The asymptotic normality of $(s, s+1)$-cores with distinct parts

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Abstract

Simultaneous core partitions are important objects in algebraic combinatorics. Recently there has been interest in studying the distribution of sizes among all (s, t)-cores for coprime s and t. Zaleski (2017) gave strong evidence that when we restrict our attention to $(s, s+1)$-cores with *distinct parts*, the resulting distribution is approximately normal. We prove his conjecture by applying the Combinatorial Central Limit Theorem and mixing the resulting normal distributions.

Mathematics Subject Classifications: 05A16, 05A17

1 Introduction

A partition of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0)$ whose parts sum to n, i.e., $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We say that n is the *size* of λ, and k is its *length*, which we will denote by $\ell(\lambda)$. For example, the partition $(4, 3, 3, 3, 2)$ has size 15 and length 5.

To each partition, we associate a diagram, known as a Ferrers diagram. The (French) Ferrers diagram of a partition λ is an arrangement of boxes which is left-justified and whose ith row from the bottom contains λ_i boxes. For example, see Figure 1.

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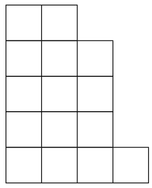

Figure 1: The Ferrers diagram of the partition $(4, 3, 3, 3, 2)$.

To each cell of a Ferrers diagram we associate a number known as the cell's *hook length*. The hook length of a cell c is the number of boxes strictly right of c (known as the *arm* of the cell) plus the number of boxes strictly above c (the *leg*) plus one. For example, the cell c indicated in Figure 2 has hook length 4. The cell marked a is the only one in the arm of c and the two cells marked ℓ form the leg of c.

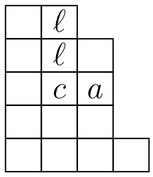

Figure 2: The arm and leg of a cell of a Ferrers diagram.

For convenience, we will sometimes write the hook length of each cell into the Ferrers diagram. We say that a partition is an s-core if none of its cells have hook-length s. A partition is an (s, t)-core if it is simultaneously an s-core and a t-core. See Figure 3. *The number of* (s, t)*-cores is finite if and only if* $\gcd(s, t) = 1$. Jaclyn Anderson [And02] gives a beautiful bijection between (s, t)-cores and certain lattice paths from $(0, 0)$ to (s, t), which proves this and much more.

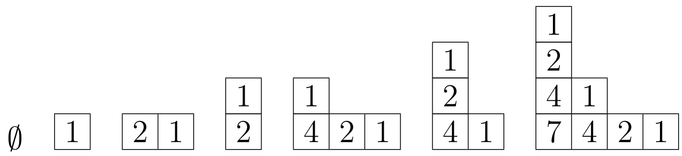

Figure 3: The Ferrers diagrams of all $(3, 5)$-cores with hook lengths indicated.

Simultaneous cores have numerous applications in algebraic combinatorics. For instance, Susanna Fishel and Monica Vazirani [FV10a, FV10b] showed that when $t = ds \pm 1$ for some $d \in \mathbb{N}$, they are naturally in bijection with certain regions of the d-Shi arrangement in type A. Drew Armstrong, Christopher Hanusa, and Brant Jones [AHJ14] extended this work to type C and related simultaneous cores to rational Catalan combinatorics. Purely enumerative questions have yielded deep connections as well. For instance, Armstrong [AHJ14] initially conjectured a simple formula for the average size of an (s, t)-core in 2011. (Here again $\gcd(s, t) = 1$, so the average is taken over the finite set of all (s, t)-cores.) Paul Johnson [Joh18] gave the first proof of Armstrong's conjecture by relating cores to polytopes.

Shalosh B. Ekhad and Doron Zeilberger [EZ15] determined the entire limit distribution obtained by fixing $t - s$, taking the size of a random (s, t)-core, normalizing, and letting $s \to \infty$. Somewhat surprisingly these distributions are *not normal* and are not known to be associated with other combinatorial problems. However, Anthony Zaleski [Zal17b] gave strong experimental evidence that if $t = s + 1$ and only cores with *distinct parts* are considered, then the resulting limit distribution *is* normal. We will prove a much stronger form below (Theorem 1).

For a positive integer s, let X_s be the random variable given by the size of an $(s, s+1)$-core with distinct parts which is chosen uniformly at random. (As mentioned above, there are only finitely many $(s, s+1)$-cores.) Let μ and σ^2 be the mean and variance of X_s. Let

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^{x} \varphi(t)\, dt$$

denote the standard normal *density function* and the standard normal *(cumulative) distribution function,* respectively.

Theorem 1. *For all positive integers s,*

$$\sup_{x \in \mathbb{R}} \left| P(X_s \leqslant \mu + x\sigma) - \Phi(x) \right| = O(1/\sqrt{s}). \tag{1}$$

Here, and throughout the paper, the implied constants in error bounds $O(.)$ are universal constants not depending on any of our parameters. That is, Theorem 1 says: There is a universal constant C_1 such that, for all s and x,

$$\left| P(X_s \leqslant \mu + x\sigma) - \Phi(x) \right| \leqslant C_1/\sqrt{s}.$$

Zaleski [Zal17a] makes a similar conjecture for the case $t = ms - 1$. Both of Zaleski's normality conjectures were supported by strong experimental evidence regarding moments. Huan Xiong and Wenston J.T. Zang [XZ19] further pursued this line of investigation for the case $t = ms \pm 1$, computing asymptotic formulas for the moments. (The enumerative properties of these families of cores have also been studied by many authors recently. For instance, Xiong [Xio18] determined the largest size of such cores, while Rishi Nath and James Sellers [NS17] developed a geometric approach to count these cores and self-conjugate cores of this type.)

Our approach here is not based on moments. Instead we apply a powerful probabilistic tool: the Combinatorial Central Limit Theorem (CCLT). Its original form is due to Wassily Hoeffding [Hoe51]. There is a stronger version due to Erwin Bolthausen [Bol84] with tail bounds. This allows us to prove Theorem 1, a strengthening of Zaleski's conjecture.

Another classical tool we will apply is Proposition 6 on page 7 about generating functions with only real roots. These two tools are named Propositions and they are numbered separately. All other statements (theorem, corollary, lemma) are labeled in one single sequence.

The rest of the paper is organized as follows. In Section 2, we review the Combinatorial Central Limit Theorem. In Section 3, we prove the following strong refinement of Theorem 1: the distribution of size among $(s, s+1)$-cores with distinct parts is *already* approximately normal when the number of parts is fixed. In Section 4, we recall that the weights needed to mix these distributions together are also approximately normal. In Section 5, we compute this mixture to prove Theorem 1. Section 6 contains the proofs of some technical lemmas used in Section 5.

