Minimum supports of eigenfunctions with the second largest eigenvalue of the Star graph[*]

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Abstract

The Star graph S_n, $n \geqslant 3$, is the Cayley graph on the symmetric group Sym_n generated by the set of transpositions $\{(12), (13), \ldots, (1n)\}$. In this work we study eigenfunctions of S_n corresponding to the second largest eigenvalue $n-2$. For $n \geqslant 8$ and $n = 3$, we find the minimum cardinality of the support of an eigenfunction of S_n corresponding to the second largest eigenvalue and obtain a characterization of eigenfunctions with the minimum cardinality of the support.

Mathematics Subject Classifications: 05C50, 05C25, 05E15, 05B30

1   Introduction

The Star graph $S_n = Cay(Sym_n, S)$, $n \geqslant 3$, is the Cayley graph on the symmetric group Sym_n with the generating set $S = \{(1\ i) \mid i \in \{2, \ldots, n\}\}$. It is a connected bipartite $(n-1)$-regular graph of order $n!$ and diameter $diam(S_n) = \lfloor \frac{3(n-1)}{2} \rfloor$ [1]. Since this graph is bipartite it does not contain odd cycles but it does contain all even l-cycles where $l = 6, 8, \ldots, n!$ (with the sole exception when $l = 4$) [9] which means that S_n is hamiltonian.

The spectrum of the Star graph is integral [5, 12]. More precisely, for $n \geqslant 3$ and for each integer $1 \leqslant k \leqslant n - 1$, the values $\pm(n-k)$ are eigenvalues of S_n; if $n \geqslant 4$, then 0 is an eigenvalue of S_n. Let $\text{mul}(\lambda)$ denote the multiplicity of an eigenvalue λ. Since the Star graph is bipartite, $\text{mul}(n-k) = \text{mul}(-n+k)$ for each integer $1 \leqslant k \leqslant n$. Moreover, $\pm(n-1)$ are simple eigenvalues of S_n. A lower bound on multiplicities of the eigenvalues was found as $\binom{n-2}{k-1}$ [12], and it was improved as $\binom{n-2}{n-k-1}\binom{n-1}{n-k}$ [5] for any $1 \leqslant k \leqslant n - 1$. Later a method for getting explicit formulas for multiplicities of eigenvalues $\pm(n-k)$ in the Star graphs S_n was suggested [2, 10], and the behavior of the eigenvalues multiplicity function of the Star graph S_n for eigenvalues $\pm(n-k)$ where $1 \leqslant k \leqslant \frac{n+1}{2}$ was investigated [11]. It was shown that the function has a polynomial behavior on n. Computational results showed that the same polynomial behavior of the eigenvalues multiplicity function occurs for any integers $n \geqslant 2$ and $1 \leqslant k \leqslant n$. Moreover, explicit formulas for calculating multiplicities of eigenvalues $\pm(n-k)$ where $2 \leqslant k \leqslant 12$ were found. In particular, $\text{mul}(n-2) = \text{mul}(2-n) = (n-1)(n-2)$.

We investigate the following problem.

Problem 1. For a graph Γ and its eigenvalue λ to find the minimum cardinality of the support of a λ-eigenfunction of Γ.

In many cases Problem 1 is directly related to the problem of finding the minimum possible difference of two combinatorial objects and the problem of finding the minimum cardinality of the bitrades. In more details, these connections are described in [13, 14]. Problem 1 was studied for the bilinear forms graphs in [17], the cubical distance-regular graphs in [16], the Doob graphs in [3], the Grassmann graphs in [14], the Hamming graphs in [13, 15, 18, 19, 20, 21], the Johnson graphs in [22] and the Paley graphs in [7]. We note that Problem 1 is completely solved for all eigenvalues only for the Hamming graph and for the Johnson graph. Moreover, even for distance-regular graphs Problem 1 for the second largest eigenvalue is solved only for the Hamming graph, the Johnson graph, the Doob graph and the cubical distance-regular graphs.

In this work we start to investigate Problem 1 for graphs which are not distance-regular and consider Problem 1 for the Star graph S_n and its eigenvalue $n - 2$. We find the minimum cardinality of the support of $(n-2)$-eigenfunctions of S_n and give a characterization of $(n-2)$-eigenfunctions with the minimum cardinality of the support for $n \geqslant 8$ and $n = 3$. We also show that for $n \geqslant 8$ and $n = 3$ an arbitrary $(n-2)$-eigenfunction of S_n with the minimum cardinality of the support is the difference of the characteristic functions of two completely regular codes of covering radius 2.

The paper is organized as follows. In Section 2, we introduce basic definitions and give some preliminary results. In Section 3, we reduce Problem 1 for the Star graph S_n and its eigenvalue $n - 2$ to some extremal problem on the set of real $n \times n$ matrices. In Section 4, we solve this extremal problem. In Section 5, for $n \geqslant 8$ and $n = 3$ we prove that the minimum cardinality of the support of an $(n - 2)$-eigenfunction of S_n is $2(n - 1)!$ and give a characterization of eigenfunctions with the minimum cardinality of the support. In Section 6, for $n \geqslant 8$ and $n = 3$ we show that an arbitrary $(n - 2)$-eigenfunction of S_n with the minimum cardinality of the support is the difference of the characteristic functions of two completely regular codes of covering radius 2.

