3-uniform hypergraphs without a cycle of length five

Beka Ergemlidze
Department of Mathematics and Statistics
University of South Florida
Tampa, Florida, U.S.A.
beka.ergemlidze@gmail.com

Ervin Győri
Alfréd Rényi Institute of Mathematics
Budapest, Hungary
gyori.ervin@renyi.mta.hu

Abhishek Methuku
School of Mathematics
University of Birmingham
Birmingham, U.K.
abhishekmethuku@gmail.com

Submitted: Jun 19, 2019; Accepted: Apr 15, 2020; Published: May 1, 2020

Abstract

In this paper we show that the maximum number of hyperedges in a 3-uniform hypergraph on n vertices without a (Berge) cycle of length five is less than $(0.254 + o(1))n^{3/2}$, improving an estimate of Bollobás and Győri.

We obtain this result by showing that not many 3-paths can start from certain subgraphs of the shadow.

Mathematics Subject Classifications: 05C65, 05D99

1 Introduction

A hypergraph $H = (V, E)$ is a family E of distinct subsets of a finite set V. The members of E are called *hyperedges* and the elements of V are called *vertices*. A hypergraph is called 3-uniform if each member of E has size 3. A hypergraph $H = (V, E)$ is called *linear* if every two hyperedges have at most one vertex in common.

A Berge cycle of length $k \geqslant 2$, denoted Berge-C_k, is an alternating sequence of distinct vertices and distinct edges of the form $v_1, h_1, v_2, h_2, \ldots, v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_k, v_1 \in h_k$. (Note that if a hypergraph does not contain a Berge-C_2, then it is linear.) This definition of a hypergraph cycle is the classical definition due to Berge. More generally, if $F = (V(F), E(F))$ is a graph and $\mathcal{Q} = (V(\mathcal{Q}), E(\mathcal{Q}))$ is a hypergraph, then we say $\mathcal{Q}$ is *Berge-F* if there is a bijection $\phi : E(F) \to E(\mathcal{Q})$ such that

$e \subseteq \phi(e)$ for all $e \in E(F)$. In other words, given a graph F we can obtain a Berge-F by replacing each edge of F with a hyperedge that contains it.

Given a family of graphs $\mathcal{F}$, we say that a hypergraph $\mathcal{H}$ is *Berge-$\mathcal{F}$-free* if for every $F \in \mathcal{F}$, the hypergraph $\mathcal{H}$ does not contain a Berge-F as a subhypergraph. The maximum possible number of hyperedges in a Berge-$\mathcal{F}$-free 3-uniform hypergraph on n vertices is the *Turán number* of Berge-$\mathcal{F}$, and is denoted by $\mathrm{ex}_3(n, \mathcal{F})$. When $\mathcal{F} = \{F\}$ then we simply write $\mathrm{ex}_3(n, F)$ instead of $\mathrm{ex}_3(n, \{F\})$.

Determining $\mathrm{ex}_3(n, \{C_2, C_3\})$ is basically equivalent to the famous $(6, 3)$-problem. This was settled by Ruzsa and Szemerédi in their classical paper [23], showing that $n^{2-\frac{c}{\sqrt{\log n}}} < \mathrm{ex}_3(n, \{C_2, C_3\}) = o(n^2)$ for some constant $c > 0$. An important Turán-type extremal result for Berge cycles is due to Lazebnik and Verstraëte [21], who studied the maximum number of hyperedges in an r-uniform hypergraph containing no Berge cycle of length less than five (i.e., girth five). They showed the following.

Theorem 1 (Lazebnik, Verstraëte [21])**.**

$$\mathrm{ex}_3(n, \{C_2, C_3, C_4\}) = \frac{1}{6}n^{3/2} + o(n^{3/2}).$$

The systematic study of the Turán number of Berge cycles started with the study of Berge triangles by Győri [15], and continued with the study of Berge five cycles by Bollobás and Győri [1] who showed the following.

Theorem 2 (Bollobás, Győri [1])**.**

$$(1 + o(1))\frac{n^{3/2}}{3\sqrt{3}} \leqslant \mathrm{ex}_3(n, C_5) \leqslant \sqrt{2}n^{3/2} + 4.5n.$$

The following example of Bollobás and Győri proves the lower bound in Theorem 2.

Bollobás-Győri Example. Take a C_4-free bipartite graph G_0 with $n/3$ vertices in each part and $(1 + o(1))(n/3)^{3/2}$ edges. In one part, replace each vertex u of G_0 by a pair of two new vertices u_1 and u_2, and add the triple u_1u_2v for each edge uv of G_0. It is easy to check that the resulting hypergraph H does not contain a Berge cycle of length 5. Moreover, the number of hyperedges in H is the same as the number of edges in G_0.

In this paper, we improve Theorem 2 as follows.

Theorem 3.

$$\mathrm{ex}_3(n, C_5) < (1 + o(1))\ 0.254n^{3/2}.$$

Roughly speaking, our main idea in proving the above theorem is to analyze the structure of a Berge-C_5-free hypergraph, and use this structure to efficiently bound the number of paths of length 3 that start from certain dense subgraphs (e.g., triangle, K_4) of the 2-shadow. This bound is then combined with the lower bound on the number of paths of length 3 provided by the Blakley-Roy inequality [2]. We prove Theorem 3 in Section 2.

Ergemlidze, Győri and Methuku [3] considered the analogous question for linear hypergraphs and proved that $\mathrm{ex}_3(n, \{C_2, C_5\}) = n^{3/2}/3\sqrt{3} + o(n^{3/2})$. Surprisingly, even though their lower bound is the same as the lower bound in Theorem 2, the linear hypergraph that they constructed in [3] is very different from the hypergraph used in the Bollobás-Győri example discussed above – the latter is far from being linear. In [3], the authors also strengthened Theorem 1 by showing that $\mathrm{ex}_3(n, \{C_2, C_3, C_4\}) \sim \mathrm{ex}_3(n, \{C_2, C_4\})$. Recently, $\mathrm{ex}_3(n, C_4)$ was studied in [5]. See [6] for results on the maximum number of hyperedges in an r-uniform hypergraph of girth six.

Győri and Lemons [16, 17] generalized Theorem 2 to Berge cycles of any given length and proved bounds on $\mathrm{ex}_r(n, C_{2k+1})$ and $\mathrm{ex}_r(n, C_{2k})$. These bounds were improved by Füredi and Özkahya [9], Jiang and Ma [19], Gerbner, Methuku and Vizer [11]. Recently Füredi, Kostochka and Luo [7] started the study of the maximum size of an n-vertex r-uniform hypergraph without any Berge cycle of length at least k. This study has been continued in [8, 18, 20, 4].

