3-uniform hypergraphs without a cycle of length five

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Submitted: Jun 19, 2019; Accepted: Apr 15, 2020; Published: May 1, 2020
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Abstract

In this paper we show that the maximum number of hyperedges in a 3-uniform hypergraph on \( n \) vertices without a (Berge) cycle of length five is less than \((0.254 + o(1))n^{3/2}\), improving an estimate of Bollobás and Győri.

We obtain this result by showing that not many 3-paths can start from certain subgraphs of the shadow.

Mathematics Subject Classifications: 05C65, 05D99

1 Introduction

A hypergraph \( H = (V,E) \) is a family \( E \) of distinct subsets of a finite set \( V \). The members of \( E \) are called hyperedges and the elements of \( V \) are called vertices. A hypergraph is called 3-uniform if each member of \( E \) has size 3. A hypergraph \( H = (V,E) \) is called linear if every two hyperedges have at most one vertex in common.

A Berge cycle of length \( k \geq 2 \), denoted Berge-\( C_k \), is an alternating sequence of distinct vertices and distinct edges of the form \( v_1, h_1, v_2, h_2, \ldots, v_k, h_k \) where \( v_i, v_{i+1} \in h_i \) for each \( i \in \{1, 2, \ldots, k-1\} \) and \( v_k, v_1 \in h_k \). (Note that if a hypergraph does not contain a Berge-\( C_2 \), then it is linear.) This definition of a hypergraph cycle is the classical definition due to Berge. More generally, if \( F = (V(F), E(F)) \) is a graph and \( Q = (V(Q), E(Q)) \) is a hypergraph, then we say \( Q \) is Berge-\( F \) if there is a bijection \( \phi : E(F) \rightarrow E(Q) \) such that
$e \subseteq \phi(e)$ for all $e \in E(F)$. In other words, given a graph $F$ we can obtain a Berge-$F$ by replacing each edge of $F$ with a hyperedge that contains it.

Given a family of graphs $\mathcal{F}$, we say that a hypergraph $\mathcal{H}$ is Berge-$\mathcal{F}$-free if for every $F \in \mathcal{F}$, the hypergraph $\mathcal{H}$ does not contain a Berge-$F$ as a subhypergraph. The maximum possible number of hyperedges in a Berge-$\mathcal{F}$-free 3-uniform hypergraph on $n$ vertices is the Turán number of Berge-$\mathcal{F}$, and is denoted by $ex_3(n, \mathcal{F})$. When $\mathcal{F} = \{F\}$ then we simply write $ex_3(n, F)$ instead of $ex_3(n, \{F\})$.

Determining $ex_3(n, \{C_2, C_3\})$ is basically equivalent to the famous $(6, 3)$-problem. This was settled by Ruzsa and Szemerédi in their classical paper [23], showing that $n^{2 - \frac{1}{\sqrt{3}}} < ex_3(n, \{C_2, C_3\}) = o(n^2)$ for some constant $c > 0$. An important Turán-type extremal result for Berge cycles is due to Lazebnik and Verstraëte [21], who studied the maximum number of hyperedges in an $r$-uniform hypergraph containing no Berge cycle of length less than five (i.e., girth five). They showed the following.

**Theorem 1** (Lazebnik, Verstraëte [21]).

$$ex_3(n, \{C_2, C_3, C_4\}) = \frac{1}{6} n^{3/2} + o(n^{3/2}).$$

The systematic study of the Turán number of Berge cycles started with the study of Berge triangles by Győri [15], and continued with the study of Berge five cycles by Bollobás and Győri [1] who showed the following.

**Theorem 2** (Bollobás, Győri [1]).

$$(1 + o(1)) \frac{n^{3/2}}{3\sqrt{3}} \leq ex_3(n, C_5) \leq \sqrt{2} n^{3/2} + 4.5n.$$

The following example of Bollobás and Győri proves the lower bound in Theorem 2.

**Bollobás-Győri Example.** Take a $C_4$-free bipartite graph $G_0$ with $n/3$ vertices in each part and $(1 + o(1))(n/3)^{3/2}$ edges. In one part, replace each vertex $u$ of $G_0$ by a pair of two new vertices $u_1$ and $u_2$, and add the triple $u_1u_2v$ for each edge $uv$ of $G_0$. It is easy to check that the resulting hypergraph $H$ does not contain a Berge cycle of length 5. Moreover, the number of hyperedges in $H$ is the same as the number of edges in $G_0$.

In this paper, we improve Theorem 2 as follows.

**Theorem 3.**

$$ex_3(n, C_5) < (1 + o(1)) 0.254n^{3/2}.$$
Ergemlidze, Győri and Methuku [3] considered the analogous question for linear hypergraphs and proved that \( \text{ex}_3(n, \{C_2, C_5\}) = n^{3/2}/3\sqrt{3} + o(n^{3/2}) \). Surprisingly, even though their lower bound is the same as the lower bound in Theorem 2, the linear hypergraph that they constructed in [3] is very different from the hypergraph used in the Bollobás-Győri example discussed above – the latter is far from being linear. In [3], the authors also strengthened Theorem 1 by showing that \( \text{ex}_3(n, \{C_2, C_3, C_4\}) \sim \text{ex}_3(n, \{C_2, C_4\}) \).

Recently, \( \text{ex}_3(n, C_4) \) was studied in [5]. See [6] for results on the maximum number of hyperedges in an \( r \)-uniform hypergraph of girth six.

Győri and Lemons [16, 17] generalized Theorem 2 to Berge cycles of any given length and proved bounds on \( \text{ex}_r(n, C_{2k+1}) \) and \( \text{ex}_r(n, C_{2k}) \). These bounds were improved by Füredi and Özkahya [9], Jiang and Ma [19], Gerbner, Methuku and Vizer [11]. Recently Füredi, Kostochka and Luo [7] started the study of the maximum size of an \( n \)-vertex \( r \)-uniform hypergraph without any Berge cycle of length at least \( k \). This study has been continued in [8, 18, 20, 4].

General results for Berge-\( F \)-free hypergraphs have been obtained in [12, 13, 10] and the Turán numbers of Berge-\( K_{2,t} \) and Berge cliques, among others, were studied in [24, 22, 11, 14, 10].

Notation

We introduce some important notations and definitions used throughout the paper.

