On the Divisibility of Character Values of the Symmetric Group

Jyotirmoy Ganguly Amritanshu Prasad*

The Institute of Mathematical Sciences (HBNI)
CIT Campus, Taramani, Chennai 600113, India

{jyotirmoy, amri}@imsc.res.in

Steven Spallone

Department of Mathematics
Indian Institute of Science Education and Research
Homi Bhabha Road, Pashan, Pune 411008, India

sspallone@iiserpune.ac.in

Submitted: Apr 26, 2019; Accepted: Mar 13, 2020; Published: Apr 3, 2020

Abstract

Fix a partition $\mu = (\mu_1, \ldots, \mu_m)$ of an integer k and positive integer d. For each $n \geqslant k$, let χ^λ_μ denote the value of the irreducible character χ^λ of S_n, corresponding to a partition λ of n, at a permutation with cycle type $(\mu_1, \ldots, \mu_m, 1^{n-k})$. We show that the proportion of partitions λ of n such that χ^λ_μ is divisible by d approaches 1 as n approaches infinity.

Mathematics Subject Classifications: 20C30, 05A16, 05A17

1 Introduction

Let k be a positive integer, and $\mu = (\mu_1, \ldots, \mu_m)$ a partition of k. For a partition λ of an integer $n \geqslant k$, let χ^λ_μ denote the value of the irreducible character of S_n corresponding to λ at an element with cycle type $(\mu_1, \ldots, \mu_m, 1^{n-k})$. The purpose of this article is to prove:

Main Theorem. For any positive integers k and d, and any partition μ of k,

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid \chi^\lambda_\mu \text{ is divisible by } d\}}{p(n)} = 1.$$

Here $p(n)$ denotes the number of partitions of n.

*Supported by a Swarnajayanti Fellowship of the Department of Science & Technology, Govt. of India.

In particular, for any integer d, the probability that an irreducible character of S_n has degree divisible by d converges to 1 as $n \to \infty$.

Recall the theorem of Lassalle [4, Theorem 6], which defines a rational number A^λ_μ such that

$$\chi^\lambda_\mu = \frac{f_\lambda}{(n)_k} A^\lambda_\mu. \tag{1}$$

Here $(n)_k = n(n-1)\cdots(n-k+1)$, and f_λ is the degree of the irreducible character of S_n corresponding to λ.

In fact, A^λ_μ is an integer for all λ, μ. This is likely well known to experts, but for the convenience of the reader we sketch a proof in Section 4.

From here, in order to prove the main theorem, we focus on the divisibility properties of f_λ. For each prime number q, let $v_q(m)$ denote the q-adic valuation of an integer m, in other words, $q^{v_q(m)}$ is the largest power of q that divides m. The main theorem will follow from the following result:

Theorem A. For every prime number q and non-negative integer m,

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid v_q(f_\lambda) \leqslant m + (q-1)\log_q n\}}{p(n)} = 0.$$

In the rest of this article, we first prove Theorem A, next show that it implies the main theorem, and then explain the integrality of A^λ_μ.

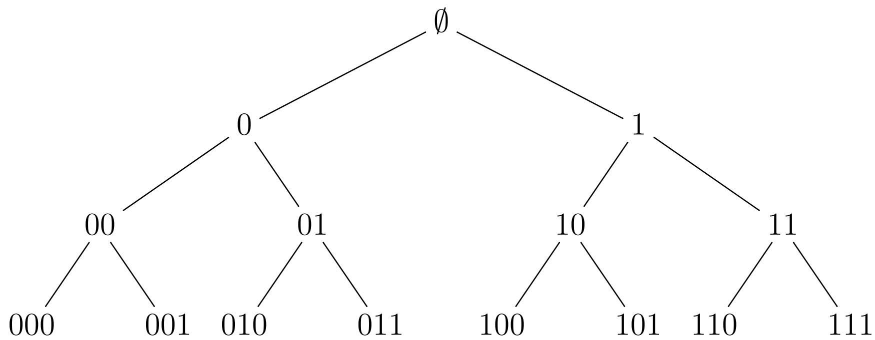
2 Proof of Theorem A

The proof of Theorem A is based on the theory of q-core towers. This construction originated in the seminal paper [5] of Macdonald, and was developed further by Olsson in [6]. We now recall the relevant aspects.

