b-invariant edges in essentially 4-edge-connected near-bipartite cubic bricks

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Abstract

A *brick* is a non-bipartite matching covered graph without non-trivial tight cuts. Bricks are building blocks of matching covered graphs. We say that an edge e in a brick G is *b-invariant* if $G - e$ is matching covered and a tight cut decomposition of $G - e$ contains exactly one brick. A 2-edge-connected cubic graph is *essentially 4-edge-connected* if it does not contain nontrivial 3-cuts. A brick G is *near-bipartite* if it has a pair of edges $\{e_1, e_2\}$ such that $G - \{e_1, e_2\}$ is bipartite and matching covered.

Kothari, de Carvalho, Lucchesi and Little proved that each essentially 4-edge-connected cubic non-near-bipartite brick G, distinct from the Petersen graph, has at least $|V(G)|$ b-invariant edges. Moreover, they made a conjecture: every essentially 4-edge-connected cubic near-bipartite brick G, distinct from K_4, has at least $|V(G)|/2$ b-invariant edges. We confirm the conjecture in this paper. Furthermore, all the essentially 4-edge-connected cubic near-bipartite bricks, the numbers of b-invariant edges of which attain the lower bound, are presented.

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1 Introduction

All graphs considered in this paper are finite and may contain multiple edges, but no loops. We will generally follow the notation and terminology used by Bondy and Murty in [1]. A graph is called *matching covered* if it is connected, has at least one edge and each of its edges is contained in some perfect matching. For the terminology that is specific to matching covered graphs, we follow Lovász and Plummer [15].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v, we use $N(v)$ to denote its neighborhood, that is $N(v) := \{u \in V(G) : uv \in E(G)\}$. For any $X, Y \subseteq V(G)$, denote by $E_G(X, Y)$ the set of edges in G with one end in X, the other in Y. We say $\partial(X) := E_G(X, \overline{X})$ is an *edge cut* of G, where $\overline{X} := V(G) - X$. An edge cut $C := \partial(X)$ of G is a *separating cut* if, for every edge e in G, there exists a perfect matching M_e that contains e satisfying $|M_e \cap C| = 1$, and is a *tight cut* if $|C \cap M| = 1$ for every perfect matching M of G. Obviously, every tight cut is separating. An edge cut $\partial(X)$ is *trivial* if either $|X| = 1$ or $|\overline{X}| = 1$. We call a matching covered graph free of non-trivial tight cuts a *brick* if it is non-bipartite, and a *brace* otherwise. Edmonds et al. [10] (also see Lovász [14], Szigeti[18] and de Carvalho et al. [8]) showed that a graph G is a brick if and only if G is 3-connected and $G - \{x, y\}$ has a perfect matching for any two distinct vertices $x, y \in V(G)$ (bicritical). Lovász [14] proved that any matching covered graph can be decomposed into a unique list of bricks and braces by a procedure called the tight cut decomposition. In particular, any two applications of the tight cut decomposition of a matching covered graph G yield the same number of bricks, which is called the *brick number* of G and denoted by $b(G)$.

A non-bipartite matching covered graph G is *near-bipartite* if it has a pair of edges e_1 and e_2 such that $G - \{e_1, e_2\}$ is bipartite and matching covered. An edge e of a matching covered graph G is *removable* if $G - e$ is also matching covered. Suppose $\{e_1, e_2\} \subseteq E(G)$. We say that $\{e_1, e_2\}$ is a *removable doubleton* of G if neither e_1 nor e_2 are removable, and $G - \{e_1, e_2\}$ is matching covered. A removable edge e of a matching covered graph G is *b-invariant* if $b(G - e) = b(G)$, and is *quasi-b-invariant* if $b(G) = 1$ and $b(G - e) = 2$. For a bipartite graph G, every removable edge is b-invariant, since $b(G) = 0$. De Carvalho, Lucchesi and Murty [3] showed that each removable edge is also b-invariant in a solid brick, where a solid brick is a brick free of non-trivial separating cuts. Confirming a conjecture of Lovász, they proved in [4] that every brick, distinct from K_4, $\overline{C}_6$ and the Petersen graph, has a b-invariant edge. De Carvalho, Lucchesi and Murty [7] showed that every solid brick G has at least $|V(G)|/2$ b-invariant edges. Based on properties of b-invariant edges, all bricks can be generated by using several operations from three basic bricks [6]. b-invariant edges have many applications in matching theory. Readers may refer to [5, 13, 16] for applications of b-invariant edges.