2 The Combinatorial Central Limit Theorem

Let $A = (a_{ij})$ be an $m \times m$ matrix of real numbers. We are interested in the random sum

$$S_A = \sum_i a_{i\pi(i)}$$

where $\pi \in S_m$ is a random permutation of $\{1, 2, \dots, m\}$ chosen uniformly from among all $m!$ permutations. Following [Bol84] we write

$$a_{i\cdot} = \frac{1}{m} \sum_j a_{ij}, \quad a_{\cdot j} = \frac{1}{m} \sum_i a_{ij}, \quad \text{and} \quad a_{\cdot\cdot} = \frac{1}{m^2} \sum_{i,j} a_{ij}$$

and doubly center A by letting

$$\dot{a}_{ij} = a_{ij} - a_{i\cdot} - a_{\cdot j} + a_{\cdot\cdot}$$

(so now all row- and column-sums are 0). Furthermore, we write

$$\mu_A = m a_{\cdot\cdot} \quad \text{and} \quad \sigma_A^2 = \frac{1}{m-1} \sum_{i,j} \dot{a}_{ij}^2$$

for the mean and variance of S_A, and consider the normalized sum

$$T_A = \frac{S_A - \mu_A}{\sigma_A} = \sum_i \widehat{a}_{i\pi(i)}$$

where

$$\widehat{a}_{ij} = \dot{a}_{ij} / \sigma_A$$

The following theorem of Bolthausen [Bol84] gives an estimate of the remainder in the Combinatorial Central Limit Theorem. When A is of rank 1, this gives a tail bound for the classical result of Abraham Wald and Jacob Wolfowitz [WW44].

Proposition 2. *There is an absolute constant K such that for all A with $\sigma_A^2 > 0$,*

$$\sup_t |P(T_A \leqslant t) - \Phi(t)| \leqslant K \sum_{i,j} |\widehat{a}_{ij}|^3 / m\,.$$

3 Normality for a fixed number of parts

Armin Straub [Str16] gave the following elegant characterization of our chosen objects: *A partition* λ *into distinct parts is an* $(s, s+1)$*-core if and only if* $\ell(\lambda) + \lambda_1 - 1 \leqslant s - 1$. Here $\ell(\lambda)$ is the number of parts in λ, and λ_1 is the size of the largest part in λ. Remark: the number $\ell(\lambda) + \lambda_1 - 1$ is sometimes called the perimeter of λ.

Let k and s be fixed non-negative integers. By the above characterization, a partition λ consisting of k distinct parts is an $(s, s+1)$-core if and only if λ_1 is at most $s - k$. We naturally associate to each such partition a vector of length $s - k$ by recording a 1 at position λ_i for $1 \leqslant i \leqslant k$ and 0 elsewhere. For example, the vector $(0, 1, 1, 0, 1, 0)$ corresponds to the $(9, 10)$-core $(5, 3, 2)$.

It is now easy to see that the number of $(s, s+1)$-cores with k distinct parts is just $\binom{s-k}{k}$. Summing shows that the total number of $(s, s+1)$-cores with any number of distinct parts is the Fibonacci number Fib_{s+1}. This fact was originally conjectured by Tewodros Amdeberhan [Amd16] and proved by Straub [Str16].

We can also see that the size of the initial core is just the sum of the positions of 1's in the resulting vector, i.e., the inner product of this vector and $(1, 2, 3, \dots, s-k)$. With this rephrasing we are able to apply the CCLT: simply take the matrix A to be the outer product of the vector $(1^k, 0^{s-2k})$ and the vector $(1, 2, 3, \dots, s-k)$.

In general, suppose $A = (a_{ij})$ is an $m \times m$ rank 1 matrix, i.e., $a_{ij} = \alpha_i x_j$ for some vectors α, x. Thus, writing $\bar{\alpha} = (\sum \alpha_i)/m$ and $\bar{x} = (\sum x_j)/m$, we have

$$\begin{aligned} \dot{a}_{ij} &= (\alpha_i - \bar{\alpha})(x_j - \bar{x}), \quad \mu_A = m\bar{\alpha}\bar{x} \\ \sigma_A^2 &= \frac{1}{m-1} \sum_{i,j} \dot{a}_{ij}^2 = \frac{m^2}{m-1} \left(\frac{1}{m} \sum_i (\alpha_i - \bar{\alpha})^2 \right) \left(\frac{1}{m} \sum_j (x_j - \bar{x})^2 \right) \end{aligned} \tag{2}$$

Let $\alpha_1 = \dots = \alpha_k = 1$, $\alpha_{k+1} = \dots = \alpha_m = 0$. Note that now S_A is the sum of the elements in a random k-subset of the list $x_1, \dots, x_m$. Here we will only need the special case $x_i = i$ for $i = 1, \dots, m$.

Theorem 3. *For the choice of parameters above and* K *as in Proposition 2, the following explicit bound holds:*

$$\sup_{x \in \mathbb{R}} |P(T_A \leqslant x) - \Phi(x)| \leqslant \left(\frac{12m^2}{k(m-k)} \right)^{3/2} \cdot \frac{K}{\sqrt{m}} \tag{3}$$

which goes to 0 when both $km^{-2/3} \to \infty$ *and* $(m-k)m^{-2/3} \to \infty$.

Proof. It is easy to see that

$$\bar{\alpha} = k/m, \ \bar{x} = (m+1)/2, \ \mu_A = \frac{m+1}{2} \cdot k, \ \sigma_A^2 = \frac{m+1}{12} \cdot k(m-k). \tag{4}$$

Using $|\dot{a}_{ij}| = |\alpha_i - \bar{\alpha}| \cdot |x_j - \bar{x}| \leqslant 1 \cdot m = m$, the right-hand side in Proposition 2 is

$$K \sum_{i,j} |\widehat{a}_{ij}|^3/m \leqslant \frac{Km^4}{\sigma_A^3} < \left(\frac{12m^2}{k(m-k)} \right)^{3/2} \cdot \frac{K}{\sqrt{m}}$$

which goes to 0 if $km^{-2/3} \to \infty$ and $(m-k)m^{-2/3} \to \infty$. □

Plugging $m = s - k$ in to (3) gives the following corollary.

Corollary 4. *Let $X_{s,k}$ be the random variable given by the size of an $(s, s+1)$-core with k distinct parts chosen uniformly at random. Let μ_k and σ_k^2 denote the mean and variance of $X_{s,k}$, respectively. Then for any $0 < k < s/2$, the normalized variable $(X_{s,k} - \mu_k)/\sigma_k$ satisfies the following.*

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{X_{s,k} - \mu_k}{\sigma_k} \leqslant x\right) - \Phi(x) \right| \leqslant \frac{12^{3/2} K (s-k)^{5/2}}{(k(s-2k))^{3/2}}$$

Hence the distribution of $(X_{s,k} - \mu_k)/\sigma_k$ tends to the standard normal distribution if $s \to \infty$ and both $ks^{-2/3} \to \infty$ and $(s - 2k)s^{-2/3} \to \infty$.

We will use Corollary 4 only when $s/4 \leqslant k \leqslant s/3$, in which case we obtain the bound

$$\sup_{x\in\mathbb{R}} |P(X_{s,k} \leqslant \mu_k + x\sigma_k) - \Phi(x)| < \frac{1000K}{\sqrt{s}}. \tag{5}$$

Remark 5. Zaleski [Zal17b] already noted that the generating function for $(s, s+1)$-cores with k distinct parts is none other than the shifted q-binomial coefficient $q^{\binom{k+1}{2}} \binom{s-k}{k}_q$.

