Incompatible double posets and double order polytopes

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Abstract
In 1986 Stanley associated to a poset the order polytope. The close interplay between its combinatorial and geometric properties makes the order polytope an object of tremendous interest. Double posets were introduced in 2011 by Malvenuto and Reutenauer as a generalization of Stanley’s labelled posets. A double poset is a finite set equipped with two partial orders. To a double poset Chappell, Friedl and Sanyal (2017) associated the double order polytope. They determined the combinatorial structure for the class of compatible double posets. In this paper we generalize their description to all double posets and we classify the 2-level double order polytopes.

Mathematics Subject Classifications: 06A07, 52B05, 52B12, 52B20

1 Introduction

A partially ordered set \((P, \preceq)\), also called poset, is a finite set \(P\) together with a reflexive, transitive and antisymmetric relation \(\preceq\). To a poset Stanley [4] associates a convex polytope, the order polytope \(\mathcal{O}(P)\), which is the set of all order-preserving functions from \(P\) into the interval \([0, 1]\):

\[
\mathcal{O}(P) = \{ f : P \to [0, 1] : a \prec b \Rightarrow f(a) \leq f(b) \}.
\]

Since the order polytope reflects many combinatorial properties of the poset, it is worth to study the geometric properties of \(\mathcal{O}(P)\). For more details about convex polytopes we refer to [6].

A double poset \(P = (P, \preceq_+, \preceq_-)\), as introduced by Malvenuto and Reutenauer [3], is a finite set \(P\) together with two partial order relations \(\preceq_+\) and \(\preceq_-\). The two underlying posets are denoted \(P_+ = (P, \preceq_+)\) and \(P_- = (P, \preceq_-)\). Chappell, Friedl, and Sanyal constructed in [2] a polytope for a double poset \(P\), the double order polytope given by

\[
\mathcal{O}(P) = \mathcal{O}(P, \preceq_+, \preceq_-) := \text{conv}\{ (2\mathcal{O}(P_+) \times \{1\}) \cup (-2\mathcal{O}(P_-) \times \{-1\}) \} \subseteq \mathbb{R}^P \times \mathbb{R}.
\]
The interplay of the two partial orders of a double poset is reflected in the geometry of its double order polytope. The reduced double order polytope is a simpler construction that captures most properties of $\mathcal{O}(P)$, and is defined as

$$
\mathcal{O}(P) := \mathcal{O}(P) \cap \{(f, t) : t = 0\} = \mathcal{O}(P^+) - \mathcal{O}(P^-) \subseteq \mathbb{R}^P.
$$

Note that here and in the following we write $Q - R$ for the Minkowski sum of polytopes $Q$ and $-R$. In [2, Thm 2.7] the authors gave a characterization of the facets of double order polytopes for the class of compatible double posets, that is, the case where $P_+$ and $P_-$ have a common linear extension. We generalize their description to all double posets in Theorem 17. We use this description to give a complete classification of 2-level polytopes among the double order polytopes. We finish by determining the vertices of reduced double order polytopes for general double posets in Corollary 22.

2 Double posets and double order polytopes

Let $(P, \preceq)$ be a poset. By adjoining a new minimum $\hat{0}$ and a new maximum $\hat{1}$ to $P$, we obtain the poset $\hat{P}$. The linear form associated to an order relation $a \prec b$ is the map $\ell_{a,b} : \mathbb{R}^P \to \mathbb{R}$ with

$$
\ell_{a,b}(f) := f(a) - f(b)
$$

for $f \in \mathbb{R}^P$. Moreover, for $a \in P$ we define $\ell_{a,\hat{1}}(f) := f(a)$ and $\ell_{\hat{0},a}(f) := -f(a)$. With these definitions it follows that a map $f : P \to \mathbb{R}$ is contained in $\mathcal{O}(P)$ if and only if

$$
\begin{align*}
\ell_{a,b}(f) &\leq 0 \quad \text{for all } a \prec b, \\
\ell_{\hat{0},a}(f) &\leq 0 \quad \text{for all } b \in P, \text{ and} \\
\ell_{a,\hat{1}}(f) &\leq 1 \quad \text{for all } a \in P.
\end{align*}
$$

A nonempty face of $\mathcal{O}(P)$ is a subset $F \subseteq \mathcal{O}(P)$ such that

$$
F = \mathcal{O}(P)^F := \{f \in \mathcal{O}(P) : \ell(f) \geq \ell(f') \text{ for all } f' \in \mathcal{O}(P)\}
$$

for some linear function $\ell \in (\mathbb{R}^P)^*$. If $F \neq \mathcal{O}(P)$, then $F$ is a proper face.

As mentioned before, the order polytope geometrically describes combinatorial features of the underlying poset. For example, the vertices of $\mathcal{O}(P)$ are in bijection to filters of $P$. Recall that a filter of $(P, \preceq)$ is a subset $J \subseteq P$ such that $a \in J$ and $a \prec b$ for $b \in P$ implies $b \in J$. Dually, an ideal is a subset $I \subseteq P$ such that $b \in I$ and $a \prec b$ for $a \in P$ implies $a \in I$.