Let k be a positive integer. Recall that a graph G is *k-edge-connected* if $|C| \geqslant k$ for every edge cut C of G. An edge cut with k edges is called a *k-cut*. A cubic graph is *essentially 4-edge-connected* if it is 2-edge-connected and the only 3-cuts are the trivial ones. Recently, Kothari, de Carvalho, Lucchesi and Little considered the property of removable edges in essentially 4-edge-connected cubic bricks, and showed that each essentially

4-edge-connected cubic non-near-bipartite brick G, distinct from the Petersen graph, has at least $|V(G)|$ b-invariant edges [12]. They also made the following conjecture in the same paper.

Conjecture 1. *[12] Every essentially 4-edge-connected cubic near-bipartite brick G, distinct from K_4, has at least $|V(G)|/2$ b-invariant edges.*

Denote by H_k the Cartesian product of a path of order $k (k \geqslant 2)$ and K_2 (the complete graph with two vertices). Suppose the four vertices with degree two of H_k are $\{u, v, x, y\}$ such that u and x lie in the same color class of H_k. By adding edges ux, vy to H_k, we get a *prism* if k is odd, and a *Möbius ladder* if k is even. Prisms and Möbius ladders are two types of cubic bricks which play an important role in generating bricks [6, 17].[1]

Kothari, de Carvalho, Lucchesi and Little [12] also point out two infinite families that attain the lower bound in Conjecture 1 exactly are: prisms of order $4k + 2$, and Möbius ladders of order $4k$, where $k \geqslant 2$. In this paper we present a proof of Conjecture 1 and characterize all the graphs that attain this lower bound. The main result is stated as follows.

Theorem 2. *Every essentially 4-edge-connected cubic near-bipartite brick G, distinct from K_4, has at least $|V(G)|/2$ b-invariant edges. Furthermore, prisms of order $4k + 2$, and Möbius ladders of order $4k$, where $k \geqslant 2$, are the only two families of graphs that attain this lower bound.*

The proof of Theorem 2 will be given in Section 3 after we present some properties concerning removable edges and removable doubletons of matching covered graphs in Section 2. We now state two theorems that will be useful in the proof of the main result.

Theorem 3. *[12] In an essentially 4-edge-connected cubic brick, each edge is either removable or otherwise participates in a removable doubleton. Moreover, each removable edge is either b-invariant or otherwise quasi-b-invariant.*

Theorem 4. *[12] Let G be an essentially 4-edge-connected cubic near-bipartite brick that has two adjacent quasi-b-invariant edges. Then G is the Cubeplex (see Fig.1 (a)).*

2 Equivalence classes in a brick

Let G be a matching covered graph. Two edges e_1, e_2 of G are *mutual dependence* if either $\{e_1, e_2\} \subseteq M$ or $\{e_1, e_2\} \cap M = \emptyset$ for every perfect matching M of G. Obviously, mutual dependence is an equivalence relation. It partitions the edge set $E(G)$ into equivalence classes[2]. Lovász proved the following attractive property of an equivalence class in a brick.

[1] By adding edges uy, vx to H_k, we also get a *prism* when k is even, and a *Möbius ladder* when k is odd. In this case, the resulting graphs are bipartite, which are two types of cubic braces, see [16] for example.

[2] This terminology follows de Carvallo et al. [2]. An equivalence class contained at least two edges is called an equivalent set in [11].

Theorem 5. *[14] Let G be a brick and let e and f be two distinct mutual dependence edges of G. Then $G - \{e, f\}$ is bipartite.*

An equivalence class in a brick contains at most two edges; and a removable doubleton in a brick is an equivalence class by Theorem 5. Obviously, the intersection of any two different equivalence classes of a brick is the empty set. Two distinct equivalence classes of a matching covered graph are *mutually exclusive* if no perfect matching contains edges in both classes. The following theorem gave a characterization of bricks with more than two mutually exclusive removable doubletons.

