Projections of antichains

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Abstract

A subset A of $\mathbb{Z}^n$ is called a weak antichain if it does not contain two elements x and y satisfying $x_i < y_i$ for all i. Engel, Mitsis, Pelekis and Reiher showed that for any weak antichain A, the sum of the sizes of its $(n-1)$-dimensional projections must be at least as large as its size $|A|$. They asked what the smallest possible value of the gap between these two quantities is in terms of $|A|$. We answer this question by giving an explicit weak antichain attaining this minimum for each possible value of $|A|$. In particular, we show that sets of the form

$$A_N = \{x \in \mathbb{Z}^n : 0 \leqslant x_j \leqslant N-1 \text{ for all } j \text{ and } x_i = 0 \text{ for some } i\}$$

minimise the gap among weak antichains of size $|A_N|$.

Mathematics Subject Classifications: 05D05

1 Introduction

A subset of $\mathbb{Z}^n$ is called a *weak antichain* if it contains no elements x and y such that for all i $x_i < y_i$. Let us denote by π_i the projection along the i^{th} coordinate, that is, $\pi_i : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ is given by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. Engel, Mitsis, Pelekis and Reiher [3] proved the following projection inequality for weak antichains (which they used to prove an analogous result about weak antichains in the continuous cube $[0,1]^n$).

Theorem 1 (Engel, Mitsis, Pelekis and Reiher [3])**.** *For every finite weak antichain A in $\mathbb{Z}^n$, we have*

$$|A| \leqslant \sum_{i=1}^{n} |\pi_i(A)|.$$

The same authors asked the following question.

Question 2. What is the smallest possible value $g(n,m)$ of $\mathrm{gap}(A) = \sum_{i=1}^{n} |\pi_i(A)| - |A|$ as A varies over weak antichains in $\mathbb{Z}^n$ of size m?

Note that the question is uninteresting for (strong) antichains A in $\mathbb{Z}^n$, as we trivially have $|\pi_i(A)| = |A|$ for all i in this case. Furthermore, a weak antichain in $\{0,1\}^n$ is just a subset of $\{0,1\}^n$ not containing both the zero vector and the vector with all entries equal to 1. So classical results about set systems (such as Sperner's theorem, see e.g. [1]) are not particularly relevant here.

In this paper we answer Question 2. To state the result, we need some definitions. Let $\mathbb{Z}_{\geqslant 0}$ denote the set of non-negative integers, and let X_n be the subset of $\mathbb{Z}^n_{\geqslant 0}$ consisting of elements that have at least one coordinate which is zero. Note that any subset of X_n is a weak antichain. For given $x, y \in X_n$, let $T = \{i : x_i \neq y_i\}$, let $x' = (x_i)_{i \in T}$, $y' = (y_i)_{i \in T}$. Write $x < y$ if $\max x' < \max y'$ or ($\max x' = \max y'$ and $\max\{i : x'_i = \max x'\} < \max\{i : y'_i = \max y'\}$). Then $<$ defines a total order on X_n. We will call this the *balanced order* on X_n.

For example, the first few elements of the balanced order on X_2 are

$$(0,0), (1,0), (0,1), (2,0), (0,2), (3,0), (0,3), (4,0), (0,4),$$

and the first few elements of the balanced order on X_3 are

$$\begin{gathered}(0,0,0), (1,0,0), (0,1,0), (1,1,0), (0,0,1), (1,0,1), (0,1,1), (2,0,0), (2,1,0), (2,0,1), \\ (0,2,0), (1,2,0), (0,2,1), (2,2,0), (0,0,2), (1,0,2), (0,1,2), (2,0,2), (0,2,2), (3,0,0).\end{gathered}$$

Theorem 3. *For every $n \geqslant 2$ and $m \geqslant 0$, the initial segment of size m of the balanced order on X_n minimises the gap among weak antichains in $\mathbb{Z}^n$ of size m. In particular, for every positive integer N, the set*

$$A_N = \{x \in \mathbb{Z}^n_{\geqslant 0} : 0 \leqslant x_i \leqslant N-1 \text{ for all } i, \text{ and } x_j = 0 \text{ for some } j\}$$

minimises the gap among weak antichains of size $|A_N| = N^n - (N-1)^n$.

In terms of asymptotic lower bounds on the gap, this gives the following result.

Theorem 4. *For every $n \geqslant 2$ and $m \geqslant 1$, we have*

$$g(n,m) \geqslant c_n m^{1-1/(n-1)},$$

where $c_n = \frac{1}{2}(n-1)n^{1/(n-1)}$. Moreover, for every $n \geqslant 2$, we have

$$g(n,m) \sim c_n m^{1-1/(n-1)} \text{ as } m \to \infty.$$

Our proofs have the following structure. Starting with any weak antichain, we modify it into a subset of X_n. This modification will be made step-by-step, and at some points our set will not be a weak antichain. However, it will always satisfy a certain weaker property, which we will call 'layer-decomposability'. Studying subsets of X_n is much simpler than studying general weak antichains, and we will finish the proof of Theorem 3 using induction on n and codimension-1 compressions. For our proof to work we will need to show that initial segments of the balanced order are extremal for another property as well. Instead of deducing the asymptotic result from Theorem 3, we will prove it directly and before Theorem 3, because its proof is simpler and motivates some of the steps in the proof of Theorem 3.

