The Ramsey number of the Fano plane versus the tight path

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Abstract

The hypergraph Ramsey number of two 3-uniform hypergraphs G and H, denoted by $R(G, H)$, is the least integer N such that every red-blue edge-coloring of the complete 3-uniform hypergraph on N vertices contains a red copy of G or a blue copy of H.

The Fano plane $\mathbb{F}$ is the unique 3-uniform hypergraph with seven edges on seven vertices in which every pair of vertices is contained in a unique edge. There is a simple construction showing that $R(G, \mathbb{F}) \geqslant 2(v(G)-1)+1$ for every connected G. Hypergraphs G for which the equality $R(G, \mathbb{F}) = 2(v(G)-1)+1$ holds are called $\mathbb{F}$-good. Conlon posed the problem to determine all G that are $\mathbb{F}$-good.

In this short paper we make progress on this problem by proving that the tight path of length n is $\mathbb{F}$-good.

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1 Introduction

Ramsey theory is one of the most intensively studied topics in combinatorics. Given two k-uniform hypergraphs G and H, we denote by $R(G, H)$ the hypergraph Ramsey number of G and H. That is, $R(G, H)$ is the least integer such that any red-blue edge-coloring of the complete k-uniform hypergraph on that many vertices contains a red G or a blue H as a subhypergraph. The existence of $R(G, H)$ is guaranteed by Ramsey's theorem [15]. Given a bounded degree k-uniform hypergraph H, it is known that the Ramsey number $R(H, H)$ is linear in the number of vertices of H [4, 6, 7, 10, 12]. However, estimating or even determining Ramsey numbers precisely is often a difficult problem.

In this short paper we will determine exactly the Ramsey number of the tight path and the Fano plane. This result is the first progress on a question asked at the AIMS workshop on hypergraph Ramsey problems in 2015 by Conlon [16]. A simple construction by Burr [1] shows that

$$R(G, H) \geqslant (\chi(H) - 1)(v(G) - 1) + \sigma(H), \tag{1}$$

provided H is connected and $v(G) \geqslant \sigma(H)$, where $\chi(H)$ is the chromatic number of H and $\sigma(H)$ is the size of the smallest color class in any $\chi(H)$-coloring of H. Following Burr and Erdős [1, 2], we will say that G is H-good, if (1) holds with equality. The intuition behind this definition was that H-good graphs tend to be poor expanders (see [5, Section 2.5] for further details). Denote by $\mathbb{F}$ the Fano plane, i.e. the unique 3-uniform hypergraph with seven edges on seven vertices in which every pair of vertices is contained in a unique edge. Conlon asked which hypergraphs are $\mathbb{F}$-good [16].

In the graph case, there are many exact Ramsey numbers known. Erdős [2] started the systematic study of cliques versus large graphs. Nikiforov and Rousseau [13] gave a new approach to provide exact result for several families of graphs. Recently, the Ramsey number of the cycle and the clique (Keevash, Long and Skokan [11]), and the Ramsey number of the clique and the hypercube (Griffiths, Morris, Fiz Pontiveros, Saxton, Skokan [8]) have been determined. For similar results we refer the interested reader to the excellent recent survey [5].

In the hypergraph case there are only few instances where the Ramsey number is known exactly. Our result is the first Ramsey-goodness-result for hypergraphs.

From now on, we consider only 3-uniform hypergraphs. Let P_n^t be the tight path on n vertices, i.e. it contains distinct vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_{n-2}$ where $e_i = \{v_i, v_{i+1}, v_{i+2}\}$.

Theorem 1. *There exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, we have $R(P_n^t, \mathbb{F}) = 2n - 1$.*

Using the definitions from above, Theorem 1 states that P_n^t is $\mathbb{F}$-good. The lower bound $R(P_n^t, \mathbb{F}) \geqslant 2n - 1$ follows easily by the following folklore construction. Write the vertex set of the complete 3-uniform hypergraph on $2n - 2$ vertices $K_{2n-2}^{(3)}$ as the disjoint union of two sets A and B with $|A| = |B| = n - 1$. Color all 3-edges that are fully contained in either A or in B red, and all other edges blue. Observe that as $|A|, |B| < n$

there is no red P_n^t in this coloring. Since the chromatic number[1] $\chi(\mathbb{F}) = 3$, in every copy of $\mathbb{F}$ there is one edge that is fully contained in either A or B, hence this coloring cannot contain a blue copy of $\mathbb{F}$ either. This establishes that $R(P_n^t, \mathbb{F}) > 2n - 2$.
There is another almost extremal example, which is very different. For simplicity, let n be divisible by 6. Write the vertex set of the complete 3-uniform hypergraph on $2n-3$ vertices $K_{2n-3}^{(3)}$ as the disjoint union of three sets A,B and C with $|A| = |B| = |C| = 2n/3 - 1$. Color red all triples of vertices $\{x, y, z\}$ such that either $x, y \in A$ and $z \in A \cup B$, or $x, y \in B$ and $z \in B \cup C$, or $x, y \in C$ and $z \in A \cup C$. Color all other triples blue. A short case analysis (see Lemma 4) shows that there is no blue Fano plane $\mathbb{F}$. The longest red tight path is obtained by alternating between taking two vertices from one of the sets and one vertex from another. Such a path can have length at most $|A| + |B|/2 + 1 \leqslant n - 1$.
The main contribution of our work is to establish the upper bound $R(P_n^t, \mathbb{F}) \leqslant 2n - 1$. In the proof of the upper bound, we will build up a picture of what a potentially bad coloring could look like and quickly realize that a bad coloring needs to be close to one of the two previous described colorings. In the remainder of the proof we then rule out these two types separately.

We remark that this proof technique also works for proving that the tight cycle is $\mathbb{F}$-good. Let $C_n^{(3)}$ be the tight cycle on n vertices, i.e. it contains distinct vertices $v_1, v_2, \dots, v_n$ and edges $e_1, e_2, \dots, e_n$ with $e_i = \{v_i, v_{i+1}, v_{i+2}\}$ where $v_{n+1} := v_1$ and $v_{n+2} := v_2$.

Theorem 2. *There exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$, we have $R(C_n^{(3)}, \mathbb{F}) = 2n - 1$.*

We choose not to present the proof of Theorem 2 since the proof is almost the same as the proof of Theorem 1 and differs only in some technicalities which do not give the reader more insight into the methods used.

The organization of the paper is as follows. In Section 2 we will show that Theorem 1 is sharp in the sense that the Ramsey number increases when one adds a small number of edges to the tight path. In Section 3 we will give some definitions and basic tools which will be needed for the proof of the upper bound in Theorem 1. The proof itself will be given in Section 4.

2 Sharpness example

The following example shows that Theorem 1 is best possible in the following sense: Let P' be the 3-uniform hypergraph obtained from the tight path P_n^t by adding three edges $\{v_2, v_3, v_6\}$, $\{v_1, v_2, v_5\}$ and $\{v_1, v_4, v_6\}$. We claim that Ramsey number of P' and $\mathbb{F}$ is bigger than the Ramsey number of P_n^t and $\mathbb{F}$.

Theorem 3. *$R(P', \mathbb{F}) > 2n$.*

Assume, for simplicity, that n is divisible by 3. Take three sets A, B, C of size $2n/3$ each. Color red all triples of vertices $\{x, y, z\}$ such that either $x, y \in A$ and $z \in A \cup B$, or

[1]Here, the chromatic number $\chi(H)$ of a hypergraph H is the smallest number k of colors for which there exists a k-coloring of the vertices of H with no monochromatic edge. This is sometimes called the weak chromatic number of H.

$x, y \in B$ and $z \in B \cup C$, or $x, y \in C$ and $z \in A \cup C$. All other triples are colored blue. The following two Lemmas check that there is no blue $\mathbb{F}$ and no red P_n^t.