General results for Berge-F-free hypergraphs have been obtained in [12, 13, 10] and the Turán numbers of Berge-$K_{2,t}$ and Berge cliques, among others, were studied in [24, 22, 11, 14, 10].

Notation

We introduce some important notations and definitions used throughout the paper.

- Length of a path is the number of edges in the path. We usually denote a path $v_0, v_1, \ldots, v_k$, simply as $v_0 v_1 \ldots v_k$.
- For convenience, an edge $\{a, b\}$ of a graph or a pair of vertices a, b is referred to as ab. A hyperedge $\{a, b, c\}$ is written simply as abc.
- For a hypergraph H (or a graph G), for convenience, we sometimes use H (or G) to denote the edge set of the hypergraph H (or G respectively). Thus the number of edges in H is $|H|$.
- Given a graph G and a subset of its vertices S, let the subgraph of G induced by S be denoted by $G[S]$.
- For a hypergraph H, let $\partial H = \{ab \mid ab \subset e \in E(H)\}$ denote its *2-shadow* graph.
- For a hypergraph H, the *neighborhood* of v in H is defined as
$$N(v) = \{x \in V(H) \setminus \{v\} \mid v, x \in h \text{ for some } h \in E(H)\}.$$
- For a hypergraph H and a pair of vertices $u, v \in V(H)$, let $\mathrm{codeg}(v, u)$ denote the number of hyperedges of H containing the pair $\{u, v\}$.

2 Proof of Theorem 3

Let H be a hypergraph on n vertices without a Berge 5-cycle and let $G = \partial H$ be the 2-shadow of H. First we introduce some definitions.

Definition 4. A pair $xy \in \partial H$ is called *thin* if $\text{codeg}(xy) = 1$, otherwise it is called *fat*.
We say a hyperedge $abc \in H$ is *thin* if at least two of the pairs ab, bc, ac are thin.

Definition 5. We say a set of hyperedges (or a hypergraph) is tightly-connected if it can be obtained by starting with a hyperedge and adding hyperedges one by one, such that every added hyperedge intersects with one of the previous hyperedges in 2 vertices.

Definition 6. A *block* in H is a maximal set of tightly-connected hyperedges.

Definition 7. For a block B, a maximal subhypergraph of B without containing thin hyperedges is called the *core* of the block.

Let K_4^3 denote the complete 3-uniform hypergraph on 4 vertices. A crown of size k is a set of $k \geqslant 1$ hyperedges of the form $abc_1, abc_2, \ldots, abc_k$. Below we define 2 specific hypergraphs:

- Let F_1 be a hypergraph consisting of exactly 3 hyperedges on 4 vertices (i.e., K_4^3 minus an edge).
- For distinct vertices a, b, c, d and o, let F_2 be the hypergraph consisting of hyperedges oab, obc, ocd and oda.

Lemma 8. *Let B be a block of H, and let $\mathcal{B}$ be a core of B. Then $\mathcal{B}$ is either $\emptyset, K_4^3, F_1, F_2$ or a crown of size k for some $k \geqslant 1$.*

Proof. If $\mathcal{B} = \emptyset$, we are done, so let us assume $\mathcal{B} \neq \emptyset$. Since B is tightly-connected and it can be obtained by adding thin hyperedges to $\mathcal{B}$, it is easy to see that $\mathcal{B}$ is also tightly-connected. Thus if $\mathcal{B}$ has at most two hyperedges, then it is a crown of size 1 or 2 and we are done. Therefore, in the rest of the proof we will assume that $\mathcal{B}$ contains at least 3 hyperedges.

If $\mathcal{B}$ contains at most 4 vertices then it is easy to see that $\mathcal{B}$ is either K_4^3 or F_1. So assume that $\mathcal{B}$ has at least 5 vertices (and at least 3 hyperedges). Since $\mathcal{B}$ is not a crown, there exists a tight path of length 3, say abc, bcd, cde. Since abc is in the core, one of the pairs ab or ac is fat, so there exists a hyperedge $h \neq abc$ containing either ab or ac. Similarly there exists a hyperedge $f \neq cde$ and f contains ed or ec. If $h = f$ then $\mathcal{B} \supseteq F_2$. However, it is easy to see that F_2 cannot be extended to a larger tightly-connected set of hyperedges without creating a Berge 5-cycle, so in this case $\mathcal{B} = F_2$. If $h \neq f$ then the hyperedges h, abc, bcd, cde, f create a Berge 5-cycle in H, a contradiction. This completes the proof of the lemma. □

Observation 9. *Let B be a block of H and let $\mathcal{B}$ be the core of B. If $\mathcal{B} = \emptyset$ then the block B is a crown, and if $\mathcal{B} \neq \emptyset$ then every fat pair of B is contained in $\partial\mathcal{B}$.*

Edge Decomposition of $G = \partial H$. We define a decomposition $\mathcal{D}$ of the edges of G into paths of length 2, triangles and K_4's such as follows:
Let B be a block of H and $\mathcal{B}$ be its core.

If $\mathcal{B} = \emptyset$, then B is a crown-block $\{abc_1, abc_2, \ldots, abc_k\}$ (for some $k \geqslant 1$); we partition ∂B into the triangle abc_1 and paths ac_ib where $2 \leqslant i \leqslant k$.

If $\mathcal{B} \neq \emptyset$, then our plan is to first partition $\partial B \setminus \partial \mathcal{B}$. If $abc \in B \setminus \mathcal{B}$, then abc is a thin hyperedge, so it contains at least 2 thin pairs, say ab and bc. We claim that the pair ac is in $\partial \mathcal{B}$. Indeed, ac has to be a fat pair, otherwise the block B consists of only one hyperedge abc, so $\mathcal{B} = \emptyset$ contradicting the assumption. So by Observation 9, ac has to be a pair in $\partial \mathcal{B}$. For every $abc \in B \setminus \mathcal{B}$ such that ab and bc are thin pairs, add the 2-path abc to the edge decomposition $\mathcal{D}$. This partitions all the edges in $\partial B \setminus \partial \mathcal{B}$ into paths of length 2. So all we have left is to partition the edges of $\partial \mathcal{B}$.

- If $\mathcal{B}$ is a crown $\{abc_1, abc_2, \ldots, abc_k\}$ for some $k \geqslant 1$, then we partition ∂B into the triangle abc_1 and paths ac_ib where $2 \leqslant i \leqslant k$.
- If $\mathcal{B} = F_1 = \{abc, bcd, acd\}$ then we partition $\partial \mathcal{B}$ into 2-paths abc, bdc and cad.
- If $\mathcal{B} = F_2 = \{oab, obc, ocd, oda\}$ then we partition $\partial \mathcal{B}$ into 2-paths abo, bco, cdo and dao.
- Finally, if $\mathcal{B} = K_4^3 = \{abc, abd, acd, bcd\}$ then we partition $\partial \mathcal{B}$ as K_4, i.e., we add $\partial \mathcal{B} = K_4$ as an element of $\mathcal{D}$.