2 Compressing to a down-set in X_n

Recall that we denote $X_n = \{x \in \mathbb{Z}^n_{\geqslant 0} : x_i = 0 \text{ for some } i\}$. In this section our aim is to prove the following lemma.

Lemma 5. *If A is a finite weak antichain in $\mathbb{Z}^n$, then there is a weak antichain $A' \subseteq X_n$ of the same size having $|\pi_i(A')| \leqslant |\pi_i(A)|$ for each i which is a down-set, i.e., if $x, y \in \mathbb{Z}^n_{\geqslant 0}$, $x_i \leqslant y_i$ for all i and $y \in A'$ then $x \in A'$.*

We start by recalling the proof of Engel, Mitsis, Pelekis and Reiher [3] that $\text{gap}(A) \geqslant 0$ for every finite weak antichain. For any finite set $A \subseteq \mathbb{Z}^n$, define the i^{th} bottom layer $B_i(A)$ to be the set of elements with minimal i^{th} coordinate, i.e.,

$$B_i(A) = \{x \in A : \text{whenever } y \in A \text{ with } y_j = x_j \text{ for all } j \neq i \text{ then } y_i \geqslant x_i\}.$$

Furthermore, define $A_1, \ldots, A_n$ inductively by setting ($A_1 = B_1(A)$ and)

$$A_i = B_i\left(A \setminus (A_1 \cup \cdots \cup A_{i-1})\right).$$

Observe that for a weak antichain we have $A = A_1 \cup \cdots \cup A_n$. Indeed, if $x \in A \setminus (A_1 \cup \cdots \cup A_n)$ then we may inductively find $x^{(i)} \in A_{n-i}$ (for all $0 \leqslant i \leqslant n-1$) such that $x^{(i)}_j < x_j$ for all $j \geqslant n-i$ and $x^{(i)}_j = x_j$ for all $j < n-i$. Then $x^{(n-1)}$ has all coordinates smaller than x, giving a contradiction.

We will call a finite set A with $A = A_1 \cup \cdots \cup A_n$ *layer-decomposable.* Note that π_i restricted to A_i is injective, hence $\sum_{i=1}^n |\pi_i(A)| \geqslant \sum_{i=1}^n |A_i| = |A|$ for all layer-decomposable sets (and in particular for all weak antichains).

Now assume $A \subseteq \mathbb{Z}^n_{\geqslant 0}$. Define the *$i$-compression* $C_i(A)$ of A by replacing each $x \in B_i(A)$ by $(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$. Note that $|C_i(A)| = |A|$.

Lemma 6. *Let $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ be any finite set. For every $i \neq j$, $\pi_j(C_i(A)) \subseteq C_i(\pi_j(A))$. In particular, $|\pi_j(C_i(A))| \leqslant |\pi_j(A)|$.*

(When considering $C_i(\pi_j(A))$, we mean compressing along the coordinate labelled by i, not along the i^{th} remaining coordinate.)

Proof. Suppose $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in \pi_j(C_i(A))$ so that there is an $x \in C_i(A)$ with k^{th} coordinate x_k for all k.

- If $x_i = 0$ then there is a $y \in B_i(A)$ with $x_k = y_k$ for all $k \neq i$. So we have $(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) \in \pi_j(A)$. But this vector and $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ agree in all entries except maybe the one labelled by i, so (since $x_i = 0$) we have $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in C_i(\pi_j(A))$.
- If $x_i \neq 0$ then $x \in A \setminus B_i(A)$ and there is a $y \in B_i(A)$ with $y_k = x_k$ for all $k \neq i$ and $y_i < x_i$. But then $\pi_j(y)$ and $\pi_j(x)$ agree in all coordinates except the i^{th} one, which shows $\pi_j(x) \notin B_i(\pi_j(A))$ and hence $\pi_j(x) \in C_i(\pi_j(A))$, as claimed. $\square$

Say that A is i-compressed if $C_i(A) = A$, i.e., $B_i(A) = \{x \in A : x_i = 0\}$.

Lemma 7. *Suppose that $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ is finite, layer-decomposable (i.e., $A = A_1 \cup \cdots \cup A_n$), and k-compressed for all $k < i$. Then $A' = C_i(A)$ satisfies the following.*

(i) A' is k-compressed for all $k \leqslant i$.

(ii) A' is layer-decomposable.

Proof. Let $j < i$. By Lemma 6, $|\pi_j(A')| \leqslant |\pi_j(A)|$. But, since $B_j(A) = \{x \in A : x_j = 0\}$, $C_i(B_j(A))$ is a subset of A' having j^{th} coordinate constant zero and j^{th} projection of size $|\pi_j(A)|$. It follows that $B_j(A') = C_i(B_j(A)) = \{x \in A' : x_j = 0\}$, giving (i).