It was this observation that lead us to study the distribution when k is fixed. By taking $s = n + m$ and $k = m$, Corollary 4 shows that the partial sums of coefficients in the q-binomial coefficient $\binom{n}{m}_q$ are approximately normally distributed. It would be interesting to see that the distribution is also locally approximately normal.

4 The distribution of the weights

Ultimately we will mix together the distributions of $X_{s,k}$ for all k with s fixed. Each distribution is weighted according to how many cores are being enumerated, namely $X_{s,k}$ gets weight

$$p_k = P(W = k) = \binom{s-k}{k} / Fib_{s+1}.$$

Here the random variable W is the number of parts in a random $(s, s+1)$-core with distinct parts.

The sequence $\binom{s-k}{k}$ appears often in combinatorics. Its generating function is

$$g_s(z) = \sum_{0\leqslant k\leqslant \frac{s}{2}} \binom{s-k}{k} z^k = \frac{1}{\sqrt{1+4z}} \left(\left(\frac{1+\sqrt{1+4z}}{2}\right)^{s+1} - \left(\frac{1-\sqrt{1+4z}}{2}\right)^{s+1} \right)$$

— see Concrete Mathematics [GKP94] by Ronald Graham, Donald Knuth, and Oren Patashnik. By differentiating it twice, we get the moments:

$$\mu(W) = \sum_k k\, p_k = \frac{5-\sqrt{5}}{10} \cdot s + O(1)$$

and

$$\sigma^2(W) = \frac{\sqrt{5}}{25} \cdot s + O(1).$$

For convenience we write

$$c_0 = (5 - \sqrt{5})/10 = 0.2764.. \quad \text{and} \quad k_0 = \lfloor c_0 s \rfloor.$$

There is a long history of normal approximations for finite non-negative real sequences whose generating functions have only real roots. The first appearance in combinatorics of a global normal law similar to (6) is a result of Lawrence Harper [Har67] studying Stirling numbers. Harper's brilliant idea was further developed and generalized in the classical paper of Ed Bender [Ben73]. For two modern results, see the paper of Joel Lebowitz, Boris Pittel, David Ruelle and Eugene Speer [LPRS16] and the paper of Marcus Michelen and Julian Sahasrabudhe [MS19].

The following proposition is from Pitman [Pit97]. It says that if a polynomial f with non-negative coefficients has only real zeros, then its coefficients are approximately normally distributed, both globally and locally. For completeness, we cite both the global and the local versions.

Proposition 6. *Let $\mathbf{p} = (p_0, p_1, \dots, p_n)$ be a sequence of non-negative real numbers summing to 1 with mean and variance*

$$\mu = \mu(\mathbf{p}) = \sum i p_i \quad \text{and} \quad \sigma^2 = \sigma^2(\mathbf{p}) = \sum (i-\mu)^2 p_i = \left(\sum i^2 p_i\right) - \mu^2.$$

Let $f(x) = \sum_k p_k x^k$ be its generating function. Write $S_k = \sum_{i=0}^k p_i$ for the partial sums. Assume all roots of the polynomial f are real. Then,

$$\max_{0 \leqslant k \leqslant n} \left| S_k - \Phi\left(\frac{k-\mu}{\sigma}\right) \right| < \frac{0.7975}{\sigma} \tag{6}$$

and there exists a universal constant C such that

$$\max_{0 \leqslant k \leqslant n} \left| \sigma p_k - \varphi\left(\frac{k-\mu}{\sigma}\right) \right| < \frac{C}{\sigma}. \tag{7}$$

Remark 7. It is obvious that if f has only real roots, then the non-negativity of the coefficients $p_0, \dots, p_n$ is equivalent to all roots of f being non-positive – another traditional way of stating the result.

Our generating function $g_s(x)$ has only real roots, since only real numbers $z \leqslant -1/4$ can satisfy

$$\left| 1 + \sqrt{1+4z} \right| = \left| 1 - \sqrt{1+4z} \right|.$$

Hence Proposition 6 applies to our sequence of weights $p_k = \binom{s-k}{k}/Fib_{s+1}$ with $n = \lfloor s/2 \rfloor$, $\mu = \mu(W)$, and $\sigma = \sigma(W)$.

The same paper [Pit97] (Formula (11) on page 284) contains exponential tail bounds for our weight distribution (phrased in the more general setup of so-called PF-distributions). Plugging in our specific parameter $\mu(W) = c_0 s + O(1)$, we get the following bound: for every $\varepsilon > 0$ there is a $\delta > 0$ and a constant $C(\varepsilon) > 0$ such that

$$\sum_{k < (c_0 - \varepsilon)s} p_k + \sum_{k > (c_0+\varepsilon)s} p_k < C(\varepsilon) e^{-\delta s} \tag{8}$$

where $p_k = P(W = k)$. We will use this tail probability estimate later with $\varepsilon = \min\{1/3 - c_0, c_0 - 1/4\} = 0.026..$

5 Proof of Theorem 1

Fix a positive integer s. Recall that X_s is the random variable given by the size of an $(s, s+1)$-core with distinct parts which is chosen uniformly at random. Zaleski [Zal17b] shows that the mean and variance of X_s are:

$$\mu = \mu(X_s) = \frac{1}{10}s^2 + O(s), \qquad \sigma^2 = \sigma^2(X_s) = \frac{2\sqrt{5}}{375}s^3 + O(s^2). \tag{9}$$

Recall also that if $0 \leqslant k \leqslant s/2$, then $X_{s,k}$ is the random variable given by the size of an $(s, s+1)$-core with k distinct parts which is chosen uniformly at random. Hence the distribution of X_s is the mixture of the distributions of the $\lfloor s/2 \rfloor + 1$ individual $X_{s,k}$.

Setting $m = s - k$ in (4) gives

$$\mu_k = \frac{1}{2}\,k\,(s+1-k), \quad \sigma_k^2 = \frac{1}{12}\,k\,(s+1-k)(s-2k). \tag{10}$$

Remark 8. Zaleski's formulas (9) could be obtained by a lengthy computation involving the generating function $g_s(z)$, (10), and the Pythagorean Theorem of Probability Theory (a.k.a. the Law of Total Variance):

$$Var\big[\xi\big] = EVar\big[\xi|\eta\big] + Var\big[E(\xi|\eta)\big].$$

Fix $x \in \mathbb{R}$. Let

$$F(x) = P(X_s \leqslant \mu + x\sigma) = EP(X_{s,k} \leqslant \mu + x\sigma). \tag{11}$$

Here the expected value E denotes the weighted sum

$$EP(X_{s,k} \leqslant \mu + x\sigma) = \sum_{0 \leqslant k \leqslant s/2} P(X_{s,k} \leqslant \mu + x\sigma)\,p_k. \tag{12}$$

For $0 < k < s/2$ we can rewrite the terms

$$P(X_{s,k} \leqslant \mu + x\sigma) = P(X_{s,k} \leqslant \mu_k + y_k\sigma_k) =: F_k(y_k), \tag{13}$$

where

$$y_k = \frac{1}{\sigma_k}\big((\mu - \mu_k) + x\sigma\big). \tag{14}$$

For $k = 0$ and $k = s/2$ (when s is even) we have $\sigma_k = 0$, so y_k is undefined. These at most two terms of the right-hand side of (12) have weight $1/Fib_{s+1}$ (each), so we will only work with integers k with $0 < k < s/2$.