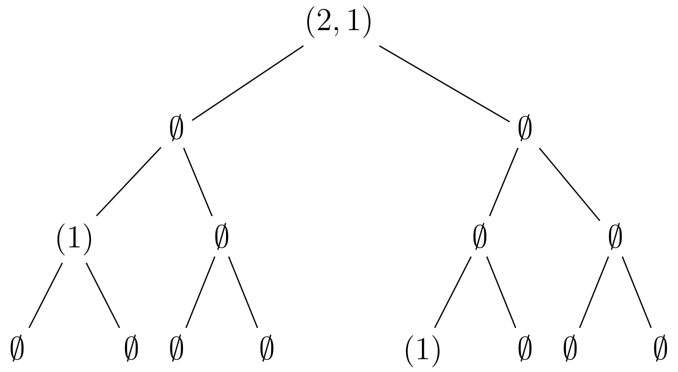
Let $[q]$ denote the set $\{0, \dots, q-1\}$. Consider the disjoint union

$$T_q = \coprod_{i=0}^{\infty} [q]^i = \{(a_1, \dots, a_i) \mid i \in \mathbf{N},\ a_i \in [q]\} \cup \{\emptyset\},$$

which can be regarded as a rooted q-ary tree with root $\emptyset$ as follows. The children of a vertex $(a_1, \dots, a_i) \in [q]^i$ are the vertices $(a_1, \dots, a_i, a_{i+1})$, where $a_{i+1} \in [q]$. For $q = 2$, rows 0 to 3 of this tree are as below. For compactness, commas and parentheses have been omitted.



A partition λ is said to be a q-core if no cell in its Young diagram has hook length divisible by q. Denote the set of all q-core partitions by C_q. The q-core tower construction associates to each partition λ of n a function $\mathcal{T}_q^\lambda : T_q \to C_q$ known as the q-core tower of λ (see [6, pages 29–30]). In particular, $\mathcal{T}_q^\lambda(\emptyset)$ is the q-core of λ (the partition obtained by removing all the q-hooks from the Young diagram of λ). This function is visualized as the rooted q-ary tree T_q with each vertex x of T_q replaced by the partition $\mathcal{T}_q^\lambda(x) \in C_q$. For example, the 2-core tower of $\lambda = (4, 3, 2, 2, 2, 2)$ is:



For a function $\mathcal{T}_q : T_q \to C_q$, define:

$$w_i(\mathcal{T}_q) = \sum_{x \in [q]^i} |\mathcal{T}_q(x)|.$$

Then the q-core tower satisfies the following constraint:

$$\sum_{i=0}^{\infty} w_i(\mathcal{T}_q^\lambda) q^i = n. \tag{2}$$

In particular, $\mathcal{T}_q^\lambda(x) = \emptyset$ for all $i > \log_q n$. This function $\lambda \mapsto \mathcal{T}_q^\lambda$ is a bijection from the set of partitions of n onto the set of all functions $\mathcal{T} : T_q \to C_q$ satisfying the condition

$$\sum_{i=0}^{\infty} w_i(\mathcal{T}) q^i = n.$$

Define the weight of a q-core tower $\mathcal{T}_q$ as:

$$w(\mathcal{T}_q) = \sum_{i=0}^{\infty} w_i(\mathcal{T}_q).$$

For a partition λ, define $w(\lambda) = w(\mathcal{T}_q^\lambda)$.

Let n be a positive integer with q-ary expansion:

$$n = a_0 + a_1 q + \cdots + a_r q^r, \text{ with } a_i \in [q] \text{ for } i = 1, \ldots, r, \text{ and } a_r > 0. \tag{$*$}$$

Define $a(n) = \sum_{i=0}^{r} a_i$.

Recall the following Theorem:

Theorem 1 ([5, Equation (3.3)]). *For any partition* λ *of* n *and any prime* q,

$$v_q(f_\lambda) = \frac{w(\lambda) - a(n)}{q-1}.$$

For example when $\lambda = (4, 3, 2, 2, 2, 2)$, a partition of 15, and $q = 2$, from the 2-core tower of λ computed above, $w_1(\lambda) = 3$, $w_2(\lambda) = 0$, $w_3(\lambda) = w_4(\lambda) = 1$ so that $w(\lambda) = 5$. On the other hand $a(15) = 4$. Therefore $v(f_\lambda) = w(\lambda) - a(n) = 1$.