We now show (ii). Since A is k-compressed for all $k < i$, induction on k gives

$$A_k = \{x \in A : x_k = 0 \text{ but } x_l \neq 0 \text{ for all } l < k\} \quad \text{for all } k < i. \tag{1}$$

Indeed, if this holds for all k' with $k' < k$, then $\bigcup_{l=1}^{k-1} A_l = \{x \in A : x_l = 0 \text{ for some } l < k\}$, so A_k contains the right hand side of (1), and every element of A_k has $x_l \neq 0$ for all $l < k$. Furthermore, if there is some $x \in A$ with $x_k > 0$ and $x_l \neq 0$ for all $l < k$, then there is some $y \in A$ with $y_k = 0$ and $y_j = x_j$ for all $j \neq i$ (as A is k-compressed). Then $y \in A_k$, so certainly $x \notin A_k$, giving the claim.

Similarly,

$$A'_k = \{x \in A' : x_k = 0 \text{ but } x_l \neq 0 \text{ for all } l < k\} \quad \text{for all } k \leqslant i. \tag{2}$$

But then we have

$$\begin{aligned} C_i(A \setminus (A_1 \cup \cdots \cup A_{i-1})) &= C_i(\{x \in A : x_k \neq 0 \text{ for all } k < i\}) \\ &= \{x \in C_i(A) : x_k \neq 0 \text{ for all } k < i\} \\ &= A' \setminus (A'_1 \cup \cdots \cup A'_{i-1}). \end{aligned}$$

(The first equality is immediate from (1). The second equality follows from the fact that C_i acts independently on each set consisting of points having a fixed value of $x_1, \ldots, x_{i-1}$. The last equality is immediate from (2).)

It follows that $\{x \in C_i(A \setminus (A_1 \cup \cdots \cup A_{i-1})) : x_i \neq 0\} = \{x \in A' \setminus (A'_1 \cup \cdots \cup A'_{i-1}) : x_i \neq 0\}$. But the left hand side is $A \setminus (A_1 \cup \cdots \cup A_i)$ and the right hand side is $A' \setminus (A'_1 \cup \cdots \cup A'_i)$ by (2). Thus $A \setminus (A_1 \cup \cdots \cup A_i) = A' \setminus (A'_1 \cup \cdots \cup A'_i)$. Hence $A_j = A'_j$ for all $j > i$. Using $A = A_1 \cup \cdots \cup A_n$, we have $A \setminus (A_1 \cup \cdots \cup A_i) = A_{i+1} \cup \cdots \cup A_n$ and so $A' \setminus (A'_1 \cup \cdots \cup A'_i) = A'_{i+1} \cup \cdots \cup A'_n$, giving (ii). $\square$

Lemma 8. *If $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ is a finite weak antichain, then $A' = C_n(C_{n-1}(\dots(C_1(A))\dots))$ satisfies*

(i) $|\pi_i(A')| \leqslant |\pi_i(A)|$ *for each* i.

(ii) A' *is* k*-compressed for all* k.

(iii) $A' = A'_1 \cup \dots \cup A'_n$.

(iv) $A'_k = \{x \in A' : x_k = 0 \text{ but } x_l \neq 0 \text{ for all } l < k\}$ *for all* k.

(v) $A' \subseteq X_n = \{x \in \mathbb{Z}^n_{\geqslant 0} : x_i = 0 \text{ for some } i\}$.

Proof. The claims (i), (ii), (iii) are immediate from Lemma 6 and Lemma 7. Claim (iv) follows from (ii) exactly as in the proof of Lemma 7. Then (v) follows from (iii) and (iv). □

Note that even though some intermediate steps $C_i(C_{i-1}(\dots(C_1(A))\dots))$ need not give weak antichains, we see that after the n^{th} compression we end up with a set which is necessarily a weak antichain.

For a set $A \subseteq \mathbb{Z}^n_{\geqslant 0}$, define the *complete* i*-compression*

$$\begin{aligned} C_i^{\text{compl}}(A) = \{&(x_1,\dots,x_{i-1},a,x_{i+1},\dots,x_n) : \\ &a \in \mathbb{Z}_{\geqslant 0} \text{ and there are at least } a+1 \text{ elements } y \text{ of } A \text{ having for all } j \neq i \ y_j = x_j\}. \end{aligned}$$

Note that $|C_i^{\text{compl}}(A)| = |A|$.

Lemma 9. *If $A \subseteq X_n$ then for any j we have $|\pi_j(C_i^{\text{compl}}(A))| \leqslant |\pi_j(A)|$.*

Proof. The proof is essentially the same as for Lemma 6. Indeed, let $j \neq i$ and suppose that $(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) \in \pi_j(C_i^{\text{compl}}(A))$. So there is an $x \in C_i^{\text{compl}}(A)$ with k^{th} coordinate x_k for all k, and hence there are $y^{(0)},\dots,y^{(x_i)} \in A$ such that $y_k^{(a)} = x_k$ for all $k \neq i$ and all $0 \leqslant a \leqslant x_i$, and $y_i^{(0)} < y_i^{(1)} < \dots < y_i^{(x_i)}$. But then $(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n) \in C_i^{\text{compl}}(\pi_j(A))$. It follows that $\pi_j(C_i^{\text{compl}}(A)) \subseteq C_i^{\text{compl}}(\pi_j(A))$, giving the result.