- Length of a path is the number of edges in the path. We usually denote a path \( v_0, v_1, \ldots, v_k \), simply as \( v_0v_1 \ldots v_k \).
- For convenience, an edge \( \{a, b\} \) of a graph or a pair of vertices \( a, b \) is referred to as \( ab \). A hyperedge \( \{a, b, c\} \) is written simply as \( abc \).
- For a hypergraph \( H \) (or a graph \( G \)), for convenience, we sometimes use \( H \) (or \( G \)) to denote the edge set of the hypergraph \( H \) (or \( G \) respectively). Thus the number of edges in \( H \) is \( |H| \).
- Given a graph \( G \) and a subset of its vertices \( S \), let the subgraph of \( G \) induced by \( S \) be denoted by \( G[S] \).
- For a hypergraph \( H \), let \( \partial H = \{ab \mid ab \subset e \in E(H)\} \) denote its 2-shadow graph.
- For a hypergraph \( H \), the neighborhood of \( v \) in \( H \) is defined as
  \[
  N(v) = \{x \in V(H) \setminus \{v\} \mid v, x \in h \text{ for some } h \in E(H)\}.
  \]
- For a hypergraph \( H \) and a pair of vertices \( u, v \in V(H) \), let \( \text{codeg}(v, u) \) denote the number of hyperedges of \( H \) containing the pair \( \{u, v\} \).

2 Proof of Theorem 3

Let \( H \) be a hypergraph on \( n \) vertices without a Berge 5-cycle and let \( G = \partial H \) be the 2-shadow of \( H \). First we introduce some definitions.
Definition 4. A pair $xy \in \partial H$ is called thin if $\text{codeg}(xy) = 1$, otherwise it is called fat.

We say a hyperedge $abc \in H$ is thin if at least two of the pairs $ab, bc, ac$ are thin.

Definition 5. We say a set of hyperedges (or a hypergraph) is tightly-connected if it can be obtained by starting with a hyperedge and adding hyperedges one by one, such that every added hyperedge intersects with one of the previous hyperedges in 2 vertices.

Definition 6. A block in $H$ is a maximal set of tightly-connected hyperedges.

Definition 7. For a block $B$, a maximal subhypergraph of $B$ without containing thin hyperedges is called the core of the block.

Let $K_3^4$ denote the complete 3-uniform hypergraph on 4 vertices. A crown of size $k$ is a set of $k \geq 1$ hyperedges of the form $abc_1, abc_2, \ldots, abc_k$. Below we define 2 specific hypergraphs:

- Let $F_1$ be a hypergraph consisting of exactly 3 hyperedges on 4 vertices (i.e., $K_3^4$ minus an edge).
- For distinct vertices $a, b, c, d$ and $o$, let $F_2$ be the hypergraph consisting of hyperedges $oab, obc, ocd$ and $oda$.

Lemma 8. Let $B$ be a block of $H$, and let $B$ be a core of $B$. Then $B$ is either $\emptyset, K_3^4, F_1, F_2$ or a crown of size $k$ for some $k \geq 1$.

Proof. If $B = \emptyset$, we are done, so let us assume $B \neq \emptyset$. Since $B$ is tightly-connected and it can be obtained by adding thin hyperedges to $B$, it is easy to see that $B$ is also tightly-connected. Thus if $B$ has at most two hyperedges, then it is a crown of size 1 or 2 and we are done. Therefore, in the rest of the proof we will assume that $B$ contains at least 3 hyperedges.

If $B$ contains at most 4 vertices then it is easy to see that $B$ is either $K_3^4$ or $F_1$. So assume that $B$ has at least 5 vertices (and at least 3 hyperedges). Since $B$ is not a crown, there exists a tight path of length 3, say $abc, bcd, cde$. Since $abc$ is in the core, one of the pairs $ab$ or $ac$ is fat, so there exists a hyperedge $h \neq abc$ containing either $ab$ or $ac$. Similarly there exists a hyperedge $f \neq cde$ and $f$ contains $ed$ or $ec$. If $h = f$ then $B \supseteq F_2$. However, it is easy to see that $F_2$ cannot be extended to a larger tightly-connected set of hyperedges without creating a Berge 5-cycle, so in this case $B = F_2$. If $h \neq f$ then the hyperedges $h, abc, bcd, cde, f$ create a Berge 5-cycle in $H$, a contradiction. This completes the proof of the lemma.

Observation 9. Let $B$ be a block of $H$ and let $B$ be the core of $B$. If $B = \emptyset$ then the block $B$ is a crown, and if $B \neq \emptyset$ then every fat pair of $B$ is contained in $\partial B$.

Edge Decomposition of $G = \partial H$. We define a decomposition $D$ of the edges of $G$ into paths of length 2, triangles and $K_4$'s such as follows:

Let $B$ be a block of $H$ and $B$ be its core.
If $B = \emptyset$, then $B$ is a crown-block $\{abc_1, abc_2, \ldots, abc_k\}$ (for some $k \geq 1$); we partition $\partial B$ into the triangle $abc_1$ and paths $ac\_b$ where $2 \leq i \leq k$.

If $B \neq \emptyset$, then our plan is to first partition $\partial B \setminus \partial B$. If $abc \in B \setminus B$, then $abc$ is a thin hyperedge, so it contains at least 2 thin pairs, say $ab$ and $bc$. We claim that the pair $ac$ is in $\partial B$. Indeed, $ac$ has to be a fat pair, otherwise the block $B$ consists of only one hyperedge $abc$, so $B = \emptyset$ contradicting the assumption. So by Observation 9, $ac$ has to be a pair in $\partial B$. For every $abc \in B \setminus B$ such that $ab$ and $bc$ are thin pairs, add the 2-path $abc$ to the edge decomposition $D$. This partitions all the edges in $\partial B \setminus \partial B$ into paths of length 2. So all we have left is to partition the edges of $\partial B$.

- If $B$ is a crown $\{abc_1, abc_2, \ldots, abc_k\}$ for some $k \geq 1$, then we partition $\partial B$ into the triangle $abc_1$ and paths $ac\_b$ where $2 \leq i \leq k$.
- If $B = F_1 = \{abc, bcd, acd\}$ then we partition $\partial B$ into 2-paths $abc, bcd$ and $cad$.
- If $B = F_2 = \{oab, obc, ocd, oda\}$ then we partition $\partial B$ into 2-paths $obo, bco, cdo$ and $daa$.
- Finally, if $B = K_4^3 = \{abc, abd, acd, bcd\}$ then we partition $\partial B$ as $K_4$, i.e., we add $\partial B = K_4$ as an element of $D$.

Clearly, by Lemma 8 we have no other cases left. Thus all of the edges of the graph $G$ are partitioned into paths of length 2, triangles and $K_4$'s.