**Definition 1.** A (closed) face partition of a face $F \subseteq \mathcal{O}(P)$ is a partition of $\hat{P}$ into nonempty and pairwise disjoint blocks $B_1, \ldots, B_k \subseteq \hat{P}$ such that

$$
F = \{f \in \mathcal{O}(P) : f \text{ is constant on } B_i \text{ for } i = 1, \ldots, k\}
$$

and for any $i \neq j$ there is a $f \in F$ such that $f(B_i) \neq f(B_j)$. The reduced face partition of $F$ is $\mathcal{B}(F) := \{B_i : |B_i| > 1\}$.
In [2, Prop 2.1] the following description for the normal cone of a nonempty face \( F \subseteq \mathcal{O}(P) \) with a reduced face partition \( \mathcal{B}(F) = \{B_1, \ldots, B_k\} \) is given:

\[
N_P(F) = \text{cone}\{\ell_{a,b} : [a, b] \subseteq B_i \text{ for some } i = 1, \ldots, k\}.
\] (2)

We will need the following consequences that were noted in [2].

**Corollary 2.** Let \( F \subseteq \mathcal{O}(P) \) be a nonempty face with reduced face partition \( \{B_1, \ldots, B_k\} \). Then for every \( \ell \in \text{relint} \, N_P(F) \) and \( p \in P \) the following hold:

(i) if \( p \in \min(B_i) \) for some \( i \), then \( \ell_p > 0 \);
(ii) if \( p \in \max(B_i) \) for some \( i \), then \( \ell_p < 0 \);
(iii) if \( p \notin \bigcup_i B_i \), then \( \ell_p = 0 \).

If \( P \) is a polytope and \( \dim(P) = d \), then we call the \((d - 1)\)-dimensional faces **facets**. Maximizing the linear functions \( \ell(f, t) = t \) and \( \ell(f, t) = -t \) over \( \mathcal{O}(P) \subseteq \mathbb{R}^P \times \mathbb{R} \) one obtains the facets \( 2\mathcal{O}(P_+) \times \{1\} \) and \(-2\mathcal{O}(P_-) \times \{-1\}\). We call the remaining facets **vertical**. They are in bijection with the facets of \( \tilde{\mathcal{O}}(P) \). A facet of the reduced double order polytope is a face of the form \( F = F_+ - F_- \) such that there is a linear function \( \ell \in (\mathbb{R}^P)^* \), where \( F_+ = \mathcal{O}(P_+)^\ell \) and \( F_- = \mathcal{O}(P_-)^{-\ell} \).

**Definition 3.** A linear function \( \ell \in (\mathbb{R}^P)^* \) is called **rigid** for \( \mathcal{O}(P) \) if it satisfies

\[
\text{relint} \, N_{P_+}(F_+) \cap \text{relint} \, -N_{P_-}(F_-) = \mathbb{R}_{>0} \cdot \ell
\]
for a pair of faces \((F_+, F_-)\). Note that \( F = F_+ - F_- \) is necessarily a facet of \( \tilde{\mathcal{O}}(P) \).

**Definition 4.** An **alternating chain** \( C \) of a double poset \( P = (P, \preceq_+, \preceq_-) \) is a finite sequence of distinct elements

\[
\hat{0} = p_0 \prec_\sigma p_1 \prec_{-\sigma} p_2 \prec_\sigma \cdots \prec_{\pm\sigma} p_k = \hat{1},
\]
where \( \sigma \in \{\pm\} \). If \( k \) is odd, then we additionally require that \( p_{k-1} \not\sim_\sigma p_1 \). For an alternating chain \( C \), we define a linear function \( \ell_C \) by

\[
\ell_C(f) := \sigma(-f(p_1) + f(p_2) - \cdots + (-1)^{k-1}f(p_{k-1})).
\]
If \( k = 1 \), then \( \ell_C \equiv 0 \). If \( k > 1 \), then \( C \) is a **proper** alternating chain. Let \( \text{sign}(C) = \tau \in \{\pm\} \) be the sign of an alternating chain \( C \) if \( p_{k-1} \prec_\tau p_k \) is the last relation in \( C \).

**Definition 5.** An **alternating cycle** \( C \) of \( P \) is a sequence of elements of \( P \) of length \( 2k \) of the form

\[
p_0 \prec_\sigma p_1 \prec_{-\sigma} p_2 \prec_\sigma \cdots \prec_{-\sigma} p_{2k} = p_0,
\]
where \( \sigma \in \{\pm\} \) and \( p_i \neq p_j \) for \( 0 \leq i < j < 2k \). Similarly the linear function associated to \( C \) is defined by

\[
\ell_C(f) := \sigma(f(p_0) - f(p_1) + f(p_2) - \cdots + (-1)^{2k-1}f(p_{2k-1})).
\]
Note that any cyclic shift yields an alternating cycle with the same linear function \( \ell_C \). Hence, we identify an alternating cycle with all its cyclic shifts.
we conclude

Furthermore we can write

For

Due to the fact that

Then we can write the linear function associated to

Proof. Let

Then

Let

be the principal filter generated by

The linear function

can be written in terms of the linear form of the order relation

Then

Let

be an alternating cycle and

We can write

Remark 6. Our definition of alternating chains differs slightly from the one given in [2] in that we require

for a chain of odd length. Without that condition, alternating cycles would yield alternating chains with the same linear function.

The following technical fact will be of importance later.