Theorem 6. *[2] If a brick G has three mutually exclusive removable doubletons, then either G is K_4 or G is $\overline{C}_6$, up to multiple edges joining vertices of both triangles of $\overline{C}_6$.*

An ear decomposition of a matching covered graph G is *optimal* if, among all ear decompositions of G, it uses the least possible number of double ears (for the definition of the ear decomposition, see page 174 in [15]).

Theorem 7. *[5] The number of double ears in an optimal ear decomposition of a matching covered graph G is $b(G) + p(G)$, where $p(G)$ is the number of bricks of G whose are isomorphic to the Petersen graph (up to multiple edges).*

It can be checked that the number of double ears in an optimal ear decomposition of a near-bipartite graph is 1. The following proposition is implied by Theorem 7.

Proposition 8. *If G is a near-bipartite graph, then $b(G) = 1$.*

We say a bipartite graph $G(A, B)$ is *balanced* if $|A| = |B|$. A bipartite graph with a perfect matching is always balanced. For the equivalence classes in a bipartite graph, we have the following result, which is an immediate consequence of Lemma 4.1 in [2].

Lemma 9. *Suppose e_1 and e_2 are equivalent in a matching covered bipartite graph G. Then $\{e_1, e_2\}$ forms a 2-edge-cut which separates G into two balanced components.*

The following Dulmage-Mendelsohn decomposition of a bipartite graph with a perfect matching will be used later.

Theorem 10. *[9] Let $G(A, B)$ be a bipartite graph with a perfect matching. An edge e of G does not lie in any perfect matching of G if, and only if, there exists a partition (A_1, A_2) of A and a partition (B_1, B_2) of B such that $|A_1| = |B_1|$, $e \in E_G(A_2, B_1)$ and $E_G(A_1, B_2) = \emptyset$.*

An equivalence class with two edges in a brick is not always a removable doubleton, that is a brick contained two distinct equivalent edges could be non-near-bipartite. For example, it can be checked that $\{e_1, e_2\}$ is the only equivalence class with two edges in the graph in Figure 1 (b); after removing e_1 and e_2, no perfect matching in the remaining graph would contain any red edge. So this graph is not near-bipartite. But for cubic bricks, this result is true as shown in the following proposition.

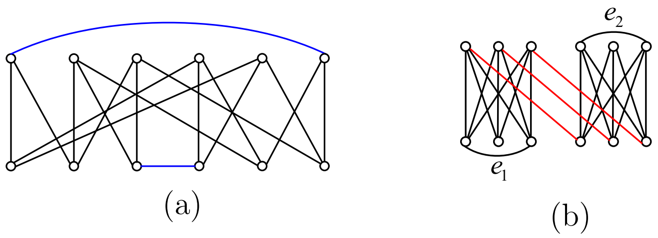

Figure 1: (a) The Cubeplex; (b) a non-near-bipartite brick with two distinct equivalent edges.

Proposition 11. *Let G be a cubic brick. If there exist two edges e and f such that $G - \{e, f\}$ is bipartite, then G is near-bipartite.*

Proof. Since G is a brick, G is not bipartite and is matching covered. By the definition of a near-bipartite graph, we need to show that $G - \{e, f\}$ is matching covered to complete the proof. Suppose the two color classes of $G - \{e, f\}$ are A and B. Obviously, two ends of e lie in the same color class of $G - \{e, f\}$, say A. So two ends of f belong to B. Suppose to the contrary that $G - \{e, f\}$ is not matching covered. By Theorem 10, A and B can be partitioned into A_i and B_i ($i = 1, 2$), respectively, such that $|A_1| = |B_1|$, $|E_{G-\{e,f\}}(A_1, B_2)| \geqslant 1$ and $|E_{G-\{e,f\}}(A_2, B_1)| = 0$. Recalling that G is cubic, by calculating the sum of degrees of the vertices in A_i and B_i, we have either $E_G(A_1, A_2) = \{e\}$, $|E_G(A_1, B_2)| = 1$, $E_G(B_1, A_2 \cup B_2) = \emptyset$ or $E_G(B_1, B_2) = \{f\}$, $|E_G(A_1, B_2)| = 1$, $E_G(A_2, A_1 \cup B_1) = \emptyset$. We have $|E_G(A_1 \cup B_1, A_2 \cup B_2)| = 2$ in both cases. Therefore G is not 3-connected, contradicting the fact that G is a brick. □

The next two propositions concern the equivalence classes in a near-bipartite cubic brick. The first one was already known to de Carvalho, Kothari, Lucchesi and Murty in 2014.