Theorem 1 can be used to find constraints on partitions with small values of $v_q(f_\lambda)$. Suppose that $v_q(f_\lambda) \leqslant b$. By Theorem 1, this is equivalent to

$$w(\lambda) \leqslant a(n) + b(q-1).$$

The expansion $(*)$ implies that $r \leqslant \log_q n < r+1$, so that $a(n) \leqslant (q-1)(r+1) \leqslant (q-1)(\log_q n + 1)$. So if $v_q(f_\lambda) \leqslant b$, then

$$w(\lambda) \leqslant (q-1)(\log_q n + 1 + b).$$

Thus an upper bound for the number $p_b(n)$ of partitions λ of n such that $v_q(f_\lambda) \leqslant b$ can be obtained by counting the number of q-core towers with weight $(q-1)(\log_q n + 1 + b)$ or less. The total number of vertices in the first $r+1$ rows of T_q, i.e., in $\coprod_{i=0}^{r} [q]^i$, is:

$$1 + q + \cdots + q^r = \frac{q^{r+1} - 1}{q-1} < qn,$$

since $q^r \leqslant n$. Let $c_q(n)$ denote the number of q-core partitions of n. Set $N_b = (q-1)(\log_q n + b + 1)$. Let $\tilde{c}_q(n)$ denote $\max\{c_q(i) \mid 1 \leqslant i \leqslant n\}$. There are $\binom{w+N-1}{w}$ ways to distribute the weight w into N vertices of T_q. Thus

$$\begin{aligned} p_b(n) &\leqslant \tilde{c}_q(N_b)^{N_b} \binom{qn + N_b}{N_b} \\ &\leqslant \tilde{c}_q(N_b)^{N_b} (qn + N_b)^{N_b} \end{aligned}$$

It is known that, for every integer q, there exists a polynomial $f_q(n)$ such that $\tilde{c}_q(n) \leqslant f_q(n)$ for all $n \geqslant 0$. Indeed, for $q = 2$, it is well-known that $c_2(n) \leqslant 1$, and for $q = 3$, using a formula of Granville and Ono [2, Section 3, p. 340], $c_3(n) \leqslant 3n + 1$. For $q \geqslant 4$, the existence of $f_q(n)$ follows from Anderson [1, Corollary 7].

We get:

$$p_b(n) \leqslant f_q(N_b)^{N_b} (qn + N_b)^{N_b},$$

whence

$$\log_q p_b(n) \leqslant N_b[\log_q f_q(N_b) + \log_q(qn + N_b)].$$

Taking $b = m + (q-1)\log_q n$ gives $N_b = (q-1)(q \log_q n + m + 1)$. Thus $\log_q p_b(n) = o(n^\epsilon)$ for every $\epsilon > 0$. On the other hand, the Hardy-Ramanujan asymptotic [3] for $p(n)$ implies that $\log_q p(n)$ grows faster than $n^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$. Thus Theorem A follows.

3 Proof of the Main Theorem

The identity (1) implies that

$$v_q(\chi_\mu^\lambda) \geqslant v_q(f_\lambda) - v_q((n)_k).$$

Using Legendre's formula on the valuation of a factorial, that $v_q(n!) = \frac{n-a(n)}{q-1}$, we have:

$$v_q((n)_k) = v_q\left(\frac{n!}{(n-k)!}\right) = \frac{k + a(n-k) - a(n)}{q-1} \leqslant k + (q-1)\log_q n.$$

Hence if $v_q(f_\lambda) \geqslant m + (q-1)\log_q n$, then $v_q(\chi_\mu^\lambda) \geqslant (m-k)$. Thus taking $m = k+b$ in Theorem A tells us that

$$\lim_{n\to\infty} \frac{\#\{\lambda \vdash n \mid v_p(\chi_\mu^\lambda) \leqslant b\}}{p(n)} = 0.$$

From this the main theorem follows.

