A graph-theoretic approach to Wilf's conjecture

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Abstract

Let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity $m = \min(S \setminus \{0\})$ and conductor $c = \max(\mathbb{N} \setminus S) + 1$. Let P be the set of primitive elements of S, and let L be the set of elements of S which are smaller than c. A longstanding open question by Wilf in 1978 asks whether the inequality $|P||L| \geqslant c$ always holds. Among many partial results, Wilf's conjecture has been shown to hold in case $|P| \geqslant m/2$ by Sammartano in 2012. Using graph theory in an essential way, we extend the verification of Wilf's conjecture to the case $|P| \geqslant m/3$. This case covers more than 99.999% of numerical semigroups of genus $g \leqslant 45$.

Mathematics Subject Classifications: 05C25, 11B75, 20M14

1 Introduction

Denote $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{N}_+ = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$. For $a, b \in \mathbb{Z}$, let $[a, b[= \{z \in \mathbb{Z} \mid a \leqslant z < b\}$ and $[a, \infty[= \{z \in \mathbb{Z} \mid a \leqslant z\}$ denote the integer intervals they span. A *numerical semigroup* is a subset $S \subseteq \mathbb{N}$ containing 0, stable under addition and with finite complement in $\mathbb{N}$. Equivalently, it is a subset $S \subseteq \mathbb{N}$ of the form $S = \langle a_1, \dots, a_n\rangle = \mathbb{N}a_1 + \dots + \mathbb{N}a_n$ where $\gcd(a_1, \dots, a_n) = 1$. The set $\{a_1, \dots, a_n\}$ is then called a *system of generators* of S, and the smallest such n is called the *embedding dimension* of S.

For a numerical semigroup S, its *gaps* are the elements of $\mathbb{N} \setminus S$, its *genus* is $g = |\mathbb{N} \setminus S|$, its *multiplicity* is $m = \min S^*$ where $S^* = S \setminus \{0\}$, its *Frobenius number* is $f = \max \mathbb{Z} \setminus S$ and its *conductor* is $c = f + 1$. Thus $[c, \infty[\subseteq S$ and c is minimal for this property. As in [9], we denote $L = S \cap [0, c[$.

We partition S^* as $S^* = P \sqcup D$, where $D = S^* + S^* = \{x + y \mid x, y \in S^*\}$ is the set of *decomposable* elements of S^*, and $P = S^* \setminus D$ is the set of *primitive elements* of S^*. As easily seen, P is finite since $P \subseteq [m, c + m[$. Moreover $S = \langle P\rangle$ since every element of S^*

is a sum of primitive elements, and P is the unique *minimal system of generators* of S. Thus $|P|$ equals the embedding dimension of S.

In 1978 Wilf asked, in equivalent terms, whether the inequality

$$|P||L| \geqslant c \tag{1}$$

always holds [22]. Wilf's conjecture, as it is now known, has been verified in several cases, including when $|P| \leqslant 3$, or $c \leqslant 3m$, or $m \leqslant 18$, or $|L| \leqslant 12$, or $|P| \geqslant m/2$. See Delgado [5] for an extensive recent survey of partial results on Wilf's conjecture, and [1, 2, 4, 9, 11, 13, 14, 15, 17, 20, 21, 22] for some relevant papers. The verification in case $|P| \geqslant m/2$ is due to Sammartano [20] in 2012. Our purpose in this paper is to extend it to the case $|P| \geqslant m/3$.

Theorem 1. *Let S be a numerical semigroup with multiplicity m and minimal generating set P. If $|P| \geqslant m/3$ then S satisfies Wilf's conjecture.*

This result was first presented in 2017 at a conference in Umeå [10]. The present proof is a streamlined version of the original unpublished one.

As later noted by Manuel Delgado, who attended the Umeå conference, an overwhelming majority of numerical semigroups satisfies the condition of Theorem 1. Specifically, among all 23 022 228 615 numerical semigroups of genus $g \leqslant 45$, the proportion of those satisfying $|P| \geqslant m/3$ exceeds 99.999%. It is likely, though it remains to be proved, that this proportion tends to 1 as g goes to infinity, in complete analogy to Zhai's result regarding the condition $c \leqslant m/3$, where c is the conductor [23, 16]. In addition, Delgado discovered that the condition of Theorem 1 is well suited to efficiently trim the tree of numerical semigroups while probing certain open problems concerning them [6]. In particular, this will lead to significant advances on the verification of Wilf's conjecture by computer. While the first such major effort reached genus $g = 50$ [1], and the current published verification record stands at genus $g = 60$ [14], Delgado and Fromentin *have now verified Wilf's conjecture up to genus* $g = 80$, and aim to reach genus $g = 100$ before publishing their result [7].

1.1 Contents

In Section 2, we introduce the depth and total depth functions on a numerical semigroup. In Section 3, we construct a map $S \mapsto G(S)$ associating to every numerical semigroup S a finite graph $G(S)$ whose properties play a key role in this paper. Those properties, combining algebra and graph theory, are developed in Section 4. Section 5 is devoted to proving Theorem 1. In the last Section 6, we take a closer look at the map $S \mapsto G(S)$ by considering its range and fibers.

2 The depth functions δ and τ

Throughout this section, let $S \subseteq \mathbb{N}$ be a numerical semigroup with multiplicity m and conductor c.

Definition 2. The *depth* of S is the integer $q = \lceil c/m \rceil$. We denote it by $\mathrm{depth}(S)$.

See also [12]. More generally, we define the *depth function* $\delta\colon S \to \mathbb{Z}$ on S as follows.

Definition 3. For all $x \in S$, let $\delta(x) \in \mathbb{Z}$ denote the unique integer such that

$$x + \delta(x)m \ \in \ [c, c+m[.$$

We call $\delta(x)$ the *depth* of x.

For instance, assuming $S \neq \mathbb{N}$, the elements of $[c, c+m[$ have depth 0, those in $[c+m, \infty[$ have negative depth while those in $S \cap [0, c[$ have positive depth. The largest depth in S is attained by 0, namely $\delta(0) = \mathrm{depth}(S) = \lceil c/m \rceil$.

Notation 4. *Let $q = \mathrm{depth}(S) = \lceil c/m \rceil$. We set $\rho = qm - c$. Thus $\rho \in [0, m[$ and $c = qm - \rho$.*

As in [9], we denote

$$S_i = S \cap [im - \rho, im + m - \rho[\tag{2}$$

for all $i \geqslant 0$. This yields the partition $S = \bigsqcup_{i \geqslant 0} S_i$. In particular, we have $S_0 = \{0\}$, $m \in S_1$ and $c \in S_q$. More generally, we have

$$S_i = \{x \in S \mid \delta(x) = q - i\} \tag{3}$$

as easily verified. Note also the equality

$$L = S_0 \sqcup S_1 \sqcup \cdots \sqcup S_{q-1}. \tag{4}$$

The following was shown in [9]. Its verification is straightforward.

Proposition 5. *Let S be a numerical semigroup. For all $0 \leqslant i \leqslant j$ such that $j \geqslant 1$, we have*

$$S_i + S_j \subset S_{i+j-1} \sqcup S_{i+j} \sqcup S_{i+j+1}. \tag{5}$$

Moreover, if $\rho = 0$ then

$$S_i + S_j \subseteq S_{i+j} \sqcup S_{i+j+1}. \tag{6}$$

These set addition properties may be translated in terms of the depth function δ as follows. The rightmost inequality will be used throughout the paper.

Proposition 6. *Let S be a numerical semigroup of depth $q \geqslant 1$. For all $x, y \in S$, we have*

$$\delta(x+y) + q + 1 \ \geqslant\ \delta(x) + \delta(y) \ \geqslant\ \delta(x+y) + q - \min(\rho, 1). \tag{7}$$

Proof. As observed in (3), for all $x \in S$ we have

$$x \in S_i \iff \delta(x) = q - i.$$

Let $x, y \in S$, and assume $x \in S_i$, $y \in S_j$. Then $\delta(x) = q - i$, $\delta(y) = q - j$, and so $\delta(x) + \delta(y) - q = q - i - j$. The addition properties (5) and (6) now yield

$$q - i - j - 1 \leqslant \delta(x + y) \leqslant q - i - j + \min(\rho, 1),$$

whence

$$\delta(x) + \delta(y) - q - 1 \leqslant \delta(x + y) \leqslant \delta(x) + \delta(y) - q + \min(\rho, 1).$$

This is equivalent to (7), as desired. □

Definition 7. Let $A \subset S$ be a finite subset. We define the *total depth* of A as

$$\tau(A) \ = \ \sum_{x \in A} \delta(x).$$

In Theorem 37 and in Section 5, we use graph-theoretical tools to estimate the total depth $\tau(X)$ of X, the set of nonzero Apéry elements of S, as a step towards proving Theorem 1. The key idea is to exploit (7) by forming suitable pairs $\{x, y\}$ of elements of X.

2.1 The number $W(S)$ and Apéry elements

Let $S \subseteq \mathbb{N}$ be a numerical semigroup of multiplicity m and conductor c. As above, we partition $S^* = P \sqcup D$ into primitive and decomposable elements, and we set $L = S \cap [0, c[$. We shall use the following notation from [9].

Notation 8. $W(S) = |P||L| - c$.

Thus, Wilf's conjecture amounts to state that $W(S) \geqslant 0$ holds for every numerical semigroup S. In this paper, as in [9], we focus on estimating $W(S)$ from below. For this purpose, we need the nonzero Apéry elements of S. The set

$$\mathrm{Ap}(S) = \{s \in S \mid s - m \notin S\},$$

called the *Apéry set* of S[1], is central in the theory of numerical semigroups. It has m elements, one in each class mod m, actually its least member belonging to S. As is well known and easy to see, the smallest and largest elements of $\mathrm{Ap}(S)$ are 0 and $c + m - 1$, respectively. The additive properties of $\mathrm{Ap}(S) \setminus \{0\}$ play a key role in this paper.

Notation 9. *We denote by* $X = \mathrm{Ap}(S) \setminus \{0\}$ *the set of nonzero Apéry elements.*

[1]Or more precisely, the *Apéry set of S with respect to m.*

Proposition 10. *The following hold.*

- $\delta(x) \geqslant 0$ *for all* $x \in X$.
- $m = |P| + |X \cap D|$.
- $|L| \ = \ q + \tau(X)$.

Proof.