**Observation 10.**

(a) If $D$ is a triangle that belongs to $D$, then there is a hyperedge $h \in H$ such that $D = \partial h$.

(b) If $abc$ is a 2-path that belongs to $D$, then $abc \in H$. Moreover $ac$ is a fat pair.

(c) If $D$ is a $K_4$ that belongs to $D$, then there exists $F = K_4^3 \subseteq H$ such that $D = \partial F$.

Let $\alpha_1 |G|$ and $\alpha_2 |G|$ be the number of edges of $G$ that are contained in triangles and 2-paths of the edge-decomposition $D$ of $G$, respectively. So $(1 - \alpha_1 - \alpha_2) |G|$ edges of $G$ belong to the $K_4$’s in $D$.

**Claim 11.** We have,

$$|H| = \left(\frac{\alpha_1}{3} + \frac{\alpha_2}{2} + \frac{2(1 - \alpha_1 - \alpha_2)}{3}\right) |G|.$$  

**Proof.** Let $B$ be a block with the core $B$. Recall that for each hyperedge $h \in B \setminus B$, we have added exactly one 2-path or a triangle to $D$.

Moreover, because of the way we partitioned $\partial B$, it is easy to check that in all of the cases except when $B = K_4^3$, the number of hyperedges of $B$ is the same as the number of elements of $D$ that $\partial B$ is partitioned into; these elements being 2-paths and triangles. On the other hand, if $B = K_4^3$, then the number of hyperedges of $B$ is 4 but we added only one element to $D$ (namely $K_4$).
This shows that the number of hyperedges of \( H \) is equal to the number of elements of \( \mathcal{D} \) that are 2-paths or triangles plus the number of hyperedges which are in copies of \( K^3_3 \) in \( H \), i.e., 4 times the number of \( K^3_3 \)'s in \( \mathcal{D} \). Since \( \alpha_1 |G| \) edges of \( G \) are in 2-paths, the number of elements of \( \mathcal{D} \) that are 2-paths is \( \alpha_1 |G|/2 \). Similarly, the number of elements of \( \mathcal{D} \) that are triangles is \( \alpha_2 |G|/3 \), and the number of \( K^3_3 \)'s in \( \mathcal{D} \) is \( (1 - \alpha_1 - \alpha_2) |G|/6 \). Combining this with the discussion above finishes the proof of the claim. \( \square \)

The link of a vertex \( v \) is the graph consisting of the edges \( \{uv | uvw \in E\} \) and is denoted by \( L_v \).

**Claim 12.** \( |L_v| \leq 2|N(v)| \).

**Proof.** First let us notice that there is no path of length 5 in \( L_v \). Indeed, otherwise, there exist vertices \( v_0, v_1, \ldots, v_5 \) such that \( vv_{i-1}v_i \in E \) for each \( 1 \leq i \leq 5 \) which means there is a Berge 5-cycle in \( H \) formed by the hyperedges containing the pairs \( v_0v_1, v_1v_2, v_2v_3, v_3v_4, v_4v_5 \), a contradiction. So by the Erdős-Gallai theorem \( |L_v| \leq \frac{d-2}{2} |N(v)| \), proving the claim. \( \square \)

**Lemma 13.** Let \( v \in V(H) \) be an arbitrary vertex, then the number of edges in \( G[N(v)] \) is less than \( 8|N(v)| \).

**Proof.** Let \( G_v \) be a subgraph of \( G \) on a vertex set \( N(v) \), such that \( xy \in G_v \) if and only if there exists a vertex \( z \neq v \) such that \( xyz \in E \). Then each edge of \( G[N(v)] \) belongs to either \( L_v \) or \( G_v \), so \( |G[N(v)]| \leq |L_v| + |G_v| \). Combining this with Claim 12, we get \( |G[N(v)]| \leq |G_v| + 2|N(v)| \). So it suffices to prove that \( |G_v| < 6|N(v)| \).

First we will prove that there is no path of length 12 in \( G_v \). Let us assume by contradiction that \( P = v_0, v_1, \ldots, v_5 \) is a path in \( G_v \). Since for each pair of vertices \( v_i, v_{i+1} \), there is a hyperedge \( v_iv_i+1 \) in \( H \) where \( x \neq v \), we can conclude that there is a subsequence \( u_0, u_1, \ldots, u_6 \) of \( v_0, v_1, \ldots, v_{12} \) and a sequence of distinct hyperedges \( h_1, h_2, \ldots, h_6 \) such that \( u_{i-1}u_i \subset h_i \) and \( v \notin h_i \) for each \( 1 \leq i \leq 6 \). Since \( u_0, u_3, u_6 \in N(v) \) there exist hyperedges \( f_1, f_2, f_3 \in H \) such that \( vu_0 \subset f_1, vu_3 \subset f_2 \) and \( uu_6 \subset f_3 \). Clearly, either \( f_1 \neq f_2 \) or \( f_2 \neq f_3 \). In the first case the hyperedges \( f_1, h_1, h_2, h_3, f_2 \), and in the second case the hyperedges \( f_2, h_4, h_5, h_6, f_3 \) form a Berge 5-cycle in \( H \), a contradiction.

Therefore, there is no path of length 12 in \( G_v \), so by the Erdős-Gallai theorem, the number of edges in \( G_v \) is at most \( \frac{12-1}{2} |N(v)| < 6|N(v)| \), as required. \( \square \)

## 2.1 Relating the hypergraph degree to the degree in the shadow

For a vertex \( v \in V(H) = V(G) \), let \( d(v) \) denote the degree of \( v \) in \( H \) and let \( d_G(v) \) denote the degree of \( v \) in \( G \) (i.e., \( d_G(v) = \) the degree in the shadow).

Clearly \( d_G(v) \leq 2d(v) \). Moreover, \( d(v) = |L_v| \) and \( d_G(v) = |N(v)| \). So by Claim 12, we have

\[
\frac{d_G(v)}{2} \leq d(v) \leq 2d_G(v).
\] (1)

Let \( \overline{d} \) and \( \overline{d}_G \) be the average degrees of \( H \) and \( G \) respectively.

Suppose there is a vertex \( v \) of \( H \), such that \( d(v) < \overline{d}/3 \). Then we may delete \( v \) and all the edges incident to \( v \) from \( H \) to obtain a graph \( H' \) whose average degree is more than
Lemma 15. For any vertex \( v \in V(G) \) and a set \( M \subseteq N(v) \), let \( \mathcal{P} \) be the set of the good 2-paths \( vxy \) such that \( x \in M \). Let \( M' = \{ y \mid vxy \in \mathcal{P} \} \) then \(|\mathcal{P}| < 2|M'| + 48d_G(v)\).