2 Preliminaries

2.1 Star graph

Let G be a group and S be an inverse-closed identity-free generating set in G. The *Cayley graph on G with the generating set S* (denoted by $Cay(G, S)$) is the graph whose vertices are the elements of G, and any two elements $x, y \in G$ are adjacent in $Cay(G, S)$ whenever $xy^{-1} \in S$. For a positive integer $n \geqslant 3$, the *Star graph* (denoted by S_n) is the Cayley graph on the symmetric group Sym_n with the generating set $S = \{(1\ i) \mid i \in \{2, \ldots, n\}\}$.

2.2 Eigenfunctions of graphs

Let $\Gamma = (V, E)$ be a graph with the adjacency matrix $A(\Gamma)$. The set of neighbors of a vertex x is denoted by $N(x)$. Let λ be an eigenvalue of the matrix $A(\Gamma)$. A function $f : V \longrightarrow \mathbb{R}$ is called a λ-*eigenfunction* of Γ if $f \not\equiv 0$ and the equality

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

holds for any $x \in V$. Note that the vector of values of a λ-eigenfunction is an eigenvector of $A(\Gamma)$ with eigenvalue λ. The *support* of a function $f : V \longrightarrow \mathbb{R}$ is the set $Supp(f) = \{x \in V \mid f(x) \neq 0\}$. For a function $f : V \longrightarrow \mathbb{R}$ denote $E(f) = \{y \in \mathbb{R} \mid y = f(x), x \in V\}$.

Let $u \in \{1, \ldots, n\}$ and $v, w \in \{2, \ldots, n\}$, where $v \neq w$. We define the function $f_u^{v,w} : Sym_n \longrightarrow \mathbb{R}$ by the following rule:

$$f_u^{v,w}(\pi) = \begin{cases} 1, & \text{if } \pi(v) = u; \\ -1, & \text{if } \pi(w) = u; \\ 0, & \text{otherwise.} \end{cases}$$

The following result is a particular case of Proposition 1 proved in [8].

Lemma 2. *Let $u \in \{1, \ldots, n\}$ and $v, w \in \{2, \ldots, n\}$, where $v \neq w$. Then for $n \geqslant 3$, the function $f_u^{v,w}$ is an $(n - 2)$-eigenfunction of S_n.*

Denote
$$\mathcal{F}_2 = \{f_u^{2,w} \mid u \in \{2, \ldots, n\}, w \in \{3, \ldots, n\}\}.$$
The following lemma was proved in [8].

Lemma 3 ([8], Lemma 15)**.** *For $n \geqslant 3$, the set $\mathcal{F}_2$ forms a basis of the eigenspace of S_n with eigenvalue $n - 2$.*

Denote
$$\mathcal{F} = \{f_u^{v,w} \mid u \in \{1, \ldots, n\}, v, w \in \{2, \ldots, n\}, v \neq w\}.$$
In Section 5 we prove that if f is an $(n-2)$-eigenfunction of S_n, then $|Supp(f)| \geqslant 2(n-1)!$. Moreover, we prove that $|Supp(f)| = 2(n-1)!$ if and only if $f = c \cdot \tilde{f}$, where c is a constant and $\tilde{f} \in \mathcal{F}$.

2.3 Matrices

Let $M = (m_{i,j})$ be a real $n \times n$ matrix. We say that M is *special* if M is non-zero and the following conditions hold:

1. $m_{i,1} = 0$ for any $i \in \{1, \ldots, n\}$.

2. $m_{1,j} = 0$ for any $j \in \{1, \ldots, n\}$.

3. $\sum_{j=1}^{n} m_{i,j} = 0$ for any $i \in \{1, \ldots, n\}$.

Example 4. The matrix
$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & -1 & -2 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$
is special.

Let $M = (m_{i,j})$ be a real $n \times n$ matrix and let X be a subset of Sym_n. Denote
$$g_M(n) = |\{\pi \in Sym_n \mid \sum_{i=1}^{n} m_{i,\pi(i)} \neq 0\}|$$
and
$$g_M(X) = |\{\pi \in X \mid \sum_{i=1}^{n} m_{i,\pi(i)} \neq 0\}|.$$

For an $n \times n$ matrix M and $\alpha \in \{1, \ldots, n\}$ denote by R_α the α-th row of M.

2.4 Equitable partitions and completely regular codes

Let $\Gamma = (V, E)$ be a graph. An ordered partition $(C_1, \ldots, C_r)$ of V is called *equitable* if for any $i, j \in \{1, \ldots, r\}$ there is $S_{i,j}$ such that any vertex of C_i has exactly $S_{i,j}$ neighbors in C_j. The matrix $S = (S_{i,j})_{i,j \in \{1,\ldots,r\}}$ is called the *quotient matrix* of the equitable partition. A set $C \subseteq V$ is called a *completely regular code* in Γ if the partition $(C^{(0)}, \ldots, C^{(\rho)})$ is equitable, where $C^{(d)}$ is the set of vertices at distance d from C and ρ (*the covering radius of C*) is the maximum d for which $C^{(d)}$ is nonempty. In other words, a subset of V is a completely regular code in Γ if the distance partition with respect to the subset is equitable. For more information on equitable partitions and completely regular codes see [4, 6].

3 Correspondence between the values of an $(n-2)$-eigenfunction of S_n and the diagonals of an $n \times n$ matrix

In this section, for an arbitrary $(n-2)$-eigenfunction f of S_n, we introduce a special $n \times n$ matrix $M(f)$ and match the permutations from Sym_n with diagonals of $M(f)$ in such a way that the value of f on a permutation π is the sum of elements of the corresponding diagonal of $M(f)$.