Clearly, by Lemma 8 we have no other cases left. Thus all of the edges of the graph G are partitioned into paths of length 2, triangles and K_4's.

Observation 10.

(a) If D is a triangle that belongs to $\mathcal{D}$, then there is a hyperedge $h \in H$ such that $D = \partial h$.

(b) If abc is a 2-path that belongs to $\mathcal{D}$, then $abc \in H$. Moreover ac is a fat pair.

(c) If D is a K_4 that belongs to $\mathcal{D}$, then there exists $F = K_4^3 \subseteq H$ such that $D = \partial F$.

Let $\alpha_1 |G|$ and $\alpha_2 |G|$ be the number of edges of G that are contained in triangles and 2-paths of the edge-decomposition $\mathcal{D}$ of G, respectively. So $(1 - \alpha_1 - \alpha_2)|G|$ edges of G belong to the K_4's in $\mathcal{D}$.

Claim 11. *We have,*

$$|H| = \left(\frac{\alpha_1}{3} + \frac{\alpha_2}{2} + \frac{2(1 - \alpha_1 - \alpha_2)}{3} \right) |G| .$$

Proof. Let B be a block with the core $\mathcal{B}$. Recall that for each hyperedge $h \in B \setminus \mathcal{B}$, we have added exactly one 2-path or a triangle to $\mathcal{D}$.

Moreover, because of the way we partitioned $\partial \mathcal{B}$, it is easy to check that in all of the cases except when $\mathcal{B} = K_4^3$, the number of hyperedges of $\mathcal{B}$ is the same as the number of elements of $\mathcal{D}$ that $\partial \mathcal{B}$ is partitioned into; these elements being 2-paths and triangles. On the other hand, if $\mathcal{B} = K_4^3$, then the number of hyperedges of $\mathcal{B}$ is 4 but we added only one element to $\mathcal{D}$ (namely K_4).

This shows that the number of hyperedges of H is equal to the number of elements of $\mathcal{D}$ that are 2-paths or triangles plus the number of hyperedges which are in copies of K_4^3 in H, i.e., 4 times the number of K_4's in $\mathcal{D}$. Since $\alpha_1 |G|$ edges of G are in 2-paths, the number of elements of $\mathcal{D}$ that are 2-paths is $\alpha_1 |G| /2$. Similarly, the number of elements of $\mathcal{D}$ that are triangles is $\alpha_2 |G| /3$, and the number of K_4's in $\mathcal{D}$ is $(1 - \alpha_1 - \alpha_2) |G| /6$. Combining this with the discussion above finishes the proof of the claim. □

The link of a vertex v is the graph consisting of the edges $\{uw \mid uvw \in H\}$ and is denoted by L_v.

Claim 12. $|L_v| \leqslant 2 |N(v)|$.

Proof. First let us notice that there is no path of length 5 in L_v. Indeed, otherwise, there exist vertices $v_0, v_1, \ldots, v_5$ such that $vv_{i-1}v_i \in H$ for each $1 \leqslant i \leqslant 5$ which means there is a Berge 5-cycle in H formed by the hyperedges containing the pairs $vv_1, v_1v_2, v_2v_3, v_3v_4, v_4v$, a contradiction. So by the Erdős-Gallai theorem $|L_v| \leqslant \frac{5-1}{2} |N(v)|$, proving the claim. □

Lemma 13. *Let $v \in V(H)$ be an arbitrary vertex, then the number of edges in $G[N(v)]$ is less than $8 |N(v)|$.*

Proof. Let G_v be a subgraph of G on a vertex set $N(v)$, such that $xy \in G_v$ if and only if there exists a vertex $z \neq v$ such that $xyz \in H$. Then each edge of $G[N(v)]$ belongs to either L_v or G_v, so $|G[N(v)]| \leqslant |L_v| + |G_v|$. Combining this with Claim 12, we get $|G[N(v)]| \leqslant |G_v| + 2 |N(v)|$. So it suffices to prove that $|G_v| < 6 |N(v)|$.

First we will prove that there is no path of length 12 in G_v. Let us assume by contradiction that $P = v_0, v_1, \ldots, v_{12}$ is a path in G_v. Since for each pair of vertices v_i, v_{i+1}, there is a hyperedge $v_iv_{i+1}x$ in H where $x \neq v$, we can conclude that there is a subsequence $u_0, u_1, \ldots, u_6$ of $v_0, v_1, \ldots, v_{12}$ and a sequence of distinct hyperedges $h_1, h_2, \ldots, h_6$, such that $u_{i-1}u_i \subset h_i$ and $v \notin h_i$ for each $1 \leqslant i \leqslant 6$. Since $u_0, u_3, u_6 \in N(v)$ there exist hyperedges $f_1, f_2, f_3 \in H$ such that $vu_0 \subset f_1$, $vu_3 \subset f_2$ and $vu_6 \subset f_3$. Clearly, either $f_1 \neq f_2$ or $f_2 \neq f_3$. In the first case the hyperedges f_1, h_1, h_2, h_3, f_2, and in the second case the hyperedges f_2, h_4, h_5, h_6, f_3 form a Berge 5-cycle in H, a contradiction.

Therefore, there is no path of length 12 in G_v, so by the Erdős-Gallai theorem, the number of edges in G_v is at most $\frac{12-1}{2} |N(v)| < 6 |N(v)|$, as required. □

2.1 Relating the hypergraph degree to the degree in the shadow

For a vertex $v \in V(H) = V(G)$, let $d(v)$ denote the degree of v in H and let $d_G(v)$ denote the degree of v in G (i.e., $d_G(v)$ is the degree in the shadow).

Clearly $d_G(v) \leqslant 2d(v)$. Moreover, $d(v) = |L_v|$ and $d_G(v) = |N(v)|$. So by Claim 12, we have

$$\frac{d_G(v)}{2} \leqslant d(v) \leqslant 2d_G(v). \tag{1}$$

Let $\overline{d}$ and $\overline{d}_G$ be the average degrees of H and G respectively.

Suppose there is a vertex v of H, such that $d(v) < \overline{d}/3$. Then we may delete v and all the edges incident to v from H to obtain a graph H' whose average degree is more than

$3(n\overline{d}/3 - \overline{d}/3)/(n-1) = \overline{d}$. Then it is easy to see that if the theorem holds for H', then it holds for H as well. Repeating this procedure, we may assume that for every vertex v of H, $d(v) \geqslant \overline{d}/3$. Therefore, by (1), we may assume that the degree of every vertex of G is at least $\overline{d}/6$.