Lemma 4. *The coloring described above does not contain a blue* $\mathbb{F}$.

Proof. Let F be a blue Fano plane in the previously described coloring. For $i, j, k \in [7]$ with $i + j + k = 7$, we say that a Fano plane is of type (i, j, k) if i vertices come from A, j vertices come from B and k vertices come from C. Let $a_1, \ldots, a_i$ be the vertices in A; $b_1, \ldots, b_j$ be the vertices in B; and $c_1, \ldots, c_k$ the vertices in C. Since the chromatic number of the Fano plane is three, F needs to have one vertex from each set. Thus, F has to be of type $(5, 1, 1), (4, 2, 1), (4, 1, 2), (3, 2, 2)$ or $(3, 3, 1)$ up to rotation. Since every 5-subset of vertices in a Fano plane contains an edge, F cannot be of type $(5, 1, 1)$. If F is of type $(4, 2, 1)$, then $b_1b_2c_1$ forms an edge or there is an edge inside A, because F has chromatic number three. However, by construction of the coloring both of these edge are red. If F is of type $(4, 1, 2)$, then again $c_1c_2b_1$ forms an edge or there is an edge inside A. Since the edge inside A is red, $c_1c_2b_1$ forms an edge. In a Fano plane every pair is in exactly one edge, thus b_1 cannot be in any further blue edge. This contradicts that every vertex is in three edges. Now, let F be of type $(3, 2, 2)$. There has to be an edge inside $\{b_1, b_2, c_1, c_2\}$. Without loss of generality let this edge be $c_1c_2b_1$. Now, b_1 can only be in one further edge (one containing b_2). This again contradicts that every vertex is in three edges. Finally, let F be of type $(3, 3, 1)$. Again, there has to be an edge inside $\{b_1, b_2, b_3, c_1\}$. Without loss of generality let this edge be $b_1b_2c_1$. Now, c_1 can only be in one further edge (one containing b_3), contradicting that c_1 needs to have degree three. □

Lemma 5. *The previously described coloring does not contain a red* P_n^t.

Proof. Let us assume there is an embedding of a red P'. The first three vertices of such an embedding cannot come from different sets A, B and C. Without loss of generality, let A be the set which contains at least two of them. The only way to embed a red copy of P_n^t is to use all vertices of A and $n/3$ vertices of B. Since between 2 vertices from B there has to be at least 2 vertices from A, the only way for an embedding of P_n^t to start is with the first 6 vertices having the following patterns: $AABAAB$, $ABAABA$, $ABAAAB$, $BAABAA$, $BAAABA$ or $BAAAAB$. However, regardless of which pattern we use, the resulting red tight path cannot be extended to a red copy of P': one of $\{v_2, v_3, v_6\}$, $\{v_1, v_2, v_5\}$ and $\{v_1, v_4, v_6\}$ would be of the form BBA and therefore blue. □

3 Preparations

Let the hyperedges of $\mathcal{H} := K_{2n-1}^{(3)}$ be two-colored with colors red and blue, without a blue $\mathbb{F}$. In the proof we will build up a picture of how this bad coloring could potentially look like over a sequence of Lemmas and eventually rule out its existence entirely.
Our starting point in the proof of Theorem 1 will be an upper bound on the off-diagonal hypergraph Ramsey numbers. We choose to use an upper bound from [3], but any weaker bound would suffice.

Theorem 6 (Conlon–Fox–Sudakov [3]). *There exists $C > 0$ so that for every integer $s \geqslant 4$ and sufficiently large t,*

$$R(K_s^{(3)}, K_t^{(3)}) \leqslant 2^{Ct^{s-2}\log t}.$$

In the proof we will make use of the following definitions.

Definition 7. Given two disjoint sets A, B of vertices in $\mathcal{H}$, we say four vertices $a_1, a_2 \in A$, $b_1, b_2 \in B$ form a red *butterfly* if there exists $i, j \in \{1, 2\}$ such that the two hyperedges $a_1a_2b_i, a_jb_1b_2$ are red.

We denote with $|\overrightarrow{AB}|_r$ ($|\overrightarrow{AB}|_b$) the number of red (blue) hyperedges in $\mathcal{H}$ of the form ab_1b_2 with $a \in A, b_1, b_2 \in B$. Given three disjoint sets A, B, C of vertices in $\mathcal{H}$, we denote with $|ABC|_r$ ($|ABC|_b$) the number of red (blue) hyperedges of the form abc with $a \in A, b \in B, c \in C$.

For $W \subset V(\mathcal{H}), v \notin W$, denote $G_{v,W}^{blue}$ the blue link graph of v in W, i.e. the graph on W with ab being an edge iff abv is blue in $\mathcal{H}$. Analogously, $G_{v,W}^{red}$ defines the red link graph.

For $t \in \mathbb{N}$, we define the complete directed bipartite graph $\overrightarrow{K}_{t,t}$ to be the directed graph on vertex set $A \cup B$ with $|A| = |B| = t$, A and B disjoint, and the arc set $\{ab \mid a \in A, b \in B\}$.

The following theorem is a directed version of the Kövári–Sós–Turán Theorem [14].

Theorem 8. *Let $t, m \in \mathbb{N}$. Define D to be a digraph with vertex set $A \cup B$, where A and B are disjoint, and $|A| = |B| = m$. If the number of arcs from A to B is at least $C'm^{2-1/t}$ for C' being a constant large enough only depending on t, then D contains a directed $\overrightarrow{K}_{t,t}$ from A to B.*

The following tools consisting of the next two Lemmas will be used multiple times in the main proof.

Lemma 9. *Let the hyperedges of $\mathcal{H} := K_{2n-1}^{(3)}$ be two-colored with colors red and blue, without a blue $\mathbb{F}$. Further, let $m \in \mathbb{N}$ be big enough and $A, B, C \subseteq V(\mathcal{H})$ be disjoint sets such that $|A| = |B| = |C| = m$. Assume that there are at most 1000 vertex-disjoint red butterflies connecting each pair of the three sets A, B, C.*

Then there exists an absolute constant $t > 0$ such that

$$|\overrightarrow{AB}|_r, |\overrightarrow{BC}|_r, |\overrightarrow{CA}|_r \leqslant m^{3-1/t} \quad \text{or} \quad |\overrightarrow{BA}|_r, |\overrightarrow{CB}|_r, |\overrightarrow{AC}|_r \leqslant m^{3-1/t}.$$

Proof. Removing at most 4000 vertices from each set, we end up with sets $A_1 \subset A, A_2 \subset B, A_3 \subset C$ so that there are no red butterflies connecting them. Note that if two vertices $a \in A_i$ and $b \in A_j$ are not contained in any red butterfly, then either all hyperedges $\{abx : x \in A_i\}$ or all hyperedges $\{aby : y \in A_j\}$ are blue.

Create a digraph $\overrightarrow{G}$ with $V(\overrightarrow{G}) = A_1 \cup A_2 \cup A_3$ as follows. We have $\overrightarrow{uv} \in E(\overrightarrow{G})$ for $u \in A_i, v \in A_j$ with $i \neq j$ if the set of hyperedges $\{uvy : y \in A_j\}$ is entirely blue. Note that some edges might be oriented in both ways.

Let t be the bipartite Ramsey number for $K_{4,4}$. That is, t is the least integer such that every two-coloring of the edges of $K_{t,t}$ contains a monochromatic copy of $K_{4,4}$. Note that by Irving [9], we have $t \leqslant 48$.