Lemma 7. If

is a proper alternating chain and

the linear function associated to

then

More precisely the following hold:

(i) if

then

and

(ii) if

then

and

Proof. Since the proof works analogously, we only consider the case of an alternating chain with

The following technical fact will be of importance later.

Lemma 8. Let

be an alternating cycle and

the linear function associated to

Then

Proof. Let

Then we can write the linear function associated to

in terms of the linear form of the order relation

Then we have

Furthermore we can write

Analogously it follows


The following Proposition was stated by Chappell, Friedl and Sanyal in [2].

**Proposition 9.** Let $P = (P, \preceq_+, \preceq_-)$ be a double poset. If $\ell$ is a rigid linear function for $\overline{O}(P)$, then $\ell = \mu C$ for some alternating chain or alternating cycle $C$ and $\mu > 0$.

**Definition 10.** A double poset $P = (P, \preceq_+, \preceq_-)$ is called compatible if $P_+ = (P, \preceq_+)$ and $P_- = (P, \preceq_-)$ have a common linear extension. Otherwise, $P$ is incompatible.

In case $P$ is a compatible double poset, it was shown in [2, Thm 2.7] that the linear functions $\ell_C$ associated to proper alternating chains $C$ are in bijection to rigid linear functions of $O(P)$. Recall that a linear extension of $(P, \preceq)$ is a injective and order-preserving map $l : P \rightarrow [n]$ where $n = |P|$. 

**Proposition 11.** A double poset $P = (P, \preceq_+, \preceq_-)$ is compatible if and only if it has no alternating cycles.

**Proof.** If $P$ is compatible, then $P_+$ and $P_-$ have a common linear extension $l : P \rightarrow [n]$, where $n = |P|$. Suppose there is an alternating cycle $p_0 \prec_\sigma p_1 \prec_- \sigma p_2 \prec_\sigma \cdots \prec_- \sigma p_{2k} = p_0$. Then $l$ has to satisfy

$$l(p_0) < l(p_1) < l(p_2) < \cdots < l(p_{2k-1}) < l(p_{2k}).$$

Since $p_0 = p_{2k}$ this contradicts $l(p_0) < l(p_{2k})$.

Let $P$ be a double poset without alternating cycles and $|P| = n$. Let $M = \max(P_+) \cap \max(P_-)$. We claim that $M \neq \emptyset$. Otherwise, for every $p \in \max(P_+)$, there is a $q \in P \setminus \max(P_+)$ with $p \prec_- q$. And for any such $q$ there is a $q' \in P \setminus \max(P_-)$ with $q \prec_+ q'$. Repeating yields an alternating chain or cycle. Since $|P| < \infty$ and there are no alternating cycles in $P$, it has to be a finite sequence, and hence there is a $p \in P$ for which $p \in \max(P_+)$ and $p \in \max(P_-)$. We can construct a map $l : P \rightarrow \{1, \ldots, n\}$ that is strictly order preserving for $\prec_+$ and $\prec_-$ by induction on $n$. For $n = 1$, let $P = \{p\}$ and $l(p) = 1$. For $n > 1$, pick a $p \in M$ and define $l(p) = n$. By induction, there is a map $l : P \setminus \{p\} \rightarrow \{1, \ldots, n-1\}$ that is strictly order preserving for $\prec_+$ and $\prec_-$. Any map that is constructed in this way, gives us a common linear extension for $P_+$ and $P_-$ and hence $P$ is compatible. 

The next example, taken from [2], illustrates that for incompatible double posets not every alternating chain or cycle corresponds to a facet of the double order polytope.

**Example 12.** Let $(P, \preceq)$ be a poset and $\preceq^{op}$ the opposite order of $\preceq$. Then $P = (P, \preceq_+, \preceq_-)$ with $\preceq_+ = \preceq$ and $\preceq_- = \preceq^{op}$ is an incompatible double poset. Since $O(P_+) = 1 - O(P_-)$, where $1 : \mathbb{R}^P \rightarrow \mathbb{R}$ is the function $1(p) = 1$ for all $p \in P$, we conclude, that the double order polytope is a prism over $O(P_+)$. Hence the vertical facets of $O(P)$ are prisms over the facets of $O(P_+)$. Thus the number of facets of $\overline{O}(P)$ equals the number of facets of $O(P_+)$, and these are in bijection to the minima, maxima, and cover relations of $P_+$. For any $p \in P$ we have the alternating chains $\hat{0} \prec_+ p \prec_- \hat{1}$ and $\hat{0} \prec_- p \prec_+ \hat{1}$. Furthermore any cover relation $p \prec_\sigma q$ gives rise to the alternating cycle $p \prec_\sigma q \prec_- \sigma p$. Hence, there are more alternating chains and cycles than facets.
In the next section, we determine the facets of the reduced double order polytope for general posets.

3 Facets and 2-levelness

Let \( P = (P, \preceq_+, \preceq_-) \) be a double poset.