Proof. Let \( B_\mathcal{P} = \{xy \mid x \in M, y \in M', xy \in G\} \) be a bipartite graph, clearly \(|B_\mathcal{P}| = |\mathcal{P}|\). Let \( E = \{xyz \in H \mid x, y \in N(v), \text{codeg}(x, y) \leq 2\} \). By Lemma 13, \(|E| \leq 2 \cdot 8|N(v)|\) so the number of edges of 2-shadow of \( E \) is \(|\partial E| \leq 48|N(v)|\). Let \( B = \{xy \in B_\mathcal{P} \mid \exists z \in V(H), xyz \in H \setminus E\} \). Then clearly,

\[
|B| \geq |B_\mathcal{P}|-|\partial E| \geq |\mathcal{P}|-48|N(v)| = |\mathcal{P}|-48d_G(v). \tag{2}
\]

Let \( d_B(x) \) denote the degree of a vertex \( x \) in the graph \( B \).

Claim 16. For every \( y \in M' \) such that \( d_B(y) = k \geq 3 \), there exists a set of \( k - 2 \) vertices \( S_y \subseteq M' \) such that \( \forall w \in S_y \), we have \( d_B(w) = 1 \). Moreover, \( S_y \cap S_z = \emptyset \) for any \( y \neq z \in M' \) (with \( d_B(y), d_B(z) \geq 3 \)).

Proof. Let \( yx_1, yx_2, \ldots, yx_k \in B \) be the edges of \( B \) incident to \( y \). For each \( 1 \leq j \leq k \) let \( f_j \in H \) be a hyperedge such that \( vx_j \in f_j \). For each \( yx_i \in B \) clearly there is a hyperedge \( yx_iw_i \in H \setminus E \).

We claim that for each \( 1 \leq i \leq k \), \( w_i \in M' \). It is easy to see that \( w_i \in N(v) \) or \( w_i \in M' \) (because \( vx_iw_i \) is a 2-path in \( G \)). Assume for a contradiction that \( w_i \in N(v) \), then since \( yx_iw_i \notin E \) we have, \( \text{codeg}(x_i, w_i) \geq 3 \). Let \( f \in H \) be a hyperedge such that \( vv_i \subset f \). Now take \( j \neq i \) such that \( x_j \neq w_i \). If \( f_j \neq f \) then since \( \text{codeg}(x_i, w_i) \geq 3 \) there exists a hyperedge \( h \supset x_iw_i \) such that \( h \neq f \) and \( h \neq x_iw_iy \), then the hyperedges \( f, f_h, x_iw_iy, yx_jw_j, f_j \) form a Berge 5-cycle. So \( f_j = f \), therefore \( f_j \neq f_i \). Similarly in this case, there exists a hyperedge \( h \supset x_iw_i \) such that \( h \neq f_i \) and \( h \neq x_iw_iy \), therefore the hyperedges \( f_i, f_h, x_iw_iy, yx_jw_j, f_j \) form a Berge 5-cycle, a contradiction. So we proved that \( w_i \in M' \) for each \( 1 \leq i \leq k \).

Claim 17. For all but at most 2 of the \( w_i \)'s (where \( 1 \leq i \leq k \)), we have \( d_B(w_i) = 1 \).

Proof. If \( d_B(w_i) = 1 \) for all \( 1 \leq i \leq k \) then we are done, so we may assume that there is \( 1 \leq i \leq k \) such that \( d_B(w_i) \neq 1 \).

For each \( 1 \leq i \leq k \), \( w_i \in M' \) and \( x_iw_i \in \partial(H \setminus E) \) (because \( x_iw_iy \in H \setminus E \)), so it is clear that \( d_B(w_i) \geq 1 \). So \( d_B(w_i) > 1 \). Then there is a vertex \( x \in M \setminus \{x_i\} \) such that \( w_ix \in B \). Let \( f, h \in H \) be hyperedges with \( w_ix \in h \) and \( xv \in f \). If there are \( j, l \in \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( x, x_j \) and \( x_l \) are all different from each other, then
clearly, either \( f \neq f_j \) or \( f \neq f_l \), so without loss of generality we may assume \( f \neq f_j \). Then the hyperedges \( f, h, w_i x_j y, y w_j x_i, f_j \) create a Berge cycle of length 5, a contradiction. So there are no \( j, l \in \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( x, x_j \) and \( x_l \) are all different from each other. Clearly this is only possible when \( k < 4 \) and there is a \( j \in \{1, 2, 3\} \setminus \{i\} \) such that \( x = x_j \). Let \( l \in \{1, 2, 3\} \setminus \{i, j\} \). If \( f_j \neq f_l \) then the hyperedges \( f_j, h, w_i x_j y, y w_j x_i, f_l \) form a Berge 5-cycle. Therefore \( f_j = f_l \). So we proved that \( d_B(w_i) \neq 1 \) implies that \( k = 3 \) and for \( \{j, l\} = \{1, 2, 3\} \setminus \{i\} \), we have \( f_j = f_l \). So if \( d_B(w_i) \neq 1 \) and \( d_B(w_j) \neq 1 \) we have \( f_j = f_l \) and \( f_i = f_l \), which is impossible. So \( d_B(w_j) = 1 \). So we proved that if for any \( 1 \leq i \leq k, \) \( d_B(w_i) \neq 1 \) then \( k = 3 \) and all but at most 2 of the vertices in \( \{w_1, w_2, w_3\} \) have degree 1 in the graph \( B \), as desired.

We claim that for any \( i \neq j \) where \( d_B(w_i) = d_B(w_j) = 1 \) we have \( w_i \neq w_j \). Indeed, if there exists \( i \neq j \) such that \( w_i = w_j \) then \( w_i x_j \) and \( w_i x_i \) are both adjacent to \( w_i \) in the graph \( B \) which contradicts to \( d_B(w_i) = 1 \). So using the above claim, we conclude that the set \( \{w_1, w_2, \ldots, w_k\} \) contains at least \( k - 2 \) distinct elements with each having degree one in the graph \( B \), so we can set \( S_y \) to be the set of these \( k - 2 \) elements. (Then of course \( \forall w_i \in S_y \) we have \( d_B(w_i) = 1 \).