Let f be an $(n-2)$-eigenfunction of S_n. By Lemma 3, there exist the numbers $\mu_i^j(f) \in \mathbb{R}$, where $i \in \{2, \ldots, n\}$ and $j \in \{3, \ldots, n\}$, such that

$$f = \sum_{\substack{j \in \{3,\ldots,n\} \\ i \in \{2,\ldots,n\}}} \mu_i^j(f) \cdot f_i^{2,j}.$$

We define the matrix $M(f) = (m_{i,j}(f))_{i,j \in \{1,\ldots,n\}}$ by the following rule:

$$m_{i,j}(f) = \begin{cases} -\mu_i^j(f), & \text{if } i > 1 \text{ and } j > 2; \\ \sum_{s=3}^{n} \mu_i^s(f), & \text{if } i > 1 \text{ and } j = 2; \\ 0, & \text{if } i = 1 \text{ or } j = 1. \end{cases} \tag{1}$$

Lemma 5. *Let f be an $(n-2)$-eigenfunction of S_n. Then*

$$f(\pi) = \sum_{i=1}^{n} m_{i,\pi^{-1}(i)}(f)$$

for any $\pi \in Sym_n$.

Proof. For $i \in \{2, \ldots, n\}$ denote

$$f_i = \sum_{j=3}^{n} \mu_i^j(f) \cdot f_i^{2,j}.$$

By the definition of $f_u^{v,w}$ we have

$$f_i(\pi) = \begin{cases} -\mu_i^j(f), & \text{if } \pi(j) = i \text{ and } j \in \{3,\dots,n\}; \\ \sum_{s=3}^n \mu_i^s(f), & \text{if } \pi(2) = i; \\ 0, & \text{if } \pi(1) = i. \end{cases} \qquad (2)$$

Using the equalities (1) and (2), we see that $f_i(\pi) = m_{i,\pi^{-1}(i)}(f)$ for any $i \in \{2,\dots,n\}$ and $\pi \in Sym_n$. Then

$$f(\pi) = \sum_{i=2}^n f_i(\pi) = \sum_{i=2}^n m_{i,\pi^{-1}(i)}(f)$$

for any $\pi \in Sym_n$. Since $m_{1,\pi^{-1}(1)}(f) = 0$ due to the definition of $M(f)$, we obtain that

$$f(\pi) = \sum_{i=1}^n m_{i,\pi^{-1}(i)}(f). \qquad \qquad \square$$

Lemma 6. *Let f be an $(n-2)$-eigenfunction of S_n. Then*

$$E(f) = \left\{ \sum_{i=1}^n m_{i,\pi(i)}(f) \mid \pi \in Sym_n \right\}.$$

Proof. We note that

$$\left\{ \sum_{i=1}^n m_{i,\pi(i)}(f) \mid \pi \in Sym_n \right\} = \left\{ \sum_{i=1}^n m_{i,\pi^{-1}(i)}(f) \mid \pi \in Sym_n \right\}.$$

Then by Lemma 5 we have

$$E(f) = \left\{ \sum_{i=1}^n m_{i,\pi(i)}(f) \mid \pi \in Sym_n \right\}. \qquad \qquad \square$$

Using Lemma 6 and the definition of $g_M(n)$, we immediately obtain the following result.

Lemma 7. *Let f be an $(n-2)$-eigenfunction of S_n. Then $|Supp(f)| = g_{M(f)}(n)$.*

4   Extremal problem on the set of all special $n \times n$ matrices

In view of Lemma 7, in order to solve Problem 1 for the Star graph S_n and its eigenvalue $n - 2$, it suffices to find the minimum value of $g_{M(f)}(n)$, where n is fixed and f is an arbitary $(n-2)$-eigenfunction of S_n. Since the matrix $M(f)$ is special, in this section we focus on the following extremal problem formulated for the class of special matrices.

Problem 8. Given a positive integer n, to find the minimum value of $g_M(n)$ for the class of special $n \times n$ matrices M.

In this section, we solve Problem 8 (see Theorem 1) and prove that $g_M(n) \geqslant 2(n-1)!$ holds for any special $n \times n$ matrix M with $n \geqslant 8$ or $n = 3$. We then prove that this bound is tight and classify the special matrices in the equality case. This finally leads to a solution of Problem 1 for the Star graph S_n and its eigenvalue $n - 2$ (see Theorem 20).

Let $M = (m_{i,j})$ be a real $n \times n$ matrix and let $(A_1, \ldots, A_t)$ be a partition of $\{1, \ldots, n\}$, where $t \geqslant 2$. Let $\alpha, \beta \in \{1, \ldots, n\}$ and $\alpha \neq \beta$. We say that R_α and R_β have the $(A_1, \ldots, A_t)$-*property* if for any $k, m \in \{1, \ldots, t\}$, $k \neq m$, and for any $a \in A_k$, $b \in A_m$ the condition

$$m_{\alpha,a} + m_{\beta,b} \neq m_{\alpha,b} + m_{\beta,a}$$

holds.