Our ultimate goal is to show that $F(x)$ is approximately $\Phi(x)$ with an error bound $O(1/\sqrt{s})$ uniformly for $x \in \mathbb{R}$. We will accomplish this with a sequence of approximations $Q_1, \ldots, Q_7$ and several lemmas. Each subsequent Q introduces an error of only $O(1/\sqrt{s})$. The proofs of these lemmas will be put off to Section 6.

Let

$$Q_1 = \sum_{0<k<s/2} p_k F_k(y_k). \quad \text{Then,} \quad |F(x) - Q_1| \leqslant 2/Fib_{s+1}.$$

Let $I = \mathbb{Z} \cap (s/4, s/3)$, $J = \mathbb{Z} \cap (0, s/2) - I$, and

$$Q_2 = \sum_{0<k<s/2} \Phi(y_k) p_k. \tag{15}$$

Note that by the CCLT (5),

$$\left|P(X_{s,k} \leqslant \mu_k + y\sigma_k) - \Phi(y)\right| = O(1/\sqrt{s}) \tag{16}$$

uniformly for $k \in I$ and $y \in \mathbb{R}$. Hence,

$$\left|P(X_{s,k} \leqslant \mu_k + y_k\sigma_k) - \Phi(y_k)\right| = O(1/\sqrt{s}) \tag{17}$$

uniformly for $k \in I$ and $x \in \mathbb{R}$. On the other hand, for $k \in J$ the weights p_k are exponentially small in s by (8). Since both $P(X_{s,k} \leqslant \mu_k + y_k\sigma_k)$ and $\Phi(y_k)$ are between 0 and 1 and the weights p_k are non-negative and sum to at most 1, we have

$$|Q_1 - Q_2| = \sum_{0<k<s/2} p_k \cdot \left|P(X_{s,k} \leqslant \mu_k + y_k\sigma_k) - \Phi(y_k)\right| = O(1/\sqrt{s}).$$

Now we must approximate $\Phi(y_k)$ and p_k. We start with approximating y_k. For $k \in \mathbb{Z}$, write

$$y_k^* = ax + bt_k \quad \text{where} \quad a = \sqrt{8/5},\ b = -\sqrt{3/5},\ \text{and } t_k = 5^{3/4}\,(k - k_0)/\sqrt{s}.$$

The next lemma says that y_k is well approximated by the arithmetic progression $y_k^* = ax + bt_k$ in the relevant range of k. We also write

$$dt_k = t_k - t_{k-1} = 5^{3/4}/\sqrt{s}.$$

The quantity dt_k (which is independent of k) will be used as a mesh size in approximating integrals. We will also see (41) that σ_k is roughly constant when k is close to k_0.

Lemma 9. *For all integers k with $0 < k < s/2$,*

$$|y_k - y_k^*| = \frac{1}{\sqrt{s}} \cdot O(1 + |xt_k| + t_k^2). \tag{18}$$

We will also show in the last section that Lemma 9 implies the following statement.

Corollary 10. *For all integers k with $0 < k < s/2$ we have*

$$|\Phi(y_k) - \Phi(y_k^*)| = O\left(\frac{1}{\sqrt{s}}\left(1 + t_k^2\right)\right) \tag{19}$$

uniformly for $x \in \mathbb{R}$.

Hence,

$$Q_2 = \sum_{0<k<s/2} \Phi(y_k) p_k = \sum_{0<k<s/2} \Phi(y_k^*) p_k + \frac{1}{\sqrt{s}} \cdot O\left(\sum_{0<k<s/2} \left(1 + t_k^2\right) p_k\right). \tag{20}$$

Lemma 11. *There exists a universal constant K_0 such that for all $s \in \mathbb{N}$,*

$$\sum_{0 \leqslant k \leqslant s/2} (1 + t_k^2)\, p_k \;\leqslant\; K_0. \tag{21}$$

Thus,

$$Q_2 = \sum_{0<k<s/2} \Phi(y_k) p_k = \sum_{0<k<s/2} \Phi(y_k^*) p_k + O\left(\frac{1}{\sqrt{s}}\right). \tag{22}$$

Let

$$Q_3 = \sum_{0<k<s/2} \Phi(y_k^*) p_k. \quad \text{Then, } \; |Q_2 - Q_3| = O\left(\frac{1}{\sqrt{s}}\right). \tag{23}$$

It would be natural to use the local approximation (7) for the weights p_k at this point. However, it would be harder to deal with the accumulation of errors. So instead we will apply the following version of summation by parts and use the global approximation (6).

Lemma 12. *Let $m \leqslant n$ be integers. Suppose $(U_k : m \leqslant k \leqslant n+1)$ and $(V_k : m-1 \leqslant k \leqslant n)$ are two (finite) real sequences. Then,*

$$\sum_{k=m}^{n} U_k (V_k - V_{k-1}) = \sum_{k=m}^{n} (U_k - U_{k+1}) V_k \;+\; \big[U_{n+1} V_n - U_m V_{m-1}\big]. \tag{24}$$

(Lemma 12 can be verified easily by comparing the two sides term by term.)

Write $dU_k = U_k - U_{k+1}$ $(m \leqslant k \leqslant n)$ and $dV_k = V_k - V_{k-1}$ $(m \leqslant k \leqslant n)$. Thus (24) becomes

$$\sum_{k=m}^{n} U_k \, dV_k = \sum_{k=m}^{n} dU_k \, V_k \;+\; \big[U_{n+1} V_n - U_m V_{m-1}\big]. \tag{25}$$

Note also: for all $m \leqslant k \leqslant n$,

$$U_k = U_{n+1} + \sum_{k \leqslant i \leqslant n} dU_i \quad \text{and} \quad V_k = V_{m-1} + \sum_{m \leqslant i \leqslant k} dV_i.$$

Corollary 13. *Let $m \leqslant n$ be integers. Suppose $(U_k : m \leqslant k \leqslant n+1)$, $(U'_k : m \leqslant k \leqslant n+1)$, $(V_k : m-1 \leqslant k \leqslant n)$, and $(V'_k : m-1 \leqslant k \leqslant n)$ are real sequences. Define dU_k, dU'_k, dV_k, dV'_k as in Lemma 12. Write*

$$\delta_U = \sup_{m \leqslant k \leqslant n} |U_k - U'_k|, \quad \delta_V = \sup_{m \leqslant k \leqslant n} |V_k - V'_k|. \tag{26}$$

Then,

$$\begin{aligned} &\Big| \sum_{k=m}^{n} U_k dV_k - \sum_{k=m}^{n} U'_k dV'_k \Big| \\ &\leqslant \delta_U \sum |dV'_k| + \delta_V \sum |dU_k| + |U_{n+1} V_n - U_m V_{m-1}| + \big|U_{n+1} V'_n - U_m V'_{m-1}\big| . \end{aligned} \tag{27}$$

This simple corollary of Lemma 12 will be proved in the last section.