Now we have to prove that for each \( z \neq y \) we have \( S_y \cap S_z = \emptyset \). Assume by contradiction that \( w_i \in S_z \cap S_y \) for some \( z \neq y \). That is, there is some hyperedge \( u w_i z \in H \setminus E \) where \( u \in M \), moreover \( u = x_i \) otherwise \( d_B(w_i) > 1 \). So we have a hyperedge \( x_i w_i z \in H \setminus E \) for some \( z \in M \setminus \{y\} \). Let \( j, l \in \{1, 2, \ldots, k\} \setminus \{i\} \) such that \( j \neq l \). Recall that \( x_j v \subset f_j \) and \( x_l v \subset f_l \). Clearly either \( f_j \neq f_l \) or \( f_l \neq f_i \) so without loss of generality we can assume \( f_j \neq f_l \). Then it is easy to see that the hyperedges \( f_j, x_j y, x_j y, x_j w_l z, x_j w_l x_i, f_i \) are all different and they create a Berge 5-cycle \( (x_j y, y, x_j w_l z, x_j w_l x_i \because x_j \neq w_i) \).

For each \( x \in M' \) with \( d_B(x) = k \geq 3 \), let \( S_x \) be defined as in Claim 16. Then the average of the degrees of the vertices in \( S_y \cup \{x\} \) in \( B \) is \((k + |S_x|)/(k - 1) = (2k - 2)(k - 1) = 2 \). Since the sets \( S_x \cup x \) (with \( x \in M' \), \( d_B(x) \geq 3 \)) are disjoint, we can conclude that average degree of the set \( M' \) is at most \( 2 \). Therefore \( 2 |M'| \geq |B| \). So by (2) we have \( |M'| \geq |B| > |P| - 48d_G(V) \), which completes the proof of the lemma.

**Claim 18.** We may assume that the maximum degree in the graph \( G \) is less than \( 160\sqrt{n} \) when \( n \) is large enough.

**Proof.** Let \( v \) be an arbitrary vertex with \( d_G(v) = C \bar{d} \) for some constant \( C > 0 \). Let \( P \) be the set of the good 2-paths starting from the vertex \( v \). Then applying Lemma 15 with \( M = N(v) \) and \( M' = \{y \mid vxy \in P\} \), we have \( |P| < 2 |M'| + 48d_G(v) < 2n + 48 \cdot C \bar{d} \). Since the minimum degree \( \delta(G) \) is at least \( \bar{d}/6 \), the number of (ordered) 2-paths starting from \( v \) is at least \( d(v) \cdot (\bar{d}/6 - 1) = C \bar{d} \cdot (\bar{d}/6 - 1) \). Notice that the number of (ordered) bad 2-paths starting at \( v \) is the number of 2-paths \( vxy \) such that \( x, y \in N(v) \). So by Lemma 13, this is at most \( 2 \cdot 8 |N(v)| = 16C \bar{d} \), so the number of good 2-paths is at least \( C \bar{d} \cdot (\bar{d}/6 - 17) \). So \( |P| \geq C \bar{d} \cdot (\bar{d}/6 - 17) \). Thus we have

\[
C \bar{d} \cdot (\bar{d}/6 - 17) \leq |P| < 2n + 48C \bar{d}.
\]
So $C\bar{d}/6 - 65 < 2n$. Therefore, $6C(\bar{d}/6 - 65)^2 < 2n$, i.e., $\bar{d} < 6n/\sqrt{3C} + 390$, so $|H| = n\bar{d}/3 \leq 2n\sqrt{n/3C} + 130n$. If $C \geq 36$ we get that $|H| \leq \frac{2n^{3/2}}{\sqrt{3}} + 130n = \frac{n^{3/2}}{\sqrt{3}} + O(n)$, proving Theorem 3. So we may assume $C < 36$.

Theorem 2 implies that

$$|H| = n\bar{d}/3 \leq \sqrt{2}n^{3/2} + 4.5n,$$

so $\bar{d} \leq 3\sqrt{2}/\sqrt{n} + 13.5$. So combining this with the fact that $C < 36$, we have $d_G(v) = C\bar{d}/3 < 108\sqrt{2}/\sqrt{n} + 486 < 160\sqrt{n}$ for large enough $n$.

Combining Lemma 15 and Claim 18, we obtain the following.

**Lemma 19.** For any vertex $v \in V(G)$ and a set $M \subseteq N(v)$, let $P$ be the set of good 2-paths $vxy$ such that $x \in M$. Let $M' = \{y \mid vxy \in P\}$ then $|P| < 2|M'| + 7680\sqrt{n}$ when $n$ is large enough.

**Definition 20.** A 3-path $x_0, x_1, x_2, x_3$ is called good if both 2-paths $x_0, x_1, x_2$ and $x_1, x_2, x_3$ are good 2-paths.

**Claim 21.** The number of (ordered) good 3-paths in $G$ is at least $n\bar{d}_G^3 - C_0n^{3/2}\bar{d}_G$ for some constant $C_0 > 0$ (for large enough $n$).

**Proof.** First we will prove that the number of (ordered) 3-walks that are not good 3-paths is at most $5440n^{3/2}\bar{d}_G$.

For any vertex $x \in V(H)$ if a path $yxz$ is a bad 2-path then $zy$ is an edge of $G$, so the number of (ordered) bad 2-paths whose middle vertex is $x$, is at most 2 times the number of edges in $G[N(x)]$, which is less than $2 \cdot 8|N(x)| = 16d_G(x)$ by Lemma 13. The number of 2-walks which are not 2-paths and whose middle vertex is $x$ is exactly $d_G(x)$. So the total number of (ordered) 2-walks that are not good 2-paths is at most $\sum_{x \in V(H)}17d_G(x) = 17n\bar{d}_G$.

Notice that, by definition, any (ordered) 3-walk that is not a good 3-path must contain a 2-walk that is not a good 2-path. Moreover, if $yxz$ is a 2-walk that is not a good 2-path, then the number of 3-walks in $G$ containing it is at most $d_G(x) + d_G(z) < 320\sqrt{n}$ (for large enough $n$) by Claim 18. Therefore, the total number of (ordered) 3-walks that are not good 3-paths is at most $17n\bar{d}_G \cdot 320\sqrt{n} = 5440n^{3/2}\bar{d}_G$.

By the Blakley-Roy inequality, the total number of (ordered) 3-walks in $G$ is at least $n\bar{d}_G^3$. By the above discussion, all but at most $5440n^{3/2}\bar{d}_G$ of them are good 3-paths, so letting $C_0 = 5440$ proves the proof of the claim.

**Claim 22.** Let $\{a, b, c\}$ be the vertex set of a triangle that belongs to $D$. (By Observation 10 (a) $abc \in H$.) Then the number of good 3-paths whose first edge is $ab$, $bc$ or $ca$ is at most $8n + C_1\sqrt{n}$ for some constant $C_1$ and for large enough $n$.

