Cooperative colorings of trees and of bipartite graphs

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Abstract

Given a system $(G_1, \ldots, G_m)$ of graphs on the same vertex set V, a cooperative coloring is a choice of vertex sets $I_1, \ldots, I_m$, such that I_j is independent in G_j and $\bigcup_{j=1}^m I_j = V$. For a class $\mathcal{G}$ of graphs, let $m_{\mathcal{G}}(d)$ be the minimal m such that every m graphs from $\mathcal{G}$ with maximum degree d have a cooperative coloring. We prove that $\Omega(\log\log d) \leqslant m_{\mathcal{T}}(d) \leqslant O(\log d)$ and $\Omega(\log d) \leqslant m_{\mathcal{B}}(d) \leqslant O(d/\log d)$, where $\mathcal{T}$ is the class of trees and $\mathcal{B}$ is the class of bipartite graphs.

Mathematics Subject Classifications: 05C15, 05C69

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1 Introduction

A set of vertices in a graph is called *independent* if no two vertices in it form an edge. A *coloring* of a graph G is a covering of $V(G)$ by independent sets. Given a system $(G_1, \ldots, G_m)$ of graphs on the same vertex set V, a *cooperative coloring* is a choice of vertex sets $\{I_j \subseteq V : j \in [m]\}$ such that I_j is independent in G_j and $\bigcup_{j=1}^{m} I_j = V$. If all G_j's are the same graph G, then a cooperative coloring is just a proper vertex coloring of G by m independent sets.

A basic fact about vertex coloring is that every graph G of maximum degree d is $(d+1)$-colorable. It is therefore natural to ask whether $d+1$ graphs, each of maximum degree d, always have a cooperative coloring. This was shown to be false:

Theorem 1 (Theorem 5.1 of Aharoni, Holzman, Howard and Sprüssel [AHHS15])**.** *For every $d \geqslant 2$, there exist $d+1$ graphs of maximum degree d that do not have a cooperative coloring.*

Using the fundamental result on independent transversals of Haxell [Hax01, Theorem 2], it can be shown that $2d$ graphs of maximum degree d always have a cooperative coloring. Let $m(d)$ be the minimal m such that every m graphs of maximum degree d have a cooperative coloring. By the above, $m(1) = 2$ and

$$d + 2 \leqslant m(d) \leqslant 2d, \text{ for every } d \geqslant 2. \tag{1}$$

The theorem of Loh and Sudakov [LS07, Theorem 4.1] on independent transversals in locally sparse graphs implies that $m(d) = d + o(d)$. Neither the lower bound nor the upper bound in (1) has been improved for general d; even $m(3)$ is not known. However, restricting the graphs to specific classes, better upper bounds can be obtained.

Definition 2. For a class $\mathcal{G}$ of graphs, denote by $m_{\mathcal{G}}(d)$ the minimal m such that every m graphs belonging to $\mathcal{G}$, each of maximum degree at most d, have a cooperative coloring.

For example, the following was proved:

Theorem 3 (Corollary 3.3 of Aharoni et al. [ABZ07] and Theorem 6.6 of Aharoni et al. [AHHS15])**.** *Let $\mathcal{C}$ be the class of chordal graphs and let $\mathcal{P}$ be the class of paths. Then $m_{\mathcal{C}}(d) = d + 1$ for all d, and $m_{\mathcal{P}}(2) = 3$.*

In this paper, we prove some bounds on $m_{\mathcal{G}}(d)$ for two more classes:

Theorem 4. *Let $\mathcal{T}$ be the class of trees, and let $\mathcal{B}$ be the class of bipartite graphs. Then for $d \geqslant 2$,*

$$\begin{aligned} \log_2 \log_2 d \leqslant\, & m_{\mathcal{T}}(d) \leqslant (1 + o(1)) \log_{4/3} d, \\ \log_2 d \leqslant\, & m_{\mathcal{B}}(d) \leqslant (1 + o(1)) \frac{2d}{\ln d}. \end{aligned}$$

