On the Divisibility of Character Values of the Symmetric Group

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Abstract

Fix a partition $\mu = (\mu_1, \ldots, \mu_m)$ of an integer $k$ and positive integer $d$. For each $n \geq k$, let $\chi_\mu^\lambda$ denote the value of the irreducible character $\chi^\lambda$ of $S_n$, corresponding to a partition $\lambda$ of $n$, at a permutation with cycle type $(\mu_1, \ldots, \mu_m, 1^{n-k})$. We show that the proportion of partitions $\lambda$ of $n$ such that $\chi_\mu^\lambda$ is divisible by $d$ approaches 1 as $n$ approaches infinity.

Mathematics Subject Classifications: 20C30, 05A16, 05A17

1 Introduction

Let $k$ be a positive integer, and $\mu = (\mu_1, \ldots, \mu_m)$ a partition of $k$. For a partition $\lambda$ of an integer $n \geq k$, let $\chi_\mu^\lambda$ denote the value of the irreducible character of $S_n$ corresponding to $\lambda$ at an element with cycle type $(\mu_1, \ldots, \mu_m, 1^{n-k})$. The purpose of this article is to prove:

Main Theorem. For any positive integers $k$ and $d$, and any partition $\mu$ of $k$,

$$\lim_{n \to \infty} \frac{#\{\lambda \vdash n \mid \chi_\mu^\lambda \text{ is divisible by } d\}}{p(n)} = 1.$$ 

Here $p(n)$ denotes the number of partitions of $n$.

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In particular, for any integer \( d \), the probability that an irreducible character of \( S_n \) has degree divisible by \( d \) converges to 1 as \( n \to \infty \).

Recall the theorem of Lassalle [4, Theorem 6], which defines a rational number \( A^\lambda_\mu \) such that
\[
\chi^\lambda_\mu = \frac{f_\lambda}{(n)^k} A^\lambda_\mu.
\]
Here \( (n)^k = n(n-1) \cdots (n-k+1) \), and \( f_\lambda \) is the degree of the irreducible character of \( S_n \) corresponding to \( \lambda \).

In fact, \( A^\lambda_\mu \) is an integer for all \( \lambda, \mu \). This is likely well known to experts, but for the convenience of the reader we sketch a proof in Section 4.

From here, in order to prove the main theorem, we focus on the divisibility properties of \( f_\lambda \). For each prime number \( q \), let \( v_q(m) \) denote the \( q \)-adic valuation of an integer \( m \), in other words, \( q^{v_q(m)} \) is the largest power of \( q \) that divides \( m \). The main theorem will follow from the following result:

**Theorem A.** For every prime number \( q \) and non-negative integer \( m \),
\[
\lim_{n \to \infty} \frac{\# \{ \lambda \vdash n \mid v_q(f_\lambda) \leq m + (q - 1) \log_q n \}}{p(n)} = 0.
\]

In the rest of this article, we first prove Theorem A, next show that it implies the main theorem, and then explain the integrality of \( A^\lambda_\mu \).

## 2 Proof of Theorem A

The proof of Theorem A is based on the theory of \( q \)-core towers. This construction originated in the seminal paper [5] of Macdonald, and was developed further by Olsson in [6]. We now recall the relevant aspects.