Now define

$$F_k^* = \begin{cases} 1 & \text{if } k \leqslant 0, \\ \Phi(y_k^*) = \Phi(ax + bt_k) & \text{if } 0 < k < s/2, \\ 0 & \text{if } k \geqslant s/2. \end{cases} \tag{28}$$

Then,

$$Q_3 = \sum_{0<k<s/2} F_k^* p_k = \sum_{0\leqslant k\leqslant s/2} F_k^* p_k - p_0 = \sum_{0\leqslant k\leqslant s/2} F_k^* p_k - (1/Fib_{s+1}). \tag{29}$$

Let

$$Q_4 = \sum_{0\leqslant k\leqslant s/2} F_k^* p_k \tag{30}$$

Thus,

$$Q_3 = Q_4 - (1/Fib_{s+1}) = Q_4 + O(1/\sqrt{s}). \tag{31}$$

Note: The doubly infinite sequence $(y_k^* : k \in \mathbb{Z}) = (ax + bt_k : k \in \mathbb{Z})$ is monotone decreasing, so $(\Phi(y_k^*) : k \in \mathbb{Z})$ is monotone decreasing. Hence $(F_k^* : k \in \mathbb{Z})$ is also monotone decreasing. Consequently, the numbers

$$f_k = F_k^* - F_{k+1}^* \quad (k \in \mathbb{Z}) \tag{32}$$

are non-negative and add up to 1.

We apply Corollary 13 with $m = 0$, $n = \lfloor s/2 \rfloor$, $U_k = F_k^*$, $U_k' = \Phi(ax + bt_k)$, $V_k = S_k = \sum_{i=0}^k p_i, V_k' = \Phi(t_k)$. Note that for us: $U_m = U_0 = 1$, $U_{n+1} = F^*_{\lfloor s/2 \rfloor+1} = 0$, $V_{m-1} = S_{-1} = 0$. Hence,

$$\Big| \sum_{0\leqslant k\leqslant s/2} U_k dV_k - \sum_{0\leqslant k\leqslant s/2} U_k' dV_k' \Big| \leqslant \delta_U \sum |dV_k'| + \delta_V \sum |dU_k| + \Phi(t_{-1}). \tag{33}$$

Plugging in our values, we get $\delta_U = 1 - \Phi(ax + bt_0)$ if s is odd, and when s is even, $\delta_U = \max\{1 - \Phi(ax + bt_0), \Phi(ax + bt_n)\}$. In both cases, δ_U is exponentially small in s.

Now let's estimate δ_V. The discussion after Proposition 6 mentioned that its assumptions are satisfied for W, hence one can apply the inequality 6 to get

$$\delta_V < 0.7975/\sigma(W) = O(1/\sqrt{s}).$$

Also, both $dU_k (= f_k)$ and $dV_k' (= \Phi(t_k) - \Phi(t_{k-1}))$ are non-negative, hence

$$\sum_{0\leqslant k\leqslant s/2} |dU_k| = \sum_{0\leqslant k\leqslant s/2} dU_k = U_0 - U_{n+1} = F_0^* - F_{n+1}^* = 1 - 0 = 1$$

and

$$\sum_{0\leqslant k\leqslant s/2} |dV_k'| = \sum_{0\leqslant k\leqslant s/2} dV_k' = \Phi(t_n) - \Phi(t_{-1}) \leqslant 1.$$

Thus, (33) becomes

$$\Big| \sum_{0\leqslant k\leqslant s/2} U_k dV_k \;-\; \sum_{0\leqslant k\leqslant s/2} U'_k dV'_k \Big| \;\leqslant\; \delta_U + \delta_V \;+\; \Phi(t_{-1}) \;\leqslant\; \frac{K_1}{\sqrt{s}} \tag{34}$$

for some universal constant K_1.

Recall that

$$Q_4 = \sum_{0\leqslant k\leqslant s/2} F_k^* p_k = \sum_{0\leqslant k\leqslant s/2} U_k dV_k.$$

Let

$$Q_5 \;=\; \sum_{0\leqslant k\leqslant s/2} U'_k dV'_k \;=\; \sum_{0\leqslant k\leqslant s/2} \Phi(ax+bt_k)\big[\Phi(t_k)-\Phi(t_{k-1})\big]. \tag{35}$$

Thus, by (34),

$$|Q_4 - Q_5| \;\leqslant\; K_1/\sqrt{s}.$$

Lemma 14. *For all integers $k \in \mathbb{Z}$,*

$$\Phi(t_k) - \Phi(t_{k-1}) \;=\; \varphi(t_k)dt_k + \frac{1}{\sqrt{s}}\, O\big(|\varphi'(t_k)|dt_k\big) + O(1/s^{3/2}). \tag{36}$$

Applying Lemma 14, we get

$$\begin{aligned} Q_5 \;&=\; \sum_{0\leqslant k\leqslant s/2} \Phi(ax+bt_k)\,[\Phi(t_k)-\Phi(t_{k-1})] \\ &=\; \sum_{0\leqslant k\leqslant s/2} \Phi(ax+bt_k)\,\varphi(t_k)\,dt_k \;+\; \frac{1}{\sqrt{s}}\cdot O\left(\sum_{0\leqslant k\leqslant s/2} |\varphi'(t_k)|\,dt_k\right) + O(1/\sqrt{s}) \qquad (37) \\ &=\; \sum_{0\leqslant k\leqslant s/2} \Phi(ax+bt_k)\,\varphi(t_k)\,dt_k \;+\; O(1/\sqrt{s}). \end{aligned}$$

For the last line we used the fact that the $O(\sum \ldots)$ term is a (partial) Riemann-sum for the convergent integral $\int_{-\infty}^{\infty} |\varphi'(t)|dt$. The bounded non-negative function $|\varphi'(t)|$ is made up of four monotone pieces, and our mesh size is $dt_k = O(1/\sqrt{s})$.

The sum in the last line of (37) can be extended for all integers k with an error of only $O(1/\sqrt{s})$. This is because

$$\sum_{k<0} \Phi(ax+bt_k)\,\varphi(t_k)\,dt_k < \sum_{k<0} \varphi(t_k)\,dt_k$$

and the right-hand side is a Riemann sum for the function $\varphi(t)$ integrated from $-\infty$ to $-5^{3/4}k_0/\sqrt{s}$. This integral is exponentially small in s. Since on this domain $\varphi(t)$ is monotone increasing and is between 0 and $1/\sqrt{2\pi}$, the Riemann sum approximation itself only introduces an error at most $dt_k/\sqrt{2\pi} = O(1/\sqrt{s})$. The same applies to the sum $\sum_{k>s/2} \Phi(ax+bt_k)\,\varphi(t_k)\,dt_k$.