[Alternatively, we can deduce Lemma 9 from Lemma 6 by applying C_i to A then $A \setminus B_i(A)$ then $A \setminus (B_i(A) \cup B_i(A \setminus B_i(A)))$ and so on.] □

Proof of Lemma 5. We may assume that $A \subseteq \mathbb{Z}^n_{\geqslant 0}$. By Lemma 8, we may also assume $A \subseteq X_n$. Keep applying complete compressions while it changes our set. These do not increase any projection by Lemma 9, and keeps our set a subset of X_n. Note that if $A' \neq C_i^{\text{compl}}(A')$ then $\sum_{x \in C_i^{\text{compl}}(A')} \sum_j x_j < \sum_{x \in A'} \sum_j x_j$, so the process must terminate. So the set A' we end up with must have $C_i^{\text{compl}}(A') = A'$ for all i, so it must be a down-set. □

3 The asymptotic result

We now show how Lemma 5 can be used to prove the asymptotic version of our theorem. The proof of the exact version (Theorem 3) in the next section will be independent of this section, but the proof below motivates some of the steps in the proof of Theorem 3. Recall that $g(n,m)$ denotes the smallest possible value of $\text{gap}(A) = \sum_{i=1}^{n} |\pi_i(A)| - |A|$ as A varies over weak antichains of size m in $\mathbb{Z}^n$, and our aim is to prove the following result.

Theorem 4. *For every $n \geqslant 2$ and $m \geqslant 1$, we have*

$$g(n,m) \geqslant c_n m^{1-1/(n-1)},$$

where $c_n = \frac{1}{2}(n-1)n^{1/(n-1)}$. Moreover, for every $n \geqslant 2$, we have

$$g(n,m) \sim c_n m^{1-1/(n-1)} \text{ as } m \to \infty.$$

Proof. By Lemma 5, it suffices to consider sets $A \subseteq X_n$ which are down-sets. We prove the result by induction on n. The case $n = 2$ is trivial, since the gap is exactly 1 for any down-set in X_2. Now assume $n \geqslant 3$ and the result holds for $n-1$.

Define, for every $a \in \mathbb{Z}_{\geqslant 0}$,

$$L_a = \{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{Z}_{\geqslant 0}^{n-1} : (x_1, x_2, \dots, x_{n-1}, a) \in A \text{ and } x_i = 0 \text{ for some } i < n\}.$$

Let $K = \pi_n(A) \setminus L_0$. Note that A can be written as a disjoint union of $K \times \{0\}$ and the $L_a \times \{a\}$. Also, $L_0 \supseteq L_1 \supseteq L_2 \supseteq \dots$, and each L_a is a subset of X_{n-1} (and in particular is a weak antichain). Note furthermore that $|\pi_i(A)| = \sum_{a \geqslant 0} |\pi_i(L_a)|$ for all $i < n$, and $|\pi_n(A)| = |K| + |L_0|$. It follows that

$$\begin{aligned} \sum_{i=1}^{n} |\pi_i(A)| - |A| &= \sum_{i=1}^{n-1} |\pi_i(L_0)| + \sum_{a \geqslant 1} \left(\sum_{i=1}^{n-1} |\pi_i(L_a)| - |L_a| \right) \\ &\geqslant |L_0| + \sum_{a \geqslant 0} g(n-1, |L_a|) \\ &\geqslant |L_0| + \sum_{a \geqslant 0, L_a \neq \emptyset} c_{n-1} |L_a|^{1-1/(n-2)}. \end{aligned}$$

Write $|L_0| = x$. Since $|L_a| \leqslant x$ for each a, we have $|L_a|^{1-1/(n-2)} \geqslant |L_a| x^{-1/(n-2)}$. It follows that

$$\sum_{i=1}^{n} |\pi_i(A)| - |A| \geqslant x + c_{n-1} \left(\sum_{a \geqslant 0} |L_a| \right) x^{-1/(n-2)}.$$

Note that $\sum_{a \geqslant 0} |L_a| = m - |K|$. By the (discrete) Loomis–Whitney inequality [4] (see [2] for a generalisation), and the inequality between the arithmetic and geometric mean,

$$|K|^{n-2} \leqslant \prod_{i=1}^{n-1} |\pi_i(K)| \leqslant \left(\frac{\sum_{i=1}^{n-1} |\pi_i(K)|}{n-1} \right)^{n-1}.$$