- As $\max X = c + m - 1$, it follows that $X \subset [m, c+m[$. The conclusion follows from the definition of δ.
- We have $|X| = |X \cap P| + |X \cap D|$. The definitions imply that $|X| = m - 1$ and $P \setminus X = \{m\}$, so $|X \cap P| = |P| - 1$. The stated formula follows.
- Let $a \in L$ be minimal in its class mod m. Then either $a = 0$ or $a \in X$. Moreover $a + im \in L$ if and only if $i \in [0, \delta(a)[$. Hence

  $$|L \cap (a + m\mathbb{N})| \ = \ \delta(a).$$

  Now $\delta(0) = q$, so that $\tau(L \cap m\mathbb{N}) = q$. Summing over all $x \in X$, i.e. over all nonzero classes mod m, we cover all of L and the claimed formula follows. □

Corollary 11. *We have* $W(S) \ = \ |P|\tau(X) - |X \cap D|q + \rho$.

Proof. By definition, $W(S) = |P||L| - c = |P||L| - qm + \rho$. Since $|L| = q + \tau(X)$ and $m = |P| + |X \cap D|$ by Proposition 10, the stated formula follows. □

Our proof strategy for Theorem 1 will be to use graphs to estimate $\tau(X)$ from below using (7) and simultaneously estimate $|X \cap D|$ from above, thereby leading to show $W(S) \geqslant 0$ for the numerical semigroups under consideration. For this purpose, the following considerations will be useful. First, here is an analogue, in additive notation, of the notion of proper divisor.

Definition 12. Let $b \in S^*$. A *summand* of b is any $a \in S^*$ such that $b \in a + S^*$, i.e. such that there exists $s \in S^*$ with $b = a + s$.

As a matter of notation, given $a, b \in S$, it is customary to write $a \preceq b$ whenever $b - a \in S$. The following additive property is well known and crucial.

Lemma 13. *Let* $x \in X \cap D$. *If* $x = a + b$ *with* $a, b \in S^*$, *then* $a, b \in X$. *That is, any summand of a nonzero Apéry element is a nonzero Apéry element.*

Proof. If $a \notin X$, then $a = a' + m$ for some $a' \in S^*$. Hence $x = a' + b + m$, whence $x \notin X$ since $a' + b \in S^*$. □

3 The associated graph

In this section, we define a map $S \mapsto G(S)$ associating to every numerical semigroup S a finite graph $G(S)$. Properties of $G(S)$ will then be shown to have a direct bearing on the parameters $\tau(X)$ and $|X \cap D|$ involved in Corollary 11 and hence on Wilf's conjecture.

Definition 14. Let $S \subseteq \mathbb{N}$ be a numerical semigroup. The graph $G = G(S)$ associated to S is defined as follows.

- The edge set $E(G)$ consists of all subsets $\{x, y\} \subseteq X$ such $x + y \in X$. The equality $x = y$ is allowed.
- The vertex set $V(G)$ consists of all endvertices of the edges. Thus, an element $x \in X$ belongs to $V(G)$ if and only if there exists $y \in X$ such that $x + y \in X$.

Remark 15. More generally, one may associate a graph $G(A)$ to any finite (or not) subset A of a commutative monoid $(M, +)$. The edges of $G(A)$ are all subsets $\{x, y\} \subseteq A$ such that $x + y \in A$, and its vertices are all endvertices of the edges. This graph carries much information on the additive properties of A. For a numerical semigroup S, the graph $G(S)$ is obtained in this general form by taking $G(S) = G(A)$, where $A = X$ is the set of nonzero Apéry elements of S.

By construction, the graph $G(S)$ has no isolated vertices. More generally, it follows from the definition that $G(S)$ is a *loopy graph* as defined below.

Definition 16. A *loopy graph* is a finite graph with no isolated vertices, no multiple edges but possibly with loops.

We shall further need the following definitions/notation.

Definition 17. In a loopy graph, an edge with equal endvertices is a *loop*, otherwise it is a *true edge*. A vertex is *loopy* if it supports a loop, or *nonloopy* otherwise. The *loopy-complete graph* on n vertices, denoted LK_n, is the graph obtained from the complete graph K_n by attaching a loop to every vertex.

Notation 18. *For a loopy graph G, we denote by $\lambda(G)$ its number of loops. It coincides with its number of loopy vertices since G has no multiple edges.*

For example, Figure 1 displays $G(S)$ for $S = \langle 12, 13, 14, 15, 17, 19, 20, 21\rangle$. Here $|P| = 8$, $m = 12$ and $X = \{13, 14, 15, 17, 19, 20, 21, 28, 30, 34, 35\}$. The simple NE-SE path of length 6 visits the vertices $19, \mathbf{15}, 20, \mathbf{14}, 21, 13, \mathbf{17}$. In bold are the three loopy vertices $\mathbf{15}, \mathbf{14}, \mathbf{17}$, exactly those $x \in X$ such that $2x \in X$.

3.1 Vertex-maximal matchings

Let $G = (V, E)$ be a loopy graph. A *matching* M in G is a subgraph consisting of mutually nonadjacent edges. Loops are allowed in M.

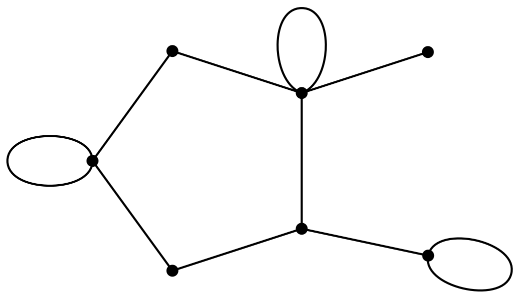


Figure 1: The graph $G(S)$ associated to $S = \langle 12, 13, 14, 15, 17, 19, 20, 21 \rangle$.

Definition 19. The *vertex-maximal matching number of* G is the maximum number of vertices touched by a matching M in G. We denote this number by $\mathrm{vm}(G)$. In formula:

$$\mathrm{vm}(G) = \max_{M \subseteq G} |V(M)|$$

where M runs over all matchings of G.

Definition 20. A *vertex-maximal matching of* G is a matching touching $\mathrm{vm}(G)$ vertices. An edge in G is *active* if it is contained in a vertex-maximal matching of G, and *passive* otherwise. We denote by $E^+ \subseteq E$ the set of active edges.

A loop needs not be active in general. However, a vertex-maximal matching contains all the loopy vertices, as easily seen. Moreover, we have $\mathrm{vm}(G) \geqslant \lambda(G)$, since any set of ℓ loops in G is a matching with ℓ vertices.

Proposition 21. *Let* G *be a loopy graph with* $\mathrm{vm}(G) = k$ *and such that* G *is edge-maximal for this property. Let* $\ell = \lambda(G)$*. Then* G *contains* LK_ℓ*.*

Proof. As mentioned above, every vertex-maximal matching in G contains all of its ℓ loopy vertices[2]. Assume that x, y are nonadjacent loopy vertices. Then, as easily seen, adding the edge $\{x, y\}$ to G does not increase $\mathrm{vm}(G)$. This contradicts the edge-maximality of G with respect to $\mathrm{vm}(G)$. Hence $G \supseteq LK_\ell$. □

An interesting general question, with direct implications for the present approach to Wilf's conjecture, is the following.

Question 22. Given integers $n \geqslant k \geqslant 1$, let G be a loopy graph on n vertices and such that $\mathrm{vm}(G) = k$. What is the maximum number of edges allowed in G?

For instance, consider a loopy graph G with $(n, k) = (5, 4)$. While the non-complying graph LK_5 has 15 edges, we show in Proposition 61 that G has at most 10 edges, and this is optimal as witnessed by the complying graph K_5.

[2] But again, not necessarily all of its *loops*.

If $k = 2r$ is even, $k \geqslant 2$ and $n \geqslant k+2$, is the optimal upper bound given by

$$|E(G)| \leqslant \binom{r+1}{2} + r(n-r) \ ?$$

Equality is achieved by the complying graph $G = LK_r \vee \overline{K_{n-r}}$, the *join* [3] of LK_r and the empty graph $\overline{K_{n-r}}$ on $n-r$ vertices. Recall that $G_1 \vee G_2$ is obtained by adding to $G_1 \sqcup G_2$ all possible edges between $V(G_1)$ and $V(G_2)$.

A similar construction can be made for k odd.

3.2 The weight of edges

Let $G = G(S)$ be the graph associated to a numerical semigroup $S \subseteq \mathbb{N}$. As usual, we denote by $D, X \subset S^*$ the sets of decomposable and nonzero Apéry elements, respectively.

Definition 23. Let $e = \{x, y\} \in E(G)$. The *weight* of e is defined as $\mathrm{wt}(e) = x + y$.

By construction, this yields a map $\mathrm{wt}\colon E(G) \to X \cap D$.

Proposition 24. *The map* $\mathrm{wt}\colon E(G) \to X \cap D$ *is onto.*

Proof. For every $z \in X \cap D$, there exist $x, y \in X$ such that $z = x + y$. Thus $\{x, y\}$ is an edge of G and has weight z. □

It follows that

$$|X \cap D| \leqslant |E(G)|. \tag{8}$$

Here is a useful formula for the difference $|E(G)| - |X \cap D|$.

Proposition 25. *We have*

$$|X \cap D| = |E(G)| - \sum_{z \in X \cap D} (|\,\mathrm{wt}^{-1}(z)| - 1).$$

Proof. The fibers of wt constitute a partition of $E(G)$. Thus

$$|E(G)| = \sum_{z \in X \cap D} |\,\mathrm{wt}^{-1}(z)|.$$

Note that $|\,\mathrm{wt}^{-1}(z)| \geqslant 1$ for all $z \in X \cap D$ since w is onto. Subtracting 1 to each such summand yields

$$|E(G)| = |X \cap D| + \sum_{z \in X \cap D} (|\,\mathrm{wt}^{-1}(z)| - 1). \quad \square$$

In particular, the larger $|V \cap D|$ is, the farther away $|X \cap D|$ will be from $|E(G)|$. For instance, if there is at least one fiber of cardinality more than 1, then $|X \cap D| < |E(G)|$.

Remark 26. If all edge weights are distinct, then wt is a bijection and hence $|X \cap D| = |E(G)|$.

Lemma 27. *Distinct adjacent edges have distinct weights. Similarly, distinct loops have distinct weights.*

Proof. Distinct adjacent edges are of the form $\{x, y\}, \{x, z\}$ with $y \neq z$, whence $x + y \neq x + z$. Distinct loops are of the form $\{x, x\}, \{y, y\}$ with $x \neq y$, implying $2x \neq 2y$. □

3.3 Normal and weak edges

We use the same notation as above.

Lemma 28. *Let $\{x, y\}$ be an edge in G. Then $\delta(x) + \delta(y) \geqslant q - \min(\rho, 1)$.*

Proof. We have $x + y \in X$ by hypothesis. The inequality now directly follows from (7) and Proposition 10. $\square$

Definition 29. An edge $\{x, y\}$ in G is *weak* if $\delta(x)+\delta(y) = q-1$, and *normal* otherwise, i.e. if $\delta(x) + \delta(y) \geqslant q$.

Remark 30. If $\rho = 0$ then all edges of G are normal. This follows from the above lemma.