Remark 5. Let $\mathcal{F}$ be the class of forests. It is evident that $m_{\mathcal{F}}(d) \geqslant m_{\mathcal{T}}(d)$ as $\mathcal{F} \supset \mathcal{T}$. Conversely, when $d \geqslant 2$, given $m = m_{\mathcal{T}}(d)$ forests $F_1, \ldots, F_m$ of maximum degree d, we can add edges to F_i to obtain a tree F_i' of maximum degree d, and the cooperative coloring for $F_1', \ldots, F_m'$ is also a cooperative coloring for $F_1, \ldots, F_m$. Therefore $m_{\mathcal{F}}(d) = m_{\mathcal{T}}(d)$ for $d \geqslant 2$.

The notions of cooperative coloring and of list coloring have a common generalization: given a system $(G_1, \ldots, G_m)$ of graphs with vertex sets $V_1, \ldots, V_m$ (which are not necessarily the same vertex set), a *cooperative list coloring* is then a choice of independent sets in G_i whose union equals $V := V_1 \cup \cdots \cup V_m$. The notion of cooperative coloring is obtained by taking $V_i = V$, and list colorings are formed when G_i is an induced subgraph of the same graph G for all i. The upper bounds in Theorem 4 generalize to cooperative list colorings. For example, our proof of Theorem 4 for bipartite graphs readily gives the following result.

Theorem 6. *For every system $(G_1, \ldots, G_m)$ of bipartite graphs with maximum degree d with vertex sets $V_1, \ldots, V_m$, there is a cooperative list coloring if for every $v \in V_1 \cup \cdots \cup V_m$, the number of its occurrences in $V_1, \ldots, V_m$, that is $|\{i \in [m] : v \in V_i\}|$, is at least $(1 + o(1))\frac{2d}{\ln d}$.*

A conjecture of Alon and Krivelevich [AK98, Conjecture 5.1] states that the choice number of any bipartite graph with maximum degree d is at most $O(\log d)$ (see [AR08] for a result in this direction). This conjecture would follow if the term $(1 + o(1))\frac{2d}{\ln d}$ in Theorem 6 was strengthened to $\Omega(\log d)$.

The rest of the paper is organized as follows. In Section 2 and Section 3, we prove Theorem 4 for trees and bipartite graphs respectively. In Section 4 we discuss a further generalization of cooperative colorings.

2 Trees

Proof of the lower bound on $m_{\mathcal{T}}(d)$. Note that the system $\mathcal{T}_2$, consisting of two paths in Figure 1 (one in thin red, the other in bold blue), does not have a cooperative coloring.

Suppose now that $\mathcal{S} = (F_1, F_2, \ldots, F_m)$ is a system of forests on a vertex set V, not having a cooperative coloring. We shall construct a system $Q(\mathcal{S})$ of $m + 1$ new forests $F_1', F_2', \ldots, F_m', F_{m+1}'$, again not having a cooperative coloring.

The vertex set common to the new forests is $V' = (V \cup \{z\}) \times V$, namely the vertex set consists of $|V| + 1$ copies of V. For every $u \in V \cup \{z\}$ and every $i \in [m]$, take a copy

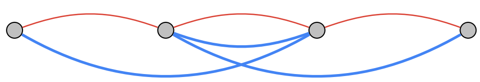

Figure 1: Construction of two paths without a cooperative coloring.

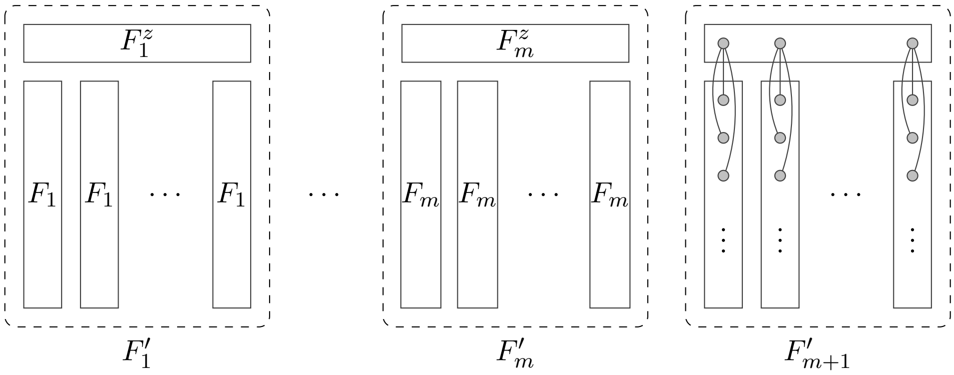

Figure 2: Construction of $Q(\mathcal{S}) = (F_1', \ldots, F_m', F_{m+1}')$ from $\mathcal{S} = (F_1, \ldots, F_m)$.