Example 9. Let us consider the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix}.$$

1. The rows R_1 and R_2 have (A_1, A_2, A_3)-property, where $A_1 = \{1, 4\}$, $A_2 = \{2\}$ and $A_3 = \{3\}$.

2. The rows R_1 and R_3 have (A_1, A_2)-property, where $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$.

3. The rows R_3 and R_4 have (A_1, A_2)-property, where $A_1 = \{1, 2\}$ and $A_2 = \{3, 4\}$.

In Lemmas 10 and 11, we obtain lower bounds for $g_M(n)$, where M is an arbitrary real matrix having a pair of rows with the $(A_1, \ldots, A_t)$-property.

Let $\{a_1, \ldots, a_\ell\}$ and $\{\alpha_1, \ldots, \alpha_\ell\}$ be subsets of $\{1, \ldots, n\}$ for some $1 \leqslant \ell \leqslant n$. Denote by $S^{\alpha_1, \ldots, \alpha_\ell}_{a_1, \ldots, a_\ell}$ the set of permutations $\pi \in Sym_n$ such that $\pi(\alpha_i) = a_i$ for any $i \in \{1, \ldots, \ell\}$.

Lemma 10. *Let $M = (m_{i,j})$ be a real $n \times n$ matrix. Let $\{c_1, \ldots, c_h, a, b\}$, $\{\gamma_1, \ldots, \gamma_h, \alpha, \beta\}$ be subsets of $\{1, \ldots, n\}$ for some $0 \leqslant h \leqslant n-2$. Suppose R_α and R_β have the $(A_1, \ldots, A_t)$-property and $a \in A_k$, $b \in A_m$ for some $k, m \in \{1, \ldots, t\}$ and $k \neq m$. Then*

$$g_M\left(S^{\gamma_1, \ldots, \gamma_h, \alpha, \beta}_{c_1, \ldots, c_h, a, b} \cup S^{\gamma_1, \ldots, \gamma_h, \alpha, \beta}_{c_1, \ldots, c_h, b, a}\right) \geqslant (n - h - 2)!.$$

Proof. Denote $Y_1 = S^{\gamma_1, \ldots, \gamma_h, \alpha, \beta}_{c_1, \ldots, c_h, a, b}$ and $Y_2 = S^{\gamma_1, \ldots, \gamma_h, \alpha, \beta}_{c_1, \ldots, c_h, b, a}$. We note that for any permutation $\pi \in Y_1$ there is a unique permutation $\pi' \in Y_2$ such that $\pi(s) = \pi'(s)$ for any $s \in \{1, 2, \ldots, n\} \setminus \{\alpha, \beta\}$. Then

$$\sum_{i=1}^{n} m_{i,\pi(i)} - \sum_{i=1}^{n} m_{i,\pi'(i)} = m_{\alpha,a} + m_{\beta,b} - m_{\alpha,b} - m_{\beta,a}$$

for any $\pi \in Y_1$. Since R_α and R_β have the $(A_1, \ldots, A_t)$-property, $a \in A_k$ and $b \in A_m$, we have

$$\sum_{i=1}^{n} m_{i,\pi(i)} \neq \sum_{i=1}^{n} m_{i,\pi'(i)}.$$

Therefore, $\sum_{i=1}^{n} m_{i,\pi(i)} \neq 0$ or $\sum_{i=1}^{n} m_{i,\pi'(i)} \neq 0$ for any $\pi \in Y_1$. Using the equality

$$|Y_1| = |Y_2| = (n - h - 2)!,$$

we obtain

$$g_M(Y_1 \cup Y_2) \geqslant (n - h - 2)!. \qquad \square$$

Lemma 11. *Let M be a real $n \times n$ matrix, $\alpha, \beta \in \{1, \ldots, n\}$ and $\alpha \neq \beta$. Suppose R_α and R_β have the $(A_1, \ldots, A_t)$-property. Then*

$$g_M(n) \geqslant \left(\sum_{1 \leqslant k < m \leqslant t} |A_k| \cdot |A_m| \right) \cdot (n - 2)!.$$

Proof. Let $k, m \in \{1, \ldots, t\}$ and $k < m$. Let us consider arbitrary $a \in A_k$ and $b \in A_m$. Lemma 10 implies that $g_M(S_{a,b}^{\alpha,\beta} \cup S_{b,a}^{\alpha,\beta}) \geqslant (n - 2)!$. Denote

$$X = \bigcup_{\substack{1 \leqslant k < m \leqslant t \\ a \in A_k, b \in A_m}} \left(S_{a,b}^{\alpha,\beta} \cup S_{b,a}^{\alpha,\beta} \right).$$

Then we have

$$g_M(X) = \sum_{\substack{1 \leqslant k < m \leqslant t \\ a \in A_k, b \in A_m}} g_M(S_{a,b}^{\alpha,\beta} \cup S_{b,a}^{\alpha,\beta}) \geqslant \left(\sum_{1 \leqslant k < m \leqslant t} |A_k| \cdot |A_m| \right) \cdot (n - 2)!.$$

Thus, we obtain

$$g_M(n) \geqslant g_M(X) \geqslant \left(\sum_{1 \leqslant k < m \leqslant t} |A_k| \cdot |A_m| \right) \cdot (n - 2)!. \qquad \square$$

Lemma 12. *Let $n = n_1 + \ldots + n_t$, where $n_i \in \mathbb{N}$ for any $i \in \{1, \ldots, t\}$, $n_1 \geqslant \ldots \geqslant n_t$, $n \geqslant 7$ and $t \geqslant 3$. Then either*

$$\sum_{1 \leqslant k < m \leqslant t} n_k n_m > 2(n - 1)$$

or

$$t = 3, \ n_1 = n - 2 \ and \ n_2 = n_3 = 1.$$

Proof. Denote

$$S = \sum_{1 \leqslant k < m \leqslant t} n_k n_m.$$

We consider three cases.