Notation 31. *We denote by $E_0(G)$ and $E_1(G)$ the set of* weak *and* normal *edges of G, respectively. Thus*

$$E(G) = E_0(G) \sqcup E_1(G).$$

Lemma 32. *If $\{x, y\} \in E_0(G)$, then $\delta(x + y) = 0$.*

Proof. Indeed, by hypothesis we have $x + y \in X$ and $\delta(x) + \delta(y) = q - 1$. The former implies $\delta(x + y) \geqslant 0$ by Proposition 10, and the latter implies $\rho \geqslant 1$ and $\delta(x + y) = 0$ by (7). $\square$

Proposition 33. *Let $S \subseteq \mathbb{N}$ be a numerical semigroup. Let*

$$X_0 = \{z \in X \cap D \mid \exists x, y \in X,\ z = x + y,\ \delta(x) + \delta(y) \ = \ \delta(z) + q - 1\}. \tag{9}$$

Then $|X_0| \leqslant \rho$.

Proof. Let $z = x + y \in X_0$, and assume $x \in S_i, y \in S_j$. Then $\delta(x) + \delta(y) \ = \ \delta(z) + q - 1$ if and only if $z \in S_{i+j-1}$. Now, by the definition of the S_i, we have

$$(S_i + S_j) \cap S_{i+j-1} \subseteq [(i + j)m - 2\rho, (i + j)m - \rho[.$$

Thus, the only classes mod m for which such a deficit may occur are those in $[-2\rho, -\rho[$. And since there is only one element of X per class mod m, the statement follows. $\square$

Corollary 34. *We have $\rho \geqslant |\operatorname{wt}(E_0(G))|$.*

Proof. Let $X_0 \subseteq X \cap D$ be as defined in (9). It suffices to show

$$\operatorname{wt}(E_0(G)) \subseteq X_0, \tag{10}$$

and the conclusion will follow from Proposition 33. Let $e = \{x, y\} \in E_0(G)$. Then $\delta(x) + \delta(y) = q - 1$ by hypothesis. Let $z = \operatorname{wt}(e) = x + y$. Then $z \in X \cap D$ by definition, and $\delta(z) = 0$ by Lemma 32. Therefore $z \in X_0$ and we are done. $\square$

3.4 The normality number

We keep using the same notation as above.

Definition 35. The *normality number* of the graph $G = G(S)$ is defined as

$$\nu = \nu(G) = \max_{M \subseteq G} \#\{\text{endvertices of all normal edges in } M\},$$

where M runs over all vertex-maximal matchings in G. Thus $0 \leqslant \nu \leqslant \mathrm{vm}(G)$.

Recall from Section 3.1 that an edge is *active* if it belongs to a vertex-maximal matching, and that we denote by $E^+ \subseteq E$ the subset of active edges. The partition $E = E_0 \sqcup E_1$ into weak and normal edges induces a corresponding partition on E^+.

Notation 36. *We denote by $E_0^+ \subseteq E_0$ the subset of active weak edges, and by $E_1^+ \subseteq E_1$ the subset of active normal edges. Thus $E^+ = E_0^+ \sqcup E_1^+$.*

The interest of this partition is that *only active edges* are actually involved in the definition of the normality number $\nu(G)$. That is, we have

$$\nu(G) = \max_{M \subseteq G} \#\{\text{endvertices of } E(M) \cap E_1^+\}, \tag{11}$$

where M runs over all vertex-maximal matchings in G.

3.5 A lower bound on $\tau(X)$

We now have all the ingredients at hand to formulate our key lower bound on $\tau(X)$ and hence on $W(S)$. We keep using the same notation as above.

Theorem 37. *Let $G = G(S)$, $n = |V(G)|$ and $k = \mathrm{vm}(G)$. Then*

$$\tau(X) \geqslant \big(k(q-1) + \nu\big)/2 + (n-k).$$

Proof. Let $M \subseteq G$ be a vertex-maximal matching, and set $V_M = V(M)$. Thus $|V_M| = k$. Moreover, by (11), we may assume that the number of vertices touched by the normal edges of M is maximal, i.e. is equal to $\nu = \nu(G)$.

We have $\tau(X) \geqslant \tau(V)$ since $V \subset X$. We now evaluate $\tau(V)$ from below. Let $\overline{V}_M = V \setminus V_M$. Then $|\overline{V}_M| = n - k$. We have $\tau(V) = \tau(V_M) + \tau(\overline{V}_M)$. Since $V \subset L$ and since $\delta(a) \geqslant 1$ for all $a \in L$, we have

$$\tau(\overline{V}_M) \geqslant |\overline{V}_M| = n - k.$$

We now estimate $\tau(V_M)$. For that, we need to count the edges of M by distinguishing the nonloops and the loops, and the weak and the normal ones. Let r_0, t_0 denote the number of weak nonloops and loops in M, respectively. Similarly, let r_1, t_1 denote the number of normal nonloops and loops in M, respectively. Thus

$$k = 2(r_0 + r_1) + t_0 + t_1, \ \nu = 2r_1 + t_1.$$

For every edge $\{x, y\}$ in M, we have $\delta(x)+\delta(y) = q-1$ if it is weak, while $\delta(x)+\delta(y) \geqslant q$ if it is normal. It follows that

$$\begin{aligned} \tau(V_M) &\geqslant r_0(q-1) + r_1 q + t_0(q-1)/2 + t_1 q/2 \\ &= \big((2r_0 + t_0 + 2r_1 + t_1)(q-1) + 2r_1 + t_1\big)/2 \\ &= \big(k(q-1) + \nu\big)/2. \end{aligned}$$

Summarizing, we have

$$\tau(X) \geqslant \tau(V) = \tau(V_M) + \tau(\overline{V}_M) \geqslant \big(k(q-1) + \nu\big)/2 + (n-k). \quad \square$$

4 Properties of $G(S)$

Let $G(S) = G = (V, E)$ be the graph associated to the numerical semigroup S. Most results in this section, combining algebraic and graph-theoretic properties, will be used in Section 5 to prove Theorem 1.

Among the vertices in V, distinguishing between the primitive and the decomposable ones is crucial. Thus, we shall systematically consider the partition

$$V = (V \cap P) \sqcup (V \cap D).$$

In this context, we prefer using the more intuitive multiplicative notation, as the elements of $V \cap D$ are best viewed as *monomials* in $V \cap P$.

For instance, if $V \cap P = \{x_1, x_2\}$ and $V \cap D = \{2x_1, x_1 + x_2, 2x_2, 3x_1\}$ in standard additive notation, we prefer to write $V \cap D = \{x_1^2, x_1x_2, x_2^2, x_1^3\}$. In this way, we can speak of divisors, multiples, antichains under divisibility, and so on. For instance, we find it more convenient to say "x_1 divides x_1x_2" rather than "x_1 is a summand of $x_1 + x_2$" or write $x_1 \preceq (x_1 + x_2)$ in standard additive notation.

More formally, let us rename our given additive numerical semigroup S as S_0. We then embed S_0 in the one-variable polynomial ring $\mathbb{R}[Z]$, and more precisely in the semigroup ring $\mathbb{R}[S_0] \subseteq \mathbb{R}[Z]$, where

$$\mathbb{R}[S_0] = \{\sum_{a \in S_0} \lambda_a Z^a \mid \lambda_a \in \mathbb{R} \text{ for all } a \in S_0 \text{ and } \lambda_a = 0 \text{ for almost all } a\}.$$

We then set $S = \{Z^a \mid a \in S_0\}$. It is a multiplicative submonoid of $\{Z^n \mid n \in \mathbb{N}\}$ with finite complement and neutral element $Z^0 = 1$. We have a monoid isomorphism

$$\varphi \colon S_0 \to S \tag{12}$$

defined by $\varphi(a) = Z^a$ and satisfying $\varphi(a + b) = \varphi(a)\varphi(b)$ for all $a, b \in S_0$. We will refer to S as a *numerical semigroup in multiplicative notation.*

4.1 Switching to multiplicative notation

Thus, from now on in this section, S is a numerical semigroup in multiplicative notation, arising from its additive counterpart $S_0 \subseteq \mathbb{N}$ via the isomorphism φ in (12). We denote $S^* = S \setminus \{1\}$. All other usual notions related to S_0, such as the multiplicity, the conductor, the subsets L, P, D, X, V and so on, are transported via φ to S without changing notation.

For clarity, let us rewrite the weight of edges of $G = G(S)$ in multiplicative notation. The weight map $\mathrm{wt}\colon E(G) \to X \cap D$ is then defined as follows: for any edge $\{x, y\} \in E(G)$, we set

$$\mathrm{wt}(\{x, y\}) = xy.$$

Note that $xy \in X \cap D$ by construction.

A word of caution is needed here. The decomposition of an element $z \in X \cap D$ as a product of primitive elements is not unique in general. That is, z may be represented by several formally distinct monomials in P. On the other hand, we do have simplification properties such as

$$x^2 = y^2 \Rightarrow x = y \quad \text{and} \quad xz = yz \Rightarrow x = y$$

for all $x, y, z \in S$, as follows from the analogous additive properties in $S_0 \subseteq \mathbb{N}$.

4.2 Downsets

As above, let S denote a numerical semigroup in multiplicative notation.

Definition 38. Let $u \in S^*$. A *proper factor* of u is an element $v \in S^*$ such that $v \neq u$ and v divides u, i.e. such that there exists $v' \in S^*$ satisfying $u = vv'$.

Definition 39. A *downset* in S^* is a subset $I \subseteq S^*$ which is stable under taking proper factors. That is, if $u \in I$ and if $v \in S^*$ is a proper factor of u, then $v \in I$.

The following lemma is a restatement of Lemma 13 in the present context.

Lemma 40. *The subset $X \subset S^*$ is a downset.* □

Lemma 41. *The set V of vertices of G is a downset. It coincides with the set of proper factors of all elements of $X \cap D$.*

Proof. Let $x \in V$. Then there exists $y \in X$ such that $xy \in X$ and so $\{x, y\} \in E$. Actually $xy \in X \cap D$ and x is a proper factor of xy. If x' is a proper factor of x, then $x'y$ is a proper factor of xy, hence it belongs to X since X is a downset, hence $\{x', y\} \in E$. This implies $x' \in V$. Therefore V is a downset, as claimed. Let now $z \in X \cap D$, and let $x \in S^*$ be a proper factor of z. Let $y = z/x$. Then $x, y \in X$ by Lemma 40 and $\{x, y\} \in E$. Hence $x \in V$, as desired. □

Given a vertex $x \in V$, we denote as usual by $N_G(x) \subseteq V$ its set of neighbors, i.e.

$$N_G(x) = \{y \in X \mid xy \in X\} = \{y \in V \mid xy \in X\}.$$

As usual, the *degree* of vertex x is defined as $\deg(x) = |N_G(x)|$.