Thus,

$$Q_5 = \sum_{k\in\mathbb{Z}} \Phi(ax+bt_k)\,\varphi(t_k)\,dt_k \;+\; O(1/\sqrt{s}). \tag{38}$$

Let

$$Q_6 = \sum_{k\in\mathbb{Z}} \Phi(ax+bt_k)\,\varphi(t_k)\,dt_k. \quad \text{Then,} \;\; Q_5 = Q_6 + O(1/\sqrt{s}). \tag{39}$$

Define

$$Q_7 = \int_{-\infty}^{\infty} \Phi(ax+bt)\,\varphi(t)dt. \tag{40}$$

Lemma 15. *Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume*

$$V_h = \int_{-\infty}^{\infty} |\, h't)\,|\, dt < \infty.$$

Let $I_j = [\ell_j, r_j]$ $(j \in \mathbb{Z})$ be a partition of $\mathbb{R}$ into intervals of lengths not exceeding $\delta > 0$, and let $\xi_j \in I_j$ be arbitrary points. Then,

$$\left| \sum_{j\in\mathbb{Z}} h(\xi_j)\,|I_j| - \int_{-\infty}^{\infty} h(t)\,dt \right| \leqslant V_h\,\delta\,.$$

We apply this lemma to the function $h(t) = \Phi(ax+bt)\,\varphi(t)$ with $\delta = dt_k = 5^{3/4}/\sqrt{s}$. Thus, $h'(t) = \varphi(t)\cdot[b\,\varphi(ax+bt) - t\,\Phi(ax+bt)]$, whence $|h'(t)| \leqslant \varphi(t)\cdot(|b|+|t|)$. Since

$$\int_{-\infty}^{\infty} |h'(t)|\,dt < \infty$$

uniformly for $x \in \mathbb{R}$, by Lemma 15 we get

$$Q_6 = Q_7 + O(1/\sqrt{s}).$$

Lemma 16. *Let a and b be real numbers. Then for all $x \in \mathbb{R}$,*

$$\int_{-\infty}^{\infty} \Phi(ax+bt)\,\varphi(t)dt = \Phi\left(\frac{ax}{\sqrt{1+b^2}}\right).$$

We apply Lemma 16 with $a = \sqrt{8/5}$ and $b = -\sqrt{3/5}$ to obtain

$$Q_7 = \Phi(x).$$

This completes the proof of Theorem 1. Namely, we have shown that

$$F(x) \;=\; \Phi(x) \;+\; O(1/\sqrt{s})$$

uniformly in $x \in \mathbb{R}$.

6 Computational Proofs of the Lemmas

Lemma 9. For all integers k with $0 < k < s/2$,

$$|y_k - y_k^*| = \frac{1}{\sqrt{s}} \cdot O(1 + |xt| + t_k^2).$$

Proof. Recall that $\mu_k = \frac{k(s+1-k)}{2}$, $\sigma_k^2 = \frac{k(s+1-k)(s-2k)}{12}$, $k_0 = \lfloor \frac{5-\sqrt{5}}{10} s \rfloor$. Let $D_k = k - k_0$. Then

$$\frac{\sigma_k^2}{\sigma_{k_0}^2} = \frac{(k_0 + D_k)(s+1-k_0-D_k)(s-2k_0-2D_k)}{k_0(s+1-k_0)(s-2k_0)} = 1 + O\left(\frac{D_k}{s}\right). \tag{41}$$

Therefore

$$y_k = \frac{1}{\sigma_k}\big((\mu - \mu_k) + x\sigma\big) = \left[1 + O\left(\frac{D_k}{s}\right)\right] \cdot \frac{1}{\sigma_{k_0}}\big((\mu - \mu_k) + x\sigma\big).$$

Let $q = \sigma/\sigma_{k_0}$. Then

$$y_k = \left[1 + O\left(\frac{D_k}{s}\right)\right] \cdot q \cdot \left(\frac{\mu - \mu_k}{\sigma} + x\right).$$

Now note that

$$\begin{aligned}
\mu_{k_0} &= \frac{1}{2}\left(\frac{5-\sqrt{5}}{10}s\right)\left(s + 1 - \frac{5-\sqrt{5}}{10}s\right) + O(s) \\
&= \frac{1}{2}\left(\frac{5-\sqrt{5}}{10}\right)\left(1 - \frac{5-\sqrt{5}}{10}\right)s^2 + O(s) \\
&= \frac{s^2}{10} + O(s) \;=\; \mu + O(s).
\end{aligned}$$

So

$$\begin{aligned}
\mu - \mu_k &= \mu_{k_0} - \mu_k + O(s) \\
&= \frac{1}{2}\left(k_0(s+1-k_0) - k(s+1-k)\right) + O(s) \\
&= \frac{1}{2}(k - k_0)\Big(-s - 1 + (k + k_0)\Big) + O(s) \\
&= \frac{1}{2}(k - k_0)\Big(-s - 1 + (k - k_0) + 2k_0\Big) + O(s) \\
&= \frac{1}{2}D_k\Big(-s - 1 + D_k + \frac{5-\sqrt{5}}{5}s\Big) + O(s) \\
&= -\frac{\sqrt{5}}{10} \cdot sD_k + O(D_k^2) + O(s).
\end{aligned}$$

(Above and below we use the obvious inequality: $2D_k \leqslant D_k^2 + 1$.)

Therefore

$$\begin{aligned}\frac{\mu-\mu_k}{\sigma} &= \frac{-\frac{\sqrt5}{10}\cdot sD_k + O(D_k^2)+O(s)}{\sqrt{\frac{2\sqrt5}{375}s^{3/2}\left[1+O\left(\frac1s\right)\right]}}\\
&= \sqrt{\frac{375}{2\sqrt5}}\left(-\frac{\sqrt5}{10}\cdot\frac{D_k}{\sqrt s}+O\left(\frac{D_k^2}{s^{3/2}}\right)+O\left(\frac{1}{\sqrt s}\right)\right)\left[1+O\left(\frac1s\right)\right]\\
&= -\frac{3^{1/2}\,5^{3/4}}{2^{3/2}}\cdot\frac{D_k}{\sqrt s}+O\left(\frac{D_k^2}{s^{3/2}}\right)+O\left(\frac{1}{\sqrt s}\right).\end{aligned}$$

Finally, setting $t_k = 5^{3/4}D_k/\sqrt{s}$ and using $|D_k| \leqslant s$ gives

$$\begin{aligned}y_k &= \left[1+O\left(\frac{D_k}{s}\right)\right]\cdot q\cdot\left(\frac{\mu-\mu_k}{\sigma}+x\right)\\
&= \left[1+O\left(\frac{t_k}{\sqrt s}\right)\right]\cdot q\cdot\left(x-\sqrt{\frac38}\,t_k+O\left(\frac{t_k^2}{\sqrt s}\right)+O\left(\frac{1}{\sqrt s}\right)\right)\\
&= q\cdot\left(x-\sqrt{\frac38}\,t_k\right)+\frac{1}{\sqrt s}\cdot O\left(1+|xt_k|+t_k^2\right).\end{aligned}$$