In the first case we suppose that $3 \leqslant n_1 \leqslant n - 3$. Then we have

$$S \geqslant n_1(n_2 + \ldots + n_t) + n_2 n_3 \geqslant 3(n - 3) + 1 > 2(n - 1).$$

In the second case we suppose that $n_1 \geqslant n - 2$. Since $t \geqslant 3$, it is possible only if $t = 3$, $n_1 = n - 2$ and $n_2 = n_3 = 1$.

In the third case we suppose that $n_1 \leqslant 2$. Let $i = |\{1 \leqslant k \leqslant t \mid n_k = 2\}|$.

If $i \geqslant 2$, then $n_1 = n_2 = 2$ and

$$S \geqslant n_1(n_2 + \ldots + n_t) + n_2(n_3 + \ldots + n_t) = 4n - 12 > 2(n - 1).$$

If $n - i \geqslant 3$, then $n_t = n_{t-1} = n_{t-2} = 1$ and

$$S \geqslant n_t(n_1 + \ldots + n_{t-1}) + n_{t-1}(n_1 + \ldots + n_{t-2}) + n_{t-2}(n_1 + \ldots + n_{t-3}) = 3n - 6 > 2(n - 1).$$

Since $n \geqslant 7$, $i \geqslant 2$ or $n - i \geqslant 3$ and we obtain the case considered above. $\qquad\square$

Let $M = (m_{i,j})$ be a real $n \times n$ matrix and $\alpha \in \{1, \ldots, n\}$. Let $x, y \in \mathbb{R}$ and $r_1, r_2, s \in \{2, \ldots, n\}$, where $x, y \neq 0$ and $r_1 \neq r_2$. We say that R_α is the (x, r_1, r_2)-row if $m_{\alpha, r_1} = x$, $m_{\alpha, r_2} = -x$ and $m_{\alpha, j} = 0$ for any $j \in \{1, \ldots, n\} \setminus \{r_1, r_2\}$. We say that R_α is the (y, s)-row if $m_{\alpha, s} = (n - 2)y$, $m_{\alpha, 1} = 0$ and $m_{\alpha, j} = -y$ for any $j \in \{1, \ldots, n\} \setminus \{1, s\}$.

Example 13. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 \end{pmatrix}.$$

Then R_2 is the $(1, 2, 4)$-row and R_4 is the $(2, 3)$-row.

Lemma 14. *Let M be a special $n \times n$ matrix, $g_M(n) \leqslant 2(n-1)!$, $n \geqslant 7$ and $\alpha \in \{1, \ldots, n\}$. Suppose R_α is a non-zero row. Then R_α is the (x, r_1, r_2)-row or the (y, s)-row.*

Proof. Suppose that R_α consists of the distinct elements $z_1, \ldots, z_t$ each of them z_k, where $1 \leqslant k \leqslant t$, occurs n_k times in R_α. Without loss of generality, we assume that $n_1 \geqslant \ldots \geqslant n_t$. If $t \leqslant 2$, then by the definition of special matrix we obtain that all elements of R_α are zeros. So, we can assume that $t \geqslant 3$.

For $k \in \{1, \ldots, t\}$ denote

$$A_k = \{j \in \{1, \ldots, n\} \mid m_{\alpha, j} = z_k\}.$$

We note that $|A_k| = n_k$ for any $k \in \{1, \ldots, t\}$. Since M is special, all elements of R_1 are zeros. Hence R_1 and R_α have $(A_1, \ldots, A_t)$-property. Lemma 11 implies that

$$g_M(n) \geqslant \left(\sum_{1 \leqslant k < m \leqslant t} n_k n_m \right) \cdot (n - 2)!.$$

On the other hand, we have $g_M(n) \leqslant 2(n - 1)!$. So

$$\sum_{1 \leqslant k < m \leqslant t} n_k n_m \leqslant 2(n - 1).$$

Recall that in the beginning of the proof we assumed $t \geqslant 3$. Then by Lemma 12 we obtain that $t = 3$, $n_1 = n - 2$ and $n_2 = n_3 = 1$. Therefore, by the definition of special matrix we obtain that R_α is the (x, r_1, r_2)-row or the (y, s)-row. $\qquad\square$

Lemma 15. *Let M be a special $n \times n$ matrix and $g_M(n) \leqslant 2(n-1)!$, where $n \geqslant 8$. Suppose $\alpha, \beta \in \{1, \ldots, n\}$, $\alpha \neq \beta$ and $R_\alpha \neq R_\beta$. Then there exists a partition (A, B) of $\{1, \ldots, n\}$ such that R_α and R_β have the (A, B)-property and $|A| = 2$.*

Proof. Firstly, we prove that R_α and R_β have the (A, B)-property, where $|A| \in \{2, 3, 4\}$. For a set $X \subseteq \{1, \ldots, n\}$ denote $\overline{X} = \{1, \ldots, n\} \setminus X$. By Lemma 14 we have four cases for R_α and R_β.

Case 1. Suppose R_α or R_β is a zero row. Without loss of generality, we assume that R_α is a zero row. Let us consider two subcases.