**Definition 13.** Let \( \tau, \sigma \in \{\pm\} \). An alternating chain or cycle \( C \) is crossed by \( a \in P \) if there are \( i \neq j \) such that
\[
p_i \preceq_\tau a \prec_\tau p_{i+1} \text{ and } p_j \preceq_\sigma a \prec_\sigma p_{j+1}.
\]

The motivation of this definition is the following proposition. It was shown in [2, Thm 2.7] that if \( P \) is a compatible double poset, then its alternating chains are in bijection to the facets of \( \mathcal{O}(P) \). To prove it, a property of alternating chains of compatible double posets is used:

If \( p_i \prec_\sigma p_{i+1} \prec_- \cdots \prec_- p_j \prec_\tau p_{j+1} \) is part of an alternating chain \( C \) with \( \sigma, \tau \in \{\pm\} \) and \( i < j \), then there is no \( a \in P \) such that \( p_i \prec_\tau a \prec_\sigma p_{i+1} \) and \( p_j \prec_\tau a \prec_\tau p_{j+1} \). Uncrossed alternating chains and cycles of incompatible double posets fulfil this as well.

**Proposition 14.** If \( C \) is an uncrossed alternating chain or cycle, then \( \ell_C \) is rigid.

**Proof.** We only consider \( C \) to be an alternating chain of the form
\[
\hat{0} = p_0 \prec_+ p_1 \prec_- p_2 \prec_+ \cdots \prec_- p_{2k} \prec_+ p_{2k+1} = \hat{1},
\]
since the proof works analogously for the other forms of alternating chains and cycles. Then the linear function is
\[
\ell_C(f) = -f(p_1) + f(p_2) - \cdots + f(p_{2k}).
\]

Let \( F_+ = \mathcal{O}(P_+)^{\ell_C} \) and \( F_- = \mathcal{O}(P_-)^{-\ell_C} \) be the corresponding faces. If \( J \) is a filter of \( P_+ \), then \( p_{2i} \in J \) implies \( p_{2i+1} \in J \) for \( 1 \leq i \leq k \), since \( p_{2i} \prec_+ p_{2i+1} \). It follows from \( \text{sign}(C) = + \) with Lemma 7(i) that \( \max_{J \subseteq P_+} \ell_C(J) > 1 \). Thus \( 1_J \in F_+ \) if and only if \( J \) does not separate \( p_{2j} \) and \( p_{2j+1} \) for \( 1 \leq j \leq k \), because otherwise \( \ell_C(1_J) < 1 \). From Definition 1 it follows that \( p_{2j} \) and \( p_{2j+1} \) for \( 1 \leq j \leq k \) are not contained in different parts of the face partition \( B_+ \).

If \( J \) is a filter of \( P_- \), then \( p_{2i-1} \in J \) implies \( p_{2i} \in J \) for \( 1 \leq i \leq k \), since \( p_{2i-1} \prec_- p_{2i} \). It follows again with Lemma 7(i) that \( \min_{J \subseteq P_-} \ell_C(J) = 0 \). Thus a filter \( J \subseteq P_- \) is contained in \( F_- \) if and only if \( J \) does not separate \( p_{2j-1} \) and \( p_{2j} \) for \( 1 \leq j \leq k \), otherwise \( \ell_C(1_J) > 0 \). Again from Definition 1 it follows that \( p_{2j-1} \) and \( p_{2j} \) for \( 1 \leq j \leq k \) are not contained in different parts of the face partition \( B_- \).

Since \( C \) is an uncrossed alternating chain, there is no \( a \in P \) and \( i \neq j \) such that \( p_{2i} \preceq_+ a \prec_+ p_{2i+1} \) and \( p_{2j} \preceq_+ a \prec_+ p_{2j+1} \) and hence there is \( f \in F_+ \) such that \( f(p_{2i}) \neq f(p_{2j}) \).
As well, there is \( g \in F_+ \) such that \( g(p_{2i-1}) \not= g(p_{2j-1}) \) for any \( 1 \leq i \leq j \leq k \). Thus, the reduced face partitions \( B_\pm \) are

\[
B_+ = \{ [p_0, p_1], [p_2, p_3], \ldots, [p_{2k}, p_{2k+1}] \} \quad \text{and} \quad B_- = \{ [p_1, p_2], [p_3, p_4], \ldots, [p_{2k-1}, p_{2k}] \}.
\]

Let \( \ell \) be a linear function with \( \ell(\phi) = \sum_{p \in P} \ell_p(\phi)(p) \) such that \( F_+ = \mathcal{O}(P_+)^\ell \) and \( F_- = \mathcal{O}(P_-)^{-\ell} \). Since for \( i \leq k \) the element \( p_{2i} \) is a minimal and \( p_{2j-1} \) is a maximal element of \( B_+ \), it follows from Corollary 2 that \( \ell_p > 0 \) if \( p = p_{2i-1} \) and \( \ell_p < 0 \) if \( p = p_{2i} \) for \( 1 \leq i \leq k \). Since \( C \) is an uncrossed alternating chain, it follows that if \( a \in (p_i, p_{i+1})_P \) for some \( i \), then \( a \not\in [p_j, p_{j+1}]_P \) for all \( j \) and vice versa. Otherwise there would be \( p_j, p_{j+1} \) such that \( p_i \prec_j a \prec_j p_{i+1} \) and \( p_j \succeq_{j+1} a \succeq_{j+1} p_{j+1} \). That is why \( a \not\in \bigcup B_i \) for one of the face partitions \( B_+ \) or \( B_- \) and hence it follows from Corollary 2(iii) that \( \ell_a = 0 \).

