a union bound over all $v \in V_k$ and obtain that whp $\delta(B_\pi) \geqslant (\frac{1}{2}+\frac{\varepsilon}{2})mp$. □

Lemma 16. *Let $\varepsilon > 0$, $k \in \mathbb{N}$ and $p \geqslant C\log n/n$, where $C := C(\varepsilon, k) > 0$ is a sufficiently large constant. Then, a random hypergraph $H^k_{n,p}$ with high probability satisfies the following: For every k-partite, k-uniform subhypergraph $H' \subseteq H^k_{n,p}$ with parts $V(H') = V_1 \cup \ldots \cup V_k$ of the same size $m := \frac{n}{k}$, if $\delta^*_{k-1}(H') \geqslant (1/2+\varepsilon)mp$, there exists $\pi := \{\pi_1, \pi_2, \ldots, \pi_{k-1}\}$, $\pi_i : [m] \to V_i$, s.t. B_π is $(\varepsilon/2, p)$-pseudorandom.*

Proof. Let H' be such a subhypergraph. Our goal is to prove the existence of π for which B_π is $(\varepsilon/2, p)$-pseudorandom. That is, we want to show that B_π satisfies the following properties:

1. $\delta(B_\pi) \geqslant (1/2+\varepsilon/2)mp$,
2. for every $X \subseteq X_\pi$ and $Y \subseteq V_k$ with $|X|-1=|Y| \leqslant m/10$ we have $e_{B_\pi}(X,Y) \leqslant mp|X|/2$,
3. for every $X \subseteq X_\pi$ and $Y \subseteq V_k$ with $m/10 \leqslant |X|-1=|Y| \leqslant m/2$ we have $e_{B_\pi}(X,Y) \leqslant (1/2+\varepsilon/4)|X|mp$

Let π be obtained as in Lemma 15, and consider $B_\pi = (X_\pi \cup V_k, E)$. Clearly, Property 1 is satisfied by the conclusion of Lemma 15.

For Property 2, let us fix $X \subseteq X_\pi$ and $Y \subseteq V_k$ of sizes x and y respectively where $x-1=y \leqslant m/10$. We now wish to establish an upper bound for the number of edges between them. Assume towards contradiction that $e_{B_\pi}(X,Y) > mpx/2$. Observe that this translates to the following: There exist x disjoint sets $F_1,\dots,F_x$, each of size exactly $k-1$ and a set Y of size $x-1$, which is disjoint to all the F_is, such that the number of edges in $H^k_{n,p}$, of the form $F_i \cup \{a\}$ where $a \in Y$, is larger than $mpx/2$. Let us show that whp $H^k_{n,p}$ has no such sets, thereby also guaranteeing that whp no such sets exist in any subhypergraph $H' \subseteq H^k_{n,p}$.

First, let us fix such $F_1,\dots,F_x$ and Y. Observe that the expected number of edges of the form $F_i \cup \{y\}$ in $H^k_{n,p}$ is exactly xyp. Therefore, by Lemma 5 we obtain

$$\Pr[\#\text{ such edges } \geqslant xmp/2] \leqslant \left(\frac{2exyp}{xmp}\right)^{xmp/2} = \exp\left(-\frac{xmp}{2}\log\frac{m}{2ey}\right).$$

By applying the union bound over all choice of F_i's and Y we obtain that the probability for having such sets which span at least $xmp/2$ edges of the form discussed above, is at most

$$\begin{aligned}
&\sum_{x=mp/2}^{m/10}\binom{n}{k-1}^x\binom{n}{x}\exp\left(-\frac{xmp}{2}\log\frac{m}{2ey}\right)\\
&\leqslant \sum_{x=mp/2}^{m/10}\left(\frac{en}{k-1}\right)^{kx}\left(\frac{en}{x}\right)^x\exp\left(-\frac{xmp}{2}\log\left(\frac{m}{2ex}\right)\right)\\
&\leqslant \sum_{x=mp/2}^{m/10}\exp\left(kx\log\left(\frac{en}{k-1}\right)+x\log\left(\frac{en}{x}\right)-\frac{xmp}{2}\log\left(\frac{m}{2ex}\right)\right)\\
&\leqslant \sum_{x=mp/2}^{m/10}\exp\left((k+1)\,x\log n-\frac{mpx}{2}\log\left(\frac{10}{2e}\right)+O(1)\right)=o(1)
\end{aligned}$$

where the last equality holds if we pick $p = C\log n/n$ where C is a sufficiently large constant to satisfy

$$\frac{mp}{2}\log\left(\frac{10}{2e}\right) > 2(k+1)\log n$$

Therefore, whp B_π satisfies property 2.

For property 3, let us fix $X \subseteq X_\pi$ and $Y \subseteq V_k$ of sizes x and y respectively where $m/10 \leqslant x-1 = y \leqslant m/2$. We now wish to establish an upper bound for the number of edges between them. Assume towards contradiction that $e_{B_\pi}(X,Y) > (1/2+\varepsilon/4)mpx$. Observe that this translates to the following: There exist x disjoint sets $F_1,\dots,F_x$, each of size exactly $k-1$ and a set Y of size $x-1$, which is disjoint to all the F_is, such that the number of edges in $H^k_{n,p}$, of the form $F_i \cup \{a\}$ where $a \in Y$, is larger than $(1/2+\varepsilon/4)mpx$. Let us show that whp $H^k_{n,p}$ has no such sets, thereby also guaranteeing that whp no such sets exist in any subhypergraph $H' \subseteq H^k_{n,p}$.