4 Integrality of A_μ^λ

For each partition $\mu = (\mu_1, \ldots, \mu_m)$, with m positive parts, the constant A_μ^λ from (1) is given in [4, Theorem 6] as

$$A_\mu^\lambda = \sum_\varepsilon \sum_{(i_1,\ldots,i_m)} A_{i_1,\ldots,i_m}^{(\varepsilon)}(\mu) \prod_k c_{i_k}^\lambda(\mu_k). \tag{3}$$

The ε in the first sum runs over "upper triangular matrices" $\varepsilon = (\varepsilon_{ij})$ for $1 \leqslant i < j \leqslant n$ with $\varepsilon_{ij} \in \{0, 2\}$. The second sum runs over r-tuples of nonnegative integers. The quantities $c_i^\lambda(q)$, defined for nonnegative integers i, q and partitions λ, are certain rational numbers. Their "boundary values" are $c_0^\lambda(q) = -1/q$, and $c_i^\emptyset(q) = 0$ for $i > 0$. However, $c_i^\lambda(q)$ are integers for $i > 0$, which may be seen recursively by Lemma 1 on page 396 of [4]. The quantities $A_{i_1,\ldots,i_m}^{(\varepsilon)}(\mu)$ are defined in Theorem 6 on page 399 by an intricate formula.

For a given ε and $i_1, \ldots, i_m$, we argue below that each of the terms

$$A_{i_1,\ldots,i_m}^{(\varepsilon)}(\mu) \cdot \prod_k c_{i_k}^\lambda(\mu_k),$$

from (3) are integers. From this it follows that A_μ^λ is an integer.

Lemma 2. *If there exists a k with*

1. $\varepsilon_{\ell k} = 0$ *for all* $\ell < k$,

2. $\varepsilon_{k\ell} = 0$ *for all* $k < \ell$, *and*

3. $i_k = 0$,

then

$$A^{(\varepsilon)}_{i_1,\ldots,i_m}(\mu_1, \ldots, \mu_m) = 0.$$

Proof. In the definition of $A^{(\varepsilon)}_{i_1,\ldots,i_m}(\mu_1, \ldots, \mu_m)$ in Theorem 6 of Lassalle, it is expressed as a sum over certain a, b of products, including a product over certain Stirling numbers. The Stirling number corresponding to k as above is

$$s(\mu_k + 1, 0) = 0,$$

since by the given convention the a's and b's are 0. □

Let $Z = \{k : i_k = 0\}$. The product $\prod_k c^\lambda_{i_k}(\mu_k)$ is an integer multiple of $\prod_{k\in Z} \mu_k^{-1}$, and $A^{(\varepsilon)}_{i_1,\ldots,i_m}(\mu)$ is an integer multiple of

$$\prod_{i<j} \theta_{ij} = \prod_{i<j \text{ and } \varepsilon_{ij}\neq 0} \mu_i\mu_j.$$

Therefore $A^{(\varepsilon)}_{i_1,\ldots,i_m}(\mu) \cdot \prod_k c^\lambda_{i_k}(\mu_k)$ is an integer, unless for some $k \in Z$ we have $\varepsilon_{ik} = 0$ for all $i < k$ and $\varepsilon_{kj} = 0$ for all $k < j$. But then the product anyway vanishes by Lemma 2.

Since A^λ_μ is the sum over the tuples $(i_1, \ldots, i_m)$ and the (ε) of the $A^{(\varepsilon)}_{i_1,\ldots,i_m}(\mu)\cdot\prod_k c^\lambda_{i_k}(\mu_k)$, it too must be an integer.

Acknowledgements

The authors thank the referee for several helpful remarks, which helped improve the readability of this article.

References

[1] J. Anderson. An asymptotic formula for the t-core partition function and a conjecture of Stanton. *J. Number Theory*, 128(9):2591–2615, 2008.

[2] A. Granville and K. Ono. Defect zero p-blocks for finite simple groups. *Trans. Amer. Math. Soc.*, 348(1):331–347, 1996.

[3] G. H. Hardy and S. Ramanujan. Asymptotic formulae in combinatory analysis. *Proc. London Math. Soc. (2)*, 17(1):75–115, 1918.

[4] M. Lassalle. An explicit formula for the characters of the symmetric group. *Math. Ann.*, 340(2):383–405, 2008.

[5] I. G. Macdonald. On the degrees of the irreducible representations of symmetric groups. *Bull. London Math. Soc.*, 3:189–192, 1971.

[6] J. B. Olsson. McKay numbers and heights of characters. *Math. Scand.*, 38(1):25–42, 1976.