2.2 Counting paths of length 3

Definition 14. A 2-path in ∂H is called *bad* if both of its edges are contained in a triangle of ∂H, otherwise it is called *good*.

Lemma 15. *For any vertex $v \in V(G)$ and a set $M \subseteq N(v)$, let $\mathcal{P}$ be the set of the good 2-paths vxy such that $x \in M$. Let $M' = \{y \mid vxy \in \mathcal{P}\}$ then $|\mathcal{P}| < 2|M'| + 48d_G(v)$.*

Proof. Let $B_{\mathcal{P}} = \{xy \mid x \in M, y \in M', xy \in G\}$ be a bipartite graph, clearly $|B_{\mathcal{P}}| = |\mathcal{P}|$. Let $E = \{xyz \in H \mid x, y \in N(v), \operatorname{codeg}(x,y) \leqslant 2\}$. By Lemma 13, $|E| \leqslant 2 \cdot 8|N(v)|$ so the number of edges of 2-shadow of E is $|\partial E| \leqslant 48|N(v)|$. Let $B = \{xy \in B_{\mathcal{P}} \mid \exists z \in V(H), xyz \in H \setminus E\}$. Then clearly,

$$|B| \geqslant |B_{\mathcal{P}}| - |\partial E| \geqslant |\mathcal{P}| - 48|N(v)| = |\mathcal{P}| - 48d_G(v). \tag{2}$$

Let $d_B(x)$ denote the degree of a vertex x in the graph B.

Claim 16. *For every $y \in M'$ such that $d_B(y) = k \geqslant 3$, there exists a set of $k-2$ vertices $S_y \subseteq M'$ such that $\forall w \in S_y$ we have $d_B(w) = 1$. Moreover, $S_y \cap S_z = \emptyset$ for any $y \neq z \in M'$ (with $d_B(y), d_B(z) \geqslant 3$).*

Proof. Let $yx_1, yx_2, \ldots, yx_k \in B$ be the edges of B incident to y. For each $1 \leqslant j \leqslant k$ let $f_j \in H$ be a hyperedge such that $vx_j \subset f_j$. For each $yx_i \in B$ clearly there is a hyperedge $yx_iw_i \in H \setminus E$.

We claim that for each $1 \leqslant i \leqslant k$, $w_i \in M'$. It is easy to see that $w_i \in N(v)$ or $w_i \in M'$ (because vx_iw_i is a 2-path in G). Assume for a contradiction that $w_i \in N(v)$, then since $yx_iw_i \notin E$ we have, $\operatorname{codeg}(x_i, w_i) \geqslant 3$. Let $f \in H$ be a hyperedge such that $vw_i \subset f$. Now take $j \neq i$ such that $x_j \neq w_i$. If $f_j \neq f$ then since $\operatorname{codeg}(x_i, w_i) \geqslant 3$ there exists a hyperedge $h \supset x_iw_i$ such that $h \neq f$ and $h \neq x_iw_iy$, then the hyperedges $f, h, x_iw_iy, yx_jw_j, f_j$ form a Berge 5-cycle. So $f_j = f$, therefore $f_j \neq f_i$. Similarly in this case, there exists a hyperedge $h \supset x_iw_i$ such that $h \neq f_i$ and $h \neq x_iw_iy$, therefore the hyperedges $f_i, h, x_iw_iy, yx_jw_j, f_j$ form a Berge 5-cycle, a contradiction. So we proved that $w_i \in M'$ for each $1 \leqslant i \leqslant k$.

Claim 17. *For all but at most 2 of the w_i's (where $1 \leqslant i \leqslant k$), we have $d_B(w_i) = 1$.*

Proof. If $d_B(w_i) = 1$ for all $1 \leqslant i \leqslant k$ then we are done, so we may assume that there is $1 \leqslant i \leqslant k$ such that $d_B(w_i) \neq 1$.

For each $1 \leqslant i \leqslant k$, $w_i \in M'$ and $x_iw_i \in \partial(H \setminus E)$ (because $x_iw_iy \in H \setminus E$), so it is clear that $d_B(w_i) \geqslant 1$. So $d_B(w_i) > 1$. Then there is a vertex $x \in M \setminus \{x_i\}$ such that $w_ix \in B$. Let $f, h \in H$ be hyperedges with $w_ix \in h$ and $xv \in f$. If there are $j, l \in \{1, 2, \ldots, k\} \setminus \{i\}$ such that x, x_j and x_l are all different from each other, then clearly, either $f \neq f_j$ or $f \neq f_l$, so without loss of generality we may assume $f \neq f_j$. Then the hyperedges $f, h, w_i x_i y, y w_j x_j, f_j$ create a Berge cycle of length 5, a contradiction. So there are no $j, l \in \{1, 2, \dots, k\} \setminus \{i\}$ such that x, x_j and x_l are all different from each other. Clearly this is only possible when $k < 4$ and there is a $j \in \{1, 2, 3\} \setminus \{i\}$ such that $x = x_j$. Let $l \in \{1, 2, 3\} \setminus \{i, j\}$. If $f_j \neq f_l$ then the hyperedges $f_j, h, w_i x_i y, y w_l x_l, f_l$ form a Berge 5-cycle. Therefore $f_j = f_l$. So we proved that $d_B(w_i) \neq 1$ implies that $k = 3$ and for $\{j, l\} = \{1, 2, 3\} \setminus \{i\}$, we have $f_j = f_l$. So if $d_B(w_i) \neq 1$ and $d_B(w_j) \neq 1$ we have $f_j = f_l$ and $f_i = f_l$, which is impossible. So $d_B(w_j) = 1$. So we proved that if for any $1 \leqslant i \leqslant k$, $d_B(w_i) \neq 1$ then $k = 3$ and all but at most 2 of the vertices in $\{w_1, w_2, w_3\}$ have degree 1 in the graph B, as desired. □

We claim that for any $i \neq j$ where $d_B(w_i) = d_B(w_j) = 1$ we have $w_i \neq w_j$. Indeed, if there exists $i \neq j$ such that $w_i = w_j$ then $w_i x_j$ and $w_i x_i$ are both adjacent to w_i in the graph B which contradicts to $d_B(w_i) = 1$. So using the above claim, we conclude that the set $\{w_1, w_2, \dots, w_k\}$ contains at least $k-2$ distinct elements with each having degree one in the graph B, so we can set S_y to be the set of these $k-2$ elements. (Then of course $\forall w_i \in S_y$ we have $d_B(w_i) = 1$.)