F_i^u of F_i on the vertex set $\{(u, v) : v \in V\}$. Let F_i' consist of $|V| + 1$ disjoint copies of F_i:

$$F_i' := \bigcup_{u \in V \cup \{z\}} F_i^u, \qquad \text{for all } i \in [m].$$

To these we add the $(m+1)$st forest F_{m+1}' obtained by joining (z, u) to (u, v) for all $u, v \in V$. So F_{m+1}' is a disjoint union of stars, each with $|V|$ leaves.

Assume that there is a cooperative coloring $(I_1, I_2, \ldots, I_m, I_{m+1})$ for the system $Q(\mathcal{S})$. Since the forests $F_1^u, F_2^u, \ldots, F_m^u$ do not have a cooperative coloring, I_{m+1} must contain a vertex from $\{u\} \times V$ for all $u \in V \cup \{z\}$. In particular, I_{m+1} contains a vertex $(z, u) \in I_{k+1}$ for some $u \in V$ and a vertex (u, v) for some $v \in V$. Since (z, u) is connected in F_{m+1}' to (u, v), this is contrary to our assumption that I_{m+1} is independent.

Note that $|V'| = |V|^2 + |V| \leqslant 2\,|V|^2$. Note also that the maximum degree of $Q(\mathcal{S})$ is attained in F_{m+1}', and it is equal to $|V|$. Recursively define the system $\mathcal{T}_m := Q(\mathcal{T}_{m-1})$ consisting of m forests for $m \geqslant 3$. Because the base $\mathcal{T}_2$ has 4 vertices, one can check inductively that $|V(\mathcal{T}_m)|$ is at most $2^{3 \cdot 2^{m-2} - 1}$ using $|V(\mathcal{T}_m)| \leqslant 2\,|V(\mathcal{T}_{m-1})|^2$. Thus the maximum degree of $\mathcal{T}_m$ is at most $2^{3 \cdot 2^{m-3} - 1} \leqslant 2^{2^{m-1}}$.

Given the maximum degree $d \geqslant 2$, choose $m := \lceil \log_2 \log_2 d \rceil$. By the choice of m, the maximum degree of $\mathcal{T}_m$ is at most $2^{2^{m-1}} \leqslant d$. By adding a few edges between the leaves in each forest of $\mathcal{T}_m$, we can obtain a system of m trees of maximum degree d that does not have a cooperative coloring. This means $m_{\mathcal{T}}(d) > m > \log_2 \log_2 d$. $\square$

Proof of the upper bound on $m_{\mathcal{T}}(d)$. Let $(T_1, T_2, \ldots, T_m)$ be a system of trees of maximum degree d. We shall find a cooperative coloring by a random construction if $m \geqslant (1 + o(1)) \log_{4/3} d$.

Choose arbitrarily for each tree T_i a root so that we can specify the parent or a sibling of a vertex that is not the root of T_i. For each T_i, choose independently a random vertex set S_i, in which each vertex is included in S_i independently with probability $1/2$. Set

$$R_i := \{v \in S_i : \text{the parent of } v \text{ is not in } S_i, \text{ or } v \text{ is a root}\}.$$

Since among any two adjacent vertices in T_i one is the parent of the other, R_i is independent in T_i.