**Proof.** For each $\{x, y\} \subset \{a, b, c\}$, let $S_{xy} = N(x) \cap N(y) \setminus \{a, b, c\}$. For each $x \in \{a, b, c\}$, let $S_x = N(x) \setminus (N(y) \cup N(z) \cup \{a, b, c\})$ where $\{y, z\} = \{a, b, c\} \setminus \{x\}$.
For each $x \in \{a, b, c\}$, let $\mathcal{P}_x$ be the set of good 2-paths $xuv$ where $u \in S_x$. Let $S'_x = \{v \mid xuv \in \mathcal{P}_x\}$. For each $\{x, y\} \subset \{a, b, c\}$, let $\mathcal{P}_{xy}$ be the set of good 2-paths $xuv$ and $yuv$ where $u \in S_{xy}$. Let $S'_{xy} = \{v \mid xuv \in \mathcal{P}_{xy}\}$.

Let $\{x, y\} \subset \{a, b, c\}$ and $z = \{a, b, c\} \setminus \{x, y\}$. Notice that each 2-path $yuv \in \mathcal{P}_{xy}$ ($xuv \in \mathcal{P}_{xy}$), is contained in at most one good 3-path $zuv$ (respectively $zxu$) whose first edge is in the triangle $abc$. Indeed, since $u \in S_{xy}$, $xyuv$ (respectively $yxuv$) is not a good 3-path. Therefore, the number of good 3-paths whose first edge is in the triangle $abc$, and whose third vertex is in $S_{xy}$ is at most $|\mathcal{P}_{xy}|$. The number of paths in $\mathcal{P}_{xy}$ that start with the vertex $x$ is less than $2|S'_{xy}| + 7680\sqrt{n}$, by Lemma 19. Similarly, the number of paths in $\mathcal{P}_{xy}$ that start with the vertex $y$ is less than $2|S'_{xy}| + 7680\sqrt{n}$. Since every path in $\mathcal{P}_{xy}$ starts with either $x$ or $y$, we have $|\mathcal{P}_{xy}| < 4|S'_{xy}| + 15360\sqrt{n}$. Therefore, for any $\{x, y\} \subset \{a, b, c\}$, the number of good 3-paths whose first edge is in the triangle $abc$ and whose third vertex is in $S_{xy}$ is less than $4|S'_{xy}| + 15360\sqrt{n}$.

In total, the number of good 3-paths whose first edge is in the triangle $abc$ and whose third vertex is in $S_{ab} \cup S_{bc} \cup S_{ac}$ is at most

$$4(|S'_{ab}| + |S'_{bc}| + |S'_{ac}|) + 46080\sqrt{n}. \quad (4)$$

Let $x \in \{a, b, c\}$ and $\{y, z\} = \{a, b, c\} \setminus \{x\}$. For any 2-path $xuv \in \mathcal{P}_x$ there are 2 good 3-paths with the first edge in the triangle $abc$, namely $yxuv$ and $zxuv$. So the total number of 3-paths whose first edge is in the triangle $abc$ and whose third vertex is in $S_a \cup S_b \cup S_c$ is $2(|\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c|)$, which is at most

$$4(|S'_{a}| + |S'_{b}| + |S'_{c}|) + 46080\sqrt{n}, \quad (5)$$

by Lemma 19.

Now we will prove that every vertex is in at most 2 of the sets $S'_a, S'_b, S'_c, S'_{ab}, S'_{bc}, S'_{ac}$. Let us assume by contradiction that a vertex $v \in V(G) \setminus \{a, b, c\}$ is in at least 3 of them. We claim that there do not exist 3 vertices $u_a \in N(a) \setminus \{b, c\}$, $u_b \in N(b) \setminus \{a, c\}$ and $u_c \in N(c) \setminus \{a, b\}$ such that $xuv$ is a good 3-path for each $x \in \{a, b, c\}$. Indeed, otherwise, consider hyperedges $h_a, h'_a$ containing the pairs $au_a$ and $a_v$ respectively (since $au_a$ is a good 2-path, note that $h_a \neq h'_a$), and hyperedges $h_b, h'_b, h'_c, h_c, h'_c$ containing the pairs $bu_b, cu_c, u_c v$ respectively. Then either $h'_a \neq h'_b$ or $h'_b \neq h'_c$, say $h'_a \neq h'_b$ without loss of generality. Then the hyperedges $h_a, h'_a, h'_b, h'_c, abc$ create a Berge 5-cycle in $H$, a contradiction, proving that it is impossible to have 3 vertices $u_a \in N(a) \setminus \{b, c\}$, $u_b \in N(b) \setminus \{a, b\}$ and $u_c \in N(c) \setminus \{a, b\}$ with the above mentioned property. Without loss of generality let us assume that there is no vertex $u_a \in N(a) \setminus \{b, c\}$ such that $au_a v$ is a good 2-path – in other words, $v \notin S'_a \cup S'_{ab} \cup S'_{ac}$. However, since we assumed that $v$ is contained in at least 3 of the sets $S'_a, S'_b, S'_c, S'_{ab}, S'_{bc}, S'_{ac}$, we can conclude that $v$ is contained in all 3 of the sets $S'_b, S'_c, S'_{bc}$, i.e., there are vertices $u_b \in S_b, u_c \in S_c, u \in S_{bc}$ such that $uvu_b, vu_c, vuv, vuc$ are good 2-paths. Using a similar argument as before, if $vu \in h$, $vu_b \in h_b$ and $vu_c \in h_c$, without loss of generality we can assume that $h \neq h_b$, so the hyperedges $abc, h, h_b$ together with hyperedges containing $uc$ and $ub$ form a Berge 5-cycle in $H$, a contradiction.
So we proved that
\[2|S'_a \cup S'_b \cup S'_c \cup S'_{ab} \cup S'_{bc} \cup S'_{ac}| \geq |S'_a| + |S'_b| + |S'_c| + |S'_{ab}| + |S'_{bc}| + |S'_{ac}|\]

This together with (4) and (5), we get that the number of good 3-paths whose first edge is in the triangle abc is at most
\[8|S'_a \cup S'_b \cup S'_c \cup S'_{ab} \cup S'_{bc} \cup S'_{ac}| + 92160\sqrt{n} < 8n + C_1\sqrt{n}\]

for \(C_1 = 92160\) and large enough \(n\), finishing the proof of the claim. \(\square\)

**Claim 23.** Let \(P = abc\) be a 2-path and \(P \in D\). (By Observation 10 (b) \(abc \in H\).) Then the number of good 3-paths whose first edge is \(ab\) or \(bc\) is at most \(4n + C_2\sqrt{n}\) for some constant \(C_2 > 0\) and large enough \(n\).