But $\sum_{i=1}^{n-1} |\pi_i(K)| \leqslant |L_0|$ since we may assign to $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}) \in \pi_i(K)$ the value $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_{n-1}) \in L_0$, giving an injective function from the disjoint union of the projections to L_0. It follows that

$$|K|^{n-2} \leqslant \left(\frac{x}{n-1}\right)^{n-1}$$

and so

$$\begin{aligned}\sum_{i=1}^{n} |\pi_i(A)| - |A| &\geqslant x + c_{n-1}\left(m - \frac{1}{(n-1)^{1+1/(n-2)}}x^{1+1/(n-2)}\right)x^{-1/(n-2)} \\ &= \left(1 - \frac{c_{n-1}}{(n-1)^{1+1/(n-2)}}\right)x + c_{n-1}mx^{-1/(n-2)}.\end{aligned}$$

Differentiation shows that this is minimised at

$$x = \left(\frac{c_{n-1}m}{(n-2)\left(1 - \frac{c_{n-1}}{(n-1)^{1+1/(n-2)}}\right)}\right)^{1-1/(n-1)},$$

giving

$$\sum_{i=1}^{n} |\pi_i(A)| - |A| \geqslant \left(1 - \frac{c_{n-1}}{(n-1)^{1+1/(n-2)}}\right)^{\frac{1}{n-1}} (n-1)(n-2)^{1/(n-1)-1}(c_{n-1}m)^{1-1/(n-1)}.$$

But $c_{n-1} = \frac{1}{2}(n-2)(n-1)^{1/(n-2)}$, so

$$1 - \frac{c_{n-1}}{(n-1)^{1+1/(n-2)}} = \frac{n}{2(n-1)}$$

and so

$$\left(1 - \frac{c_{n-1}}{(n-1)^{1+1/(n-2)}}\right)^{1/(n-1)} (n-1)(n-2)^{1/(n-1)-1}c_{n-1}^{1-1/(n-1)} = \frac{1}{2}(n-1)n^{1/(n-1)} = c_n,$$

giving $g(n, m) \geqslant c_n m^{1-1/(n-1)}$, as claimed.

It remains to show that for any fixed n we have $g(n, m) \leqslant (1 + o(1))c_n m^{1-1/(n-1)}$. Let

$$A_N = \{(x_1, \dots, x_n) \in \mathbb{Z}^n_{\geqslant 0} :\ 0 \leqslant x_i \leqslant N - 1 \text{ for all } i, \text{ and there is a } j \text{ such that } x_j = 0\}.$$

Note that A_N has $|\pi_i(A_N)| = N^{n-1}$ for each i, so

$$\sum_{i=1}^{n} |\pi_i(A_N)| = nN^{n-1}.$$

Moreover, it has size

$$m_N = |A_N| = N^n - (N-1)^n = nN^{n-1} - \binom{n}{2} N^{n-2} + O(N^{n-3}).$$

Now pick N such that $m_N \leqslant m < m_{N+1}$, and consider the weak antichain given as follows. Let B be an arbitrary subset of $\{0\} \times [N, N + \lfloor (m_{N+1} - m_N)^{1/(n-1)} \rfloor]^{n-1}$ of size $m - m_N$. Note that B has gap at most

$$(n-1)(m_{N+1} - m_N + 1)^{(n-2)/(n-1)} = O(N^{(n-2)^2/(n-1)})$$

Put $A = A_N \cup B$. So A has size m and gap equal to the sum of gaps of A_N and B, so A has gap at most

$$\binom{n}{2} N^{n-2} + O(N^{n-3}) + O(N^{(n-2)^2/(n-1)}) = \binom{n}{2} N^{n-2}(1 + o(1)).$$

But $m = nN^{n-1}(1 + o(1))$, so the gap is $c_n m^{1-1/(n-1)}(1 + o(1))$, as required. □

4 The exact result

Recall that we defined a total order (called the balanced order) on X_n as follows. Given $x, y \in X_n$, let $T = \{i : x_i \neq y_i\}$, let $x' = (x_i)_{i \in T}$, $y' = (y_i)_{i \in T}$. Write $x < y$ if $\max x' < \max y'$ or ($\max x' = \max y'$ and $\max\{i : x'_i = \max x'\} < \max\{i : y'_i = \max y'\}$). To see that this really is a total order, we need to show that if $x < y$ and $y < z$, then $x < z$. Set $M_x = \max x$ and $i_x = \max\{i : x_i = M_x\}$, and define M_y, M_z, i_y, i_z similarly. If $M_x < M_y$ or $M_y < M_z$, then $M_x < M_z$ and so $x < z$. If $M_x = M_y = M_z$ and either $i_x < i_y$ or $i_y < i_z$, then $i_x < i_z$, so $x < z$ again follows. Finally, if $M_x = M_y = M_z$ and $i_x = i_y = i_z$ then $x < z$ follows from induction on n.

Recall that the result we are trying to prove is the following.

Theorem 3. *For every $n \geqslant 2$ and $m \geqslant 0$, the initial segment of size m of the balanced order on X_n minimises the gap among weak antichains in $\mathbb{Z}^n$ of size m. In particular, for every positive integer N, the set*

$$A_N = \{x \in \mathbb{Z}^n_{\geqslant 0} : 0 \leqslant x_i \leqslant N - 1 \text{ for all } i, \text{ and } x_j = 0 \text{ for some } j\}$$

minimises the gap among weak antichains of size $|A_N| = N^n - (N-1)^n$.