We shall show that with positive probability the sets R_i form a cooperative coloring. For each vertex v, let B_v be the event that $v \notin \bigcup_{i=1}^{m} R_i$. If v is the root of T_i, then $\Pr(v \in R_i) = 1/2$; otherwise $\Pr(v \in R_i) = 1/4$. In any case, $\Pr(v \notin R_i) \leqslant 3/4$, and so $\Pr(B_v) \leqslant (3/4)^m$. Notice that B_v is only dependent on the events B_u for u that is the parent, a sibling or a child of v in some T_i. Since the degree of v is at most d, it follows that B_v is dependent on less than $2md$ other events. By the symmetric version of the Lovász Local Lemma (see for example [AS16, Chapter 5]), if

$$e \times \left(\frac{3}{4}\right)^m \times 2md \leqslant 1, \tag{2}$$

then with positive probability no B_v occurs, meaning that the sets R_i form a cooperative coloring. The inequality (2) indeed holds under the assumption that $m \geqslant (1+o(1)) \log_{4/3} d$. □

3 Bipartite graphs

Proof of the lower bound on $m_{\mathcal{B}}(d)$. Given d, take $m = \lceil \log_2 d \rceil$. Let the vertex set be $\{0,1\}^m$, and for $j \in [m]$ let G_j be the complete bipartite graph between V_j^0 and V_j^1 where

$$V_j^k = \{v \in \{0,1\}^m : v_j = k\}, \quad \text{for } k \in \{0,1\}.$$

Note that the degree of G_j is $2^{m-1} \leqslant d$.

Suppose that $I_1, \ldots, I_m$ are independent sets in $G_1, \ldots, G_m$ respectively. As each G_j is a complete bipartite graph, $I_j \subseteq V_j^{k_j}$ for some $k_j \in \{0,1\}$. Thus $(1-k_1, \ldots, 1-k_m)$ is not in any I_j, and so $I_1, \ldots, I_m$ do not form a cooperative coloring. This means $m_{\mathcal{B}}(d) > m \geqslant \log_2 d$. □

Proof of the upper bound on $m_{\mathcal{B}}(d)$. Let $\mathcal{G} = (G_1, \ldots, G_m)$ be a system of bipartite graphs on the same vertex set V with maximum degree d. By a semi-random construction, we shall find a cooperative coloring if $m \geqslant (1+\varepsilon)\frac{2d}{\ln d}$ for fixed $\varepsilon > 0$ and d sufficiently large. We may assume that $m = O(d)$ because of (1). For each $j \in [m]$, let (L_j, R_j) be a bipartition of G_j. Define $J_L(v) := \{j \in [m] : v \in L_j\}$ and $J_R(v) := \{j \in [m] : v \in R_j\}$ for each vertex $v \in V$, and let $A := \{v \in V : |J_L(v)| \geqslant m/2\}$. Set $B := V \setminus A$. Clearly, we have

$$|J_L(a)| \geqslant m/2, \quad \text{for all } a \in A; \tag{3a}$$
$$|J_R(b)| \geqslant m/2, \quad \text{for all } b \in B. \tag{3b}$$

Consider the following random process.

1. For each $a \in A$, choose $j = j(a) \in J_L(a)$ uniformly at random, and put a in the set I_j.
2. For each $b \in B$, choose arbitrarily $j \in J_R(b) \setminus \{j(a) : a \in A, (a,b) \in E(G_j)\} =: J_R'(b)$ as long as it is possible, and put b in the set I_j.

For any $a, a' \in A \cap I_j$, $a, a' \in L_j$ and so $(a, a') \notin G_j$. This means $A \cap I_j$ is independent, and similarly $B \cap I_j$ is independent. For any $b \in B \cap I_j$ and $(a, b) \in E(G_j)$, by the definition of $J'_R(b)$, $j(a) \neq j$ and so $a \notin I_j$. Therefore I_j is independent for all $j \in [m]$.

To prove the existence of a cooperative coloring it suffices to show that $J'_R(b)$ is nonempty for all $b \in B$ with positive probability. For a vertex $b \in B$, let E_b be the contrary event, that is, the event that $J'_R(b)$ is empty.