Suppose between A_1 and A_2, both the left and right density is at least $C'm^{-1/t}$ for C' being a constant large enough. By Theorem 8 we can find complete directed $\overrightarrow{K}_{t,t}$'s in both directions. That is, there are sets $D_1, D_2 \subset A_1$ and $E_1, E_2 \subset A_2$ of sizes t each such that the edges $\{\overrightarrow{ab} : a \in D_1, b \in E_1\}$ and $\{\overrightarrow{ba} : a \in D_2, b \in E_2\}$ are all present in $\overrightarrow{G}$.

Let $x \in D_1$ and set $N_x := N^+(x) \cap A_3$, where $N^+(x)$ denotes the out-neighborhood of x in $\overrightarrow{G}$. We claim $|N_x| < t$. Suppose towards contradiction that $|N_x| \geqslant t$. Since $|N_x| \geqslant t$ and $|E_1| \geqslant t$, there is a directed $\overrightarrow{K}_{4,4}$ between N_x and E_1 in $\overrightarrow{G}$. That is, we can find $S = \{s_1, \ldots, s_4\} \subset N_x$ and $T = \{t_1, \ldots, t_4\} \subseteq E_1$ such that all edges of the form $\{\overrightarrow{st} : s \in S, t \in T\}$ or all edges of the form $\{\overrightarrow{ts} : s \in S, t \in T\}$ are present in $\overrightarrow{G}$. Without loss of generality let all edges of the form $\{\overrightarrow{st} : s \in S, t \in T\}$ be present. However, this is impossible as the hyperedges $\{xs_1s_2, xt_1t_2, xt_3t_4, s_1t_1t_4, s_1t_2t_3, s_2t_1t_3, s_2t_2t_4\}$ form a blue Fano plane in $\mathcal{H}$. Thus we conclude $|N_x| < t$.

Since x was an arbitrary vertex in D_1, every vertex in D_1 has an out-neighbourhood in A_3 of size at most t. Repeating the same argument after replacing D_1 by E_2, we get that every vertex in E_2 has an out-neighbourhood in A_3 of size at most t. So by removing at most $2t^2$ vertices from A_3 we get a set A_3' of the property that all edges from A_3' to D_1 and all edges from A_3' to E_2 are present in $\overrightarrow{G}$. By the choice of t, we can find sets $W_1 \subset D_1$ and $W_2 \subset E_2$ of sizes at least four such that without loss of generality (W_1, W_2) forms a directed $K_{4,4}$. Hence we have found a 4-blowup of a transitive triangle. But this is impossible, as letting $W_1 = \{v_1, v_2, v_3, v_4\}$, $W_2 = \{w_1, w_2, w_3, w_4\}$, $a \in A_3'$ the hyperedges $\{av_1v_2, aw_1w_2, aw_3w_4, v_1w_1w_4, v_1w_2w_3, v_2w_1w_3, v_2w_2w_4\}$ are all blue in $\mathcal{H}$ and form a Fano plane.

This proves that in $\overrightarrow{G}$ between A_1 and A_2, in one of the directions the density has to be less than $C'm^{-1/t}$. Repeating this argument for the other two pairs, we get that between any pair of sets from A_1, A_2, A_3, in one of the directions the density has to be less than $C'm^{-1/t}$ whereas the density in the other direction has to be at least $1 - C'm^{-1/t}$. The majority orientation forms a transitive triangle or an oriented 3-cycle. Suppose now that the majority orientation forms a transitive triangle. Pick four vertices from each set at random. Then the probability that the 12 vertices do not form a 4-blowup of the transitive triangle is at most $48C'm^{-t}$. Therefore, there exists a 4-blowup of a transitive triangle in $\overrightarrow{G}$, giving a blue Fano plane in $\mathcal{H}$. Thus, the majority orientation has to form a 3-cycle.

Without loss of generality let $A_1 \to A_2 \to A_3 \to A_1$ be the majority orientation in $\overrightarrow{G}$. Then the density between A_1 and A_2 is at least $1 - C'm^{-1/t}$ in $\overrightarrow{G}$, then this implies $|\overrightarrow{A_1A_2}|_r \leqslant C'm^{3-1/t}$ and as we only deleted at most 12000 vertices in the beginning, also $|\overrightarrow{AB}|_r \leqslant 2C'm^{3-1/t}$. Repeating this argument for the other pairs gives us $|\overrightarrow{BC}|_r \leqslant 2C'm^{3-1/t}$ and $|\overrightarrow{CA}|_r \leqslant 2C'm^{3-1/t}$. By choosing t slightly bigger we get rid of the constant $2C'$ for large enough m. □

Lemma 9 can be improved in the following way.

Lemma 10. *Let the hyperedges of $\mathcal{H} := K_{2n-1}^{(3)}$ be two-colored with colors red and blue, without a blue $\mathbb{F}$. Further, let $m \in \mathbb{N}$ be big enough and $A, B, C \subseteq V(\mathcal{H})$ be disjoint sets such that $|A| = |B| = |C| = m$. Assume that there are at most 1000 vertex-disjoint red butterflies connecting each pair of the three sets A, B, C.*

Then there exists an absolute constant $t > 0$ such that

$$|\overrightarrow{AB}|_r, |\overrightarrow{BC}|_r, |\overrightarrow{CA}|_r, |\overrightarrow{BA}|_b, |\overrightarrow{CB}|_b, |\overrightarrow{AC}|_b \leqslant m^{3-1/t}$$

or

$$|\overrightarrow{BA}|_r, |\overrightarrow{CB}|_r, |\overrightarrow{AC}|_r, |\overrightarrow{AB}|_b, |\overrightarrow{BC}|_b, |\overrightarrow{CA}|_b \leqslant m^{3-1/t}.$$

Proof. Applying Lemma 9, we get a positive constant t' such that w.l.o.g.

$$|\overrightarrow{AB}|_r, |\overrightarrow{BC}|_r, |\overrightarrow{CA}|_r \leqslant m^{3-1/t'}.$$

Let $t = 3t'$. For the sake of contradiction, say that $|\overrightarrow{AC}|_b \geqslant m^{3-1/t}$. Define

$$Z_1 := \left\{ v \in A \,\middle|\, e(G_{v,B}^{blue}) \geqslant \frac{99}{100}\binom{m}{2} \right\} \quad \text{and} \quad Z_1' := \left\{ v \in A \,\middle|\, e(G_{v,C}^{blue}) \geqslant \frac{1}{2}m^{2-1/t} \right\}.$$

Then $|A \setminus Z_1| \leqslant 600m^{1-1/t'}$, as otherwise $|\overrightarrow{AB}|_r \geqslant 600m^{1-1/t'}\frac{1}{100}\binom{m}{2} > 2m^{3-1/t'}$. Also $|Z_1'| \geqslant 800m^{1-1/t'}$, as otherwise $|\overrightarrow{AC}|_b < 800m^{1-1/t'}m^2 + \frac{1}{2}m^{2-1/t}m \leqslant m^{3-1/t}$. Thus, one can choose a vertex $v \in Z_1 \cap Z_1'$. Let

$$Y_1 := \left\{ w \in B \,\middle|\, e(G_{w,C,r}) \leqslant 10m^{2-1/t'} \right\}.$$

Then $|Y_1| \geqslant 4/5m$ as otherwise $|\overrightarrow{BC}|_r \geqslant 2m^{3-1/t'}$. Because of the size of Y_1, $G_{v,B}^{blue}$ has to contain an edge inside Y_1. Let w_1w_2 be such an edge. The number of 4-tuples (a, b, c, d) of distinct vertices $a, b, c, d \in C$ with $ab, cd \in E(G_{v,C}^{blue})$ is at least

$$\sum_{ab \in E(G_{v,C}^{blue})} \big(e(G_{v,C}^{blue}) - deg(a) - deg(b)\big) \geqslant e(G_{v,C}^{blue})(e(G_{v,C}^{blue}) - 2m) \geqslant \frac{1}{5}m^{4-2/t}.$$