Now we have to prove that for each $z \neq y$ we have $S_y \cap S_z = \emptyset$. Assume by contradiction that $w_i \in S_z \cap S_y$ for some $z \neq y$. That is, there is some hyperedge $u w_i z \in H \setminus E$ where $u \in M$, moreover $u = x_i$ otherwise $d_B(w_i) > 1$. So we have a hyperedge $x_i w_i z \in H \setminus E$ for some $z \in M' \setminus \{y\}$. Let $j, l \in \{1, 2, \dots, k\} \setminus \{i\}$ such that $j \neq l$. Recall that $x_j v \subset f_j$ and $x_l v \subset f_l$. Clearly either $f_j \neq f_i$ or $f_l \neq f_i$ so without loss of generality we can assume $f_j \neq f_i$. Then it is easy to see that the hyperedges $f_j, x_j w_j y, y x_i w_i, w_i z x_i, f_i$ are all different and they create a Berge 5-cycle ($x_j w_j y \neq y x_i w_i$ because $x_j \neq w_i$). □

For each $x \in M'$ with $d_B(x) = k \geqslant 3$, let S_x be defined as in Claim 16. Then the average of the degrees of the vertices in $S_x \cup \{x\}$ in B is $(k + |S_x|)/(k-1) = (2k-2)(k-1) = 2$. Since the sets $S_x \cup x$ (with $x \in M'$, $d_B(x) \geqslant 3$) are disjoint, we can conclude that average degree of the set M' is at most 2. Therefore $2|M'| \geqslant |B|$. So by (2) we have $2|M'| \geqslant |B| > |\mathcal{P}| - 48 d_G(V)$, which completes the proof of the lemma. □

Claim 18. *We may assume that the maximum degree in the graph G is less than $160\sqrt{n}$ when n is large enough.*

Proof. Let v be an arbitrary vertex with $d_G(v) = C\overline{d}$ for some constant $C > 0$. Let $\mathcal{P}$ be the set of the good 2-paths starting from the vertex v. Then applying Lemma 15 with $M = N(v)$ and $M' = \{y \mid vxy \in \mathcal{P}\}$, we have $|\mathcal{P}| < 2|M'| + 48 d_G(v) < 2n + 48 \cdot C\overline{d}$. Since the minimum degree in G is at least $\overline{d}/6$, the number of (ordered) 2-paths starting from v is at least $d(v) \cdot (\overline{d}/6 - 1) = C\overline{d} \cdot (\overline{d}/6 - 1)$. Notice that the number of (ordered) bad 2-paths starting at v is the number of 2-paths vxy such that $x, y \in N(v)$. So by Lemma 13, this is at most $2 \cdot 8|N(v)| = 16C\overline{d}$, so the number of good 2-paths is at least $C\overline{d} \cdot (\overline{d}/6 - 17)$. So $|\mathcal{P}| \geqslant C\overline{d} \cdot (\overline{d}/6 - 17)$. Thus we have

$$C\overline{d} \cdot (\overline{d}/6 - 17) \leqslant |\mathcal{P}| < 2n + 48C\overline{d}.$$

So $C\overline{d}(\overline{d}/6-65) < 2n$. Therefore, $6C(\overline{d}/6-65)^2 < 2n$, i.e., $\overline{d} < 6\sqrt{n/3C}+390$, so $|H| = n\overline{d}/3 \leqslant 2n\sqrt{n/3C}+130n$. If $C \geqslant 36$ we get that $|H| \leqslant \frac{n^{3/2}}{3\sqrt{3}}+130n = \frac{n^{3/2}}{3\sqrt{3}}+O(n)$, proving Theorem 3. So we may assume $C < 36$.

Theorem 2 implies that

$$|H| = n\overline{d}/3 \leqslant \sqrt{2}n^{3/2}+4.5n, \tag{3}$$

so $\overline{d} \leqslant 3\sqrt{2}\sqrt{n}+13.5$. So combining this with the fact that $C < 36$, we have $d_G(v) = C\overline{d} < 108\sqrt{2}\sqrt{n}+486 < 160\sqrt{n}$ for large enough n. □

Combining Lemma 15 and Claim 18, we obtain the following.

Lemma 19. *For any vertex $v \in V(G)$ and a set $M \subseteq N(v)$, let $\mathcal{P}$ be the set of good 2-paths vxy such that $x \in M$. Let $M' = \{y \mid vxy \in \mathcal{P}\}$ then $|\mathcal{P}| < 2|M'| + 7680\sqrt{n}$ when n is large enough.*

Definition 20. A 3-path x_0, x_1, x_2, x_3 is called *good* if both 2-paths x_0, x_1, x_2 and x_1, x_2, x_3 are good 2-paths.

Claim 21. *The number of (ordered) good 3-paths in G is at least $n\overline{d}_G^3 - C_0 n^{3/2}\overline{d}_G$ for some constant $C_0 > 0$ (for large enough n).*

Proof. First we will prove that the number of (ordered) 3-walks that are not good 3-paths is at most $5440n^{3/2}\overline{d}_G$.

For any vertex $x \in V(H)$ if a path yxz is a bad 2-path then zy is an edge of G, so the number of (ordered) bad 2-paths whose middle vertex is x, is at most 2 times the number of edges in $G[N(x)]$, which is less than $2 \cdot 8|N(x)| = 16d_G(x)$ by Lemma 13. The number of 2-walks which are not 2-paths and whose middle vertex is x is exactly $d_G(x)$. So the total number of (ordered) 2-walks that are not good 2-paths is at most $\sum_{x \in V(H)} 17d_G(x) = 17n\overline{d}_G$.

Notice that, by definition, any (ordered) 3-walk that is not a good 3-path must contain a 2-walk that is not a good 2-path. Moreover, if xyz is a 2-walk that is not a good 2-path, then the number of 3-walks in G containing it is at most $d_G(x)+d_G(z) < 320\sqrt{n}$ (for large enough n) by Claim 18. Therefore, the total number of (ordered) 3-walks that are not good 3-paths is at most $17n\overline{d}_G \cdot 320\sqrt{n} = 5440n^{3/2}\overline{d}_G$.

By the Blakley-Roy inequality, the total number of (ordered) 3-walks in G is at least $n\overline{d}_G^3$. By the above discussion, all but at most $5440n^{3/2}\overline{d}_G$ of them are good 3-paths, so letting $C_0 = 5440$ completes the proof of the claim. □

Claim 22. *Let $\{a, b, c\}$ be the vertex set of a triangle that belongs to $\mathcal{D}$. (By Observation 10 (a) $abc \in H$.) Then the number of good 3-paths whose first edge is ab, bc or ca is at most $8n + C_1\sqrt{n}$ for some constant C_1 and for large enough n.*

Proof. For each $\{x, y\} \subset \{a, b, c\}$, let $S_{xy} = N(x) \cap N(y) \setminus \{a, b, c\}$. For each $x \in \{a, b, c\}$, let $S_x = N(x) \setminus (N(y) \cup N(z) \cup \{a, b, c\})$ where $\{y, z\} = \{a, b, c\} \setminus \{x\}$.