If $A \subseteq X_n$, we define the balanced-i-compression $\mathcal{C}_i^<(A)$ as follows. For each a, write

$$L_a^i(A) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X_{n-1} : (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \in A\}.$$

Also write

$$K^i(A) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{Z}^n_{>0} : (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \in A\}.$$

(Here $\mathbb{Z}_{>0}$ denotes the set of positive integers.) Note that for each $a > 0$, $L^i_a(A)$ corresponds to all points of A with i^{th} coordinate equal to a, but for $a = 0$ such points are partitioned into $L^i_0(A)$ and $K^i(A)$ according to whether or not they have another zero coordinate. We define $A' = \mathcal{C}^<_i(A)$ to be the set for which $L^i_a(A')$, $K^i(A')$ are given as follows. Let $L^i_a(A')$ be the initial segment of the balanced order on X_{n-1} of size $|L^i_a(A)|$ for each a, and let $K^i(A')$ be the first $|K^i(A)|$ elements of the ordering $\prec$ on $\mathbb{Z}^{n-1}_{>0}$ given by $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \prec (y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ if and only if $(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) < (y_1, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_n)$ (in the balanced order) on X_n. (Note that this is independent of the choice of i, and in fact the relation $\prec$ is given by the same rules as the balanced order.) Observe that $|A'| = |A|$.

It is not immediately clear that $\mathcal{C}^<_i(A)$ is a down-set when A is a down-set. For this we will have to establish another extremal property of initial segments. For $A \subseteq X_n$, write

$$S(A) = \{(x_1, \dots, x_n) \in \mathbb{Z}^n_{>0} : \text{ for all } j \text{ we have } (x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) \in A\}.$$

We will prove that initial segments maximise $|S(A)|$ and minimise the gap by induction on the dimension n. The following lemma will be essential in the induction step.

Lemma 10. *Let $n \geqslant 3$. Suppose that initial segments I of the balanced order maximise $|S(I)|$ among down-sets in X_{n-1} of given size. Then whenever A is a down-set in X_n and $i \in \{1, \dots, n\}$, then $A' = \mathcal{C}^<_i(A)$ satisfies the following.*

(i) *A' is a down-set.*

(ii) $|S(A')| \geqslant |S(A)|$.

(iii) *If it is also true that initial segments of the balanced order minimise the gap among subsets of X_{n-1} of given size, then* $\operatorname{gap}(A') \leqslant \operatorname{gap}(A)$.

Proof. (i) It is clear that $L^i_0(A') \supseteq L^i_1(A') \supseteq \dots$, and that the $L^i_a(A')$ and $K^i(A')$ are down-sets (in X_{n-1} and $\mathbb{Z}^{n-1}_{>0}$, respectively), since initial segments of the balanced order are down-sets. So it remains to show that $K^i(A') \subseteq S(L^i_0(A'))$. Note that we know this is true for A instead of A' since A is a down-set.

We claim that if I is an initial segment of the balanced order on X_{n-1}, then $S(I)$ is an initial segment of the ordering $\prec$ of $\mathbb{Z}^{n-1}_{>0}$ defined earlier. To see this, suppose that $x, y \in \mathbb{Z}^{n-1}_{>0}$, $y \in S(I)$ and $x \prec y$, we want to show that $x \in S(I)$. Let $T = \{j : x_j \neq y_j\}$ and $k = \min\{l \in T : y_l = \min_{j \in T} y_j\}$. Then we have the following, for each j.

- If $j \notin T$ then $(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n-1}) < (y_1, \dots, y_{j-1}, 0, y_{j+1}, \dots, y_{n-1})$.

- If $j \in T$ then $(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_{n-1}) \leqslant (y_1, \dots, y_{k-1}, 0, y_{k+1}, \dots, y_{n-1})$. Indeed, let us write $\bar{x}, \bar{y}$ for these vectors (respectively) and let $\bar{T} = \{l : \bar{x}_l \neq \bar{y}_l\}$. Note that $\bar{T} = T$ if $k \neq j$ and $\bar{T} = T \setminus \{j\}$ otherwise. It $\bar{T} = \emptyset$ then $\bar{x} = \bar{y}$, now assume $\bar{T} \neq \emptyset$. So $\max_{l \in \bar{T}} \bar{x}_l \leqslant \max_{l \in T} x_l \leqslant \max_{l \in T} y_l = \max_{l \in \bar{T}} \bar{y}_l$ and if we have equality throughout then $\max\{l \in \bar{T} : \bar{y}_l = \max_{s \in \bar{T}} \bar{y}_s\} = \max\{l \in T : y_l = \max_{s \in T} y_s\} \geqslant \max\{l \in T : x_l = \max_{s \in T} x_s\} \geqslant \max\{l \in \bar{T} : \bar{x}_l = \max_{s \in \bar{T}} \bar{x}_s\}$. These imply $\bar{x} \leqslant \bar{y}$.