Lemma 42. *Let $u \in V$. Then $N_G(u)$ is a downset.*

Proof. Let $v \in N_G(u)$. Then $uv \in X$. Let w be a proper factor of v. Then $w \in V$ and $v = v'w$ for some $v' \in V$. Hence $uv'w \in X$, implying $uw \in X$, implying in turn $w \in N_G(u)$. □

4.3 More vertex properties

Lemma 43. *We have $|P| \geqslant |V \cap P| + 1$.*

Proof. Indeed, with m denoting as usual the multiplicity of S, we have $m \in P \setminus V$ since $m \notin X$. □

The next result helps locate in V the proper factors of the vertices in $V \cap D$, if any.

Proposition 44. *Let $v_1 \neq v_2 \in V$. If v_1 divides v_2, then $\deg(v_1) > \deg(v_2)$.*

Proof. Let $w \in V$ be such that $v_2 = v_1 w$. Let $t = \deg(v_2)$ and denote $N_G(v_2) = \{z_1, \dots, z_t\}$. Since $z_i v_2 = z_i w v_1 \in X$ for all i by hypothesis, and since X is a downset, it follows that

$$\{w, z_1 w, \dots, z_t w, z_1, \dots, z_t\} \subseteq N_G(v_1).$$

If there are no repeated elements, then $\deg(v_1) \geqslant 2t + 1$. But in any case, we get $\deg(v_1) \geqslant t + 1$ since $w, z_1 w, \dots, z_t w$ are pairwise distinct. □

Corollary 45. *All vertices in G of maximal degree belong to $V \cap P$. Moreover, for any $r \geqslant 1$, the subset of vertices of G of degree r forms an antichain under divisibility.* □

4.4 On loopy and nonloopy vertices

Definition 46. Let $z \in S^*$. We define the *length* of z to be the largest integer $t \geqslant 1$ such that $z = x_1 \dots x_t$ with $x_1, \dots, x_t \in S^*$. We then write $t = \operatorname{len}(z)$.

In particular, $\operatorname{len}(z) = 1$ if and only if $z \in P$. Since X is a downset, it follows that if $z \in X$, then $\operatorname{len}(z)$ coincides with the largest integer $t \geqslant 1$ such that $z = x_1 \dots x_t$ with $x_1, \dots, x_t \in X$.

Proposition 47. *All vertices in $V \cap D$ of maximal length are nonloopy.*

Proof. Let $u \in V \cap D$ be of maximal length, say $t \geqslant 2$. Let $x \in V \cap P$ be a proper factor of u, say $u = xv$ with $v \in X$. Assume for a contradiction that u is loopy. Then $u^2 \in X$. Since $u^2 = xvu$ and $v \in X$, it follows that $xu \in V \cap D$ and $\operatorname{len}(xu) \geqslant t+1$. This contradicts the maximality of t. Therefore u is a nonloopy vertex of G, as claimed. □

Corollary 48. *If all vertices in G are loopy, then $V \cap D = \emptyset$, i.e. $V \subset P$.* □

Lemma 49. *Let $y \in V$ be a nonloopy vertex. Then y divides none of its neighbors in G.*

Proof. Let $z \in N_G(y)$, and assume that $z = yz'$ with $z' \neq 1$. We have $yz \in X$ since y, z are neighbors. Hence $y^2z' \in X$, implying $y^2 \in X$ and thus contradicting that y is a nonloopy vertex. □

Lemma 50. *Every proper factor of a loopy vertex is loopy.*

Proof. Let $u \in V$ and assume that u is loopy. Hence $u^2 \in X$. Let $v \in V$ be a proper factor of u. Since X is stable under taking proper factors, it follows that $v^2 \in X$. Whence v is loopy. □

Lemma 51. *If $\lambda(G) = 1$, then the unique loopy vertex $u \in V$ is primitive.*

Proof. We have $u^2 \in X$ since u is loopy. If $u \in D$, then $u = ab$ with $a, b \in X$. Therefore $a^2 \in X$, so that a is also a loopy vertex, and we are done since $a \neq u$. □

4.5 More on $V \cap P$ and $V \cap D$

Proposition 52. *We have $|V \cap D| \geqslant \deg(u)$ for all $u \in V \cap D$. If $V \cap D = \{u\}$, then $N_G(u) = \{x\}$ for some $x \in V \cap P$, and $u = x^2$.*

Proof. Let $u \in V \cap D$. We have $u = wv$ for some $w \in V$. Let $t = \deg(u)$ and denote

$$N_G(u) = \{z_1, \ldots, z_t\}.$$

Since $z_iu = z_iwv \in X \cap D$ for all i by hypothesis, it follows that

$$\{z_1w, \ldots, z_tw\} \subseteq V \cap D,$$

whence $|V \cap D| \geqslant t$. Assume now $V \cap D = \{u\}$ with $u = wv$ as above. Since $|V \cap D| = 1$, it follows from the above that $t = 1$, whence $N_G(u) = \{z_1\}$. Thus $z_1wv \in X \cap D$, implying $\{z_1w, z_1v, wv\} \subseteq V \cap D$. Therefore $z_1w = z_1v = wv$, whence $z_1 = w = v$ and $u = z_1^2$. Moreover $z_1 \in P$, for if z_1 had proper factors in V, this would imply $z_1 \in V \cap D$, contradicting the equality $V \cap D = \{z_1^2\}$. □

Proposition 53. *We have $|X \cap D| \leqslant |E(G)| - \deg(u)$ for all $u \in V \cap D$ such that $u \neq x^2$ with $x \in P$.*

Proof. Let $u \in V \cap D$ be such that $u \neq x^2$ with $x \in P$. Let $x \in V \cap P$ be a primitive factor of u, so that $u = wx$ for some $w \in V$ with $w \neq x$. Set $t = \deg(u)$ and $N_G(u) = \{z_1, \ldots, z_t\}$. Then $z_iu = z_iwx \in X \cap D$ for all i. For all i, the edges $\{z_ix, w\}$ and $\{x, z_iw\}$ are distinct since $x \notin \{z_ix, w\}$ but have the same weight z_iwx. Since $z_iwx \neq z_jwx$ for $i \neq j$, it follows from Proposition 25 that $|X \cap D| \leqslant |E(G)| - \deg(u)$ as desired. □

Proposition 54. *If $|X \cap D| = |E(G)|$, then any edge $\{u, v\}$ not contained in $V \cap P$ is of the form $\{x, x^2\}$ with $x \in V \cap P$ and x^2 a leaf with unique neighbor x.*

Proof. By Proposition 25, the hypothesis $|X \cap D| = |E(G)|$ implies that *distinct edges have distinct weights.* Let $\{u, v_1v_2\}$ be an edge with $v_1, v_2 \in V$. Thus $uv_1v_2 \in X$, so that $\{u, v_1v_2\}$, $\{v_1, uv_2\}$ and $\{v_2, uv_1\}$ are all edges in G with same weight uv_1v_2. Hence these edges coincide, so that $u = v_1 = v_2$ and the edge is $\{u, u^2\}$. Thus $u^3 \in X$. Now if u were not primitive, say if $u = u_1u_2$ with $u_1, u_2 \in V$, then $u_1^3u_2^3 \in X$, and this would yield at least two distinct edges with same weight, e.g. $\{u_1, u_1^2u_2^3\}$ and $\{u_1^2, u_1u_2^3\}$. Hence $u \in V \cap P$, as claimed. Finally, let $v \in V$ be a neighbor of u^2. Then $u^2v \in X$, yielding two edges with same weight, namely $\{u, uv\}$ and $\{u^2, v\}$. Hence $\{u, uv\} = \{u^2, v\}$, implying $u = v$. Thus $N_G(u^2) = \{u\}$, as claimed. □

5 Proof of main theorem

Let S be a numerical semigroup in multiplicative notation, arising from a classical numerical semigroup $S_0 \subseteq \mathbb{N}$ via the isomorphism (12). The following notation will be used throughout Section 5.

Notation 55. *The symbols $m, c, q, \rho, P, D, L, X$ usually associated to S_0 will also denote the corresponding objects in S transported from S_0 via* (12). *Further, we denote $G(S) = G = (V, E)$ the graph associated to S, and we set*

$$n = |V|,\ k = \mathrm{vm}(G),\ \nu = \nu(G),\ \lambda = \lambda(G). \tag{13}$$

Note that by definition, we have $\lambda \leqslant k \leqslant n$. This section is devoted to proving Theorem 1, i.e. that Wilf's conjecture holds when $|P| \geqslant m/3$. We shall do so by proving $W(S) \geqslant 0$ in that case using Theorem 37 as a main tool. In fact, we shall almost always obtain the stronger estimate $W(S) \geqslant 1$. See [17, Question 8] or [9, Page 2] for a conjecture about the rare occurrences of the equality $W(S) = 0$.

The proof is divided into several subcases depending mainly on the values of k and λ. Recall that Wilf's conjecture has been shown to hold when $|P| \leqslant 3$ or $q \leqslant 3$, in [13] and [9], respectively. Therefore, throughout the proof, we freely assume $|P| \geqslant 4$ and $q \geqslant 4$, even though these hypotheses may be dispensed of in most subcases.

5.1 A reduction

We first reduce the proof of Theorem 1 to the case $\tau(X) \leqslant 2q - 1$.

Lemma 56. *If Wilf's conjecture holds in case $\tau(X) \leqslant 2q - 1$, then Wilf's conjecture holds in case $|P| \geqslant m/3$.*

Proof. We have $W(S) = |P||L| - c = |P||L| - qm + \rho$. Assume $|P| \geqslant m/3$.

Case I. Assume $|L| \geqslant 3q$. Then $|P||L| \geqslant (m/3)(3q) = mq = c + \rho$. Therefore $W(S) \geqslant \rho$ and we are done.

Case II. Assume $|L| \leqslant 3q - 1$. Since $|L| = q + \tau(X)$, it follows that $\tau(X) \leqslant 2q - 1$. Since Wilf's conjecture is assumed to hold in this case, the proof is complete. □

Proposition 57. *If $\tau(X) \leqslant 2q-1$ and $q \geqslant 4$, then $k \leqslant 4$.*

Proof. We have $2q-1 \geqslant \tau(X) \geqslant k(q-1)/2$, where the second inequality follows from Theorem 37. If $k \geqslant 5$, then $2q-1 \geqslant 5(q-1)/2$, implying $3 \geqslant q$, contrary to our assumption $q \geqslant 4$. $\square$

Thus, we need only examine the cases $k = 0, 1, 2, 3, 4$ to complete the proof of Theorem 1, i.e. that $W(S) \geqslant 0$ in all cases under consideration. We start with $0 \leqslant k \leqslant 2$.