Let \( [q] \) denote the set \( \{0, \ldots, q-1\} \). Consider the disjoint union
\[
T_q = \prod_{i=0}^{\infty} [q]^i = \{ (a_1, \ldots, a_i) \mid i \in \mathbb{N}, a_i \in [q] \} \cup \{\emptyset\},
\]
which can be regarded as a rooted \( q \)-ary tree with root \( \emptyset \) as follows. The children of a vertex \( (a_1, \ldots, a_i) \in [q]^i \) are the vertices \( (a_1, \ldots, a_i, a_{i+1}) \), where \( a_{i+1} \in [q] \). For \( q = 2 \), rows 0 to 3 of this tree are as below. For compactness, commas and parentheses have been omitted.

```
    0
   / \  \
 00  01 10  11
|
|   101 110 111
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A partition $\lambda$ is said to be a $q$-core if no cell in its Young diagram has hook length divisible by $q$. Denote the set of all $q$-core partitions by $C_q$. The $q$-core tower construction associates to each partition $\lambda$ of $n$ a function $T^\lambda_q : T_q \to C_q$ known as the $q$-core tower of $\lambda$ (see [6, pages 29–30]). In particular, $T^\lambda_q(\emptyset)$ is the $q$-core of $\lambda$ (the partition obtained by removing all the $q$-hooks from the Young diagram of $\lambda$). This function is visualized as the rooted $q$-ary tree $T_q$ with each vertex $x$ of $T_q$ replaced by the partition $T^\lambda_q(x) \in C_q$. For example, the 2-core tower of $\lambda = (4, 3, 2, 2, 2)$ is:

For a function $T_q : T_q \to C_q$, define:

$$w_i(T_q) = \sum_{x \in [q]^i} |T_q(x)|.$$  

Then the $q$-core tower satisfies the following constraint:

$$\sum_{i=0}^{\infty} w_i(T^\lambda_q)q^i = n. \quad (2)$$

In particular, $T^\lambda_q(x) = \emptyset$ for all $i > \log_q n$. This function $\lambda \mapsto T^\lambda_q$ is a bijection from the set of partitions of $n$ onto the set of all functions $T : T_q \to C_q$ satisfying the condition

$$\sum_{i=0}^{\infty} w_i(T)q^i = n.$$  

Define the weight of a $q$-core tower $T_q$ as:

$$w(T_q) = \sum_{i=0}^{\infty} w_i(T_q).$$

For a partition $\lambda$, define $w(\lambda) = w(T^\lambda_q)$.

Let $n$ be a positive integer with $q$-ary expansion:

$$n = a_0 + a_1q + \cdots + a_rq^r, \text{ with } a_i \in [q] \text{ for } i = 1, \ldots, r, \text{ and } a_r > 0. \quad (*)$$

Define $a(n) = \sum_{i=0}^{r} a_i$.

Recall the following Theorem:
Theorem 1 ([5, Equation (3.3)]). For any partition \( \lambda \) of \( n \) and any prime \( q \),
\[
v_q(f_\lambda) = \frac{w(\lambda) - a(n)}{q - 1}.
\]

For example when \( \lambda = (4, 3, 2, 2, 2, 2) \), a partition of 15, and \( q = 2 \), from the 2-core tower of \( \lambda \) computed above, \( w_1(\lambda) = 3 \), \( w_2(\lambda) = 0 \), \( w_3(\lambda) = w_4(\lambda) = 1 \) so that \( w(\lambda) = 5 \).

On the other hand \( a(15) = 4 \). Therefore \( v(f_\lambda) = w(\lambda) - a(n) = 1 \).

Theorem 1 can be used to find constraints on partitions with small values of \( v_q(f_\lambda) \).
Suppose that \( v_q(f_\lambda) \leq b \). By Theorem 1, this is equivalent to
\[
w(\lambda) \leq a(n) + b(q - 1).
\]

The expansion (\( * \)) implies that \( r \leq \log_q n < r + 1 \), so that \( a(n) \leq (q - 1)(r + 1) \leq (q - 1)(\log_q n + 1) \). So if \( v_q(f_\lambda) \leq b \), then
\[
w(\lambda) \leq (q - 1)(\log_q n + 1 + b).
\]

Thus an upper bound for the number \( p_b(n) \) of partitions \( \lambda \) of \( n \) such that \( v_q(f_\lambda) \leq b \) can be obtained by counting the number of \( q \)-core towers with weight \( (q - 1)(\log_q n + 1 + b) \) or less. The total number of vertices in the first \( r + 1 \) rows of \( T_q \), i.e., in \( \prod_{i=0}[q]^i \), is:
\[
1 + q + \cdots + q^r = \frac{q^{r+1} - 1}{q - 1} < qn,
\]

since \( q^r \leq n \). Let \( c_q(n) \) denote the number of \( q \)-core partitions of \( n \). Set \( N_b = (q - 1)(\log_q n + b + 1) \). Let \( \tilde{c}_q(n) \) denote \( \max\{c_q(i) \mid 1 \leq i \leq n\} \). There are \( \binom{w + N - 1}{w} \) ways to distribute the weight \( w \) into \( N \) vertices of \( T_q \).

Thus
\[
p_b(n) \leq \tilde{c}_q(N_b)^{N_b} \left( qn + N_b \right)^{N_b} \leq \tilde{c}_q(N_b)^{N_b} (qn + N_b)^{N_b}
\]