1.1. R_β is the (x, r_1, r_2)-row. Let $A = \{r_1, r_2\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

1.2. R_β is the (y, s)-row. Let $A = \{1, s\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

Case 2. Suppose R_α is the (x_1, r_1, r_2)-row and R_β is the (x_2, r_3, r_4)-row. Let us consider four subcases depending on $|\{r_1, r_2\} \cap \{r_3, r_4\}|$.

2.1. $\{r_1, r_2\} \cap \{r_3, r_4\} = \emptyset$. Let $A = \{r_1, r_2, r_3, r_4\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 4$.

2.2. $|\{r_1, r_2\} \cap \{r_3, r_4\}| = 1$ and $x_1 \neq x_2$. Without loss of generality, we assume that $r_1 = r_3$. Let $A = \{r_1, r_2, r_4\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 3$.

2.3. $|\{r_1, r_2\} \cap \{r_3, r_4\}| = 1$ and $x_1 = x_2$. Without loss of generality, we assume that $r_1 = r_3$. Let $A = \{r_2, r_4\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

2.4. $|\{r_1, r_2\} \cap \{r_3, r_4\}| = 2$. Without loss of generality, we assume that $r_1 = r_3$ and $r_2 = r_4$. Since $R_\alpha \neq R_\beta$, we have $x_1 \neq x_2$. Let $A = \{r_1, r_2\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

Case 3. Suppose R_α is the (y_1, s_1)-row and R_β is the (y_2, s_2)-row. Let us consider three subcases.

3.1. $s_1 \neq s_2$ and $y_1 \neq y_2$. Let $A = \{1, s_1, s_2\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 3$.

3.2. $s_1 \neq s_2$ and $y_1 = y_2$. Let $A = \{s_1, s_2\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

3.3. $s_1 = s_2$. Since $R_\alpha \neq R_\beta$, we have $y_1 \neq y_2$. Let $A = \{1, s_1\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

Case 4. Suppose R_α is the (x, r_1, r_2)-row and R_β is the (y, s)-row (the case when R_α is the (y, s)-row and R_β is the (x, r_1, r_2)-row is similar). Let us consider three subcases.

4.1. $s \notin \{r_1, r_2\}$. Let $A = \{1, r_1, r_2, s\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 4$.

4.2. $s \in \{r_1, r_2\}$ and $x \neq (n-1)y$. Without loss of generality, we assume that $s = r_1$. Let $A = \{1, r_1, r_2\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 3$.

4.3. $s \in \{r_1, r_2\}$ and $x = (n-1)y$. Without loss of generality, we assume that $s = r_1$. Let $A = \{1, r_2\}$ and $B = \overline{A}$. Then R_α and R_β have the (A, B)-property and $|A| = 2$.

Thus, we prove that R_α and R_β have the (A, B)-property, where $|A| \in \{2, 3, 4\}$. Hence by Lemma 11 we obtain that $g_M(n) \geqslant |A| \cdot |B| \cdot (n-2)!$. On the other hand, we have

$g_M(n) \leqslant 2(n-1)!$. Therefore $|A| \cdot |B| \leqslant 2(n-1)$. So, $|A| = 2$ and the cases when $|A| \in \{3,4\}$ do not hold. The lemma is proved. $\qquad \square$

Let M be an $n \times n$ matrix and $\theta \in \{1, \ldots, n\}$. We say that M is θ-*uniform* if $R_\alpha = R_\beta$ for any $\alpha, \beta \in \{1, \ldots, n\} \setminus \theta$.

Lemma 16. *Let M be a special $n \times n$ matrix and $g_M(n) \leqslant 2(n-1)!$, where $n \geqslant 8$ or $n = 3$. Then M is θ-uniform for some $\theta \in \{1, \ldots, n\}$.*

Proof. Firstly, let us consider the case $n = 3$. Since M is special, any non-zero row of M is the $(x, 2, 3)$-row. Hence either M is θ-uniform for some $\theta \in \{1, 2, 3\}$ or R_2 and R_3 are $(x_1, 2, 3)$-row and $(x_2, 2, 3)$-row, where $x_1 \neq x_2$. In the last case we have $g_M(n) = 6$ and we obtain a contradiction with $g_M(n) \leqslant 2(n-1)!$. In what follows, in this lemma we assume that $n \geqslant 8$.

Suppose that M is not θ-uniform for any $\theta \in \{1, \ldots, n\}$. Then there exist four rows R_α, R_β, R_γ and R_δ such that $R_\alpha \neq R_\beta$ and $R_\gamma \neq R_\delta$.

Since $R_\alpha \neq R_\beta$, by Lemma 15 we obtain that R_α and R_β have (A, B)-property, where $|A| = 2$. Let $a \in A$ and $b \in B$. Lemma 10 implies that $g_M(S_{a,b}^{\alpha,\beta} \cup S_{b,a}^{\alpha,\beta}) \geqslant (n-2)!$. Denote

$$X_1 = \bigcup_{a \in A, b \in B} (S_{a,b}^{\alpha,\beta} \cup S_{b,a}^{\alpha,\beta}).$$

Then we have

$$g_M(X_1) = \sum_{a \in A, b \in B} g_M(S_{a,b}^{\alpha,\beta} \cup S_{b,a}^{\alpha,\beta}) \geqslant |A| \cdot |B| \cdot (n-2)! = 2(n-2)(n-2)! \qquad (3)$$