The number of 4-tuples (a, b, c, d) of distinct vertices $a, b, c, d \in C$ with $ad \notin E(G_{w_1,C}^{blue})$ or $bc \notin E(G_{w_1,C}^{blue})$ is at most $e(G_{w_1,C}^{red})m^2 + m^2e(G_{w_1,C}^{red}) \leqslant 20m^{4-1/t'}$. Similarly, the number of 4-tuple (a, b, c, d) of distinct vertices $a, b, c, d \in C$ with $ac \notin E(G_{w_2,C}^{blue})$ or $bd \notin E(G_{w_2,C}^{blue})$ is at most $20m^{4-1/t'}$. Since $20m^{4-1/t'} + 20m^{4-1/t'} < \frac{1}{5}m^{4-2/t}$, there exists $a, b, c, d \in C$ such that $ab, cd \in E(G_{v,C}^{blue})$; $ad, bc \in E(G_{w_1,C}^{blue})$ and $ac, bd \in E(G_{w_2,C}^{blue})$. Thus, the hyperedges $vw_1w_2, vab, vcd, w_1ad, w_1bc, w_2ac, w_2bd$ form a blue Fano plane; a contradiction, therefore we conclude that $|\overrightarrow{AC}|_b \leqslant m^{3-1/t}$. Similarly, we get $|\overrightarrow{CB}|_b \leqslant m^{3-1/t}$ and $|\overrightarrow{BA}|_b \leqslant m^{3-1/t}$. □

4 Proof of Theorem 1

4.1 Set up of the proof

For the sake of contradiction, assume that there is a red-blue edge-coloring of $\mathcal{H} := K_{2n-1}^{(3)}$ without a blue $\mathbb{F}$ and without a red P_n^t. Fix such a coloring. Let $\varepsilon > 0$ be a sufficiently small constant and assume that n is sufficiently large. Set

$$m = \left\lceil \varepsilon \sqrt[5]{\frac{\log n}{\log \log n}} \right\rceil .$$

Observe that $m^5 \log m \leqslant \frac{\varepsilon^5}{5} \log n$, hence we have by Theorem 6

$$R(K_m^{(3)}, K_7^{(3)}) \leqslant 2^{Cm^5 \log m} \leqslant n^{\varepsilon^4}.$$

Since $\mathcal{H}$ contains no blue $\mathbb{F}$, it cannot contain a blue $K_7^{(3)}$ and we conclude that it contains a red $K_m^{(3)}$, call it D_1. Set $\mathcal{H}_1 := \mathcal{H} \setminus V(D_1)$ and find a red $K_m^{(3)}$, call it D_2, in $\mathcal{H}_1$. Repeating this process, setting $\mathcal{H}_{i+1} := \mathcal{H}_i \setminus V(D_i)$, we can find a red copy of $K_m^{(3)}$ in $\mathcal{H}_{i+1}$, calling it D_{i+1}, as long as $|V(\mathcal{H}_i)| \geqslant n^{\varepsilon^4}$. At the end of this process we end up with a collection of vertex-disjoint red $K_m^{(3)}$-s $D_1, D_2, \dots, D_d$, and a set J of remaining vertices with $|J| \leqslant n^{\varepsilon^4}$.

Create a graph G_1 with $V(G_1) = \{D_1, \dots, D_d\}$, by connecting D_i, D_j if in $\mathcal{H}$ there are at least 1000 vertex-disjoint red butterflies between them. The vertices of G_1 will be called blobs. The next two lemmas give information on the structure of G_1.

Lemma 11. *The complement of G_1 contains no K_4.*

Proof. For the sake of contradiction, assume that there are 4 blobs A_1, A_2, A_3 and A_4 which form a K_4 in the complement of G_1. Define a directed graph D with vertex set $V(D) = \{A_1, A_2, A_3, A_4\}$ and an edge from blob A_i to A_j $(i \neq j)$ iff $|\overrightarrow{A_iA_j}|_r \leqslant m^{3-1/t}$ with t from Lemma 10. Applying Lemma 10 on all subsets of size 3 of the 4 blobs gives that every edge in D is oriented in exactly one direction. This means that D is a tournament. However, a tournament on 4 vertices contains a transitive triangle and Lemma 10 says this cannot happen. □

Lemma 12. *G_1 has one of the following forms:*

(i) $V(G_1) = \{A_1, \dots, A_a, B_1, \dots, B_b\}$ such that $A_1, \dots, A_a$ and $B_1, \dots, B_b$ form vertex-disjoint paths or

(ii) $V(G_1) = \{A_1, \dots, A_a, B_1, \dots, B_b, C_1, \dots, C_c\}$ such that $A_1, \dots, A_a$; $B_1, \dots, B_b$ and $C_1, \dots, C_c$ form vertex-disjoint paths.

Proof. Let $A_1, \dots, A_a$ be a longest path in G_1. If $V(G_1) = \{A_1, \dots, A_a\}$ then one can find a red P_n^t in $\mathcal{H}$ just by jumping from red blob to red blob along the path using the red butterflies. Let $B_1, \dots, B_b$ be a longest path in G_1 on the vertices $V(G_1) \setminus \{A_1, \dots, A_a\}$. If $V(G_1) = \{A_1, \dots, A_a, B_1, \dots, B_b\}$ we are in case (i). Otherwise we can take the longest path $C_1, \dots, C_c$ in $V(G_1) \setminus \{A_1, \dots, A_a, B_1, \dots, B_b\}$. In this case $V(G_1) = \{A_1, \dots, A_a, B_1, \dots, B_b, C_1, \dots, C_c\}$ as otherwise any blob in $V(G_1) \setminus \{A_1, \dots, A_a, B_1, \dots, B_b, C_1, \dots, C_c\}$ would form a K_4 in the complement of G_1 together with A_1, B_1 and C_1. This is not possible by Lemma 11. □

In the next two Subsections the two cases from Lemma 12 will be handled separately. The strategy is to build a long red tight path using these two or three blocks.

Remark 13. For the proof of Theorem 2 one can use the fact that when a long path say starting in A_1 and ending in A_a is found, it is clear that some of the vertices inside the block can be used to close the cycle.

4.2 The two paths case

In this case G_1 is the vertex-disjoint union of two paths, i.e. in $\mathcal{H}$ we have vertex-disjoint red K_m-s $\{A_1, \dots, A_a, B_1, \dots, B_b\}$ and a set J of junk vertices with $|J| \leqslant n^{\varepsilon^4}$. For every i, j there are at least 1000 red butterflies between A_i and A_{i+1} and also between B_j and B_{j+1}. Slightly abusing notation, let $P_1 = \cup_i A_i$ and $P_2 = \cup_j B_j$. Note that if $|P_i| \geqslant n$ for some $i \in \{1, 2\}$ then we can embed the tight path P_n^t into P_i just by walking through each blob and jumping from blob to blob by using the hyperedges from the red butterflies. We know that $|P_1| + |P_2| + n^{\varepsilon^4} \geqslant |V(\mathcal{H})| = 2n - 1$. So $n - n^{\varepsilon^4} \leqslant |P_i| \leqslant n - 1$ for $i = 1, 2$.

Definition 14. A red triple triangle between A_i and B_j is a set of vertices $w, x, y, z \in A_i$ and $v \in B_j$ (or $w, x, y, z \in B_j$ and $v \in A_i$) so that wxv, xyv, yzv is red in $\mathcal{H}$.

Observe that when we have a red triple triangle between A_i and B_j we can find a red tight path of length $m + 1$ by swallowing one additional vertex from B_j using the red triple triangle. If there is no red triple triangle between two blobs then there also have to be few red hyperedges between the blobs.