It is known that, for every integer \( q \), there exists a polynomial \( f_q(n) \) such that \( \tilde{c}_q(n) \leq f_q(n) \) for all \( n \geq 0 \). Indeed, for \( q = 2 \), it is well-known that \( c_2(n) \leq 1 \), and for \( q = 3 \), using a formula of Granville and Ono [2, Section 3, p. 340], \( c_3(n) \leq 3n + 1 \). For \( q \geq 4 \), the existence of \( f_q(n) \) follows from Anderson [1, Corollary 7].

We get:
\[
p_b(n) \leq f_q(N_b)^{N_b} (qn + N_b)^{N_b},
\]

whence
\[
\log_q p_b(n) \leq N_b[\log_q f_q(N_b) + \log_q (qn + N_b)].
\]

Taking \( b = m + (q - 1)\log_q n \) gives \( N_b = (q - 1)(\log_q n + m + 1) \). Thus \( \log_q p_b(n) = o(n^\epsilon) \) for every \( \epsilon > 0 \). On the other hand, the Hardy-Ramanujan asymptotic [3] for \( p(n) \) implies that \( \log_q p(n) \) grows faster than \( n^{1/2 - \epsilon} \) for any \( \epsilon > 0 \). Thus Theorem A follows.
3 Proof of the Main Theorem

The identity (1) implies that
\[ v_q(\chi^\lambda) \geq v_q(f_\lambda) - v_q((n)_k). \]

Using Legendre’s formula on the valuation of a factorial, that \( v_q(n!) = \frac{n-a(n)}{q-1} \), we have:
\[ v_q((n)_k) = v_q\left(\frac{n!}{(n-k)!}\right) = \frac{k + a(n-k) - a(n)}{q-1} \leq k + (q-1) \log_q n. \]

Hence if \( v_q(f_\lambda) \geq m + (q-1) \log_q n \), then \( v_q(\chi^\lambda) \geq (m-k) \). Thus taking \( m = k + b \) in Theorem A tells us that
\[ \lim_{n \to \infty} \frac{\#\{\lambda \vdash n \mid v_p(\chi^\lambda) \leq b\}}{p(n)} = 0. \]

From this the main theorem follows.

4 Integrality of \( A^\lambda_\mu \)

For each partition \( \mu = (\mu_1, \ldots, \mu_m) \), with \( m \) positive parts, the constant \( A^\lambda_\mu \) from (1) is given in [4, Theorem 6] as
\[ A^\lambda_\mu = \sum_{\varepsilon} \sum_{(i_1, \ldots, i_m)} A^{(e)}_{i_1, \ldots, i_m}(\mu) \prod_k c^\lambda_{i_k}(\mu_k). \]  

The \( \varepsilon \) in the first sum runs over “upper triangular matrices” \( \varepsilon = (\varepsilon_{ij}) \) for \( 1 \leq i < j \leq n \) with \( \varepsilon_{ij} \in \{0, 2\} \). The second sum runs over \( r \)-tuples of nonnegative integers. The quantities \( c^\lambda_i(q) \), defined for nonnegative integers \( i, q \) and partitions \( \lambda \), are certain rational numbers. Their “boundary values” are \( c^\lambda_0(q) = -1/q \), and \( c^\lambda_i(q) = 0 \) for \( i > 0 \). However, \( c^\lambda_i(q) \) are integers for \( i > 0 \), which may be seen recursively by Lemma 1 on page 396 of [4].

The quantities \( A^{(e)}_{i_1, \ldots, i_m}(\mu) \) are defined in Theorem 6 on page 399 by an intricate formula.

For a given \( \varepsilon \) and \( i_1, \ldots, i_m \), we argue below that each of the terms
\[ A^{(e)}_{i_1, \ldots, i_m}(\mu) \cdot \prod_k c^\lambda_{i_k}(\mu_k), \]
from (3) are integers. From this it follows that \( A^\lambda_\mu \) is an integer.

Lemma 2. If there exists a \( k \) with
1. \( \varepsilon_{i_k} = 0 \) for all \( \ell < k \);
2. \( \varepsilon_{k\ell} = 0 \) for all \( k < \ell \), and
3. \( i_k = 0 \),

then

\[ A_{i_1, \ldots, i_m}^{(e)}(\mu_1, \ldots, \mu_m) = 0. \]

Proof. In the definition of \( A_{i_1, \ldots, i_m}^{(e)}(\mu_1, \ldots, \mu_m) \) in Theorem 6 of Lassalle, it is expressed as a sum over certain \( a, b \) of products, including a product over certain Stirling numbers. The Stirling number corresponding to \( k \) as above is

\[ s(\mu_k + 1, 0) = 0, \]

since by the given convention the \( a \)'s and \( b \)'s are 0. \( \square \)

Let \( Z = \{ k : i_k = 0 \} \). The product \( \prod_k c_{i_k}^\lambda(\mu_k) \) is an integer multiple of \( \prod_{k \in Z} \mu_k^{-1} \), and \( A_{i_1, \ldots, i_m}^{(e)}(\mu) \) is an integer multiple of

\[ \prod_{i<j} \theta_{ij} = \prod_{i<j \text{ and } \varepsilon_{ij} \neq 0} \mu_i \mu_j. \]

Therefore \( A_{i_1, \ldots, i_m}^{(e)}(\mu) \cdot \prod_k c_{i_k}^\lambda(\mu_k) \) is an integer, unless for some \( k \in Z \) we have \( \varepsilon_{ik} = 0 \) for all \( i < k \) and \( \varepsilon_{kj} = 0 \) for all \( k < j \). But then the product anyway vanishes by Lemma 2.

Since \( A_{\mu}^\lambda \) is the sum over the tuples \((i_1, \ldots, i_m)\) and the \((\varepsilon)\) of the \( A_{i_1, \ldots, i_m}^{(e)}(\mu) \cdot \prod_k c_{i_k}^\lambda(\mu_k) \), it too must be an integer.

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References