Using that $y \in S(I)$ and that I is an initial segment, the above shows that $x \in S(I)$.

So $S(L_0^i(A'))$ and $K^i(A')$ are both initial segments. But $|S(L_0^i(A'))| \geqslant |S(L_0^i(A))| \geqslant |K^i(A)| = |K^i(A')|$, proving (i).

(ii) Note that for any $B \subseteq X_n$ and $a > 0$ we have

$$\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{Z}_{\geqslant 0}^{n-1} : (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \in S(B)\} \\ = K^i(B) \cap S(L_a^i(B)).$$

But for each a we have $|S(L_a^i(A))| \leqslant |S(L_a^i(A'))|$, and $K^i(A'), S(L_a^i(A'))$ are nested (since both of them are initial segments of $\prec$). This implies that

$$|K^i(A) \cap S(L_a^i(A))| \leqslant \min(|K^i(A)|, |S(L_a^i(A))|) \\ \leqslant \min(|K^i(A')|, |S(L_a^i(A'))|) = |K^i(A') \cap S(L_a^i(A'))|.$$

We get (ii) by summing over all values of a.

(iii) For any down-set $B \subseteq X_n$, we have $|\pi_j(B)| = \sum_{a \geqslant 0} |\pi_j(L_a^i(B))|$ for all $j \neq i$ and $|\pi_i(B)| = |L_0^i(B)| + |K^i(B)|$. It follows that $\operatorname{gap}(B) = |L_0^i(B)| + \sum_{a \geqslant 0} \operatorname{gap}(L_a^i(B))$. (Note that on the right hand side we have the gaps of $(n-1)$-dimensional sets.) But then (iii) follows trivially from the assumption that initial segments of the balanced order minimise the gap on X_{n-1}. □

The following lemma will be useful when considering sets satisfying $\mathcal{C}_i^<(A) = A$ for all i.

Lemma 11. *Suppose $n \geqslant 3$ and $A \subseteq X_n$ is a down-set having $\mathcal{C}_i^<(A) = A$ for all i. Assume that $x < y$ with $x \notin A$ and $y \in A$. Then*

(i) x has a unique coordinate which is zero.

(ii) if $x_l = y_l$ for some l, then $x_l = y_l = 0$ and y has at least one other coordinate which is zero.

Proof. Assume first that $x_l = y_l$ for some l. Since $\mathcal{C}_l^<(A) = A$, it must be the case that $x_l = y_l = 0$ and exactly one of x, y have a zero coordinate not at the l^{th} position. It follows that if we write $i = \max\{j : y_j = \max y\}$ then $y_i \neq x_i$. Using $y > x$, we get that

$$y_i \geqslant x_j \text{ for all } j \text{, and} \tag{3}$$

$$y_i > x_j \text{ for all } j \geqslant i. \tag{4}$$

Pick some $k \neq i, l$. Then the vector y' obtained by replacing the k^{th} coordinate of y by 0 is in A (since A is a down-set), and we have $y' > x$ (by (3) and (4)). By the same argument as above, we deduce from $x_l = y'_l$ and $\mathcal{C}_l^<(A) = A$ that $x_l = y'_l = 0$, and – since $y'_k = 0$ – that it must be the case that x has no zero coordinates other than the l^{th} one. Hence $x_l = y_l = 0$, $x_s \neq 0$ for all $s \neq l$, and there is an $s \neq l$ such that $y_s = 0$. This proves the lemma in this case.

Now assume that $x_l \neq y_l$ for all l. Writing $i = \max\{j : y_j = \max y\}$ again, (3) and (4) still hold. We only need to show that x has at most one coordinate which is zero. Assume that $x_k = x_l = 0$ with $k \neq l$, we may assume that $l \neq i$ (otherwise swap k and l). Let y' be obtained from y by replacing the l^{th} coordinate by 0. Then $y' \in A$ (since A is a down-set) and $y' > x$ (by (3) and (4)). But also $y'_l = x_l$, so by the first case (applied to x and y') we know that x has exactly one zero coordinate, giving a contradiction. $\square$

Lemma 12. *For every $n \geqslant 2$, initial segments I of the balanced order maximise $|S(I)|$ among down-sets in X_n of given size.*

Proof. We prove the statement by induction on n. If $n = 2$, then any down-set in X_n of size m is of the form $B_N = \{(i, 0) : i \in \mathbb{Z}_{\geqslant 0}, i \leqslant N\} \cup \{(0, i) : i \in \mathbb{Z}_{\geqslant 0}, i \leqslant m - 1 - N\}$ for some $0 \leqslant N \leqslant m - 1$ integer. We have $S(B_N) = \{(i, j) \in \mathbb{Z}^2_{>0} : 1 \leqslant i \leqslant N, 1 \leqslant j \leqslant m - 1 - N\}$, so $|S(B_N)| = N(m - 1 - N)$. Over the integers, this attains a maximum at $N = \lceil (m - 1)/2 \rceil$, which corresponds to the initial segment of the balanced ordering.