But q is essentially a constant. That is,

$$\begin{aligned}q^2=\frac{\sigma^2}{\sigma_{k_0}^2} &= \frac{\frac{2\sqrt5}{375}s^3+O(s^2)}{\frac{1}{12}k_0(s+1-k_0)(s-2k_0)+O(s^2)}\\
&= \frac{\frac{2\sqrt5}{375}s^3+O(s^2)}{\frac{1}{12}c_0(1-c_0)(1-2c_0)s^3\left[1+O\left(\frac1s\right)\right]}\\
&= \frac85+O\left(\frac1s\right).\end{aligned}$$

So $q=\sqrt{8/5}+O(1/s)$. Therefore

$$y_k=\left(\sqrt{\frac85}\,x-\sqrt{\frac35}\,t_k\right)+\frac{1}{\sqrt s}\cdot O\left(1+|xt|+t_k^2\right)=y_k^*+\frac{1}{\sqrt s}\cdot O\left(1+|xt|+t_k^2\right). \qquad \square$$

Corollary 10. For all integers k with $0<k<s/2$ we have

$$|\Phi(y_k)-\Phi(y_k^*)| = O\left(\frac{1}{\sqrt s}\left(1+t_k^2\right)\right)$$

uniformly for $x\in\mathbb{R}$.

Proof.

Let K_2 be the implied constant in (18). Let $\varepsilon_1=\sqrt{2/3}$, $x_0=16K_2/a$, and $s_0=(8K_2/a)^4$.

Special case I: $|t_k|\geqslant s^{1/4}$.

Then $1+t_k^2>s^{1/2}$, so $\frac{1}{\sqrt s}(1+t_k^2)>1$. Hence (19) is automatically true (independent of the value of x).

Special case II: $|t_k| \geqslant \varepsilon_1|x|$.
Then $1 + |xt_k| + t_k^2 \leqslant 1 + (\frac{1}{\varepsilon_1} + 1)t_k^2 < 3(1 + t_k^2)$.
Special case III: $|x| \leqslant x_0$.
Then $1 + |xt_k| + t_k^2 \leqslant 1 + x_0|t_k| + t_k^2 \leqslant (1 + x_0/2)(1 + t_k^2) = O(1 + t_k^2)$.

For the rest of this proof we will assume k is an integer with $0 < k < s/2$ satisfying:

$$x > x_0, \quad |t_k| < \varepsilon_1|x|, \quad \text{and} \quad |t_k| < s^{1/4}.$$

We will first show that both y_k and y_k^* are between $\frac{1}{4}\,ax$ and $\frac{7}{4}\,ax$. This will allow us to apply the Mean Value Theorem to prove the corollary.

Recall that $a = \sqrt{8/5}$, $b = -\sqrt{3/5}$, and $t_k = 5^{3/4}\,(k - k_0)/\sqrt{s}$. Thus,

$$|bt_k| = \sqrt{3/5}\,|t_k| < \sqrt{3/5}\,\varepsilon_1|x| = \frac{1}{2}|ax|.$$

Consequently,

$$y_k^* = ax + bt_k \text{ is between } \frac{1}{2}\,ax \text{ and } \frac{3}{2}\,ax, \text{ whence } |y_k^*| > \frac{1}{2}\,a|x|.$$

Now we estimate y_k:

$$|y_k^* - y_k| \leqslant \frac{K_2}{\sqrt{s}} \cdot (1 + |xt_k| + t_k^2) = \frac{K_2}{\sqrt{s}} \cdot (1 + t_k^2) + \frac{K_2}{\sqrt{s}} \cdot |xt_k|.$$

The first term on the right-hand side is estimated as

$$\frac{K_2}{\sqrt{s}}\,(1 + t_k^2) < \frac{K_2}{\sqrt{s}}\,(1 + s^{1/2}) = K_2\,(1 + s^{-1/2}) \leqslant 2K_2 \leqslant \frac{1}{8}\,a|x|$$

for $x \geqslant x_0$.

For the second term we have

$$\frac{K_2}{\sqrt{s}} \cdot |xt_k| < \frac{K_2}{\sqrt{s}} \cdot |x|s^{1/4} = \frac{K_2}{s^{1/4}} \cdot |x| \leqslant \frac{1}{8}\,a|x|$$

for $s \geqslant s_0$.

Consequently,

$$|y_k^* - y_k| < \frac{1}{4}\,a|x|, \text{ and thus } y_k \text{ is between } \frac{1}{4}\,ax \text{ and } \frac{7}{4}\,ax \text{ as desired.}$$

By the Mean Value Theorem, there is a ξ between y_k and y_k^* such that $\Phi(y_k^*) - \Phi(y_k) = \varphi(\xi)\,(y_k^* - y_k)$. As we showed above, ξ is between $\frac{1}{4}\,ax$ and $\frac{7}{4}\,ax$, and hence

$$|\xi| > \frac{1}{4}\,a|x| > \frac{a}{4\varepsilon_1}\,|t_k|.$$

Consequently, since φ is monotone,

$$\varphi(\xi) = \varphi(|\xi|) < \varphi\left(\frac{1}{4}\,a|x|\right) \quad \text{and} \quad \varphi(\xi) < \varphi\left(\frac{a}{4\varepsilon_1}\,|t_k|\right).$$

We obtain:

$$\begin{aligned}|\Phi(y_k^*) - \Phi(y_k)| &= \varphi(\xi)\,|y_k^* - y_k| \leqslant \varphi(\xi)\,\frac{K_2}{\sqrt{s}} \cdot (1 + |xt_k| + t_k^2) \\ &< \frac{K_2}{\sqrt{s}} \cdot \left[(1+t_k^2)\,\varphi\left(\frac{a}{4\varepsilon_1}\,|t_k|\right) + \varepsilon_1 x^2\,\varphi\left(\frac{1}{4}\,a|x|\right)\right].\end{aligned}$$

Since the quantity in square brackets is bounded uniformly in $k \in \mathbb{Z}$ and $x \in \mathbb{R}$, Corollary 10 is proved. □

Lemma 11. There exists a universal constant K_0 such that for all $s \in \mathbb{N}$,

$$\sum_{0 \leqslant k \leqslant s/2} (1 + t_k^2)\, p_k \;\leqslant\; K_0.$$

Proof. By the definition of t_k, we have

$$t_k^2 = \frac{5^{3/2}}{s}\,(k-k_0)^2 \leqslant \frac{25}{s} \cdot \left[(k-\mu(W))^2 + (\mu(W)-k_0)^2\right] = \frac{25}{s}\,(k-\mu(W))^2 + O(1/s).$$

Here we used $(\alpha-\gamma)^2 \;\leqslant\; 2[(\alpha-\beta)^2 + (\beta-\gamma)^2]$. Hence,

$$\sum_{0 \leqslant k \leqslant s/2} t_k^2\, p_k \leqslant \frac{25}{s} \sum_{0 \leqslant k \leqslant s/2} (k-\mu(W))^2 p_k + O(1) = 25 \cdot \frac{\sigma^2(W)}{s} + O(1) = O(1)$$