Lemma 15. *If there is no red triple triangle between A_i and B_j, then $|A_iB_j|_r + |B_jA_i|_r \leqslant 20m^2$.*

Proof. Pick any vertex $v \in B_j$ and consider its red link graph in A_i. If v is not in a red triple triangle, then the red link graph does not contain a path of length 3, hence the number of edges in this link graph is at most $10m$. So the number of red hyperedges between B_j and A_i, assuming that there are no red triple triangles, is at most $20m^2$. □

Lemma 16. *$V(\mathcal{H})$ can be decomposed as $V(\mathcal{H}) = A \cup B \cup J'$ with $|A|, |B| \geqslant n - n^{\varepsilon^3}$, $|J'| \leqslant n^{\varepsilon^3}$ such that there are at most $500n^3/m$ blue hyperedges inside A and respectively in B.*

Proof. Consider the bipartite graph G_2, with vertex sets $\{A_1, \dots, A_a\}$ and $\{B_1, \dots, B_b\}$. Connect A_iB_j by an edge iff between A_i and B_j there is a red triple triangle in $\mathcal{H}$. Let M be a largest matching in G_2. Then we can embed into $\mathcal{H}$ a tight red path of length $|P_i| + |M|/2$ for some $i \in \{1, 2\}$, because at least half of the triple triangles represented by

edges from the matching have to go in the same direction. In particular since $|P_i| \geqslant n - n^{\varepsilon^4}$ for $i = 1, 2$ this implies $|M| \leqslant 2n^{\varepsilon^4}$. Put all blobs covered by M into J and get a new rubbish set J'. We will have $|J'| \leqslant |J| + 2|M|m \leqslant n^{\varepsilon^3}$ vertices.
The subgraph of G_2 on the blobs which have not been removed spans an independent set. Let A be the set of vertices in P_1 which have not been removed and let B be the vertices in P_2 which have not been removed. The following argument shows that for three different blobs A'_1, A'_2, A'_3 from $\{A_1, \dots, A_a\}$ which have not been removed, $|A'_1 A'_2 A'_3|_b \leqslant 400m^2$. For contradiction, assume there are more than that many blue hyperedges. Take a blob B_i which has not been removed from $\{B_1, \dots, B_b\}$. By Lemma 15

$$|\overrightarrow{A'_1 B_i}|_r, |\overrightarrow{A'_2 B_i}|_r, |\overrightarrow{A'_3 B_i}|_r, |\overrightarrow{B_i A'_1}|_r, |\overrightarrow{B_i A'_2}|_r, |\overrightarrow{B_i A'_3}|_r \leqslant 20m^2.$$

Picking at random one vertex each of A'_1, A'_2 and A'_3, and 4 vertices from B_i, these vertices do not form a blue Fano plane with probability at most $1 - 400m^{-1} + 6 \cdot 50m^{-1}$, thus, there has to exist a blue Fano plane. We conclude $|A'_1 A'_2 A'_3|_b \leqslant 400m^2$. Therefore there are at most $400m^2(n/m)^3 + m^3(n/m)^2 \leqslant 500n^3/m$ blue hyperedges inside A. Similarly, this holds for B. ☐

Definition 17. Call a vertex $v \in B$ *special* if $e(G^{red}_{v,A}) \geqslant \frac{1}{5}\varepsilon^5\binom{n}{2}$. Similarly, call a vertex $v \in A$ *special* if $e(G^{red}_{v,B}) \geqslant \frac{1}{5}\varepsilon^5\binom{n}{2}$.

Lemma 18. *Let $V(\mathcal{H}) = A \cup B \cup J'$ be the decomposition from Lemma 16. If there are at least $nm^{-1/20}$ special vertices in A or B, then one can find a red P^t_n in $\mathcal{H}$.*

Proof. Suppose there are at least $nm^{-1/20}$ special vertices in w.l.o.g. B. We now show that we can absorb enough of these special vertices from B to find a tight red path of length n. Let $a, b, c, d \in A, v \in B$. A tuple (c, d) is called *reachable* from (a, b) if both abc and bcd are red. Further, a tuple (c, d) is called *reachable* from (a, b) via v if all abv, bvc, vcd are red. A tuple (a, b) is called *open* if there exists at most $n^2m^{-1/4}$ tuples (c, d) with $c, d \in A$ such that (c, d) is not reachable from (a, b). Define O to be the set of all open tuples. As there are at most $500n^3/m$ blue hyperedges inside A, $|O| \geqslant |B|(|B| - 1) - n^2m^{-1/4}$. Call a tuple (a, b) *good* for $v \in B$ if there exists at least $\varepsilon^{100}n^2$ tuples $(c, d), c, d \in A$ such that (c, d) is reachable from (a, b) via v. Denote *Good*(v) the set of all tuples being good for v. For $v \in B$ special, $|Good(v)| \geqslant \varepsilon^{100}n^2$, because otherwise the number of P_4's in $G^{blue}_{v,B}$ would be at most $2\varepsilon^{100}n^4$. However, since $e(G^{blue}_{v,B}) \geqslant \frac{1}{5}\varepsilon^5\binom{n}{2}$, the number of P_4's in $G^{blue}_{v,B}$ is more than $2\varepsilon^{100}n^4$.

We will now walk along the red hyperedges step by step adding in each step 5 vertices to the tight path. Let $v_1, v_2, \dots$ be the special vertices in B. Let $(a_1, b_1) \in Good(v_1)$ and (c_1, d_1) be an open tuple such that (c_1, d_1) is reachable from (a_1, b_1) via v_1. We begin the walk with a_1, b_1, v_1, c_1, d_1. Now take a look at step i. Assume we already have defined $a_1, b_1, v_1, c_1, d_1, a_2, \dots, a_{i-1}, b_{i-1}, v_{i-1}, c_{i-1}, d_{i-1}$ with (c_{i-1}, d_{i-1}) being open. Pick $(a_i, b_i) \in Good(v_i)$ such that (a_i, b_i) is reachable from (c_{i-1}, d_{i-1}). For $i \leqslant nm^{-1/20}$, this is possible, because

$$|Good(v_i)| - n^2m^{-1/4} - 5ni \geqslant \frac{\varepsilon^{100}}{2}n^2.$$

Now pick $(c_i, d_i) \in O$ such that (c_i, d_i) is reachable from (a_i, b_i) via v_1. For $i \leqslant nm^{-1}$, this is possible, because

$$\varepsilon^{100}n^2 - n^2m^{-1/4} - 5ni \geqslant \frac{\varepsilon^{100}}{2}n^2.$$

Now enlarge the path with $a_ib_iv_ic_id_i$. After $i \leqslant nm^{-1/20}$ steps we end up with a path of length $5nm^{-1/20}$ such that the last two vertices form an open tuple. Now, we just keep picking open tuples and walk from an open tuple to an open tuple. We can keep doing this until at most $nm^{-1/10}$ vertices are not used inside A. This means we found a tight red path of length at least

$$n - n^{\varepsilon^4} - nm^{-1/10} + nm^{-1/20} \geqslant n. \qquad \square$$

Lemma 19. *Let $V(\mathcal{H}) = A \cup B \cup J'$ be the decomposition from Lemma 16. If there are at most $nm^{-1/20}$ special vertices in A and B, then the vertex set $V(\mathcal{H})$ can be decomposed into $V(\mathcal{H}) = A' \cup B' \cup J''$ with $|A'|, |B'| \geqslant n - \varepsilon n, |J''| \leqslant \varepsilon n$ such that $\mathcal{H}[A']$ and $\mathcal{H}[B']$ are entirely red, for all $v \in A'$ $e(G^{red}_{v,B'}) \leqslant \frac{1}{5}\varepsilon^5 n^2$ and for all $v \in B'$ $e(G^{red}_{v,A'}) \leqslant \frac{1}{5}\varepsilon^5 n^2$.*