Proposition 12. *Suppose $\mathscr{E}$ is a set of removable doubletons of an essentially 4-edge-connected cubic brick G and $|\mathscr{E}| \geqslant 2$. Then G can be decomposed into balanced bipartite vertex-induced subgraphs G_i ($i = 1, 2, \ldots, |\mathscr{E}|$) satisfying $E_G(G_j, G_k)$ is a removable doubleton of G that belongs to $\mathscr{E}$ if $|j - k| \equiv 1 \pmod{|\mathscr{E}|}$; otherwise, $E_G(G_j, G_k) = \emptyset$. Furthermore, if $\mathscr{E}$ is the set of all the removable doubletons in G, then $|V(G_i)| \neq 4$ for every $i \in \{1, 2, \ldots, |\mathscr{E}|\}$.*

Proof. Suppose $\mathscr{E} := \{\{e_i, e_i'\} : i = 1, 2, \ldots, s\}$, where $s = |\mathscr{E}|$. Let $G' := G - \{e_1, e_1'\}$. By Theorem 5, G' is bipartite. By Proposition 11, G is near-bipartite. So G' is matching covered. Note that every perfect matching of G' is also a perfect matching of G and $\{e_i, e_i'\}$ is a removable doubleton of G. Then, for $i = 2, 3, \ldots, s$, e_i and e_i' are equivalent in G'. By Lemma 9, for $i = 2, 3, \ldots, s$, $\{e_i, e_i'\}$ is an edge-cut separating G' into two balanced components, denoted by H_i and H_i'.

Claim. For any $j, k \in \{2, 3, \ldots, s\} (j \neq k)$, $V(H_j) \subset V(H_k)$ or $V(H_j) \subset V(H_k')$.

Proof of Claim. Suppose to the contrary that each of the four sets $V(H_j) \cap V(H_k)$, $V(H_j') \cap V(H_k)$, $V(H_j') \cap V(H_k')$ and $V(H_j) \cap V(H_k')$ is not empty. Let $U_1 := V(H_j) \cap V(H_k)$, $U_2 := V(H_j') \cap V(H_k)$, $U_3 := V(H_j') \cap V(H_k')$ and $U_4 := V(H_j) \cap V(H_k')$.

Noticing that both $|V(H_j)|$ and $|V(H_k)|$ are even, the parities of the cardinalities of those four sets U_1, U_2, U_3 and U_4 are the same. Recalling that G' is matching covered, it is 2-connected. Therefore, $|E_{G'}(U_i, \overline{U_i})| \geqslant 2$ for $i = 1, 2, 3, 4$. Noting that $|E_{G'}(V(H_j), V(\overline{H}_j))| = 2 = |E_{G'}(V(H_k), V(\overline{H}_k))|$, we have $|E_{G'}(U_i, \overline{U_i})| = 2$ for $i = 1, 2, 3, 4$. Note that an edge in G contributes at most 2 in $\sum_{i=1}^{4} |E_G(U_i, \overline{U_i})|$. Therefore, $\sum_{i=1}^{4} |E_G(U_i, \overline{U_i})| \leqslant \sum_{i=1}^{4} |E_{G'}(U_i, \overline{U_i})| + 2 \times 2 = 12$.