For each $x \in \{a, b, c\}$, let $\mathcal{P}_x$ be the set of good 2-paths xuv where $u \in S_x$. Let $S'_x = \{v \mid xuv \in \mathcal{P}_x\}$. For each $\{x, y\} \subset \{a, b, c\}$, let $\mathcal{P}_{xy}$ be the set of good 2-paths xuv and yuv where $u \in S_{xy}$. Let $S'_{xy} = \{v \mid xuv \in \mathcal{P}_{xy}\}$.

Let $\{x, y\} \subset \{a, b, c\}$ and $z = \{a, b, c\} \setminus \{x, y\}$. Notice that each 2-path $yuv \in \mathcal{P}_{xy}$ ($xuv \in \mathcal{P}_{xy}$), is contained in at most one good 3-path $zyuv$ (respectively $zxuv$) whose first edge is in the triangle abc. Indeed, since $u \in S_{xy}$, $xyuv$ (respectively $yxuv$) is not a good 3-path. Therefore, the number of good 3-paths whose first edge is in the triangle abc, and whose third vertex is in S_{xy} is at most $|\mathcal{P}_{xy}|$. The number of paths in $\mathcal{P}_{xy}$ that start with the vertex x is less than $2\left|S'_{xy}\right| + 7680\sqrt{n}$, by Lemma 19. Similarly, the number of paths in $\mathcal{P}_{xy}$ that start with the vertex y is less than $2\left|S'_{xy}\right| + 7680\sqrt{n}$. Since every path in $\mathcal{P}_{xy}$ starts with either x or y, we have $|\mathcal{P}_{xy}| < 4\left|S'_{xy}\right| + 15360\sqrt{n}$. Therefore, for any $\{x, y\} \subset \{a, b, c\}$, the number of good 3-paths whose first edge is in the triangle abc, and whose third vertex is in S_{xy} is less than $4\left|S'_{xy}\right| + 15360\sqrt{n}$.

In total, the number of good 3-paths whose first edge is in the triangle abc and whose third vertex is in $S_{ab} \cup S_{bc} \cup S_{ac}$ is at most

$$4(|S'_{ab}| + |S'_{bc}| + |S'_{ac}|) + 46080\sqrt{n}. \tag{4}$$

Let $x \in \{a, b, c\}$ and $\{y, z\} = \{a, b, c\} \setminus \{x\}$. For any 2-path $xuv \in \mathcal{P}_x$ there are 2 good 3-paths with the first edge in the triangle abc, namely $yxuv$ and $zxuv$. So the total number of 3-paths whose first edge is in the triangle abc and whose third vertex is in $S_a \cup S_b \cup S_c$ is $2(|\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c|)$, which is at most

$$4(|S'_a| + |S'_b| + |S'_c|) + 46080\sqrt{n}, \tag{5}$$

by Lemma 19.

Now we will prove that every vertex is in at most 2 of the sets $S'_a, S'_b, S'_c, S'_{ab}, S'_{bc}, S'_{ac}$. Let us assume by contradiction that a vertex $v \in V(G) \setminus \{a, b, c\}$ is in at least 3 of them. We claim that there do not exist 3 vertices $u_a \in N(a) \setminus \{b, c\}$, $u_b \in N(b) \setminus \{a, c\}$ and $u_c \in N(c) \setminus \{a, b\}$ such that xu_xv is a good 3-path for each $x \in \{a, b, c\}$. Indeed, otherwise, consider hyperedges h_a, h'_a containing the pairs au_a and u_av respectively (since au_av is a good 2-path, note that $h_a \neq h'_a$), and hyperedges h_b, h'_b, h_c, h'_c containing the pairs bu_b, u_bv, cu_c, u_cv respectively. Then either $h'_a \neq h'_b$ or $h'_a \neq h'_c$, say $h'_a \neq h'_b$ without loss of generality. Then the hyperedges $h_a, h'_a, h'_b, h_b, abc$ create a Berge 5-cycle in H, a contradiction, proving that it is impossible to have 3 vertices $u_a \in N(a) \setminus \{b, c\}$, $u_b \in N(b) \setminus \{a, c\}$ and $u_c \in N(c) \setminus \{a, b\}$ with the above mentioned property. Without loss of generality let us assume that there is no vertex $u_a \in N(a) \setminus \{b, c\}$ such that au_av is a good 2-path – in other words, $v \notin S'_a \cup S'_{ab} \cup S'_{ac}$. However, since we assumed that v is contained in at least 3 of the sets $S'_a, S'_b, S'_c, S'_{ab}, S'_{bc}, S'_{ac}$, we can conclude that v is contained in all 3 of the sets S'_b, S'_c, S'_{bc}, i.e., there are vertices $u_b \in S_b, u_c \in S_c, u \in S_{bc}$ such that vu_bb, vu_cc, vub, vuc are good 2-paths. Using a similar argument as before, if $vu \in h$, $vu_b \in h_b$ and $vu_c \in h_c$, without loss of generality we can assume that $h \neq h_b$, so the hyperedges abc,h,h_b together with hyperedges containing uc and u_bb form a Berge 5-cycle in H, a contradiction.

So we proved that

$$2\,|S'_a \cup S'_b \cup S'_c \cup S'_{ab} \cup S'_{bc} \cup S'_{ac}| \geqslant |S'_a| + |S'_b| + |S'_c| + |S'_{ab}| + |S'_{bc}| + |S'_{ac}|$$

This together with (4) and (5), we get that the number of good 3-paths whose first edge is in the triangle abc is at most

$$8\,|S'_a \cup S'_b \cup S'_c \cup S'_{ab} \cup S'_{bc} \cup S'_{ac}| + 92160\sqrt{n} < 8n + C_1\sqrt{n}$$

for $C_1 = 92160$ and large enough n, finishing the proof of the claim. □

Claim 23. *Let $P = abc$ be a 2-path and $P \in \mathcal{D}$. (By Observation 10 (b) $abc \in H$.) Then the number of good 3-paths whose first edge is ab or bc is at most $4n + C_2\sqrt{n}$ for some constant $C_2 > 0$ and large enough n.*

Proof. First we bound the number of 3-paths whose first edge is ab. Let $S_{ab} = N(a) \cap N(b)$. Let $S_a = N(a) \setminus (N(b) \cup \{b\})$ and $S_b = N(b) \setminus (N(a) \cup \{a\})$. For each $x \in \{a, b\}$, let $\mathcal{P}_x$ be the set of good 2-paths xuv where $u \in S_x$, and let $S'_x = \{v \mid xuv \in \mathcal{P}_x\}$. The set of good 3-paths whose first edge is ab is $\mathcal{P}_a \cup \mathcal{P}_b$, because the third vertex of a good 3-path starting with an edge ab can not belong to $N(a) \cap N(b)$ by the definition of a good 3-path.