Proof. We can remove all special vertices from A and B and add them to the junk set J'. So we obtain A', B', J'' so that for each $v \in A'$ the red link graph in B' has at most $\frac{1}{5}\varepsilon^5 n^2$ edges, for each $w \in B'$ the red link graph of w in A' has at most $\frac{1}{5}\varepsilon^5 n^2$ edges and $|J''| \leqslant |J'| + 2nm^{-1/20} \leqslant \varepsilon n$.
Suppose abc is a blue hyperedge in A'. Let $G^{blue}_{a,B'}, G^{blue}_{b,B'}, G^{blue}_{c,B'}$ be the blue link graphs in B'. By the previous observation $e(G^{blue}_{a,B'} \cap G^{blue}_{b,B'} \cap G^{blue}_{c,B'}) \geqslant \frac{9}{10}\binom{n}{2}$ and thus $G^{blue}_{a,B'} \cap G^{blue}_{b,B'} \cap G^{blue}_{c,B'}$ contains a K_4. The four vertices from the K_4 together with a, b, c contain a blue copy of the Fano plane in $\mathcal{H}$. Hence $\mathcal{H}[A']$ is entirely red. The same holds for $\mathcal{H}[B']$. $\square$

Lemma 20. *In the setting of Lemma 19, we can decompose $J'' = J_1 \cup J_2$ such that for all $v \in J_1$ $e(G^{blue}_{v,A'}) \leqslant \varepsilon n^2$ and all $v \in J_2$ $e(G^{blue}_{v,B'}) \leqslant \varepsilon n^2$.*

Proof. Let $V(\mathcal{H}) = A' \cup B' \cup J''$ be the decomposition from Lemma 19. We actually will prove that if a rubbish vertex $v \in J''$ has $e(G^{blue}_{v,A'}) \geqslant \varepsilon n^2$, then it cannot have a blue hyperedge into B'. Indeed suppose there are $a, b \in B'$ with abv blue. Then both a and b are part of at least $\binom{|A'|}{2} - \frac{1}{5}\varepsilon^5 n^2$ blue hyperedges with the other two vertices being in A' by the final statement in Lemma 19. Therefore there are at least $\binom{|A'|}{2} - \frac{2}{5}\varepsilon^5 n^2$ pairs (c, d) with $c, d \in A'$ and cda, cdb blue, call this set of edges S. Now suppose v leads at least εn^2 blue hyperedges into A', i.e. $e(G^{blue}_{v,A'}) \geqslant \varepsilon n^2$ and hence $G^{blue}_{v,A'}$ contains a path P of length $\varepsilon^2 n$. The restriction of S onto the vertex set of $P = \{p_1, p_2, \ldots\}$ contains at least $(1 - \varepsilon)\binom{|P|}{2}$ edges and hence contains four vertices $p_i, p_{i+1}, p_j, p_{j+1}$ with $p_ip_j, p_ip_{j+1}, p_{i+1}p_j, p_{i+1}p_{j+1}$ in S. Then $ap_ip_{j+1}, avb, ap_jp_{i+1}, p_{j+1}bp_{i+1}, p_{j+1}vp_j, p_ivp_{i+1}, p_ip_jb$ form a blue Fano plane. Hence we can split up the junk set $J'' = J_1 \cup J_2$ such that for all $v \in J_1$ $e(G^{blue}_{v,A'}) \leqslant \varepsilon n^2$ and all $v \in J_2$ $e(G^{blue}_{v,B'}) \leqslant \varepsilon n^2$. $\square$

Lemma 21. *In the setting of Lemma 19, we can find a red P^t_n in $\mathcal{H}$.*

Proof. Let $J'' = J_1 \cup J_2$ be the decomposition of the junk vertices from Lemma 20. Set $A^* = A' \cup J_1$ and $B^* = B' \cup J_2$. Then either $|A^*| \geqslant n$ or $|B^*| \geqslant n$. W.l.o.g. $|A^*| \geqslant n$. Now one can find a red tight path of length n inside A^*. Let $v_1, v_2, \ldots$ be the vertices from J_1. Call a tuple $(a,b), a, b \in A'$ *good** for $v \in J_1$ if the red link graph of v in A' contains at least $\frac{9}{10}n^2$ tuples $(c,d), c, d \in A'$ such that abv, bvc, vcd are red in $\mathcal{H}$. Since the blue link graph of v contains at most εn^2 edges, for each $v \in J_1$ the number of good* tuples is at least $\frac{9}{10}n^2$. Now start with an arbitrary good* tuple (a_1, b_1) for v_1. The start of the walk is a_1, b_1, v_1. Now assume we already have chosen $a_1, b_1, v_1, \ldots, a_{i-1}, b_{i-1}, v_{i-1}$ such that (a_{i-1}, b_{i-1}) is good* for v_{i-1}. Take a good* tuple (a_i, b_i) for v_i of unused vertices such that $b_{i-1}v_{i-1}a_i, v_{i-1}a_ib_i$ are red. This is possible for all $i \leqslant \varepsilon n$, because

$$\frac{9}{10}n^2 - \frac{1}{10}n^2 - 3in > 0.$$

Enlarge the path by $a_ib_iv_i$. After all vertices from J_1 are used, just walk through A' until all vertices in A' are used. This is possible, because all hyperedges inside A' are red. Thus, we find a red tight path of length $|A^*| \geqslant n$. □

4.3 The three paths case

Let A, B, C with $A = \{A_1, A_2, \ldots, A_a\}, B = \{B_1, \ldots, B_b\}$ and $C = \{C_1, \ldots, C_c\}$ be the three paths in G_1. There cannot be an edge between blobs of different paths, otherwise we can reduce this case to the two path case from Section 4.2 in the following way. Without loss of generality, assume that there is an edge between A_j and B_k. Split up each blob from A and B into two blobs of equal size (if m is odd one vertex ends up in J) in such a way that blobs coming from consecutive blobs still have at least 100 disjoint red butterflies between them and such that there are also still at least 100 disjoint red butterflies between the blobs coming from A_j and B_k. Now, when one constructs the graph of all new blobs, where two blobs are adjacent with each other when they have at least 100 disjoint red butterflies between them, then this graph can be decomposed into two paths. We already handled this case in Subsection 4.2. Therefore we can assume that there are no edges between blobs of different paths in G_1.

Using Lemma 10 it follows that w.l.o.g. $|\overrightarrow{A_iB_j}|_r, |\overrightarrow{B_jC_k}|, |\overrightarrow{C_kA_i}|_r \leqslant m^{3-1/t}$ for all i, j, k. Let $P_1 = \cup A_i, P_2 = \cup B_i$ and $P_3 = \cup C_i$. Clearly, $|P_1|, |P_2|, |P_3| < n$ as otherwise one could find a red tight path of length n just by going through a blob and then jumping to the next by using a red butterfly and so on.

Definition 22. For $X, Y \subset V(\mathcal{H})$ and $0 \leqslant \alpha \leqslant 1$, denote $G(X, Y, \alpha)$ the graph with vertex set X, and ab is an edge iff the number of red hyperedges abc with c from Y is at least $\alpha|Y|$.

Lemma 23. *There exists a constant C' such that*

$$\frac{2}{3}n - C'nm^{-1/t} \leqslant |P_l| \leqslant \frac{2}{3}n + C'nm^{-1/t}$$

for $l = 1, 2, 3$.