Since we assumed \( F_+ = \mathcal{O}(P_+)^\ell \) and \( F_- = \mathcal{O}(P_-)^{-\ell} \), it follows that \( \ell \in N_{\ell}(F_+) \) and \(-\ell \in N_{\ell}(F_-) \). As Equation 2 states we can write

\[
N_{\ell}(F_+) = \text{cone} \{ \ell_{p_0, p_1}, \ell_{p_2, p_3}, \ldots, \ell_{p_{2k}, p_{2k+1}} \} \quad \text{and} \quad N_{\ell}(F_-) = \text{cone} \{ \ell_{p_1, p_2}, \ell_{p_3, p_4}, \ldots, \ell_{p_{2k-1}, p_{2k}} \}.
\]

So \( \ell \in \text{relint} \ N_{\ell}(F_+) \cap \text{relint} \ N_{\ell}(F_-) \) satisfies \( \ell_{p_i} + \ell_{p_{i+1}} = 0 \) for all \( 1 \leq i \leq 2k \) and therefore \( \ell = \mu\ell_C \) for some \( \mu > 0 \).

The following decomposition of crossed alternating chains and cycles will be important.

**Proposition 15.** Let \( P \) be a double poset.

(i) If \( C \) is an alternating cycle crossed by \( a \), then there are two alternating cycles \( C_1 \) and \( C_2 \) such that \( \ell_C = \ell_{C_1} + \ell_{C_2} \).

(ii) If \( C \) is an alternating chain crossed by \( a \), then there is a proper alternating chain \( C_1 \) and an alternating cycle \( C_2 \) such that \( \ell_C = \ell_{C_1} + \ell_{C_2} \) and \( \text{sign}(C) = \text{sign}(C_1) \).
Proof. (i) Let $C$ be a crossed alternating cycle and $i < j$:

$$ p_0 <_+ \cdots <_{-r} p_i <_r p_{i+1} <_{-r} \cdots <_{-\sigma} p_j <_{\sigma} p_{j+1} <_{-\sigma} \cdots <_{-} p_{2k} = p_0. $$

(1) If $\tau = \sigma$, then

$$ p_0 <_+ \cdots <_{-r} p_i <_r p_{j+1} <_{-r} \cdots <_{-} p_{2k} = p_0 \text{ and } p_{i+1} <_{-r} p_{i+2} <_{-r} \cdots <_{-r} p_j <_r p_{i+1} \text{ are the two alternating cycles } C_1 \text{ and } C_2. $$

(2) If $\tau = -\sigma$, then $C_1$ is given by

$$ p_0 <_+ \cdots <_{-r} p_i <_r a <_{-r} p_{j+1} <_{-r} \cdots <_{-} p_{2k} = p_0 \text{ in case } p_i \neq a; \text{ or } $$

$$ p_0 <_+ \cdots <_{-r} p_i <_r p_{j+1} <_{-r} \cdots <_{-} p_{2k} = p_0 \text{ in case } p_i = a, \text{ and } C_2 \text{ is given by } $$

$$ p_i <_{-r} p_{i+1} <_{-r} p_{i+2} <_{-r} \cdots <_{r} p_j <_{-r} p_i. $$

(ii) We only consider the case where $C$ is a crossed alternating chain starting with $+$ and $i < j$:

$$ \hat{\alpha} = p_0 <_+ \cdots <_{-r} p_i <_r p_{i+1} <_{-r} \cdots <_{-\sigma} p_j <_{\sigma} p_{j+1} <_{-\sigma} \cdots <_{-} p_k = \hat{1}. $$

(3) If $\tau = \sigma$, then

$$ \hat{\alpha} = p_0 <_+ \cdots <_{-r} p_i <_r p_{j+1} <_{-r} \cdots <_{-} p_k = \hat{1} \text{ is the alternating cycle } C_1 \text{ and } $$

$$ p_{i+1} <_{-r} p_{i+2} <_{-r} \cdots <_{-r} p_j <_{r} p_{i+1} \text{ is the alternating cycle } C_2. $$

(4) If $\tau = -\sigma$, then

$$ \hat{\alpha} = p_0 <_{a} \cdots <_{-r} p_i <_{r} a <_{-r} p_{j+1} <_{-r} \cdots <_{-\sigma} p_k = \hat{1} \text{ is the alternating chain } C_1 \text{ in case } $$

$$ p_i \neq a; \text{ and } \hat{\alpha} = p_0 <_{a} \cdots <_{-r} p_{i-1} <_{r} p_{j+1} <_{r} \cdots <_{-\sigma} p_k = \hat{1} \text{ is the alternating chain } C_1 \text{ in case } $$

$$ p_i = a, \text{ and } a <_{r} p_{i+1} <_{r} \cdots <_{r} p_j <_{-r} a \text{ is the alternating cycle } C_2 \text{ in both cases}. \qedhere

Corollary 16. Let $P = (P, \preceq_+, \preceq_-)$ be a double poset and $C$ an alternating cycle or chain. If there is an $a \in P$ such that $C$ is crossed by $a$, then $\ell_C$ is not rigid.

Proof. Assume that $F = \mathcal{O}(P)^{\ell_C}$ is a facet. It follows from Proposition 15, that there are proper alternating chains or cycles $C_1$ and $C_2$ such that $\ell_C = \ell_{C_1} + \ell_{C_2}$ and one of the following holds:

(i) $C, C_1$ and $C_2$ are alternating cycles;

(ii) $C$ and $C_1$ are alternating chains that satisfy $\text{sign}(C) = \text{sign}(C_1)$, $C_2$ is an alternating cycle.