On the other hand, we have $\sum_{i=1}^{4} |E_G(U_i, \overline{U_i})| \geqslant 16$. If $|U_1|$ is even, then $|E_G(U_i, \overline{U_i})| \geqslant 4$ since G is essentially 4-edge-connected for $i = 1, 2, 3, 4$, therefore, $\sum_{i=1}^{4} |E_G(U_i, \overline{U_i})| \geqslant 16$. Now we consider when $|V(H_j) \cap V(H_k)|$ is odd. If three vertex sets in U_1, U_2, U_3 and U_4 are singleton, without loss of generality, suppose $|U_1| = |U_2| = |U_3| = 1$. Then $|E_G(U_4, \overline{U_4})| = 3$ since G is cubic. A contradiction with the fact that G is essentially 4-edge-connected. So at most two vertex sets in U_1, U_2, U_3 and U_4 are singleton. Again, by G is cubic, $\sum_{i=1}^{4} |E_G(U_i, \overline{U_i})| \geqslant 3+3+5+5 = 16$. This contradiction shows the Claim holds. □

Now, we can choose $i_1 \in \{2, 3, \ldots, s\}$ such that $V(H_{i_1}) \subset V(H_i)$ for every $i \in \{2, 3, \ldots, s\} \setminus \{i_1\}$; for $j = 2, 3, \ldots, s-1$, let $i_j \in \{2, 3, \ldots, s\} \setminus \{i_1, i_2, \ldots, i_{j-1}\}$ such that $V(H_{i_j}) \subset V(H_i)$ for every $i \in \{2, 3, \ldots, s\} \setminus \{i_1, i_2, \ldots, i_j\}$ (exchange the notation of H_i with H'_i if needed). Let $G_1 = H_{i_1}$, $G_j = H_{i_j} - V(H_{i_{j-1}})$ for $j = 2, 3, \ldots, s-1$ and $G_s = H'_{i_{s-1}}$. Again, since G is essentially 4-edge-connected, $E_G(G_1, G_s) = \{e_1, e'_1\}$. Therefore $G_i (i = 1, 2, \ldots, s)$ is the subgraph we need.

If $\mathscr{E}$ is the set of all the removable doubletons in G, suppose to the contrary that there exists some i such that $|V(G_i)| = 4$ (assume that A_i and B_i are the color classes of G_i). Then G_i is a 4-cycle, denoted by $a_1 b_1 a_2 b_2 a_1$, where $a_1, a_2 \in A_i$ and $b_1, b_2 \in B_i$. Therefore $G - \{a_1 b_2, a_2 b_1\}$ is a bipartite graph with color classes $\cup_{j=1}^{i-1} A_j \cup \{a_1, b_2\} \cup (\cup_{j=i+1}^{s} B_j)$ and $\cup_{j=1}^{i-1} B_j \cup \{b_1, a_2\} \cup (\cup_{j=i+1}^{s} A_j)$. By Proposition 11, $\{a_1 b_2, a_2 b_1\}$ constitutes a removable doubleton of G which is not in $\mathscr{E}$. A contradiction. So the result follows. □

Proposition 13. *Suppose $\{e_1, e'_1\}$ and $\{e_2, e'_2\}$ are removable doubletons of a cubic brick G. If both e_1 and e_2 are incident with v_0, then e'_1 and e'_2 are adjacent, and $v_0 u_0 \in E(G)$, where u_0 is the common vertex of e'_1 and e'_2.*

Proof. Suppose $e'_1 = u_0 u_1, e'_2 = u'_0 u_2$ and $N(v_0) = \{v_1, v_2, v_3\}$, where $e_1 = v_0 v_1, e_2 = v_0 v_2$. We will show that $u_0 = u'_0 = v_3$ to complete the proof. By Proposition 12, G can be decomposed into balanced bipartite vertex-induced subgraphs $G_i(A_i, B_i)$ $(i = 1, 2)$ such that $v_0 \in A_1$, $u_0, u'_0, v_3 \in B_1$, $v_1, u_2 \in A_2$, $v_2, u_1 \in B_2$.

Note that $N(A_1 - \{v_0\}) = B_1$. Then simple counting argument shows that $V(G_1) - \{v_0\}$ is a shore of a tight cut of G. Since G is free of nontrivial tight cuts, $V(G_1) - \{v_0\}$ must be trivial. We deduce that $B_1 = \{u_0\}$ and $A_1 = \{v_0\}$. This proves Proposition 13. □

3 The proof of the main theorem

It is easy to check that for a cubic brick G, G is isomorphic to K_4 if $|V(G)| = 4$, and G is isomorphic to $\overline{C_6}$ if $|V(G)| = 6$. Noticing $\overline{C_6}$ is not essentially 4-edge-connected, in the following, we always assume that $|V(G)| \geqslant 8$. We can classify the edges of G, by