5.2 Proof in cases $k = 0, 1, 2$

$\boxed{\textbf{Case } \boldsymbol{k = 0}.}$ Then $E = \emptyset$ and so $|X \cap D| = 0$. Hence $W(S) \geqslant |P|\tau(X) + \rho \geqslant 0$.

$\boxed{\textbf{Case } \boldsymbol{k = 1}.}$ Then G consists of exactly one loopy vertex, so $n = k = |X \cap D| = 1$. Hence $\tau(X) \geqslant (q-1+\nu)/2$, yielding

$$\begin{aligned} W(S) &= |P|\tau(X) - |X \cap D|q + \rho \\ &\geqslant 4(q-1+\nu)/2 - q + \rho \\ &\geqslant q - 2 + 2\nu + \rho, \end{aligned}$$

and so $W(S) \geqslant 2$ since $q \geqslant 4$ by assumption.

$\boxed{\textbf{Case } \boldsymbol{k = 2}.}$ Then $n \geqslant 2$ and G has at most two loops, i.e. $0 \leqslant \lambda \leqslant 2$. By Theorem 37, we have $\tau(X) \geqslant q - 1 + \nu/2 + (n-2)$, whence

$$W(S) \geqslant |P|(q - 1 + \nu/2 + (n-2)) - |X \cap D|q + \rho. \tag{14}$$

Assume first $|X \cap D| \leqslant 3$. Then using $|P| \geqslant 4$, we have

$$\begin{aligned} W(S) &\geqslant 4(q - 1 + \nu/2 + (n-2)) - 3q + \rho \\ &= q + 4(n-3) + 2\nu + \rho. \end{aligned}$$

Since $n \geqslant 2$ and $q \geqslant 4$, this yields $W(S) \geqslant 0$ and we are done.

Assume now $|X \cap D| \geqslant 4$. Then $n \geqslant 3$.

- The case $\lambda = 2$ cannot occur here since it would imply $n = 2$.
- If $\lambda = 1$, let $x \in V$ be the sole loopy vertex. Since $k < 3$, all true edges are incident to x. Thus all edges of G are of the form $\{x, u\}$ with $u \in V$, and $|E| = |V| = n$. Since x is of largest degree, namely n, it follows that $x \in V \cap P$ by Corollary 45. Since all edges are pairwise adjacent, all edge weights are distinct, whence $|X \cap D| = |E| = n$ by Proposition 25. Hence $V \cap D \subseteq \{x^2\}$ by Proposition 54. It follows that $|V \cap P| \geqslant n - 1$, whence $|P| \geqslant n$ by Lemma 43. Plugging the above information on $|X \cap D|$ and $|P|$ into (14), we get

$$\begin{aligned} W(S) &\geqslant n(q - 1 + \nu/2 + (n-2)) - nq + \rho \\ &= n(n-3) + n\nu/2 + \rho, \end{aligned}$$

and we are done since $n \geqslant 3$.

• Finally, if $\lambda = 0$, then since $|E| \geqslant 4$, G must be a star at a vertex x with at least 3 legs. Hence $x \in V \cap P$. Since x is nonloopy, we have $x^2 \notin X$. The same argument as above, using that all edges of G are of the form $\{x, u\}$ with $u \in V \setminus \{x\}$, yields $|X \cap D| = |E| = n$ and $V \cap D = \emptyset$ here. Hence $|P| \geqslant n + 1$, yielding

$$\begin{aligned} W(S) &\geqslant (n+1)(q - 1 + \nu/2 + (n-2)) - nq + \rho \\ &= q + n(n-3) + n\nu/2 + \rho. \end{aligned}$$

This concludes the proof in case $k = 2$.

5.3 Proof in case $k = 3$

We start with a general remark on loopy graphs H with $\mathrm{vm}(H) = 3$.

Lemma 58. *Let H be a loopy graph such that $\mathrm{vm}(H) = 3$. Then $\lambda(H) \geqslant 1$, and either $K_3 \subset H \subseteq LK_3$, or else all true edges of H share a common vertex.*

Proof. Since $\mathrm{vm}(H)$ is odd, it follows that H has at least one loop. Since $\mathrm{vm}(H) < 4$, any two true edges are adjacent. Therefore, either H contains a triangle, in which case $|V(H)| = 3$ and $1 \leqslant \lambda(H) \leqslant 3$, or else all true edges of H share a common vertex and $1 \leqslant \lambda(H) \leqslant 2$. □

Let us go back to our graph $G = G(S)$. We have $1 \leqslant \lambda \leqslant k = 3 \leqslant n$. In the present case, it follows from Theorem 37 that

$$\tau(X) \geqslant (3(q-1) + \nu)/2 + (n-3). \tag{15}$$

We start with an easy particular case.

Proposition 59. *If $k = 3$ and $|X \cap D| \leqslant 4$, then $W(S) \geqslant 0$.*

Proof. As usual, we assume $|P|, q \geqslant 4$. By (15) we have $\tau(X) \geqslant 3(q-1)/2$. Hence

$$\begin{aligned} W(S) &\geqslant |P| 3(q-1)/2 - 4q + \rho \\ &\geqslant 6(q-1) - 4q + \rho \\ &= 2(q-3) + \rho \\ &\geqslant 2 + \rho. \end{aligned}$$ □

Thus, from now on in this section, we assume $|X \cap D| \geqslant 5$, whence in particular $|E(G)| \geqslant 5$.

• **Case $\boldsymbol{\lambda = 3}$.** Then $n = 3$ and hence $5 \leqslant |X \cap D| \leqslant |E| \leqslant 6$. By (15) we have $\tau(X) \geqslant (3(q-1) + \nu)/2$, and so

$$\begin{aligned} W(S) &= |P|\tau(X) - |X \cap D|q + \rho \\ &\geqslant |P|(3(q-1) + \nu)/2 - |X \cap D|q + \rho \end{aligned}$$

$$\geqslant \quad 6(q-1)+2\nu-|X\cap D|q+\rho.$$

$\circ$ Assume first $|X\cap D|=6$. Then $|E|=6$, so that G is isomorphic to LK_3 and so all six edges are active. (See Definition 20.) Moreover, all edge weights are distinct since $|X\cap D|=|E|$ here. The above inequalities imply

$$W(S)\geqslant -6+2\nu+\rho.$$

– If $\nu=0$, then all six edges of G are weak, whence $\rho\geqslant 6$ by Corollary 34. It follows that $W(S)\geqslant 0$ and we are done.

– If $\nu=1$ then all edges of G, except exactly one loop, are weak. Therefore $\rho\geqslant 5$, whence $W(S)\geqslant 1$.

– If $\nu=2$, then since $\nu<3=k$, all three matchings of $G=LK_3$ have a weak edge. Hence $\rho\geqslant|E_0(G)|\geqslant 3$. It follows that $W(S)\geqslant -6+4+3=1$.

– Finally, if $\nu=3$ then $W(S)\geqslant\rho$ and we are done.

$\circ$ Assume now $|X\cap D|=5$. Then $|E|=5$ or 6. We now have

$$W(S)\geqslant q-6+2\nu+\rho. \tag{16}$$

Moreover, since G coincides here with either LK_3 or LK_3 minus a true edge, all edges of G are active as easily seen.

– If $\nu=0$, all active edges are weak, whence $\rho\geqslant 4$. Hence (16) implies $W(S)\geqslant 2$ and we are done.

– If $\nu\geqslant 1$ then (16) implies $W(S)\geqslant\rho$ and we are done.

• **Case $\boldsymbol{\lambda=2}$.** Let x_1,x_2 denote the two loopy vertices. At the very least, besides its two loops, G has one true edge adjacent to exactly one of the loopy vertices, say x_1. Now, either G is contained in the graph with the edge $\{x_1,x_2\}$ plus pendant edges incident to x_1, or else G is contained in LK_3 minus one loop, in which case $n=3$ and $|E(G)|\leqslant 5$.

$\circ$ Assume first that G is contained in the graph with the edge $\{x_1,x_2\}$ plus $n-2$ pendant edges incident to x_1. Among the n vertices, at most two belong to $V\cap D$. Hence $|V\cap P|\geqslant n-2$, so that $|P|\geqslant n-1$ by Lemma 43, and more precisely $|P|\geqslant\max(n-1,4)$. We have $|E(G)|\leqslant 3+(n-2)=n+1$, so that $|X\cap D|\leqslant n+1$. By (15), it follows that

$$W(S)\geqslant\max(n-1,4)((3(q-1)+\nu)/2+(n-3))-(n+1)q+\rho.$$

– If $n\geqslant 5$, we get

$$\begin{aligned}
W(S) &\geqslant (n-1)((3(q-1)+\nu)/2+(n-3))-(n+1)q+\rho\\
&\geqslant (n-5)(q-1)/2+(n-1)(\nu/2+n-4)-2+\rho\\
&\geqslant 4(\nu/2+1)-2+\rho\\
&= 2\nu+2+\rho.
\end{aligned}$$

– If $3\leqslant n\leqslant 4$, and using $|P|\geqslant 4$, we get

$$W(S) \quad\geqslant\quad 4((3(q-1)+\nu)/2+(n-3))-(n+1)q+\rho$$

$$
\begin{aligned}
&= 6(q-1) + 2\nu + 4(n-3) - (n+1)(q-1) - (n+1) + \rho \\
&= (5-n)(q-4) + 2\nu + 6 - 4 + \rho \\
&\geqslant 2\nu + 2 + \rho.
\end{aligned}
$$

∘ Assume now that G is contained in LK_3 minus one loop. Then $n = 3$ and $|X \cap D| \leqslant |E(G)| \leqslant 5$. Moreover, as easily seen by inspection, at least 4 edges of G are active.

– If $\nu = 0$, all active edges are weak, whence $\rho \geqslant 4$. Hence, with $|X \cap D| \leqslant 5$, it follows from the above that $W(S) \geqslant 2 + \rho$ and we are done.

– Assume now $\nu \geqslant 1$. By (15), we have

$$\tau(X) \geqslant (3(q-1) + \nu)/2.$$

It follows that

$$
\begin{aligned}
W(S) &\geqslant 4((3(q-1) + \nu)/2 - 5q + \rho \\
&= 6(q-1) + 2\nu - 5q + \rho \\
&= q - 6 + 2\nu + \rho \\
&\geqslant \rho
\end{aligned}
$$

since $q \geqslant 4$ and $\nu \geqslant 1$.

• **Case $\boldsymbol{\lambda = 1}$.** Then G contains one loopy vertex x and one nonincident true edge. If G contains a triangle, then $|E| \leqslant 4$ since $k = 3$, as easily seen. This is incompatible with our current assumption $|X \cap D| \geqslant 5$.