We claim that $|S'_a \cap S'_b| \leqslant 160\sqrt{n}$. Let us assume by contradiction that $v_0, v_1, \dots v_k \in S'_a \cap S'_b$ for $k > 160\sqrt{n}$. For each vertex v_i where $0 \leqslant i \leqslant k$, there are vertices $a_i \in S_a$ and $b_i \in S_b$ such that aa_iv_i, bb_iv_i are good 2-paths. For each $0 \leqslant i \leqslant k$, the hyperedge $a_iv_ib_i$ is in H, otherwise we can find distinct hyperedges containing the pairs $aa_i, a_iv_i, v_ib_i, b_ib$ and these hyperedges together with abc, would form a Berge 5-cycle in H, a contradiction. We claim that there are $j, l \in \{0, 1, \dots, k\}$ such that $a_j \neq a_l$, otherwise there is a vertex x such that $x = a_i$ for each $0 \leqslant i \leqslant k$. Then $xv_i \in G$ for each $0 \leqslant i \leqslant k$, so we get that $d_G(x) > k > 160\sqrt{n}$ which contradicts Claim 18.

So there are $j, l \in \{0, 1, \dots, k\}$ such that $a_j \neq a_l$ and $a_jv_jb_j, a_lv_lb_l \in H$. By observation 10 (b), there is a hyperedge $h \neq abc$ such that $ac \subset h$. Clearly either $a_j \notin h$ or $a_l \notin h$. Without loss of generality let $a_j \notin h$, so there is a hyperedge h_a with $aa_j \subset h_a \neq h$. Let $h_b \supset b_jb$, then the hyperedges $abc, h, h_a, a_jv_jb_j, h_b$ form a Berge 5-cycle, a contradiction, proving that $|S'_a \cap S'_b| \leqslant 160\sqrt{n}$.

Notice that $|S'_a| + |S'_b| = |S'_a \cup S'_b| + |S'_a \cap S'_b| \leqslant n + 160\sqrt{n}$. So by Lemma 19, we have

$$|\mathcal{P}_a| + |\mathcal{P}_b| \leqslant 2(|S'_a| + |S'_b|) + 2 \cdot 7680\sqrt{n} \leqslant 2(n + 160\sqrt{n}) + 2 \cdot 7680\sqrt{n} = 2n + 15680\sqrt{n}$$

for large enough n. So the number of good 3-paths whose first edge is ab is at most $2n + 15680\sqrt{n}$. By the same argument, the number of good 3-paths whose first edge is bc is at most $2n + 15680\sqrt{n}$. Their sum is at most $4n + C_2\sqrt{n}$ for $C_2 = 31360$ and large enough n, as desired. □

Claim 24. *Let $\{a, b, c, d\}$ be the vertex set of a K_4 that belongs to $\mathcal{D}$. Let $F = K_4^3$ be a hypergraph on the vertex set $\{a, b, c, d\}$. (By Observation 10 (c) $F \subseteq H$.) Then the number of good 3-paths whose first edge belongs to ∂F is at most $6n + C_3\sqrt{n}$ for some constant $C_3 > 0$ and large enough n.*

Proof. First, let us observe that there is no Berge path of length $2, 3$ or 4 between distinct vertices $x, y \in \{a, b, c, d\}$ in the hypergraph $H \setminus F$, because otherwise this Berge path together with some edges of F will form a Berge 5-cycle in H. This implies, that there is no path of length 3 or 4 between x and y in $G \setminus \partial F$, because otherwise we would find a Berge path of length $2, 3$ or 4 between x and y in $H \setminus F$.

Let $S = \{u \in V(H) \setminus \{a, b, c, d\} \mid \exists \{x, y\} \subset \{a, b, c, d\}, u \in N(x) \cap N(y)\}$. For each $x \in \{a, b, c, d\}$, let $S_x = N(x) \setminus (S \cup \{a, b, c, d\})$. Let $\mathcal{P}_S$ be the set of good 2-paths xuv where $x \in \{a, b, c, d\}$ and $u \in S$. Let $S' = \{v \mid xuv \in \mathcal{P}_S\}$. For each $x \in \{a, b, c, d\}$, let $\mathcal{P}_x$ be the set of good 2-paths xuv where $u \in S_x$, and let $S'_x = \{v \mid xuv \in \mathcal{P}_x\}$.

Let $v \in S'$. By definition, there exists a pair of vertices $\{x, y\} \subset \{a, b, c, d\}$ and a vertex u, such that xuv and yuv are good 2-paths.

Suppose that $zu'v$ is a 2-path different from xuv and yuv where $z \in \{a, b, c, d\}$. If $u' = u$ then $z \notin \{x, y\}$ so there is a Berge 2-path between x and y or between x and z in $H \setminus F$, which is impossible. So $u \neq u'$. Either $z \neq x$ or $z \neq y$, without loss of generality let us assume that $z \neq x$. Then $zu'vux$ is a path of length 4 in $G \setminus \partial F$, a contradiction. So for any $v \in S'$ there are only 2 paths of $\mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S$ that contain v as an end vertex – both of which are in $\mathcal{P}_S$ – which means that $v \notin S'_a \cup S'_b \cup S'_c \cup S'_d$, so $S' \cap (S'_a \cup S'_b \cup S'_c \cup S'_d) = \emptyset$. Moreover,

$$|\mathcal{P}_S| \leqslant 2\,|S'|\,. \tag{6}$$

We claim that S'_a and S'_b are disjoint. Indeed, otherwise, if $v \in S'_a \cap S'_b$ there exists $x \in S_a$ and $y \in S_b$ such that vxa and vyb are paths in G, so there is a 4-path $axvyb$ between vertices of F in $G \setminus \partial F$, a contradiction. Similarly we can prove that S'_a, S'_b, S'_c and S'_d are pairwise disjoint. This shows that the sets S', S'_a, S'_b, S'_c and S'_d are pairwise disjoint. So we have

$$|S' \cup S'_a \cup S'_b \cup S'_c \cup S'_d| = |S'| + |S'_a| + |S'_b| + |S'_c| + |S'_d|\,. \tag{7}$$

By Lemma 19, we have $|\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leqslant 2(|S'_a| + |S'_b| + |S'_c| + |S'_d|) + 4 \cdot 7680\sqrt{n}$. Combining this inequality with (6), we get

$$|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leqslant 2\,|S'| + 2(|S'_a| + |S'_b| + |S'_c| + |S'_d|) + 4 \cdot 7680\sqrt{n}. \tag{8}$$

Combining (7) with (8) we get

$$|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d| \leqslant 2\,|S' \cup S'_a \cup S'_b \cup S'_c \cup S'_d| + 30720\sqrt{n} < 2n + 30720\sqrt{n}, \tag{9}$$

for large enough n.