Since $R_\gamma \neq R_\delta$, by Lemma 15 we obtain that R_γ and R_δ have (C, D)-property, where $|C| = 2$. Let $c \in C$ and $d \in D$. Let us consider arbitrary $b', b'' \in B \setminus \{c, d\}$, where $b' \neq b''$. Lemma 10 implies that

$$g_M\left(S_{b',b'',c,d}^{\alpha,\beta,\gamma,\delta} \cup S_{b',b'',d,c}^{\alpha,\beta,\gamma,\delta}\right) \geqslant (n-4)!.$$

Denote

$$X_2 = \bigcup_{\substack{c \in C, d \in D \\ b', b'' \in B \setminus \{c,d\}, b' \neq b''}} \left(S_{b',b'',c,d}^{\alpha,\beta,\gamma,\delta} \cup S_{b',b'',d,c}^{\alpha,\beta,\gamma,\delta}\right).$$

Then we have

$$\begin{aligned} g_M(X_2) = \sum_{\substack{c \in C, d \in D \\ b', b'' \in B \setminus \{c,d\}, b' \neq b''}} g_M\left(S_{b',b'',c,d}^{\alpha,\beta,\gamma,\delta} \cup S_{b',b'',d,c}^{\alpha,\beta,\gamma,\delta}\right) \geqslant \\ \geqslant |C| \cdot |D| \cdot (|B| - 2) \cdot (|B| - 3) \cdot (n-4)! = \\ = 2(n-2)(n-4)(n-5)(n-4)!. \end{aligned} \qquad (4)$$

Note that $X_1 \cap X_2 = \emptyset$. Therefore, $g_M(X_1 \cup X_2) = g_M(X_1) + g_M(X_2)$. Using (3) and (4), we obtain that $g_M(X_1 \cup X_2) > 2(n-1)!$. Thus, $g_M(n) \geqslant g_M(X_1 \cup X_2) > 2(n-1)!$ and we obtain a contradiction with $g_M(n) \leqslant 2(n-1)!$. $\qquad \square$

Let $x \in \mathbb{R}$, $x \neq 0$ and $p_1, p_2 \in \{2, \ldots, n\}$, $p_1 \neq p_2$. We say that an $n \times n$ matrix $M = (m_{i,j})$ is the (x, p_1, p_2)-*matrix* if

$$m_{i,j} = \begin{cases} x, & \text{if } j = p_1 \text{ and } i > 1; \\ -x, & \text{if } j = p_2 \text{ and } i > 1; \\ 0, & \text{otherwise.} \end{cases}$$

Example 17. The matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 3 & 0 & -3 \\ 0 & 3 & 0 & -3 \end{pmatrix}$$

is the $(3, 2, 4)$-matrix.

We say that an $n \times n$ matrix M belongs to the set $\mathcal{M}_1(n)$ if M is the (x, p_1, p_2)-matrix for some $x \in \mathbb{R}$, $x \neq 0$ and $p_1, p_2 \in \{2, \ldots, n\}$, $p_1 \neq p_2$.

Let $x \in \mathbb{R}$, $x \neq 0$ and $q_1, q_2, \tau \in \{2, \ldots, n\}$, $q_1 \neq q_2$. We say that an $n \times n$ matrix M is the (x, q_1, q_2, τ)-*matrix* if

$$m_{i,j} = \begin{cases} x, & \text{if } i = \tau \text{ and } j = q_1; \\ -x, & \text{if } i = \tau \text{ and } j = q_2; \\ 0, & \text{otherwise.} \end{cases}$$

Example 18. The matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is the $(5, 2, 4, 3)$-matrix.

We say that an $n \times n$ matrix M belongs to the set $\mathcal{M}_2(n)$ if M is the (x, q_1, q_2, τ)-matrix for some $x \in \mathbb{R}$, $x \neq 0$ and $q_1, q_2, \tau \in \{2, \ldots, n\}$, $q_1 \neq q_2$.

Now we prove the main theorem of this section.

Theorem 19. *Let M be a special $n \times n$ matrix, where $n \geqslant 8$ or $n = 3$. Then $g_M(n) \geqslant 2(n-1)!$. Moreover, $g_M(n) = 2(n-1)!$ if and only if $M \in \mathcal{M}_1(n)$ or $M \in \mathcal{M}_2(n)$.*

Proof. One can verify that if $M \in \mathcal{M}_1(n)$ or $M \in \mathcal{M}_2(n)$, then $g_M(n) = 2(n-1)!$.

Suppose that $g_M(n) \leqslant 2(n-1)!$. Let us prove that $M \in \mathcal{M}_1(n)$ or $M \in \mathcal{M}_2(n)$. Lemma 16 implies that M is θ-uniform for some $\theta \in \{1, \ldots, n\}$. We consider two cases.

Case 1. Suppose $\theta = 1$. Then $R_2 = R_3 = \ldots = R_n$. By Lemma 14, R_2 is the (x, r_1, r_2)-row or the (y, s)-row. If R_2 is the (x, r_1, r_2)-row, then M is the (x, r_1, r_2)-matrix. So, in this subcase $M \in \mathcal{M}_1(n)$. If R_2 is the (y, s)-row, then $g_M(n) = (n-1) \cdot (n-1)!$ and we obtain a contradiction with $g_M(n) \leqslant 2(n-1)!$.