Therefore G is triangle-free. Hence G consists of the loopy vertex x and a star T centered at a distinct vertex y. Since $|E| \geqslant 5$ by our current assumption, T has at least 3 pendant edges. And if T is connected to x, then the connecting edge is between y and x, for otherwise we would have $k \geqslant 4$. In any case, we have $|E| \leqslant n + 1$.

We claim that $V \subset P$. First $y \in V \cap P$ since it has maximal degree. We also have $x \in V \cap P$. For otherwise, since x is loopy, we have $x^2 \in X \cap D$, whence any proper factor of x would also be a loopy vertex in G by Lemma 50, contradicting $\lambda = 1$. The remaining vertices are all of degree 1 and connected to y, thus they form an antichain for divisibility. Hence, if any such vertex z pertained to $V \cap D$, it would be a monomial in x, y of length at least 2. Now by Lemma 49, z cannot be divisible by y. Hence z is equal to or divisible by x^2. Thus $yx^2 \in X$, implying $xy \in V$ and connected to x. But this is impossible since $N_G(x) \subseteq \{x, y\}$.

By the above and Lemma 43, it follows that $|P| \geqslant n+1$. Using $|X \cap D| \leqslant |E| \leqslant n+1$ as shown earlier, we have

$$
\begin{aligned}
W(S) &\geqslant |P|\tau(X) - |X \cap D|q + \rho \\
&\geqslant (n+1)\tau(X) - (n+1)q + \rho \\
&\geqslant (n+1)(\tau(X) - q) + \rho.
\end{aligned}
$$

But $\tau(X) > q$, since $\tau(X) \geqslant 3(q-1)/2$ and $q \geqslant 4$. Hence $W(S) \geqslant \rho \geqslant 0$.

The proof of the main theorem in the particular case $k = 3$ is now complete.

5.4 Proof in case $k = 4$

By Proposition 57, the value $k = 4$ is the largest admissible one for $k = \mathrm{vm}(G)$ under the assumption $\tau(X) \leqslant 2q - 1$.

Then $n \geqslant 4$, and the general bound $\tau(X) \geqslant (k(q-1)+\nu)/2 + (n-k)$ yields

$$\begin{aligned} \tau(X) &\geqslant 2(q-1) + \nu/2 + (n-4) \\ &= 2(q-3) + \nu/2 + n. \end{aligned}$$

This puts strong restrictions on n and ν.

Lemma 60. *Assume $\tau(X) \leqslant 2q-1$ and $k = 4$. Then $n \in \{4,5\}$ and $\nu \leqslant 2$. If $n = 5$, then $\nu = 0$ and $\tau(X) = 2q-1$.*

Proof. We have $2(q-3) + \nu/2 + n \leqslant \tau(X) \leqslant 2q-1$. Hence $\nu/2 + n \leqslant 5$. It follows that $n \leqslant 5$ and that $\nu \leqslant 2$ since $n \geqslant 4$. If $n = 5$, then $\nu = 0$ and the above bounds on $\tau(X)$ yield $2(q-3)+5 \leqslant \tau(X) \leqslant 2q-1$, whence $\tau(X) = 2q-1$. □

5.4.1 The subcase $k = 4, n = 5$

Throughout this section, we fix the following values of the various parameters and refer to these hypotheses as the *current case*:

$$n = |V(G)| = 5,\ k = \mathrm{vm}(G) = 4,\ \tau(X) \leqslant 2q-1. \tag{17}$$

Then $\nu = 0$ and $\tau(X) = 2q-1$ as seen above. In particular, the former implies that *all active edges are weak*. This will imply useful lower bounds on $\rho = qm - c$ and hence on $W(S)$.

We shall need an upper bound on the number of edges of G, actually valid in a general graph-theoretic setting.

Proposition 61. *Let $H = (V,E)$ be a loopy graph. If $|V| = 5$ and $\mathrm{vm}(H) = 4$, then $|E| \leqslant 10$.*

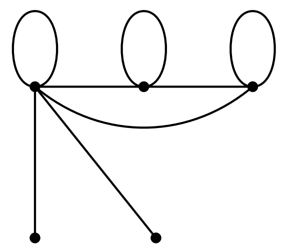
Proof. Set $V = V_1 \sqcup V_2$, where V_1 is the set of loopy vertices and $V_2 = V \setminus V_1$. Let $E = E_1 \sqcup E_2 \sqcup E_{1,2}$, where E_1 is the set of edges of the induced subgraph $H[V_1]$, E_2 is the edge set of $H[V_2]$ and $E_{1,2} = [V_1, V_2]$, the set of edges from V_1 to V_2. We further denote $H_1 = H[V_1]$, $H_2 = H[V_2]$ and $H_{1,2}$ the bipartite graph with edge set $E_{1,2}$.

The proof proceeds by fixing the loop number $\lambda = \lambda(H) = |V_1|$ and letting it assume all possible values from $\mathrm{vm}(H) = 4$ to 0.

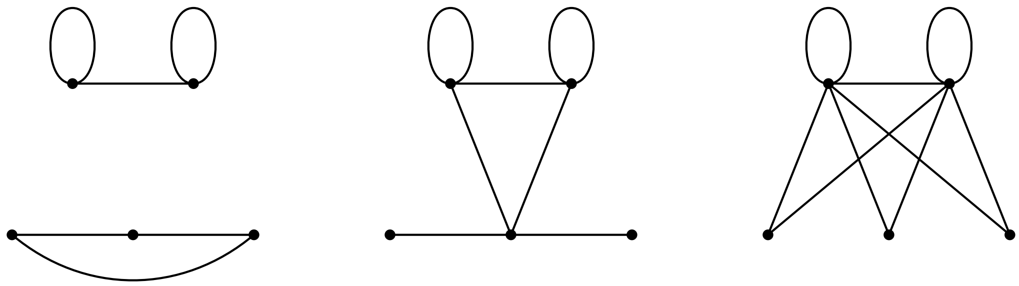
The case $\lambda(H) = 4$ is impossible. For otherwise, since V_2 would consist of a single nonisolated nonloopy vertex y_1, there would be a true edge incident with y_1 and a loopy vertex $x_1 \in V_1$. But then, that edge and the three loops at the other three vertices in V_1 would constitute a matching touching 5 vertices, contrary to the hypothesis $k = 4$.

Assume $\lambda(H) = 3$. We claim $|E| \leqslant 8$. Set $V_1 = \{x_1, x_2, x_3\}$, $V_2 = \{y_1, y_2\}$. Since $\mathrm{vm}(H_1) = 3$, we must have $\mathrm{vm}(H_2) \leqslant 1$, whence $\mathrm{vm}(H_2) = 0$ since H_2 has no loops.

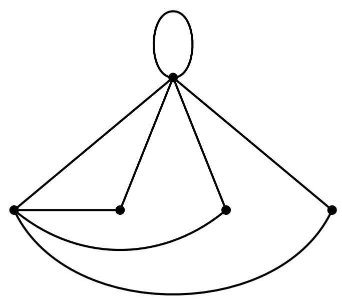
Thus y_1, y_2 are not neighbours in H, i.e. $|E_2| = 0$. Up to renumbering of V_1, we may assume $x_1 \in N_H(y_1)$. We claim then that $N_H(y_1) = N_H(y_2) = \{x_1\}$. Indeed, since y_2 is not isolated, it must have a neighbour in V_1. But if y_2 had a neighbor other than x_1, say x_2, then the edges $\{x_1, y_1\}$, $\{x_2, y_2\}$ and the loop at x_3 would yield $\mathrm{vm}(H) = 5$, contrary to the hypothesis. Therefore $N_H(y_2) = \{x_1\}$. By symmetry, we get $N_H(y_1) = \{x_1\}$ as well. Thus $|E_{1,2}| = 2$. Since $|E_1| \leqslant 6$, we conclude $|E| \leqslant 8$ in the present case. The case $|E| = 8$ is uniquely realized, up to isomorphism, by the following loopy graph:



Assume $\lambda(H) = 2$. We claim $|E| \leqslant 9$. Indeed, as easily seen, there are exactly three isomorphism classes of edge-maximal loopy graphs H with the given parameters. These classes have 6, 7 and 9 edges, respectively:



Assume $\lambda(H) = 1$. We claim $|E| \leqslant 8$. Indeed, the unique isomorphism class of edge-maximal loopy graphs H with the given parameters is the following one, with 8 edges:



Assume $\lambda(H) = 0$. Then $|E| \leqslant 10$. Indeed, the complete graph K_5 is the unique edge-maximal simple graph with the given parameters. $\square$

Let us go back to our graph $G = G(S) = (V, E)$. Since $|X \cap D| \leqslant |E|$, the above result implies $|X \cap D| \leqslant 10$. We start with a reduction to the case $|X \cap D| \in \{8, 9\}$.

Proposition 62. *In the current case* (17)*, if either* $|X \cap D| \leqslant 7$*, or* $V \subset P$*, or* $|X \cap D| \geqslant 10$*, then* S *satisfies Wilf's conjecture.*

Proof.
- Assume $|X \cap D| \leqslant 7$. We have $W(S) \geqslant |P|(2q-1) - 7q + \rho$. Our assumptions $|P|, q \geqslant 4$ further yield $W(S) \geqslant 4(2q-1) - 7q + \rho = q - 4 + \rho \geqslant \rho$ and we are done.
- Assume $V \subset P$. Then $|P| \geqslant |V| + 1 = 6$. Hence, using $|X \cap D| \leqslant 10$, we have

$$\begin{aligned} W(S) &\geqslant |P|\tau(X) - |X \cap D|q + \rho \\ &\geqslant 6(2q-1) - 10q + \rho \\ &= 2q - 6 + \rho. \end{aligned}$$

Since $q \geqslant 4$ in the current case, we get $W(S) \geqslant 2 + \rho$ and we are done.
- Assume $|X \cap D| \geqslant 10$. By Proposition 61, we have $|E| \leqslant 10$. Whence $|E| = 10$ since $|E| \geqslant |X \cap D| \geqslant 10$. Moreover, it follows from the proof of that Proposition that the only case where $|E| = 10$ is $G = K_5$. Since G is regular, it follows from Corollary 45 that $V \subset P$. Thus S satisfies Wilf's conjecture by the previous case. □

We next assume $|V \cap D| = 1$.