Proof. Let $A_i \in A, B_i \in B$. We will show that one can find a red tight path of length at least $3m/2 - 4000m^{1-1/t}$ just using A_i and B_i starting with two vertices from A_i, ending with two vertices from A_i, not using two vertices being part of a butterfly to A_{i-1} and not using two other vertices being part of a butterfly to A_{i+1}. Consider the graph $G(A_i, B_i, 0.99)$. The number of vertices in this graph with degree at most $0.9m$ is at most $2000m^{1-1/t}$ as otherwise $|\overrightarrow{B_i A_i}|_r > 2000m^{1-1/t} \cdot 0.1m \cdot 0.01m \cdot 0.5 = m^{3-1/t}$. Let $A' \subseteq A_i$ be the set of all vertices of degree at least $0.9m$ not containing the two vertices being part of a butterfly to A_{i-1} and not containing the two vertices being part of a butterfly to A_{i+1}. Then $|A'| \geqslant m - 2000m^{1-1/t} - 4$, $G(A', B_i, 0.99)$ has minimum degree at least $0.8m$ and thus there exists a Hamiltonian path $v_1, v_2, \dots, v_{A'}$ in this graph. After every second vertex in this path we will now add a vertex from B_i to find a long red tight path in $A_i \cup B_i$. Assume we already found the tight red path $v_1, v_2, w_1, v_3, v_4, w_2, \dots, v_{2j-1}, v_{2j}$. Then we can pick a vertex $w_j \in B_i$ which has not been used yet and such that $v_{2j-1}v_{2j}w_j, v_{2j}w_j, v_{2j+1}, w_j, v_{2j+1}v_{2j+2}$ are red for $j < |A'|/2$. This is possible because $m - 0.01m - 0.01m - 0.01m - j > 0$. Thus, we can find a red tight path of length at least

$$\frac{3}{2}(m - 2001m^{1-1/t}) \geqslant \frac{3m}{2} - 4000m^{1-1/t}.$$

If $|P_1| \leqslant |P_2|$, then we can find a tight red path of length at least $3/2|P_1| - 5000nm^{-1/t}$ by the following argument. Jump from blob to blob in A using the vertices from the butterflies and always absorbing the vertices from the index corresponding blob in B. When we are done with all blobs in A we stop and have found a red tight path of length at least

$$\left(\frac{3}{2}m - 4000m^{1-1/t}\right)\left\lfloor\frac{|P_1|}{m}\right\rfloor \geqslant \left(\frac{3}{2}m - 4000m^{1-1/t}\right)\left(\frac{|P_1|}{m} - 1\right) \geqslant \frac{3}{2}|P_1| - 5000nm^{-1/t}.$$

If $|P_1| \geqslant |P_2|$, then we can find a tight red path of length at least $(|P_1| + |P_2|/2) - 5000nm^{-1/t}$ by the following argument. Jump from blob to blob in A using the vertices from the butterflies and always absorbing the vertices from the index-corresponding blob in B. When we are done with all blobs in B we go back to A and walk through the remaining blobs from A using the butterflies. So we can find a tight red path of length at least

$$\left(\frac{3}{2}m - 4000m^{1-1/t}\right)\left\lfloor\frac{|P_2|}{m}\right\rfloor + |P_1| - |P_2| \geqslant |P_1| + \frac{|P_2|}{2} - 5000nm^{-1/t}.$$

We will now show that the sizes of the blocks P_1, P_2, P_3 is at most $2/3n + C'nm^{-1/t}$ and at least $2/3n - C'nm^{-1/t}$ for an absolute constant C'. W.l.o.g. let P_1 be a biggest block. If $|P_1| \leqslant 2/3n + 30000nm^{-1/t}$, then also $|P_2|, |P_3| \leqslant 2/3n + 30000nm^{-1/t}$, but since $|P_1| + |P_2| + |P_3| = 2n - 1 - |J|$, we also get $|P_1|, |P_2|, |P_3| \geqslant 2/3n - 60001nm^{-1/t}$. Assume $|P_1| \geqslant 2/3n + 30000nm^{-1/t}$. If $|P_2| \geqslant 2(n - |P_1|) + 10000nm^{-1/t}$, then we find a red tight path of length at least

$$(|P_1| + \frac{1}{2}|P_2|) - 5000nm^{-1/t} \geqslant n + 5000nm^{-1/t}.$$

Otherwise, $|P_2| \leqslant 2(n - |P_1|) + 10000nm^{-1/t}$. Then, $|P_1| + |P_2| \leqslant 2n - |P_1| + 10000nm^{-1/t}$ and thus $|P_3| \geqslant |P_1| - 10001nm^{-1/t} \geqslant 2/3n + 19999nm^{-1/t}$. But now we can find a red tight path of length at least $\frac{3}{2}|P_1| - 5000nm^{-1/t} > n$. This shows that there exists a constant C' such that

$$\frac{2}{3}n - C'nm^{-1/t} \leqslant |P_l| \leqslant \frac{2}{3}n + C'nm^{-1/t}$$

for $l = 1, 2, 3$. □

Lemma 24. *$V(\mathcal{H})$ can be decomposed as $V(\mathcal{H}) = P_1'' \cup P_2'' \cup P_3'' \cup J$ such that $|P_1''| = |P_2''| = |P_3''|$; $|J| \leqslant C''nm^{-1/t}$ for some constant $C'' > 0$ and each of $\mathcal{H}[P_1''], \mathcal{H}[P_2''], \mathcal{H}[P_3'']$ is entirely red.*

Proof. The number of vertices $v \in A_1$ with $e(G^{blue}_{v,B_1}) \leqslant 29/30\binom{m}{2}$ is at most $30m^{1-1/t}$ as otherwise $|\overrightarrow{A_1B_1}|_r \geqslant m^{3-1/t}$. This means one can move at most $30m^{1-1/t}(n/m) \leqslant 30nm^{-1/t}$ vertices v from P_1 to J (and obtain $P_1' \subseteq P_1$) such that all vertices in P_1' satisfy $e(G^{blue}_{v,B_1}) \geqslant 29/30\binom{m}{2}$. Now assume there is a blue hyperedge $v_1v_2v_3$ inside P_1'. Since $e(G^{blue}_{v_1,B_1} \cap G^{blue}_{v_2,B_1} \cap G^{blue}_{v_3,B_1}) \geqslant 27/30\binom{m}{2}$, $G^{blue}_{v_1,B_1} \cap G^{blue}_{v_2,B_1} \cap G^{blue}_{v_3,B_1}$ contains a K_4. These 4 vertices together with v_1, v_2, v_3 form a blue Fano plane. Thus P_1' is entirely red. Repeating this cleaning procedure for P_2 and P_3 one ends up with entire red blocks P_1', P_2', P_3' and a rubbish set J' of size at most $100nm^{-1/t}$. Considering that the blocks had roughly equal size, we can remove a few more vertices from the blocks and end up with entirely red blocks P_1'', P_2'', P_3'' of equal size and a rubbish set J'' of size at most $C''nm^{-1/t}$ vertices with C'' being an absolute constant. □

Lemma 25. *Let $V(\mathcal{H}) = P_1'' \cup P_2'' \cup P_3'' \cup J$ be the decomposition from Lemma 24. Then $|P_1''P_2''P_3''|_r \leqslant 7n^{3-1/t}$.*

Proof. Applying Lemma 10 gives w.l.o.g. that

$$|\overrightarrow{P_1''P_2''}|_r, |\overrightarrow{P_2''P_3''}|_r, |\overrightarrow{P_3''P_1''}|_r, |\overrightarrow{P_2''P_1''}|_b, |\overrightarrow{P_3''P_2''}|_b, |\overrightarrow{P_1''P_3''}|_b \leqslant n^{3-1/t}.$$