**Proof.** First we bound the number of 3-paths whose first edge is \(ab\). Let \(S_{ab} = N(a) \cap N(b)\). Let \(S_a = N(a) \setminus (N(b) \cup \{b\})\) and \(S_b = N(b) \setminus (N(a) \cup \{a\})\). For each \(x \in \{a, b\}\), let \(P_x\) be the set of good 2-paths \(xu\) where \(u \in S_x\), and let \(S'_x = \{v \mid xv \in P_x\}\). The set of good 3-paths whose first edge is \(ab\) is \(P_a \cup P_b\), because the third vertex of a good 3-path starting with an edge \(ab\) can not belong to \(N(a) \cap N(b)\) by the definition of a good 3-path.

We claim that \(|S'_a \cap S'_b| \leq 160\sqrt{n}\). Let us assume by contradiction that \(v_0, v_1, \ldots, v_k \in S'_a \cap S'_b\) for \(k > 160\sqrt{n}\). For each vertex \(v_i\) where \(0 \leq i \leq k\), there are vertices \(a_i \in S_a\) and \(b_i \in S_b\) such that \(aa_i v_i, bb_i v_i\) are good 2-paths. For each \(0 \leq i \leq k\), the hyperedge \(a_i v_i b_i\) is in \(H\), otherwise we can find distinct hyperedges containing the pairs \(aa_i, a_i v_i, v_i b_i, b_i b\) and these hyperedges together with \(abc\), would form a Berge 5-cycle in \(H\), a contradiction. We claim that there are \(j, l \in \{0, 1, \ldots, k\}\) such that \(a_j \neq a_l\), otherwise there is a vertex \(x\) such that \(a_i = x\) for each \(0 \leq i < k\). Then \(xv_i \in G\) for each \(0 \leq i \leq k\), so we get that \(d_G(x) > k > 160\sqrt{n}\) which contradicts Claim 18.

So there are \(j, l \in \{0, 1, \ldots, k\}\) such that \(a_j \neq a_l\) and \(a_j v_j b_j, a_l v_l b_l \in H\). By observation 10 (b), there is a hyperedge \(h \neq abc\) such that \(ac \subset h\). Clearly either \(a_j \notin h\) or \(a_l \notin h\). Without loss of generality let \(a_j \notin h\), so there is a hyperedge \(h_a\) with \(aa_j \subset h_a \neq h\). Let \(h_a \supset b_b\), then the hyperedges \(abc, h, h_a, a_j v_j b_j, h_b\) form a Berge 5-cycle, a contradiction, proving that \(|S'_a \cap S'_b| \leq 160\sqrt{n}\).

Notice that \(|S'_a| + |S'_b| = |S'_a \cup S'_b| + |S'_a \cap S'_b| \leq n + 160\sqrt{n}\). So by Lemma 19, we have
\[|P_a| + |P_b| \leq 2(|S'_a| + |S'_b|) + 2 \cdot 7680\sqrt{n} \leq 2(n + 160\sqrt{n}) + 2 \cdot 7680\sqrt{n} = 2n + 15680\sqrt{n}\]

for large enough \(n\). So the number of good 3-paths whose first edge is \(ab\) is at most \(2n + 15680\sqrt{n}\). By the same argument, the number of good 3-paths whose first edge is \(bc\) is at most \(2n + 15680\sqrt{n}\). Their sum is at most \(4n + C_2\sqrt{n}\) for \(C_2 = 31360\) and large enough \(n\), as desired. \(\square\)

**Claim 24.** Let \(\{a, b, c, d\}\) be the vertex set of a \(K_4\) that belongs to \(D\). Let \(F = K_4^3\) be a hypergraph on the vertex set \(\{a, b, c, d\}\). (By Observation 10 (c) \(F \subseteq H\).) Then the number of good 3-paths whose first edge belongs to \(\partial F\) is at most \(6n + C_3\sqrt{n}\) for some constant \(C_3 > 0\) and large enough \(n\).
Proof. First, let us observe that there is no Berge path of length 2, 3 or 4 between distinct vertices \( x, y \in \{a, b, c, d\} \) in the hypergraph \( H \setminus F \), because otherwise this Berge path together with some edges of \( F \) will form a Berge 5-cycle in \( H \). This implies, that there is no path of length 3 or 4 between \( x \) and \( y \) in \( G \setminus \partial F \), because otherwise we would find a Berge path of length 2, 3 or 4 between \( x \) and \( y \) in \( H \setminus F \).

Let \( S = \{ v \in V(H) \setminus \{a, b, c, d\} \mid \exists (x, y) \subseteq \{a, b, c, d\}, v \in N(x) \cap N(y) \} \). For each \( x \in \{a, b, c, d\} \), let \( S_x = N(x) \setminus \{S \cup \{a, b, c, d\}\} \). Let \( \mathcal{P}_S \) be the set of good 2-paths \( xuv \) where \( x \in \{a, b, c, d\} \) and \( u \in S \). Let \( S' = \{ v \mid xuv \in \mathcal{P}_S \} \). For each \( x \in \{a, b, c, d\} \), let \( \mathcal{P}_x \) be the set of good 2-paths \( xuv \) where \( u \in S_x \), and let \( S_x' = \{ v \mid xuv \in \mathcal{P}_x \} \).

Let \( v \in S' \). By definition, there exists a pair of vertices \( \{x, y\} \subseteq \{a, b, c, d\} \) and a vertex \( u \), such that \( xuv \) and \( yuv \) are good 2-paths.

Suppose that \( zuv' \) is a 2-path different from \( xuv \) and \( yuv \), where \( z \in \{a, b, c, d\} \). If \( u' = u \) then \( z \notin \{x, y\} \) so there is a Berge 2-path between \( x \) and \( y \) or between \( x \) and \( z \) in \( H \setminus F \), which is impossible. So \( u \neq u' \). Either \( z \neq x \) or \( z \neq y \), without loss of generality let us assume that \( z \neq x \). Then \( zuvx \) is a path of length 4 in \( G \setminus \partial F \), a contradiction.