For a fixed $b \in B$, let us estimate from above the probability of E_b. For every $j \in J_R(b)$, let E^j be the event that $j \notin J'_R(b)$, that is the event that $j(a) = j$ for some $a \in A$ that is a neighbor of b in G_j. For each $a \in A$ that is a neighbor of b in G_j, we have

$$\Pr(j(a) = j) = \frac{1}{|J_L(a)|} \overset{(3a)}{\leqslant} \frac{2}{m} \leqslant \frac{\ln d}{(1+\varepsilon)d}.$$

As there are at most d neighbors of b in G_j, we have for sufficiently large d that

$$1 - \Pr\left(E^j\right) \geqslant \left(1 - \frac{\ln d}{(1+\varepsilon)d}\right)^d \geqslant \exp\left(-(1-\varepsilon)\ln d\right) = d^{\varepsilon-1} \geqslant \frac{8\ln d}{m}. \tag{4}$$

We claim that the events E^j, $j \in J_R(b)$, are negatively correlated. This is easier to see with the complementary events $\bar{E}^j$, $j \in J_R(b)$. We have to show that for any choice of indices $j_1, \ldots, j_t \in J_R(b)$ there holds

$$\Pr\left(E^j \mid \bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \ldots \cap \bar{E}^{j_t}\right) \geqslant \Pr\left(E^j\right).$$

The event $\bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \cdots \cap \bar{E}^{j_t}$ means that for all $a \in A$ if a is a neighbor of b in G_{j_i} then $j(a) \neq j_i$. Then, for any $j \notin \{j_1, \ldots, j_t\}$, for those vertices $a \in A$ that are neighbors of b in G_j, knowing that $j(a) \neq j_i$ for certain $i \in [t]$ increases the probability that $j(a) = j$, and therefore increases the probability of E^j.

By the claim, the inequality (4) and the fact that $E_b = \bigcap_{j \in J_R(b)} E^j$, we have

$$\Pr(E_b) \leqslant \prod_{j \in J_R(b)} \Pr\left(E^j\right) \overset{(3b)}{\leqslant} \left(1 - \frac{8\ln d}{m}\right)^{\frac{m}{2}} \leqslant \exp\left(-\frac{8\ln d}{m} \cdot \frac{m}{2}\right) = \frac{1}{d^4}.$$

The event E_b is dependent on less than md^2 other events $E_{b'}$, since for such dependence to exist it is necessary that $b' \in B$ is at distance at most 2 from b in some graph G_j. Thus, by the Lovász Local Lemma, for the positive probability that none of E_b occurs it suffices that

$$e \times \frac{1}{d^4} \times md^2 \leqslant 1,$$

which indeed holds for d sufficiently large as $m = O(d)$. □

4 Cooperative covers

Cooperative coloring of graphs is a special case of a more general concept.

Definition 7. Given a system $(C_1, \ldots, C_n)$ of (abstract) simplicial complexes, all sharing the same vertex set V, a *cooperative cover* is a choice of faces $f_i \in C_i$ such that $\bigcup_{i=1}^n f_i = V$.

A cooperative coloring for $(G_1, \ldots, G_n)$ is the special case in which C_i is the independence complex $I(G_i)$ of G_i, that is, the collection of all independent sets in G_i.

Definition 8. Given a hypergraph C with vertex set V, the *edge covering number* $\rho(C)$ is the minimal number of hyperedges from C whose union is V. For a class $\mathcal{C}$ of simplicial complexes, let $n_{\mathcal{C}}(b)$ denote the minimal number n, such that every system $(C_1, \ldots, C_n)$ of simplicial complexes belonging to $\mathcal{C}$ on the same vertex set V satisfying $\rho(C_i) \leqslant b$ for all $i \leqslant n$, has a cooperative cover. Let $n_{\mathcal{C}}(b) = \infty$ if no such n exists.

For example, consider the class $\mathcal{I}$ of all the independence complexes of graphs. If G is bipartite, then $\rho(I(G)) \leqslant 2$. Hence the fact that $m_{\mathcal{B}}(d) \geqslant \log_2(d)$ for all $d \geqslant 2$ (see Theorem 4) implies $n_{\mathcal{I}}(2) = \infty$.