Assume $|P_1''P_2''P_3''|_r \geqslant 7n^{3-1/t}$. Pick $v_1, w_1 \in P_1''$, $v_2, w_2, x_2 \in P_2''$, $v_3 \in P_3''$ uniformly at random. The hyperedge $v_1w_1v_2$ is blue or not a proper hyperedge with probability at most $2n^{-1/t}$. Similarly, $v_2w_2v_3$ and $v_3w_2x_2$ is blue or not an hyperedge each with probability at most $2n^{-1/t}$. The hyperedge $v_1v_2v_3$ is blue with probability at most $1 - 7n^{3-1/t}$. Thus, the probability that one of the hyperedges $v_1w_1v_2, v_1v_2v_3, v_2v_3w_2, v_3w_2x_2$ is blue is at most $1 - 7n^{3-1/t} + 6n^{3-1/t} < 1$. Thus, there exists $v_1, v_2, v_3, w_1, w_2, w_3$ such that all the hyperedges $v_1w_1v_2, v_1v_2v_3, v_2v_3w_2, v_3w_2x_2$ are red. Now one can find a red tight path of length at least $|P_1''| + |P_2''| + 1 \geqslant n$ by first going through all vertices in P_1'' besides v_1 and w_1, then going along $w_1v_1v_2v_3w_1w_2$ and then through all vertices in P_2''. Recall that all hyperedges inside P_1'', P_2'' or P_3'' are red. □

Lemma 26. *There exists a decomposition of the vertices of $\mathcal{H}$ into $V(\mathcal{H}) = P_1^\dagger \cup P_2^\dagger \cup P_3^\dagger$ with $0.66n \leqslant |P_l^\dagger| \leqslant 0.67n$ for $l = 1, 2, 3$ such that all graphs $G(P_1^\dagger, P_2^\dagger, 0.98)$, $G(P_2^\dagger, P_3^\dagger, 0.98)$ and $G(P_3^\dagger, P_1^\dagger, 0.98)$ have minimum degree at least $0.39n$.*

Proof. Let $V(\mathcal{H}) = P_1'' \cup P_2'' \cup P_3'' \cup J$ be the decomposition from Lemma 24. The number of vertices in $G(P_1'', P_2'', 0.99)$ with degree less than $0.4n$ is at most $1500n^{1-1/t}$. Removing at most $1500n^{1-1/t}$ vertices from each P_1'', P_2'', P_3'' leaves us with sets P_1^*, P_2^*, P_3^* and a junk set J^* such that every vertex in the graphs $G(P_1^*, P_2^*, 0.98)$, $G(P_2^*, P_3^*, 0.98)$ and $G(P_3^*, P_1^*, 0.98)$ has minimum degree at least $0.39n$.

We now check that every vertex v in J^* has degree at least $0.39n$ in one of the graphs $G(P_1^* \cup \{v\}, P_2^*, 0.98)$, $G(P_2^* \cup \{v\}, P_3^*, 0.98)$ and $G(P_3^* \cup \{v\}, P_1^*, 0.98)$. Assume this is not the case, then there exists $v \in J^*, X_1 \subseteq P_1^*, X_2 \subseteq P_2^*$ and $X_3 \subseteq P_3^*$ with $|X_1| = |X_2| = |X_3| \geqslant 0.65n - 0.39n = 0.26n$ such that for each $x_1 \in X_1$ there are at least $0.02 \cdot 0.65n \geqslant 0.01n$ many vertices $y_2 \in P_2^*$ such that vx_1y_2 is blue. Similarly, for each $x_2 \in X_2$ there are at least $0.01n$ many vertices $y_3 \in P_3^*$ such that vx_2y_3 is blue and for each $x_3 \in X_3$ there are at least $0.01n$ many vertices $y_1 \in P_1^*$ such that vx_3y_1 is blue. Now pick $x_1 \in X_1, x_2 \in X_2, x_3 \in X_3$ independently uniformly at random. There exist random sets $Y_1 \subset P_1^*, Y_2 \subset P_2^*, Y_3 \subset P_3^*$ with $|Y_1| = |Y_2| = |Y_3| \geqslant 0.01n$ such that $vx_1y_2, vx_2y_3, vx_3y_1$ for all $y_1 \in Y_1, y_2 \in Y_2, y_3 \in Y_3$. Now pick $y_1 \in Y_1, y_2 \in Y_2, y_3 \in Y_3$ independently uniformly at random. The hyperedges $vx_1y_2, vx_2y_3, vx_3y_1$ are blue. As $|X_1X_2X_3|_r \leqslant |P_1''P_2''P_3''|_r \leqslant n^{3-1/t}$, $x_1x_2x_3$ is red with probability at most $4^3n^{-1/t}$. Since

$$|Y_1X_2Y_2|_r \leqslant |\overrightarrow{P_1''P_2''}|_r \leqslant n^{3-1/t}, \quad |Y_2X_3Y_3|_r \leqslant |\overrightarrow{P_2''P_3''}|_r \leqslant n^{3-1/t}$$

and $|Y_3X_1Y_1|_r \leqslant |\overrightarrow{P_3''P_1''}|_r \leqslant n^{3-1/t}$, the probability that each of the hyperedges $y_1x_2y_2$, $y_2x_3y_3, y_3x_1y_1$ is red is at most $C^*n^{-1/t}$ for an absolute constant C^*. Thus, with positive probability vx_1y_2, vx_2y_3, $vx_3y_1, x_1x_2x_3, y_1x_2y_2, y_2x_3y_3, y_3x_1y_1$ form a blue Fano plane. We therefore can assume that every vertex $v \in J^*$ has degree at least $0.39n$ in one of the graphs $G(P_1^* \cup \{v\}, P_2^*, 0.98)$, $G(P_2^* \cup \{v\}, P_3^*, 0.98)$ and $G(P_3^* \cup \{v\}, P_1^*, 0.98)$. Thus every vertex from J can be added to P_1^* or P_2^* or P_3^* such that one obtains blocks $P_1^\dagger, P_2^\dagger, P_3^\dagger$ ($0.66n \leqslant |P_l^\dagger| \leqslant 0.67n$ for $l = 1, 2, 3$) with $P_1^\dagger \cup P_2^\dagger \cup P_3^\dagger = [2n-1]$ in such a way that afterwards all graphs $G(P_1^\dagger, P_2^\dagger, 0.98)$, $G(P_2^\dagger, P_3^\dagger, 0.98)$ and $G(P_3^\dagger, P_1^\dagger, 0.98)$ have minimum degree at least $0.39n$. □

Lemma 27. *In the setting of Lemma 26 we can find a red P_n^t.*

Proof. Since $P_1^\dagger \cup P_2^\dagger \cup P_3^\dagger = [2n-1]$ one of the blocks has size at least $2n/3$. W.l.o.g. $|P_1^\dagger| \geqslant 2n/3$. The minimum degree of $G(P_1^\dagger, P_2^\dagger, 0.98)$ assures that this graph contains a Hamiltonian path. Label such a path $a_1, a_2, a_3, \ldots, a_{|P_1^\dagger|}$. In order to find a tight path of length n, we will add after every second vertex in this path a vertex from $P_2^\dagger$. Assume we already have found $a_1, a_2, b_1, a_3, a_4, b_2, \ldots, a_{2i-1}a_{2i}$ then we can choose b_i from $P_2^\dagger$ which has not been used so far such that $a_{2i-1}a_{2i}b_i$, $a_{2i}b_ia_{2i+1}$ and b_ia_{2i+1}, a_{2i+2} is red, because $|P_2^\dagger| - 0.02|P_2^\dagger| - 0.02|P_2^\dagger| - 0.02|P_2^\dagger| - i > 0$ for $i < 0.94|P_2^\dagger|$ and thus especially for $i < 0.5n$. Hence, we can embed a red tight path of length at least $2n/3 + n/3 = n$. □

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