Now assume that $n \geqslant 3$ and the result holds for smaller values of n. Let A be any subset of X_n, we show the initial segment I of same size has $|S(I)| \geqslant |S(A)|$. Taking a down-set A' in X_n minimising $\sum_{x \in A'}$ (position of x in the balanced order) among sets with $|A'| = |A|$ and $|S(A')| \geqslant |S(A)|$, we may assume that $\mathcal{C}_i^{<}(A) = A$ for each i (by Lemma 10). Suppose that there are $x, y \in X_n$ with $x < y$, $y \in A$ and $x \notin A$.

Take y to be maximal (in the balanced order). Let $i = \max\{j : y_j = \max y\}$. If there is an $x \notin A$ with $x < y$ and the unique zero coordinate not being at the i^{th} position, pick the minimal of these (in the balanced order). Otherwise pick $x \notin A$ which is minimal. Consider $A' = A \setminus \{y\} \cup \{x\}$. Note that A' is again a down-set.

We show that $|S(A')| \geqslant |S(A)|$. (This would give a contradiction.) If y has more than one zero coordinates, then $S(A) \setminus S(A') = \emptyset$, so the claim is clear. Otherwise y has a unique zero coordinate y_t, and we must have $x_l \neq y_l$ for all l by Lemma 11. In particular, $y_i \neq x_i$. Thus $y_i > x_l$ for all $l \geqslant i$ and $y_i \geqslant \max x$. Observe that

$$S(A) \setminus S(A') = \{(y_1, \ldots, y_{t-1}, a, y_{t+1}, \ldots, y_n) : \\ a \in \mathbb{Z}_{>0} \text{ and replacing any coordinate by 0 we get an element of } A\}.$$

Recall that there is a unique s such that $x_s = 0$. We claim that $S(A) \setminus S(A')$ is empty unless $s = i$. Indeed, suppose $s \neq i$ and $S(A) \setminus S(A')$ has an element z corresponding to $a \geqslant 1$. Let z' be obtained from z by setting the s^{th} coordinate to be zero. Then $z' \in A$, $z' > x$ (as $z'_i = y_i$ so $z'_i > x_l$ for all $l \geqslant i$ and $z' \geqslant \max x$), $x_s = z'_s = 0$ and there is a unique coordinate at which z' is zero. This contradicts Lemma 11.

So we may assume $s = i$. Note that if $a \geqslant x_t$ and the corresponding vector appears in the set above, then A has an element z with $z_i = y_i$ and $z_t = x_t \neq 0$ (using that $n \geqslant 3$ and that A is a down-set. Note that $x_t \neq 0$ since $x_l \neq y_l$ for all l.) But then $z > x$, so this contradicts Lemma 11. It follows that $|S(A) \setminus S(A')| \leqslant x_t - 1 \leqslant y_i - 1$.

Furthermore, since $i = s$,

$$S(A') \setminus S(A) = \{(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) : \\ a \in \mathbb{Z}_{>0} \text{ and replacing any coordinate by 0 we get an element of } A'\}.$$

Also, by our choice of x, any $z \notin A$ with $z < y$ has ($z_i = 0$ and) $z_l \neq 0$ for all $l \neq i$. But this easily shows that for all $1 \leqslant a \leqslant y_i - 1$, the corresponding vector lies in $S(A') \setminus S(A)$. So $|S(A') \setminus S(A)| \geqslant y_i - 1 \geqslant |S(A) \setminus S(A')|$.

So we get a contradiction, finishing the proof. $\square$

Lemma 13. *For every $n \geqslant 2$, initial segments I of the balanced order minimise* $\mathrm{gap}(I)$ *among down-sets in X_n of given size.*

Proof. Again we prove this by induction on n. The case $n = 2$ is trivial, since any down-set in X_2 has gap 1.

Now assume that $n \geqslant 3$ and the result holds for smaller values of n. Let A be any subset of X_n, we show that the initial segment of same size has a gap which is not greater. Taking a down-set A' in X_n minimising $\sum_{x \in A'}$ (position of x in the balanced order) among sets with $|A'| = |A|$ and $\mathrm{gap}(A') \leqslant \mathrm{gap}(A)$, we may assume that $\mathcal{C}_i^{<}(A) = A$ for each i (by Lemma 10 and Lemma 12). Suppose that there are $x, y \in X_n$ with $x < y$, $y \in A$ and $x \notin A$. Take y to be maximal and x to be minimal (in the balanced order). Let $A' = A \setminus \{y\} \cup \{x\}$. Note that A' is a down-set.

By Lemma 11, there is a unique s such that $x_s = 0$. Then $\pi_j(A') \setminus \pi_j(A) = \emptyset$ if $j \neq s$ and $|\pi_s(A') \setminus \pi_s(A)| = 1$. On the other hand, if t is such that $y_t = 0$ then $|\pi_t(A) \setminus \pi_t(A')| = 1$. It follows that $\mathrm{gap}(A') \leqslant \mathrm{gap}(A)$, giving a contradiction. $\square$

Proof of Theorem 3. Immediate from Lemma 13 and Lemma 5. $\square$

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