Proposition 63. *In the current case* (17)*, if* $|V \cap D| = 1$ *then* S *satisfies Wilf's conjecture.*

Proof. The hypotheses imply $|V \cap P| = 4$, whence $|P| \geqslant 5$. Moreover, by the previous result, we may assume $|X \cap D| \leqslant 9$. Then

$$\begin{aligned} W(S) &\geqslant |P|\tau(X) - |X \cap D|q + \rho \\ &\geqslant 5(2q-1) - 9q + \rho \\ &= q - 5 + \rho. \end{aligned}$$

Since $\nu = 0$, and since there is a vertex-maximal matching touching 4 vertices, it follows that there are at least two active weak edges. Corollary 34 then implies $\rho \geqslant 1$, and we conclude $W(S) \geqslant 0$ as desired. □

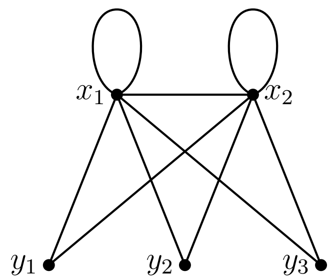
It remains to treat the case $|V \cap D| \geqslant 2$ and $|X \cap D| \in \{8, 9\}$. From here, we again proceed by descending values of $\lambda(G)$ from 4 to 0. The case $\lambda = 4$ is impossible in the present context.

Assume $\lambda = 3$. Let x_1, x_2, x_3 be the loopy vertices and y_1, y_2 the nonloopy ones. We have seen that $|E| \leqslant 8$ in this case. But since $|X \cap D| \geqslant 8$, it follows that $|X \cap D| = |E| = 8$. The only way to achieve this, up to isomorphism, is that G contains LK_3 on the vertices x_1, x_2, x_3 with y_1, y_2 linked to x_1. (See corresponding picture in the proof of Proposition 61.) We have $x_1 \in P$ since it is of highest degree. Since $|V \cap D| \geqslant 2$ by assumption, it follows from Proposition 54 that $V \cap D$ consists of leaves, each of the form x^2 with $x \in V \cap P$ as unique neighbor. Therefore $V \cap D = \{y_1, y_2\}$, and since both have x_1 as unique neighbor, this implies $y_1 = y_2 = x_1^2$, an absurdity since y_1, y_2 are distinct. Hence the present case, namely $n = 5$, $k = 4$, $|X \cap D| \geqslant 8$, $|V \cap D| \geqslant 2$ and $\lambda = 3$, cannot occur.

Assume $\lambda = 2$. Let x_1, x_2 be the loopy vertices and y_1, y_2, y_3 the nonloopy ones. We have seen that $|E| \leqslant 9$ in this case. If $|E| = 9$, then G is the join between LK_2 and $\overline{K_3}$, i.e.

$$G = LK_2 \vee \overline{K_3}$$

as pictured here:



Incidentally, note that this graph realizes the first occurrences of $W_0(S) < 0$. (See [11] for more details.) We further assume $|V \cap D| \geqslant 2$. We claim that

$$V \cap D = \{y_1, y_2, y_3\} = \{x_1^2, x_1x_2, x_2^2\}. \tag{18}$$

Indeed, by Corollary 45, the x_i belong to $V \cap P$ since they have maximal degree 5, and the y_i constitute an antichain for divisibility since they all have degree 2. Hence the vertices in $V \cap D$ are monomials in x_1, x_2. By symmetry, we may assume $y_1 \in V \cap D$ and $y_1 = x_1u$ for some $u \in V$. Since $\{x_1, y_1\} \in E$, it follows that $x_1^2u \in X$. Hence $x_1^2 \in V \cap D$. Up to symmetry again, we may assume $y_1 = x_1^2$. Since $\{x_2, y_1\} \in E$, we have $x_1^2x_2 \in X$, whence $x_1x_2 \in V \cap D$. Say $y_2 = x_1x_2$. Since $\{x_2, y_2\} \in E$, it follows that $x_1x_2^2 \in X$. Hence $x_2^2 \in V \cap D$, implying $y_3 = x_2^2$. This proves (18), as claimed. Now, even though $|E| = 9$ here, Proposition 25 implies $|X \cap D| \leqslant 7$ since two pairs of edges have the same weight, namely

$$\begin{aligned} \mathrm{wt}(\{x_1, x_1x_2\}) &= \mathrm{wt}(\{x_1^2, x_2\}), \\ \mathrm{wt}(\{x_2, x_1x_2\}) &= \mathrm{wt}(\{x_2^2, x_1\}). \end{aligned}$$

Therefore this case is settled by Proposition 62.

Assume now $|E| = |X \cap D| = 8$, and still $|V \cap D| \geqslant 2$ of course. Then G is obtained by suppressing an edge from the graph $LK_2 \vee \overline{K_3}$ above. By Proposition 54, the vertices in $V \cap D$ must all be of degree one. However, in G, at most one vertex has degree one as easily seen. Therefore this case is impossible.

Assume $\lambda = 1$. Then $|E| = |X \cap D| = 8$ again. As seen above, G is the join $LK_1 \vee T$ of a loop LK_1 with a claw T. However, this case is again made impossible by Proposition 54 since there are no vertices of degree 1.

Assume $\lambda = 0$. Again, we may assume $|X \cap D| \in \{8, 9\}$ and $|V \cap D| \geqslant 2$. We have $G \subseteq K_5$ since it has 5 vertices and no loops.

The case $G = K_5$ is impossible, for it would imply $V \subset P$, contrary to our hypotheses. Hence $|E| \in \{8, 9\}$ and G is obtained by removing 1 or 2 edges from K_5.

If $|X \cap D| = |E|$, then Proposition 54 implies that the vertices in $V \cap D$ have degree 1. But G has no vertices of degree less than 2, so this case is impossible.

It remains to consider the case $|X \cap D| = 8$, $|E| = 9$. Thus G is K_5 minus one edge, i.e. $G = K_3 \vee \overline{K_2}$. Its degree distribution is $(3, 3, 4, 4, 4)$. Hence $|V \cap P| = 3$, $|V \cap D| = 2$. Set $V \cap P = \{x_1, x_2, x_3\}$, $V \cap D = \{y_1, y_2\}$. Then y_1, y_2 are monomials in x_1, x_2, x_3. Assume y_1 is divisible by x_j for some j, so $y_1 = x_j v$ for some $v \in V$. Since y_1, x_j are neighbors, it follows that $x_j y_1 \in X$, whence $x_j^2 v \in X$, whence $x_j^2 \in X$. Therefore x_j is a loopy vertex, in contradiction with the hypothesis $\lambda = 0$. Hence this case is impossible as well.

This completes the verification of Wilf's conjecture in case $k = 4$, $n = 5$ and $\tau(X) \leqslant 2q - 1$.

5.4.2 The subcase $k = 4, n = 4$

Throughout this section, the *current case* is given by the following hypotheses:

$$n = |V(G)| = 4, \ k = \mathrm{vm}(G) = 4, \ \tau(X) \leqslant 2q - 1. \tag{19}$$

This implies

$$\tau(X) \geqslant 2(q-1) + \nu/2 \tag{20}$$

and $\nu \leqslant 2$ in this context, as seen above. We have $|E| \leqslant 10$, the number of edges of LK_4.

Proposition 64. *In the current case* (19)*, if either* $|X \cap D| \leqslant 6$ *or* $V \subset P$*, then* S *satisfies Wilf's conjecture.*

Proof. As above, we freely assume $|P|, q \geqslant 4$.

• Assume $|X \cap D| \leqslant 6$. Then

$$\begin{aligned} W(S) &\geqslant |P|(2(q-1) + \nu/2) - 6q + \rho \\ &\geqslant 8(q-1) + 2\nu - 6q + \rho \\ &= 2q - 8 + 2\nu + \rho \\ &\geqslant 2\nu + \rho \end{aligned}$$

and we are done.

• Assume $V \subset P$. Then $|P| \geqslant 5$ here. Thus

$$\begin{aligned} W(S) &\geqslant 5\big(2(q-1) + \nu/2\big) - |X \cap D|q + \rho \\ &= (10 - |X \cap D|)q + 5\nu/2 - 10 + \rho. \end{aligned}$$

We now examine separately the cases $|X \cap D| = 10, 9, 8, 7$.

∘ If $|X \cap D| = 10$, then $|E| = 10$ and $G = LK_4$. Then

$$W(S) \geqslant 5\nu/2 - 10 + \rho.$$

Since $G = LK_4$, all 10 edges are active.

– If $\nu = 0$, then all edges are weak, i.e. $E = E_0^+$. We have $\rho \geqslant \mathrm{wt}(E_0)$, and since wt is a bijection here, this implies $\rho \geqslant 10$. Hence $W(S) \geqslant 0$ if $\nu = 0$.

– If $\nu = 1$, then exactly one vertex is touched by a normal edge. Hence all edges are weak except one loop. It follows that $\rho \geqslant 9$, whence $W(S) \geqslant 5/2 - 10 + 9$, implying $W(S) \geqslant 2$.

– Finally, if $\nu = 2$, then at most 2 vertices are touched by a normal edge. Hence at most 3 edges are normal, and so at least 7 edges are weak. It follows that $\rho \geqslant 7$. Hence $W(S) \geqslant 5 - 10 + 7 = 2$. This completes the case $|X \cap D| = 10$.

∘ If $|X \cap D| = 9$, then $W(S) \geqslant q - 10 + 5\nu/2 + \rho$. Then here also, each edge is active.

– If $\nu = 0$, then all edges are weak, hence $\rho \geqslant 9$. Thus $W(S) \geqslant -6 + 9 = 3$.

– If $\nu = 1$, then exactly one loop is normal. Hence there are at least 8 weak active edges, so that $\rho \geqslant 8$. Thus $W(S) \geqslant -6 + 5/2 + 8$, implying $W(S) \geqslant 5$.

– Finally, if $\nu = 2$, then at most 2 vertices are touched by normal edges, hence at most 3 edges are normal. Hence there are at least 6 active weak edges, implying $\rho \geqslant 6$. Hence $W(S) \geqslant 5$ and we are done for the case $V \subset P$, $|X \cap D| = 9$.

∘ If $|X \cap D| = 8$, then $W(S) \geqslant 2q - 10 + 5\nu/2 + \rho \geqslant -2 + 5\nu/2 + \rho$. Then G is LK_4 with at most 2 missing edges. Then, as easily seen by examining the various possibilities for G, it is straightforward to check that G contains at least 7 active edges in each case.

– If $\nu = 0$, then the above implies $\rho \geqslant 7$, and so $W(S) \geqslant -2 + \rho \geqslant 5$.

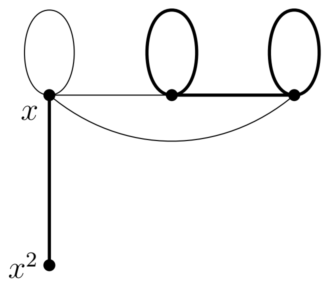
– If $\nu \geqslant 1$, then $W(S) \geqslant -2 + 5\nu/2 + \rho \geqslant 1 + \rho$ and we are done.

∘ If $|X \cap D| = 7$, then $W(S) \geqslant 3q - 10 + 5\nu/2 + \rho \geqslant 2 + 5\nu/2 + \rho$ and we are done. This completes the proof of the proposition. □

Having settled the case $|V \cap D| = 0$, we now tackle the case $|V \cap D| = 1$.