So for any \( v \in S' \) there are only 2 paths of \( \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S \) that contain \( v \) as an end vertex – both of which are in \( \mathcal{P}_S \) – which means that \( v \notin S_a' \cup S_b' \cup S_c' \cup S_d' \), so \( S' \cap (S_a' \cup S_b' \cup S_c' \cup S_d') = \emptyset \). Moreover,

\[
|\mathcal{P}_S| \leq 2|S'|. \tag{6}
\]

We claim that \( S_a' \) and \( S_b' \) are disjoint. Indeed, otherwise, if \( v \in S_a' \cap S_b' \) there exists \( x \in S_a \) and \( y \in S_b \) such that \( vxa \) and \( vyb \) are paths in \( G \), so there is a 4-path \( axvyb \) between vertices of \( F \) in \( G \setminus \partial F \), a contradiction. Similarly we can prove that \( S_a', S_b', S_c', S_d' \) are pairwise disjoint. This shows that the sets \( S', S_a', S_b', S_c' \) and \( S_d' \) are pairwise disjoint. So we have

\[
|S' \cup S_a' \cup S_b' \cup S_c' \cup S_d'| = |S'| + |S_a'| + |S_b'| + |S_c'| + |S_d'|. \tag{7}
\]

By Lemma 19, we have \(|\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leq 2(|S_a'| + |S_b'| + |S_c'| + |S_d'|) + 4 \cdot 7680 \sqrt{n}\). Combining this inequality with (6), we get

\[
|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leq 2|S'| + 2(|S_a'| + |S_b'| + |S_c'| + |S_d'|) + 4 \cdot 7680 \sqrt{n}. \tag{8}
\]

Combining (7) with (8) we get

\[
|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leq 2|S' \cup S_a' \cup S_b' \cup S_c' \cup S_d'| + 30720 \sqrt{n} < 2n + 30720 \sqrt{n}, \tag{9}
\]

for large enough \( n \).

Each 2-path in \( \mathcal{P}_S \cup \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \) can be extended to at most three good 3-paths whose first edge is in \( \partial F \). (For example, \( awu \in \mathcal{P}_a \) can be extended to \( bawu, cawu \) and \( dauv \).) On the other hand, every good 3-path whose first edge is in \( \partial F \) must contain a 2-path of \( \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S \) as a subpath. So the number of good 3-paths whose first edge is in \( \partial F \) is at most \(|\mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S| = 3(|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d|) \) which is at most \( 6n + C_3 \sqrt{n} \) by (9), for \( C_3 = 92160 \) and large enough \( n \), proving the desired claim.

\[\square\]
2.3 Combining bounds on the number of 3-paths

Recall that $\alpha_1 |G|$, $\alpha_2 |G|$, $(1-\alpha_1-\alpha_2) |G|$ are the number of edges of $G$ that are contained in triangles, 2-paths and $K_4$’s of the edge-decomposition $D$ of $G$, respectively. Then the number of triangles, 2-paths and $K_4$’s in $D$ is $\alpha_1 |G|/3$, $\alpha_2 |G|/2$ and $(1-\alpha_1-\alpha_2) |G|/6$ respectively. Therefore, using Claim 22, Claim 23 and Claim 24, the total number of (ordered) good 3-paths in $G$ is at most

$$\frac{\alpha_1}{3} |G| (8n + C_1 \sqrt{n}) + \frac{\alpha_2}{2} |G| (4n + C_2 \sqrt{n}) + \frac{(1-\alpha_1-\alpha_2)}{6} |G| (6n + C_3 \sqrt{n}) \leq$$

$$\leq |G| n \left( \frac{8\alpha_1}{3} + 2\alpha_2 + (1 - \alpha_1 - \alpha_2) \right) + (C_1 + C_2 + C_3) \sqrt{n} |G| =$$

$$= \frac{n^2 \overline{d}_G}{2} \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{3} \right) + (C_1 + C_2 + C_3) \frac{n^{3/2} \overline{d}_G}{2}.$$

Combining this with the fact that the number of good 3-paths is at least $n \overline{d}_G^3 - C_0 n^{3/2} \overline{d}_G$ (see Claim 21), we get

$$n \overline{d}_G^3 - C_0 n^{3/2} \overline{d}_G \leq \frac{n^2 \overline{d}_G}{2} \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{3} \right) + (C_1 + C_2 + C_3) \frac{n^{3/2} \overline{d}_G}{2}.$$

Rearranging and dividing by $n \overline{d}_G$ on both sides, we get

$$\overline{d}_G^2 \leq \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{3} \right) n + \frac{1}{2} \sqrt{n}((C_1 + C_2 + C_3) + 2C_0).$$

Since $(5\alpha_1 + 3\alpha_2 + 3)/6 \geq 1/2$, we may replace $1/2$ with $(5\alpha_1 + 3\alpha_2 + 3)/6$ in the above inequality to obtain

$$\overline{d}_G^2 \leq \left( \frac{5\alpha_1 + 3\alpha_2 + 3}{6} \right) n \left( 1 + \frac{(C_1 + C_2 + C_3) + 2C_0}{\sqrt{n}} \right).$$

So letting $C_4 = (C_1 + C_2 + C_3) + 2C_0$ we have,

$$\overline{d}_G \leq \sqrt{1 + \frac{C_4}{\sqrt{n}} \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}} \sqrt{n} < \left( 1 + \frac{C_4}{2\sqrt{n}} \right) \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}} \sqrt{n}, \quad (10)$$

for large enough $n$. By Claim 11, we have

$$|H| \leq \frac{\alpha_1}{3} |G| + \frac{\alpha_2}{2} |G| + \frac{2(1-\alpha_1-\alpha_2)}{3} |G| = \frac{4 - 2\alpha_1 - \alpha_2}{6} n \overline{d}_G.$$

Combining this with (10) we get

$$|H| \leq \left( 1 + \frac{C_4}{2\sqrt{n}} \right) \frac{4 - 2\alpha_1 - \alpha_2}{12} \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}} n^{3/2},$$
for sufficiently large $n$. So we have

$$\text{ex}_3(n, C_5) \leq (1 + o(1)) \frac{4 - 2\alpha_1 - \alpha_2}{12} \sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6} n^{3/2}}.$$  

The right hand side is maximized when $\alpha_1 = 0$ and $\alpha_2 = 2/3$, so we have

$$\text{ex}_3(n, C_5) \leq (1 + o(1)) \frac{4 - 2/3}{12} \sqrt{\frac{5}{6} n^{1.5}} < (1 + o(1)) 0.2536 n^{3/2}.$$  

This finishes the proof.

**Acknowledgements**

We are grateful to the two anonymous referees for their valuable comments. The research of the authors is partially supported by the National Research, Development and Innovation Office – NKFIH, grants K116769, SNN117879, KH126853. The research of the third author is also supported by IBS-R029-C1.

**References**