Let $G = \mathcal{O}(P)^{\ell_{C_1}}$ and $H = \mathcal{O}(P)^{\ell_{C_2}}$ be the faces defined by $\ell_{C_1}$ and $\ell_{C_2}$. Let $f \in \text{relint } F$. In case of (i), since $\ell_C(f) = 0$ from Lemma 8, this implies $\ell_{C_1}(f) = \ell_{C_2}(f) = 0$.

In case of (ii), since $\ell_C(f) = 1$ from Lemma 7, this implies $\ell_{C_1}(f) = 1$ and $\ell_{C_2}(f) = 0$. Thus $f \in G \cap H$. Since $f$ was in relint $F$, it follows that $F \subseteq G \cap H$. The alternating chains or cycle $C_1$ and $C_2$ have a length $k > 1$ and hence $\ell_{C_i} \neq 0$ for $i = 1, 2$. Thus, $G, H$ are proper faces and since we have assumed that $F$ is a facet, it follows that $G$ and $H$ are facets. Since $C_1$ and $C_2$ differ by at least one element it follows, that $\ell_{C_1} \neq \mu\ell_{C_2}$ for all $\mu \in \mathbb{R}_{>0}$ and hence $G \neq H$. Thus $F$ cannot be a facet and hence $\ell_C$ is not rigid. \qed
The following theorem completes the characterization of the facets of double order polytopes and follows from Proposition 14 and Corollary 16.

**Theorem 17.** Let $P = (P, \preceq_+, \preceq_-)$ be a double poset. A linear function $\ell$ is rigid if and only if $\ell \in \mathbb{R}_{>0}^{\ell_C}$ for some uncrossed alternating chain or cycle $C$. In particular, the facets of $O(P)$ are in bijection to alternating chains and cycles that are not crossed by any $a \in P$.

We now turn to the question which incompatible double order polytopes are 2-level.

**Definition 18.** A full-dimensional polytope $Q \subseteq \mathbb{R}^n$ is **2-level**, if for every facet-defining hyperplane $H$ there is some $t \in \mathbb{R}^n$ such that $H \cup (t + H)$ contains all vertices of $Q$.

2-level polytopes and compressed polytopes [5] constitute a very interesting class of polytopes in combinatorics and optimization. In particular Stanley’s order polytopes are 2-level and in [2], Chappell, Friedl and Sanyal classified the 2-level polytopes among compatible double order polytopes. To include the incompatible double order polytopes we need to determine the facet-defining inequalities of $O(P)$.

**Corollary 19.** Let $P = (P, \preceq_+, \preceq_-)$ be a double poset. Then $O(P)$ is the set of points $(f, t) \in \mathbb{R}^{P_\times \mathbb{R}}$ such that

(i) $L_C(f, t) := \ell_C(f) - \text{sign}(C) t \leq 1$ for all uncrossed alternating chains $C$ of $P$;

(ii) $L_C(f, t) := \ell_C(f) \leq 0$ for all uncrossed alternating cycles of $P$.

**Proof.** Theorem 17 says that the facet-defining inequalities of $O(P)$ are in bijection to the uncrossed alternating chains and cycles of $P$. If $C$ is an alternating cycle and $\text{sign} C =$
+, then it follows by Lemma 7 that the maximal value of \( \ell_C \) over \( 2O(P_+) \) is 2 and 0 over \(-2O(P_-)\). Since the values are exchangeable for sign \( C = - \), the facet-defining inequalities are of the form (i). If \( C \) is an alternating cycle, then it follows by Lemma 8 that maximal value of \( \ell_C \) over \( 2O(P_+) \) as well as over \(-2O(P_-)\) is 0 and hence the facet-defining inequalities are of the form (i).

**Proposition 20.** Let \( P = (P, \preceq_+, \preceq_-) \) be a double poset and \( \sigma \in \{\pm\} \). If there are \( a, b \in P \) such that \( \hat{0} \prec_{-\sigma} a \prec_{\sigma} b \prec_{-\sigma} \hat{1} \) is an uncrossed alternating chain \( C \) and it does not hold neither \( a \prec_{-\sigma} b \) nor \( b \prec_{-\sigma} a \), then \( O(P) \) is not 2-level.

**Proof.** Since \( O(P, \preceq_+, \preceq_-) \) is 2-level if and only if \( O(P, \preceq_-, \preceq_+) \) is 2-level, we only consider \( \sigma = + \).