Theorem 3, into three disjoint classes: (1) edges that participate in a removable doubleton, (2) b-invariant edges, and (3) quasi-b-invariant edges. For simplicity, we denote the three edge sets by E_1, E_2 and E_3, respectively. Therefore, $|E_1| + |E_2| + |E_3| = \frac{3}{2}|V(G)|$. We will show that $|E_2| \geqslant |V(G)|/2$ to complete the proof. Note that the Cubeplex contains $14 > 6 = |V(G)|/2$ b-invariant edges. So we suppose G is not the Cubeplex. Therefore, by Theorem 4, every vertex in G is incident with at most one quasi-b-invariant edge, that is $|E_3| \leqslant |V(G)|/2$. We will consider the following two cases depending on the number of removable doubletons.

Case 1. G has at most two removable doubletons.

This implies that $|E_1| \leqslant 4$. Recalling that $|V(G)| \geqslant 8$, $|E(G)| \geqslant 12$. So $|E_1| \leqslant |V(G)|/2$. Recall that $|E_3| \leqslant |V(G)|/2$. Hence, $|E_2| \geqslant |V(G)|/2$.

Now, we show every brick has more than $|V(G)|/2$ b-invariant edges in this case. Firstly, we claim that $|V(G)| = 8$ and $|E_1| = 4$. Otherwise, $|V(G)| > 8$, or $|E_1| = 2 < |V(G)|/2$. Since $|E_1| \leqslant 4$, we have $|E_1| < |V(G)|/2$ in either case. Since $|E_3| \leqslant |V(G)|/2$, it follows that $|E_2| > |V(G)|/2$. Therefore, G contains more than $|V(G)|/2$ b-invariant edges, a contradiction. Thus, $|V(G)| = 8$ and $|E_1| = 4$.

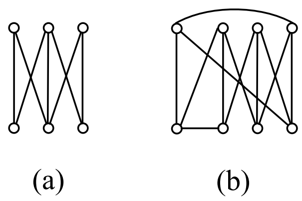

Figure 2: (a) the graph isomorphic to G_2; (b) G'.

As $|V(G)| = 8$ and $|E_1| = 4$, by Proposition 12, we may assume that $G - E_1$ contains two bipartite vertex-induced subgraphs G_1 and G_2, and $|V(G_1)| = 2$ and $|V(G_2)| = 6$. Then, G_1 is isomorphic to K_2, G_2 contains four vertices with degree two and the remaining two vertices have degree three. It is easy to check that G_2 is isomorphic the graph in Figure 2 (a). Recall that G is near-bipartite. Hence, G is isomorphic to the Möbius ladder with 8 vertices or the graph G' shown in Figure 2 (b). However, the Möbius ladder with 8 vertices has four distinct removable doubletons, and G' contains a triangle which implies that it contains a nontrivial 3-cut. So G' is not essentially 4-edge-connected, giving a contradiction. Therefore, no graph has at most two removable doubletons and exactly $|V(G)|/2$ b-invariant edges.

Case 2. G has more than two removable doubletons.

We will show that each vertex in G is incident with at least one b-invariant edge. Recall that every vertex in G is incident with at most one quasi-b-invariant edge. So, it is enough to show the following claim.

Claim 1. If every edge in $\{uu_1, uu_2\}$ participates in a removable doubleton, then uv is b-invariant in G, where $v \in N(u) - \{u_1, u_2\}$.

Proof of Claim 1. Firstly, we claim uv is removable in G. Otherwise, G contains three mutually exclusive removable doubletons, so that either G is K_4 or G is $\overline{C}_6$, up to multiple edges joining vertices of both triangles of $\overline{C}_6$ by Theorem 6, contradicting the hypothesis that $|V(G)| \geqslant 8$.