There are natural classes $\mathcal{C}$ of hypergraphs for which $n_{\mathcal{C}}$ is finite. One of these is the class of simplicial complexes associated to polymatroids, as introduced in [Edm70]. A *polymatroid* (V, r) is defined via a rank function $r\colon 2^V \to \mathbb{N}$, that is submodular, monotone increasing and is 0 on the empty set. A *k-polymatroid* is a polymatroid in which every singleton set has rank at most k. For example, a k-uniform hypergraph H endowed with the function $r(E) = |\cup E|$, for every subset of hyperedges E in H, is a k-polymatroid.

Following the notation in [LP86, Section 11], given a k-polymatroid (V, r), a set $M \subseteq V$ is called a *matching* if $r(M) = k\,|M|$. By the submodularity of the rank function r, the matchings in a k-polymatroid form a simplicial complex on V, which we call the *matching complex* of a k-polymatroid.

Theorem 9. *Let $\mathcal{M}_k$ be the class of all the matching complexes of k-polymatroids. Then $n_{\mathcal{M}_k}(b) \leqslant kb$ for every b.*

The proof uses the (homotopic) connectivity $\eta(C)$ of a complex C. We refer to [AB06, Section 2] for background. We shall use the following two topological tools. Given a complex C on V and $U \subseteq V$, we denote by $C[U]$ the simplicial subcomplex induced on U.

Theorem 10 (Topological Hall's theorem)**.** *Let C be a simplicial complex on the vertex set V and let $\bigcup_{i=1}^m W_i$ be a partition of V. If for all $I \subseteq [m]$*

$$\eta\left(C\left[\bigcup_{i\in I} W_i\right]\right) \geqslant |I|\,,$$

then C contains a face σ such that $|\sigma \cap W_i| = 1$ for all $i \in [m]$.

Theorem 11. *If C is a matching complex on V of a k-polymatroid, then the connectivity $\eta(C)$ of C is at least $\nu(C)/k$, where $\nu(C)$ is the maximal size of faces in C.*

The above formulation of Theorem 10 first appeared in [Mes01], attributed to the first author of the present paper (see the remark after Theorem 1.3 in [Mes01]). Theorem 11 is an unpublished result of the first two authors. The special case, where the k-polymatroid is the sum of k matroids on the same vertex set, is proved in [AB06, Theorem 6.5].

Proof of Theorem 9. Let $n = kb$, and let $C_1, \ldots, C_n$ be simplicial complexes associated to k-polymatroids $(V, r_1), \ldots, (V, r_n)$ on the same vertex set V such that the edge covering number of each C_i is at most b. Let C be the join of $C_1, \ldots, C_n$ on $V \times [n]$, that is,

$$C := \left\{ \bigcup_{i=1}^{n} \sigma_i \times \{i\} : \sigma_i \in C_i \text{ for all } i \in [n] \right\}.$$

A cooperative cover can be viewed as a face $\sigma \in C$ such that $|\sigma \cap (\{v\} \times [n])| = 1$ for all $v \in V$. By the topological Hall's theorem, it suffices to prove that

$$\eta\left(C[U \times [n]]\right) \geqslant |U| \text{ for all } U \subseteq V.$$

Let U be a subset of V. Note that $C_i[U]$ is the matching complex of the k-polymatroid $(U, r_i|_U)$. By Theorem 11, $\eta(C_i[U]) \geqslant \nu(C_i[U])/k$. Since $\nu(C_i[U])$ is the maximal size of faces in $C_i[U]$ and the edge covering number of $C_i[U]$ is at most b, we obtain $\nu(C_i[U])b \geqslant |U|$, and so $\eta(C_i[U]) \geqslant |U|/(kb)$. Notice that $C[U \times [n]]$ is the join of $C_1[U], \ldots, C_n[U]$. Using the superadditivity of η with respect to the join operator and Theorem 11, we obtain the required condition for the topological Hall's theorem

$$\eta(C[U \times [n]]) \geqslant \sum_{i=1}^{n} \eta(C_i[U]) \geqslant \sum_{i=1}^{n} |U|/(kb) = |U|. \qquad \square$$

Remark 12. It is of interest to explore the sharpness of this result.

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