Proposition 65. *In the current case* (19)*, if* $|V \cap D| = 1$ *then* S *satisfies Wilf's conjecture.*

Proof. Set $V \cap D = \{u\}$. It follows from Proposition 52 that $u = x^2$ with $x \in P$ as its sole neighbor. Hence u is a nonloopy vertex and $\deg(u) = 1$. The latter implies $|E| \leqslant 7$. Since $|X \cap D| \leqslant |E|$ and the case $|X \cap D| \leqslant 6$ has already been settled, it remains to examine the case $|X \cap D| = |E| = 7$. Therefore G consists of LK_3 with x as one of the vertices, to which a pendant edge is attached with endvertex $u = x^2$:



Note that G has exactly 4 active edges, the thicker ones in the picture. We have

$$W(S) \geqslant 8(q-1) + 2\nu - 7q + \rho = q - 8 + 2\nu + \rho.$$

– If $\nu = 0$ then all active edges of G are weak. Since wt is a bijection here, it follows that $\rho \geqslant 4$. Hence $W(S) \geqslant 0$, as desired.

– If $\nu = 1$ then all active edges are weak, except for one normal loop. It follows that $\rho \geqslant 3$ and that $W(S) \geqslant 1$.

– If $\nu \geqslant 2$ then $W(S) \geqslant \rho$ since $q \geqslant 4$. □

It remains to consider the cases $|V \cap D| = 2, 3$.

Proposition 66. *In the current case* (19)*, if* $|V \cap D| \geqslant 2$ *then* S *satisfies Wilf's conjecture.*

Proof. Assume first $|V \cap D| = 2$. Set $V \cap P = \{x_1, x_2\}$ and $V \cap D = \{u_1, u_2\}$. Thus u_1, u_2 are monomials in x_1, x_2. We claim that $|X \cap D| \leqslant 6$. Indeed, as V is a downset, the only possibilities up to symmetry are

$$\{u_1, u_2\} \ = \ \{x_1^2, x_1x_2\},\ \{x_1^2, x_2^2\},\ \{x_1^3, x_1^2\}.$$

Now, since all proper factors of the elements of $X \cap D$ are vertices by Lemma 41, the corresponding only possibilities for $X \cap D$ are

$$\{x_1^3, x_1^2x_2, x_1^2, x_1x_2, x_2^2\},\ \{x_1^3, x_2^3, x_1^2, x_1x_2, x_2^2\},\ \{x_1^4, x_1^3, x_1^2, x_1^2, x_1x_2, x_2^2\},$$

respectively, as is straightforward to check. For instance, if $\{u_1, u_2\} = \{x_1^2, x_1x_2\}$, then $x_1x_2^2$ cannot belong to $X \cap D$ since its proper factor x_2^2 is not in V. This concludes the proof of the claim, and hence of the case $|V \cap D| = 2$ by Proposition 64.

Assume finally $|V \cap D| = 3$. Set $V \cap P = \{x\}$. Then again, since V is a downset and made of monomials in x, it follows that $V \cap D = \{x^2, x^3, x^4\}$. Therefore $X \cap D = \{x^2, x^3, x^4, x^5\}$ and we are done again. □

This concludes our proof of Theorem 1. We close this section with a straightforward consequence.

Corollary 67. *Wilf's conjecture holds for all numerical semigroups of multiplicity* $m \leqslant 12$.

Proof. Let S be a numerical semigroup of multiplicity $m \leqslant 12$. If $|P| \leqslant 3$ then S satisfies Wilf's conjecture by [13]. If $|P| \geqslant 4$ then $|P| \geqslant m/3$ since $m \leqslant 12$, and we conclude with Theorem 1. □

Remark 68. Corollary 67 has just been improved with a *verification of Wilf's conjecture up to multiplicity* $m \leqslant 18$, by computer calculations with a specially developed algorithm based on the Kunz polytope and polyhedral geometry [2].

6 Equivalence of numerical semigroups

In this section, we investigate the range of the map $S \mapsto G(S)$ and we briefly consider its fibers.

6.1 Realizability

Given any loopy graph G, is there a numerical semigroup S such that $G(S)$ is isomorphic to G? The answer is given below.

We first recall a notation from [9]. If $x_1, \dots, x_n, t$ are positive integers, we denote by $\langle x_1, \dots, x_n \rangle_t$ the numerical semigroup defined as follows:

$$\langle x_1, \dots, x_n \rangle_t = \langle x_1, \dots, x_n \rangle \cup [t, \infty[.$$

This construction makes sense even if the x_i are not globally coprime. Note that the conductor c of $\langle x_1, \dots, x_n \rangle_t$ satisfies $c \leqslant t$.

Theorem 69. *Let $G = (V, E)$ be a loopy graph. Then there exist infinitely many numerical semigroups S such that $G(S)$ is isomorphic to G.*

Proof. Set $n = |V|$. Take m sufficiently large, and choose any integer sequence $x_1, \dots, x_n$ satisfying the following two conditions:

- $m/3 \leqslant x_1 < \cdots < x_n < (m-1)/2$,
- the $x_i + x_j$ are pairwise distinct.

Then the $n + \binom{n+1}{2}$ elements of the set

$$\{x_1, \dots, x_n\} \cup \{x_i + x_j \mid 1 \leqslant i \leqslant j \leqslant n\}$$

are *pairwise distinct mod m*. This is because $x_i \in [m/3, (m-1)/2[$ and $x_i + x_j \in [2m/3, m-1[$ for all i, j. Let

$$S_0 = \langle m, m + x_1, \dots, m + x_n \rangle_{2m}.$$

The above directly implies $G(S_0) = LK_n$. To obtain G itself, we need only erase in LK_n those edges not belonging to G. For each edge $\{m + x_i, m + x_j\}$ to be erased, it suffices to add to S_0 the new generator $m + x_i + x_j$. This will yield S such that $G(S) = G$. Details are left as an exercise to the reader. $\square$

For instance, here are realizations of the complete loopy graph LK_n as $G(S)$ for infinitely many numerical semigroups S. For a subset A in $\mathbb{Z}$ or $\mathbb{Z}/m\mathbb{Z}$, we denote $2A = A + A = \{a + b \mid a, b \in A\}$.

Example 70. The graph LK_3 is realized by the numerical semigroup

$$S = \langle m, m+1, m+3, m+7 \rangle_{2m}$$

with the condition $2(m+7) \leqslant 2m + (m-1)$, i.e. with $m \geqslant 15$. Setting $A = \{m+1, m+3, m+7\}$, and computing $A \cup 2A$ in $\mathbb{Z}/m\mathbb{Z}$, we have

$$A \cup 2A \equiv \{1,3,7\} \sqcup \{2,4,8,6,10,14\} \bmod m.$$

Since $m \geqslant 15$, these 9 elements are nonzero and pairwise distinct mod m. Moreover, $2A \subseteq [c, c+m[= [2m, 3m-1[$. Hence $G(S)$ is the loopy-complete triangle.

Example 71. More generally, the graph LK_n is realized by the numerical semigroup

$$S = \langle m, m+1, m+3, \ldots, m+2^n-1 \rangle_{2m}$$

with the condition $2(m+2^n-1) \leqslant 2m + (m-1)$, i.e. with $m \geqslant 2^{n+1}-1$. Setting $A = \{m+1, m+3, \ldots m+2^n-1\}$, and computing $A \cup 2A$ in $\mathbb{Z}/m\mathbb{Z}$, we have

$$A \cup 2A \equiv \{1,3,\ldots,2^n-1\} \sqcup \{2,4,8,\ldots,2^{n+1}-2\} \bmod m.$$

6.2 Graph-equivalence

We now briefly consider the fibers of the map $S \mapsto G(S)$.

Definition 72. Let S, S' be two numerical semigroups. We say that S, S' are *graph-equivalent* if their associated graphs $G(S), G(S')$ are isomorphic.

For instance, the class of numerical semigroups S such that $G(S) = \emptyset$ is well known. It coincides with the set of so-called *maximal embedding dimension numerical semigroups*, i.e. those for which $e = m$, where $e = |P|$ is the embedding dimension and m is the multiplicity. Indeed, we have

$$|P| = m \iff P = X \sqcup \{m\} \iff X \cap D = \emptyset,$$

where P, X are the sets of primitive and nonzero Apéry elements of S, respectively.

The following tables give, for all $1 \leqslant g \leqslant 20$,

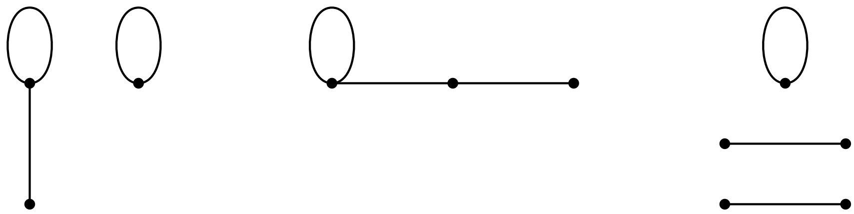
- the number n_g of numerical semigroups of genus g,
- the number γ_g of *equivalence classes* of numerical semigroups of genus g.

g	1	2	3	4	5	6	7	8	9	10	11	12
n_g	1	2	4	7	12	23	39	67	118	204	343	592
γ_g	1	1	2	3	4	6	11	15	27	41	66	115

g	13	14	15	16	17	18	19	20
n_g	1001	1693	2857	4806	8045	13467	22464	37396
γ_g	190	322	569	1014	1761	3107	5475	9621

Those values of γ_g were obtained using the function `IsomorphicGraphQ` in *Mathematica 10*. Needless to say, it would be very interesting to determine the long-term behavior of the sequence γ_g.

For instance, for $g = 7$, the 39 numerical semigroups of genus 7 regroup into $\gamma_7 = 11$ equivalence classes. The eleven nonisomorphic loopy graphs arising this way are the following ones: the empty graph, the two loopy graphs with 1 edge, the five loopy graphs with 2 edges, and three more loopy graphs with 3 edges, namely



We conclude this paper with a question. Can one show *a priori* that if a numerical semigroup S satisfies Wilf's conjecture, then so do all equivalent numerical semigroups $S' \sim S$? For instance, the less dense $G(S)$ is, the easier one may expect checking Wilf's conjecture on S will be. At any rate, the proofs in this paper show that the properties of the graphs $G(S)$ for the numerical semigroups S under consideration play a central role towards this endeavor. Moreover, as observed by an anonymous referee, a positive answer to the above question would in fact solve Wilf's conjecture. Indeed, as shown in the proof of Theorem 69, every numerical semigroup S is graph-equivalent to a numerical semigroup S_0 of depth 2, and the latter are known to satisfy Wilf's conjecture [15, 9].

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