Due to the fact that \( C \) is uncrossed, the linear function \( \ell_C \) is rigid. Then

\[
\ell_C(f, t) = f(a) - f(b) + t
\]

is a facet-defining inequality of \( O(P) \). Since \( b \not\prec_- a \), there is a filter \( J_1 \) of \( P_- \) such that \( b \in J_1 \) and \( a \not\in J_1 \). Since \( a \not\prec_- b \), there is a filter \( J_2 \) of \( P_+ \) such that \( a \in J_2 \) and \( b \not\in J_2 \). As well, there is a filter \( J_3 = \emptyset \) of \( P_- \). The vertices corresponding to these three filters let \( \ell_C(f, t) \) take three different values:

\[
\begin{align*}
\ell_C(-2J_1, -1) &= 0 - (-2) + (-1) = 1 \\
\ell_C(-2J_2, -1) &= -2 - 0 + (-1) = -3 \\
\ell_C(-2J_3, -1) &= 0 - 0 + (-1) = -1
\end{align*}
\]

Hence \( O(P) \) is not 2-level. For \( \sigma = - \), the proof works analogously. \(\square\)

**Theorem 21.** Let \( P = (P, \preceq_+, \preceq_-) \) be a double poset and \( \sigma \in \{\pm\} \). Then \( O(P) \) is 2-level if and only if for all \( a, b \in P \) such that \( a \prec_\sigma b \) is part of an uncrossed alternating chain or cycle it holds that \( a \prec_{-\sigma} b \) or \( b \prec_{-\sigma} a \).

**Proof.** Again, we consider only \( \sigma = + \). For \( \sigma = - \), the proof works analogously.

If \( b \prec_- a \), then \( a \prec_+ b \) can only be part of the alternating cycle

\[
C = a \prec_+ b \prec_- a.
\]

All other alternating chains or cycles would be crossed by \( a \). The corresponding linear function of the double order polytope

\[
\ell_C(f, t) = f(a) - f(b)
\]

defines a facet of \( O(P) \). If \( J_+ \) is a filter of \( P_+ \), then \( a \in J_+ \) implies \( b \in J_+ \) and that is why \( \ell_C(21_{J_+}, 1) = 0 \) or \( \ell_C(21_{J_+}, 1) = -2 \). If \( J_- \) is a filter of \( P_- \), then \( b \in J_- \) implies \( a \in J_- \) and that is why \( \ell_C(-21_{J_-}, -1) = 0 \) or \( \ell_C(-21_{J_-}, -1) = -2 \).

If \( a \prec_- b \), then \( a \prec_+ b \) can be part of an alternating chain or cycle \( C' \) such that \( C' \neq C \). In this case all other \( c \prec_\tau d \) in \( C' \) have to satisfy \( c \prec_{-\tau} d \), where \( \tau \in \{\pm\} \). Otherwise, if
If \( J \) is a filter of \( P_+ \) or \( P_- \), then it follows from \( p_i \in J \) that \( p_{i+1} \in J \), since \( p_i \prec_+ p_{i+1} \) and \( p_i \prec_+ p_{i+1} \) for \( i = 0, \ldots, k-1 \). Let \( \text{sign}(C') = +. \)

If \( J_+ \subseteq P_+ \), then \( \ell_{C'}(21_{J_+}) \) can only take the values 2 or 0 and if \( J_- \subseteq P_- \), then \( \ell_{C'}(-21_{J_-}) \) takes the values 0 and -2. The values are exchanged for \( \text{sign}(C') = -. \) Hence

\[
L_{C'}(f, t) = \ell_{C'}(f) - \text{sign}(C') t,
\]

where \((f, t)\) is a vertex of \( \mathcal{O}(P) \), attains only the values -1 and 1. Thus \( \mathcal{O}(P) \) is 2-level.

Assume that \( \mathcal{O}(P) \) is 2-level. If there are \( a, b \in P \) such that \( a \prec_{\sigma} b \) is part of an uncrossed alternating chain or cycle and neither \( a \prec_{-\sigma} b \) nor \( b \prec_{-\sigma} a \), then it follows by Proposition 20 that \( \mathcal{O}(P) \) is not 2-level. \( \square \)

4 Edges of general double order polytopes

In this last section we determine the vertical edges of double order polytopes. The edges of an order polytope \( \mathcal{O}(P) \) were determined by Stanley [4]: Edges correspond to pairs of filters \( J \subset J' \) such that \( J' \setminus J \) is a connected poset. The vertical edges of \( \mathcal{O}(P) \) are in bijection to the vertices of \( \mathcal{O}(P) \) and the following theorem shows that they also correspond to certain pairs of filters \((J_+, J_-)\) where \( J_+ \subseteq P_+ \) and \( J_- \subseteq P_- \).

**Theorem 22.** Let \( P = (P, \preceq_+, \preceq_-) \) be a double poset and let \( J_+ \subseteq P_+ \) and \( J_- \subseteq P_- \) be filters. Let \( I_+ := P_+ \setminus J_+ \) and \( I_- := P_- \setminus J_- \) be the corresponding ideals. Then \((21_{J_+}, 1)\) and \((-21_{J_-}, -1)\) are the endpoints of a vertical edge of \( \mathcal{O}(P) \) if and only if \( 1_{J_+} - 1_{J_-} \) is a vertex of \( \mathcal{O}(P) \) if and only if the following hold:

(i) for all \( a \in J_+ \cap J_- \) there is an alternating chain

\[
\hat{0} \prec_{-\sigma} a_1 \prec_{-\sigma} a_2 \prec_{-\sigma} \cdots \prec_{-\sigma} a_k = a \prec_{+} \hat{1},
\]

where \( a_1 \in J_\sigma \setminus J_- \) and \( a_2, \ldots, a_k \in J_+ \cap J_- \).