(where, as always, $O(1)$ is independent of s). □

Corollary 13. Let $m \leqslant n$ be integers. Suppose $(U_k : m \leqslant k \leqslant n{+}1)$, $(U'_k : m \leqslant k \leqslant n{+}1)$, $(V_k : m-1 \leqslant k \leqslant n)$, and $(V'_k : m-1 \leqslant k \leqslant n)$ are real sequences. Define dU_k, dU'_k, dV_k, dV'_k as after Lemma 12. Write

$$\delta_U = \sup_{m \leqslant k \leqslant n} |U_k - U'_k|, \quad \delta_V = \sup_{m \leqslant k \leqslant n} |V_k - V'_k|. \tag{42}$$

Then,

$$\begin{aligned}&\Big|\sum_{k=m}^{n} U_k dV_k - \sum_{k=m}^{n} U'_k dV'_k\Big| \\ &\leqslant \delta_U \sum |dV'_k| + \delta_V \sum |dU_k| + |U_{n+1}V_n - U_m V_{m-1}| + \left|U_{n+1}V'_n - U_m V'_{m-1}\right|.\end{aligned} \tag{43}$$

Proof. We start with the following four identities, the non-trivial two of which follow from applying Lemma 12 twice.

$$\sum_{k=m}^{n} U_k dV_k - \sum_{k=m}^{n} dU_k V_k \;=\; \left[U_{n+1}V_n - U_m V_{m-1}\right].$$

$$\sum_{k=m}^{n} dU_k V_k - \sum_{k=m}^{n} dU_k V'_k \;=\; \sum_{k=m}^{n} dU_k (V_k - V'_k).$$

$$\sum_{k=m}^{n} dU_k V'_k - \sum_{k=m}^{n} U_k dV'_k \;=\; -\left[U_{n+1}V'_n - U_m V'_{m-1}\right].$$

$$\sum_{k=m}^{n} U_k dV'_k - \sum_{k=m}^{n} U'_k dV'_k \;=\; \sum_{k=m}^{n} (U_k - U'_k) dV'_k.$$

Adding up these four identities we get

$$\begin{aligned}&\sum_{k=m}^{n} U_k dV_k - \sum_{k=m}^{n} U'_k dV'_k \\ &= \sum_{k=m}^{n} (U_k - U'_k) dV'_k + \sum_{k=m}^{n} dU_k (V_k - V'_k) + [U_{n+1}V_n - U_m V_{m-1}] - \big[U_{n+1}V'_n - U_m V'_{m-1}\big]\,,\end{aligned}$$

from which Corollary 13 follows. □

Lemma 14. For all integers $k \in \mathbb{Z}$,

$$\Phi(t_k) - \Phi(t_{k-1}) \;=\; \varphi(t_k) dt_k + \frac{1}{\sqrt{s}} O(|\varphi'(t_k)| dt_k) + O(1/s^{3/2})$$

where $dt_k = 5^{3/4}/\sqrt{s}$.

Proof. Let $k \in \mathbb{Z}$. There exists a ξ_k with $t_{k-1} < \xi_k < t_k$ such that

$$\begin{aligned}\Phi(t_k) - \Phi(t_{k-1}) &= \varphi(t_k)(t_k - t_{k-1}) - \frac{1}{2}\varphi'(t_k)(t_k - t_{k-1})^2 + \frac{1}{6}\varphi''(\xi_k)(t_k - t_{k-1})^3 \\ &= \varphi(t_k) dt_k - \frac{1}{2}\varphi'(t_k)(dt_k)^2 + \frac{1}{6}\varphi''(\xi_k)(dt_k)^3 \\ &= \varphi(t_k) dt_k \;+\; \frac{1}{\sqrt{s}} O(|\varphi'(t_k)| dt_k) + O(1/s^{3/2}).\end{aligned}$$

□

Lemma 15. Let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume $V_h = \int_{-\infty}^{\infty} |\, h't)\,|\, dt < \infty$. Let $I_j = [\ell_j, r_j]$ $(j \in \mathbb{Z})$ be a partition of $\mathbb{R}$ into intervals of lengths not exceeding $\delta > 0$, and let $\xi_j \in I_j$ be arbitrary points. Then,

$$\left| \sum_{j\in\mathbb{Z}} h(\xi_j)\, |I_j| \;-\; \int_{-\infty}^{\infty} h(t)\, dt \right| \leqslant V_h\, \delta\,.$$

Proof. While the statement is known in the context of total variations of functions, we give, for completeness, a simple direct proof by applying the bounded version below on each individual interval I_j.

Observation. Let h be a differentiable function on a closed interval $I = [a, b]$ $(a < b)$. Then,

$$|h(b) - h(a)| \leqslant \int_a^b |h'(t)|\, dt.$$

Indeed, by the Fundamental Theorem of Calculus,

$$\big|h(b) - h(a)\big| = \left| \int_a^b h'(t)\, dt \right| \leqslant \int_a^b |h'(t)|\, dt.$$

Bounded version. Let h be differentiable on a closed bounded interval $I = [a, b]$ $(a < b)$. Let $\xi \in I$ be arbitrary. Then,

$$D := \left| h(\xi) \cdot (b-a) - \int_a^b h(t)\, dt \right| \leqslant (b-a) \int_a^b |h'(t)|\, dt.$$

Indeed, since h is continuous on I, there exists an $\eta \in I$ such that

$$\int_a^b h(t)\, dt = h(\eta) \cdot (b-a).$$

Assume (WLOG) that $\eta \leqslant \xi$. Then, by the Observation above,

$$D = (b-a) \cdot \big| h(\xi) - h(\eta) \big| \leqslant (b-a) \int_\eta^\xi |h'(t)|\, dt \leqslant (b-a) \int_a^b |h'(t)|\, dt. \qquad \square$$

Lemma 16. Let a and b be real numbers. Then for all $x \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \Phi(ax + bt)\, \varphi(t) dt = \Phi\left(\frac{ax}{\sqrt{1+b^2}} \right).$$

Proof. One could compute the two-dimensional integral corresponding to the left hand side. We present instead a simple probabilistic proof.

Let Z_1 and Z_2 be independent standard normal variables. Define $Z_3 = Z_1 - bZ_2$. Then Z_3 is a normal random variable with 0 expectation and variance $1 + b^2$. We then have

$$\begin{aligned}
\int \Phi(ax + bt)\, \varphi(t) dt &= \int P(Z_1 \leqslant ax + bt)\, \varphi(t) dt \\
&= \int P(Z_1 \leqslant ax + bt \,|\, Z_2 = t)\, \varphi(t) dt \\
&= \int P(Z_1 \leqslant ax + bZ_2 \,|\, Z_2 = t)\, \varphi(t) dt \\
&= P(Z_1 \leqslant ax + bZ_2) \\
&= P(Z_3 \leqslant ax) = \Phi\left(\frac{ax}{\sqrt{1+b^2}} \right). \qquad \square
\end{aligned}$$

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