Note that G has more than two removable doubletons, we may assume that $\{e, e'\}$ is a removable doubleton of G such that $\{e, e'\} \cap \{uu_1, uu_2\} = \emptyset$. Now, we will show that uv is also removable in $G - \{e, e'\}$. Assume to the contrary that there exists an edge f of G that is not contained in any perfect matching of $G - \{e, e', uv\}$. Note that $G - \{e, e'\}$ is matching covered by Proposition 11. Then any perfect matching M_1 of $G - \{e, e'\}$ that contains f also contains uv. By Proposition 13, we may assume that $\{uu_1, vv_1\}$ and $\{uu_2, vv_2\}$ are two removable doubletons of G. By Proposition 12, we may assume G can be decomposed into balanced bipartite vertex-induced subgraphs $G_i(A_i, B_i)(i = 1, 2, 3)$ satisfying:
(1) $E(A_1, B_2) = vv_1$, $E(A_2, B_3) = uu_2$, $E(B_1, A_2) = uu_1$, $E(B_2, A_3) = vv_2$, $E(A_1, A_3) = e$ and $E(B_1, B_3) = e'$;
(2) $G_2 = uv$ (see Figure 3).

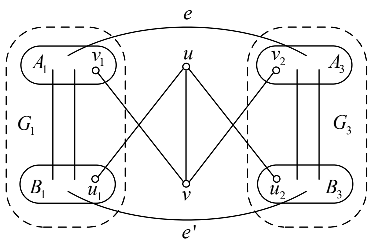

Figure 3: illustration in the proof of Claim 1.

Assume without loss of generality that $f \in E(G_1)$. Then $M_1 \cap E(G_1)$ is a perfect matching of G_1 that contains f. Let M_2 be an arbitrary perfect matching of $G - \{e, e'\}$ that contains the removable doubleton of $\{uu_2, vv_2\}$. Then $M_2 \cap E(G - V(G_1))$ is a perfect matching of $G - V(G_1)$. So, $(M_1 \cap E(G_1)) \cup (M_2 \cap E(G - V(G_1))$ is a perfect matching of $G - \{e, e'\}$ that contains f but not uv, giving a contradiction.

Finally, since uv is removable in both G and $G - \{e, e'\}$, we conclude that $G - uv$ is near-bipartite and $\{e, e'\}$ is a removable doubleton of $G - uv$. So, $b(G - uv) = 1$ by Proposition 8. Therefore, uv is b-invariant in G. □

Now, we show that all the graphs that attain this lower bound are a prism of order $4k + 2$, and a Möbius ladder of order $4k$, for some $k \geqslant 2$. Now we suppose that G is an essentially 4-edge-connected cubic near-bipartite brick with exactly $|V(G)|/2$ b-invariant edges, so that $|E_1| > 4$. We have the following claim.

Claim 2. Each component of $G - E_1$ is K_2.

Proof of Claim 2. Suppose to the contrary that there exists a component of $G - E_1$, say G_i, satisfying $|V(G_i)| > 2$. Then $|V(G_i)| \geqslant 6$ by Proposition 12. Now we consider the edge set $E(G_i)$. Note that G_i contains at most $|V(G_i)|/2$ quasi-b-invariant edges of G by Theorem 4. Since each edge of $E(G_i)$ is removable in G, G_i contains at least $|E(G_i)| - \frac{|V(G_i)|}{2}$ b-invariant edges of G. By Proposition 12, G_i and $G - V(G_i)$ are connected by two pairs of removable doubletons of G. Since $|V(G_i)| > 2$, every edge in one of those removable doubletons is not adjacent to any edge in the other removable doubleton by Proposition 13. Therefore, except four vertices, the degrees of all the other vertices in G_i are three, that is $|E(G_i)| = 3|V(G_i)|/2 - 2$. This implies that G_i contains more than $|V(G_i)|/2$ b-invariant edges. For every component G_j with two vertices, both of those two vertices are incident with two edges which lie in different removable doubletons, respectively. By Claim 1, the unique edge of the component is b-invariant. So G_j contains exactly $|V(G_j)|/2(= 1)$ b-invariant edges of G. Therefore, we can conclude that G contains more than $|V(G)|/2$ b-invariant edges if $G - E_1$ contains a component with more than one edge, giving a contradiction. □

Each vertex of G is incident with two edges in E_1 by Claim 2. Hence, G is isomorphic to a prism if $|G| = 4k + 2$, and is isomorphic to a Möbius ladder if $|G| = 4k$, for some $k \geqslant 2$. Therefore, Theorem 2 holds.

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