Case 2. Suppose $\theta > 1$. By Lemma 14, R_θ is the (x, r_1, r_2)-row or the (y, s)-row. If R_θ is the (x, r_1, r_2)-row, then M is the (x, r_1, r_2, θ)-matrix. So, in this subcase $M \in \mathcal{M}_2(n)$. If R_θ is the (y, s)-row, then $g_M(n) = (n-1) \cdot (n-1)!$ and we obtain a contradiction with $g_M(n) \leqslant 2(n-1)!$. $\qquad\square$

5 Main Theorem

In this section we prove the main theorem of this paper.

Theorem 20. *Let f be an $(n-2)$-eigenfunction of S_n, where $n \geqslant 8$ or $n = 3$. Then $|Supp(f)| \geqslant 2(n-1)!$. Moreover, $|Supp(f)| = 2(n-1)!$ if and only if $f = c \cdot \tilde{f}$, where c is a real non-zero constant and $\tilde{f} \in \mathcal{F}$.*

Proof. Lemma 7 implies that $|Supp(f)| = g_{M(f)}(n)$. We note that $M(f)$ is special. Then by Theorem 19 we obtain that $g_{M(f)}(n) \geqslant 2(n-1)!$. Therefore $|Supp(f)| \geqslant 2(n-1)!$. Moreover, $g_{M(f)}(n) = 2(n-1)!$ if and only if $M(f) \in \mathcal{M}_1(n)$ or $M(f) \in \mathcal{M}_2(n)$. We consider two cases.

Suppose that $M(f) \in \mathcal{M}_1(n)$. Then $M(f)$ is the (x, p_1, p_2)-matrix. Using Lemma 5, we have

$$f(\pi) = \begin{cases} x, & \text{if } \pi(p_2) = 1; \\ -x, & \text{if } \pi(p_1) = 1; \\ 0, & \text{otherwise.} \end{cases}$$

So, in this case $f = x \cdot f_1^{p_2, p_1}$.

Suppose that $M(f) \in \mathcal{M}_2(n)$. Then $M(f)$ is the (x, q_1, q_2, τ)-matrix. Using Lemma 5, we have

$$f(\pi) = \begin{cases} x, & \text{if } \pi(q_1) = \tau; \\ -x, & \text{if } \pi(q_2) = \tau; \\ 0, & \text{otherwise.} \end{cases}$$

So, in this case $f = x \cdot f_\tau^{q_1, q_2}$. $\qquad\square$

6 Correspondence between the extremal $(n-2)$-eigenfunctions and completely regular codes

Lemma 21. *Let $\alpha \in \{2, \ldots, n\}$, $a \in \{1, \ldots, n\}$ and $n \geqslant 3$. Then the set S_a^α is a completely regular code of covering radius 2 in S_n.*

Proof. Firstly, we note that $(S_a^\alpha)^{(1)} = S_a^1$,

$$(S_a^\alpha)^{(2)} = \bigcup_{\beta \in \{1, \ldots, n\} \setminus \{1, \alpha\}} S_a^\beta$$

and $(S_a^\alpha)^{(3)} = \emptyset$. On the other hand, the partition $((S_a^\alpha)^{(0)}, (S_a^\alpha)^{(1)}, (S_a^\alpha)^{(2)})$ has the quotient matrix

$$\begin{pmatrix} n-2 & 1 & 0 \\ 1 & 0 & n-2 \\ 0 & 1 & n-2 \end{pmatrix}.$$

So, S_a^α is a completely regular code of covering radius 2 in S_n. □

For a set $A \subseteq Sym_n$ we define the characteristic function χ_A of A in Sym_n as follows:

$$\chi_A(\pi) = \begin{cases} 1, & \text{if } \pi \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Using Theorem 20 and the definition of $f_u^{v,w}$, we immediately obtain the following result.

Lemma 22. *Let f be an $(n-2)$-eigenfunction of S_n and $|Supp(f)| = 2(n-1)!$, where $n \geqslant 8$ or $n = 3$. Then*

$$f = c \cdot (\chi_{S_u^v} - \chi_{S_u^w}),$$

where c is a real non-zero constant, $u \in \{1, \ldots, n\}$, $v, w \in \{2, \ldots, n\}$ and $v \neq w$.

Thus, Lemma 21 and Lemma 22 imply that for $n \geqslant 8$ and $n = 3$ an arbitrary $(n-2)$-eigenfunction of S_n with the minimum cardinality of the support is the difference of the characteristic functions of two completely regular codes of covering radius 2. It is very interesting that there is an analogue of this fact for the Hamming graph $H(n,q)$ (see [18], Theorem 3). Namely, an arbitrary eigenfunction of $H(n,q)$ corresponding to the second largest eigenvalue with the minimum cardinality of the support is the difference of the characteristic functions of two completely regular codes of covering radius 1.

7 Concluding remarks

The initial problem of finding $(n-2)$-eigenfunctions of S_n with the minimum size of the support is formulated for arbitrary real-valued functions from corresponding eigenspace. Surprisingly, Theorem 20 implies that such functions take only three distinct values. It is interesting that the same fact holds for the Doob graph, for the Hamming graph and for the Johnson graph (see [3, 20, 22]). But, in general case it is not true. For example, in the Petersen and Desargues graphs there are (-2)-eigenfunctions with the minimum sizes of the supports that take five distinct values (see [16], Figure 3 and 9).

We note that the restrictions for n in Theorem 20 ($n \geqslant 8$ or $n = 3$) arise from the proofs of Lemmas 12, 15 and 16.

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The mention of SB RAS N I.5.1, project No.0314-2019-0016 on page 1 was added in error.