(ii) for all \( b \in I_+ \cap I_- \) there is an alternating chain

\[
\hat{0} \prec_{-\sigma} b = b_1 \prec_{+} b_2 \prec_{-\sigma} \cdots \prec_{-\sigma} b_k \prec_{-\sigma} \hat{1},
\]

where \( b_1, b_2, \ldots, b_{k-1} \in I_+ \cap I_- \) and \( b_k \in I_- \setminus I_\sigma \).

This generalizes the result of Chappell, Friedl and Sanyal in Corollary 2.17 [2], since (i) implies that \( \min J_+ \cap \min J_- = \emptyset \) and (ii) implies that \( \max P_+ \setminus J_+ \cap \max P_- \setminus J_- = \emptyset. \)
Proof. From the definition of the reduced double order polytope

\[ O(P) \cap \{(\phi,t) : t = 0\} = (O(P_+) - O(P_-)) \times \{0\} \]

and the fact that \(1_{J_+} - 1_{J_-}\) is the midpoint between \((21_{J_+},1)\) and \((-21_{J_-},-1)\) the first equivalence follows.

To show necessity, assume that (i) is violated for some element \(a \in J_+ \cap J_-\). Let \(C\) be the union of all alternating chains

\[ \hat{0} \prec_{-\sigma} a_1 \prec_{\sigma} a_2 \prec_{-\sigma} \cdots \prec_{\pm} a_k = a \prec_+ \hat{1}, \tag{5} \]

such that \(a_1, \ldots, a_k \in J_+ \cap J_-\).

We claim that \(J_+ \setminus C\) is a filter in \(P_+\). Otherwise there is an element \(a_0 \in J_+ \setminus C\) and an element \(a_1 \in C\) such that \(a_0 \prec_+ a_1\). Since \(a_1 \in C\), there is an alternating chain of the form (5). We can assume that \(\sigma = -\). Otherwise, \(a_0 \prec_+ a_2\) and we simply delete \(a_1\) from the alternating chain. By construction \(a_0 \in J_+ \setminus J_-\) and the alternating chain \(a_0 \prec_+ a_1 \prec_{-} \cdots \prec_{\pm} a_k = a\) contradicts our assumption.

The same argument yields that \(J_- \setminus C\) is a filter in \(P_-\). Thus \(1_{J_+} - 1_{J_-} = 1_{J_+ \setminus C} - 1_{J_- \setminus C}\) and therefore \(1_{J_+} - 1_{J_-}\) cannot be a vertex of \(\overline{O}(P)\). The same argument shows necessity of (ii). Indeed, let us write \(P^{op}\) for the poset \(P\) with the opposite order relation. Filters of \(P^{op}\) are ideals in \(P\) and conversely and \(O(P^{op}) = 1 - O(P)\). In particular \(O(P^{op}) - O(P^{op}) = O(P_-) - O(P_+) = -\overline{O}(P)\). Since \(1_{J_+} - 1_{J_-}\) is a vertex of \(\overline{O}(P)\) if and only if \(1_{J_+} - 1_{J_-} = 1_{J_\perp} = 1_{J_\perp}\) is a vertex of \(-\overline{O}(P)\), condition (ii) is identical to condition (i) for the opposites of \(P_+\) and \(P_-\).

For sufficiency, let \(a \in \min J_+\). If \(a \in J_+ \setminus J_-\), then set \(\ell_{+a}(f) := f(a)\). If \(a \in J_+ \cap J_-\), then let

\[ \hat{0} \prec_{-\sigma} a_1 \prec_{\sigma} a_2 \prec_{-\sigma} \cdots \prec_{-} a_k = a \prec_+ \hat{1} \tag{6} \]

be a chain \(C\) as in (i). Note \(\text{sign}(C) = +\) since \(a \in \min J_+\). Lemma 7(i) yields that \(\ell_{+a}(1_{J_\perp}) \leq 1 = \ell_{+a}(1_{J_+})\) for every filter \(J_\perp \subseteq P_+\). Moreover, if \(\ell_{+a}(1_{J_\perp}) = 1\), then \(a \in J_\perp\). Again by Lemma 7(i), we have \(\ell_{+a}(-1_{J_\perp}) \leq 0 = \ell_{+a}(-1_{J_-})\) for all filter \(J_\perp \subseteq P_-\). Analogously, we use (ii) and define \(\ell_{+b}\) for all \(b \in \max P_+ \setminus J_+\). We set

\[ \ell_{+}(f) := \sum_{a \in \min J_+} \ell_{+a}(f) + \sum_{b \in \max P_+ \setminus J_+} \ell_{+b}(f). \]

Then \(\ell_{+}\) is maximized over \(\overline{O}(P)\) at points \(1_{J_+} - 1_{J_\perp}\) for some \(J_\perp \subseteq P_-\). Importantly, \(1_{J_+} - 1_{J_-}\) is one of the maximizers.

The same construction applied to \(J_-\) yields a function \(\ell_{-}(f)\) which is maximized over \(\overline{O}(P)\) at points \(1_{J_\perp} - 1_{J_-}\) for some \(J_\perp \subseteq P_+\). Again, \(1_{J_+} - 1_{J_-}\) is one of the maximizers. It follows that the linear function \(\ell_{+} + \ell_{-}\) is uniquely maximized \(1_{J_+} - 1_{J_-}\) over \(\overline{O}(P)\).

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References