First, let us fix such $F_1,\dots,F_x$ and Y. Observe that the expected number of edges of the form $F_i \cup \{y\}$ in $H^k_{n,p}$ is exactly xyp. Therefore, by Lemma 3 we obtain

$$\Pr[\#\text{ such edges } \geqslant (1/2+\varepsilon/4)mpx] \leqslant \exp\left(-\varepsilon^2 xyp/40\right).$$

By applying the union bound we obtain that the probability to have such sets is at most

$$\begin{aligned}
&\sum_{x=m/10}^{m/2} \binom{n}{k-1}^x \binom{n}{x} \exp\left(-\varepsilon^2 xyp/40\right)\\
&\quad\leqslant \sum_{x=m/10}^{m/2} n^{(k-1)x} n^x \exp\left(-\varepsilon^2 xyp/40\right)\\
&\quad\leqslant \sum_{x=m/10}^{m/2} \exp\left((k-1)x\log n + x\log n - \varepsilon^2 x^2 p/40\right) = o(1)
\end{aligned}$$

where the last inequality holds if we pick $p = C\log n/n$ where C is a sufficiently large constant to satisfy

$$pm\varepsilon^2/400 \geqslant 2k\log n.$$

Therefore, whp B_π satisfies property 3. We can conclude that whp B_π satisfies all three properties, and is $(\varepsilon/2,\, p)$-pseudorandom. This completes the proof. □

Now that we know we can construct an $(\varepsilon/2, p)$-pseudorandom bipartite graph B_π from every subhypergraph H with the properties outlined above, we will make use of the following lemma to show that every such B_π must also contain a perfect matching. A similar proof appears in [9].

Lemma 17. *Every (ε,p)-pseudorandom bipartite graph contains a perfect matching.*

Proof. Let $G = (A \cup B, E)$ be an (ε,p)-pseudorandom bipartite graph with $|A| = |B| = m$. If G does not contain a perfect matching, then it must violate the condition in Theorem 7. That is, without loss of generality, there exists some $X \subseteq A$ of size $x \leqslant m/2$ and $Y \subseteq B$ of size $x-1$ such that $N_G(X) \subseteq Y$. In particular, as $\delta(G) \geqslant (1/2+\varepsilon)mp$ by property 1, it follows that $e_G(X, Y) \geqslant (1/2 + \varepsilon)mpx$. In order to complete the proof we show that G does not contain two such sets for all $1 \leqslant x \leqslant m/2$.

We distinguish between three cases: First, assume $x \leqslant mp/2$. As $|Y| \leqslant x < (1/2 + \varepsilon)mp \leqslant \delta(G)$, it follows that $N_G(X) \nsubseteq Y$.

Second, assume that $mp/2 \leqslant x \leqslant m/10$. By property 2, $e_G(X, Y) \leqslant mpx/2 < (1/2+\varepsilon)mpx$, which is clearly a contradiction. Lastly, consider the case $m/10 \leqslant x \leqslant m/2$. By property 3, $e_G(X, Y) \leqslant (1/2+\varepsilon/2)xmp < (1/2+\varepsilon)mpx$, which is also a contradiction. This completes the proof. □

5 Proof of Theorem 1

Now we are ready to prove Theorem 1.

Proof. Let $k \in \mathbb{N}$, $\varepsilon > 0$ and $p \geqslant C \log n/n$, for a sufficiently large C. Observe that, by Lemma 8, whp a hypergraph $H^k_{n,p}$ satisfies

$$(1 - \varepsilon)np \leqslant \delta_{k-1}(H^k_{n,p}) \leqslant \Delta_{k-1}(H^k_{n,p}) \leqslant (1 + \varepsilon)np.$$

Let $H \subseteq H^k_{n,p}$ be any subhypergraph with $\delta_{k-1}(H) \geqslant (1/2 + \varepsilon)np$. We wish to show that H contains a perfect matching.

To this end, as was previously explained in the outline, we will construct a bipartite graph in such a way that each perfect matching of this graph corresponds to a perfect matching of H.

To do so, let $\alpha > 0$ where $(1 - \alpha)(1/2 + \varepsilon) \geqslant 1/2 + \varepsilon/2$, and let us take a partitioning $[n] = V_1 \cup \ldots \cup V_k$ into sets of the exact same size for which the following holds: For every subset $X \in \binom{[n]}{k-1}$ and for every $1 \leqslant i \leqslant k$ we have

$$d_H(X, V_i) \in (1 \pm \alpha) \cdot \frac{d_H(X)}{k}.$$

In particular, for all $X \in \binom{[n]}{k-1}$ and all $1 \leqslant i \leqslant k$, we have

$$d_H(X, V_i) \geqslant (1/2 + \varepsilon/2)mp,$$

where $m = \frac{n}{k}$. The existence of such a partitioning is guaranteed by Lemma 9.

Next, let H' be the resulting k-partite, k-uniform subhypergraph induced by the above partitioning. Recall that

$$\delta^*_{k-1}(H') := \min\{d(X, V_i) : X \in W_i, \text{ and } 1 \leqslant i \leqslant k\},$$

where $W_i = V_1 \times \ldots \times V_{i-1} \times V_{i+1} \times \ldots \times V_k$.

Clearly, $\delta^*_{k-1}(H') \geqslant (1/2+\varepsilon/2)mp$. Therefore, Lemma 16 guarantees that there exists an auxiliary bipartite graph $B_\pi(H')$ (as defined in 11) that is $(\varepsilon/4, p)$-pseudorandom. By Lemma 17, such a B_π would contain a perfect matching and therefore, by Remark 12, H' must also contain a perfect matching. This completes the proof. □

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