Each 2-path in $\mathcal{P}_S \cup \mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d$ can be extended to at most three good 3-paths whose first edge is in ∂F. (For example, $auv \in \mathcal{P}_a$ can be extended to $bauv, cauv$ and $dauv$.) On the other hand, every good 3-path whose first edge is in ∂F must contain a 2-path of $\mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S$ as a subpath. So the number of good 3-paths whose first edge is in ∂F is at most $3\,|\mathcal{P}_a \cup \mathcal{P}_b \cup \mathcal{P}_c \cup \mathcal{P}_d \cup \mathcal{P}_S| = 3(|\mathcal{P}_S| + |\mathcal{P}_a| + |\mathcal{P}_b| + |\mathcal{P}_c| + |\mathcal{P}_d|)$ which is at most $6n + C_3\sqrt{n}$ by (9), for $C_3 = 92160$ and large enough n, proving the desired claim. □

2.3 Combining bounds on the number of 3-paths

Recall that $\alpha_1 |G|$, $\alpha_2 |G|$, $(1-\alpha_1-\alpha_2)|G|$ are the number of edges of G that are contained in triangles, 2-paths and K_4's of the edge-decomposition $\mathcal{D}$ of G, respectively. Then the number of triangles, 2-paths and K_4's in $\mathcal{D}$ is $\alpha_1 |G|/3$, $\alpha_2 |G|/2$ and $(1-\alpha_1-\alpha_2)|G|/6$ respectively. Therefore, using Claim 22, Claim 23 and Claim 24, the total number of (ordered) good 3-paths in G is at most

$$\frac{\alpha_1}{3}|G|(8n + C_1\sqrt{n}) + \frac{\alpha_2}{2}|G|(4n + C_2\sqrt{n}) + \frac{(1-\alpha_1-\alpha_2)}{6}|G|(6n + C_3\sqrt{n}) \leqslant$$

$$\leqslant |G|\, n\left(\frac{8\alpha_1}{3} + 2\alpha_2 + (1-\alpha_1-\alpha_2)\right) + (C_1+C_2+C_3)\sqrt{n}\,|G| =$$

$$= \frac{n^2\overline{d}_G}{2}\left(\frac{5\alpha_1 + 3\alpha_2 + 3}{3}\right) + (C_1+C_2+C_3)\frac{n^{3/2}\overline{d}_G}{2}.$$

Combining this with the fact that the number of good 3-paths is at least $n\overline{d}_G^3 - C_0 n^{3/2}\overline{d}_G$ (see Claim 21), we get

$$n\overline{d}_G^3 - C_0 n^{3/2}\overline{d}_G \leqslant \frac{n^2\overline{d}_G}{2}\left(\frac{5\alpha_1 + 3\alpha_2 + 3}{3}\right) + (C_1+C_2+C_3)\frac{n^{3/2}\overline{d}_G}{2}.$$

Rearranging and dividing by $n\overline{d}_G$ on both sides, we get

$$\overline{d}_G^2 \leqslant \left(\frac{5\alpha_1 + 3\alpha_2 + 3}{6}\right) n + \frac{1}{2}\sqrt{n}((C_1+C_2+C_3) + 2C_0).$$

Since $(5\alpha_1 + 3\alpha_2 + 3)/6 \geqslant 1/2$, we may replace $1/2$ with $(5\alpha_1 + 3\alpha_2 + 3)/6$ in the above inequality to obtain

$$\overline{d}_G^2 \leqslant \left(\frac{5\alpha_1 + 3\alpha_2 + 3}{6}\right) n \left(1 + \frac{(C_1+C_2+C_3) + 2C_0}{\sqrt{n}}\right).$$

So letting $C_4 = (C_1+C_2+C_3) + 2C_0$ we have,

$$\overline{d}_G \leqslant \sqrt{1 + \frac{C_4}{\sqrt{n}}}\sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}}\sqrt{n} < \left(1 + \frac{C_4}{2\sqrt{n}}\right)\sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}}\sqrt{n}, \tag{10}$$

for large enough n. By Claim 11, we have

$$|H| \leqslant \frac{\alpha_1}{3}|G| + \frac{\alpha_2}{2}|G| + \frac{2(1-\alpha_1-\alpha_2)}{3}|G| = \frac{4 - 2\alpha_1 - \alpha_2}{6}\frac{n\overline{d}_G}{2}.$$

Combining this with (10) we get

$$|H| \leqslant \left(1 + \frac{C_4}{2\sqrt{n}}\right)\frac{(4 - 2\alpha_1 - \alpha_2)}{12}\sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}}n^{3/2},$$

for sufficiently large n. So we have

$$\mathrm{ex}_3(n, C_5) \leqslant (1 + o(1))\frac{(4 - 2\alpha_1 - \alpha_2)}{12}\sqrt{\frac{5\alpha_1 + 3\alpha_2 + 3}{6}}n^{3/2}.$$

The right hand side is maximized when $\alpha_1 = 0$ and $\alpha_2 = 2/3$, so we have

$$\mathrm{ex}_3(n, C_5) \leqslant (1 + o(1))\frac{4 - 2/3}{12}\sqrt{\frac{5}{6}}n^{1.5} < (1 + o(1))0.2536n^{3/2}.$$

This finishes the proof.

Acknowledgements

We are grateful to the two anonymous referees for their valuable comments.

The research of the authors is partially supported by the National Research, Development and Innovation Office – NKFIH, grants K116769, SNN117879, KH126853. The research of the third author is also supported by IBS-R029-C1.

References

[1] B. Bollobás and E. Győri. "Pentagons vs. triangles." *Discrete Mathematics*, 308 (19) (2008), 4332–4336.

[2] R. G. Blakley and P. Roy. "A Hölder type inequality for symmetric matrices with nonnegative entries." *Proceedings of the American Mathematical Society* 16.6 (1965): 1244–1245.

[3] B. Ergemlidze, E. Győri and A. Methuku. "Asymptotics for Turán numbers of cycles in 3-uniform linear hypergraphs." *Journal of Combinatorial Theory, Series A* 163 (2019), 163–181.

[4] B. Ergemlidze, E. Győri, A. Methuku, N. Salia, C. Tompkins and O. Zamora. "Avoiding long Berge cycles, the missing cases $k = r + 1$ and $k = r + 2$." *Combinatorics, Probability and Computing* (2018): 1–13.

[5] B. Ergemlidze, E. Győri, A. Methuku, N. Salia and C. Tompkins. "On 3-uniform hypergraphs avoiding a cycle of length four." *Manuscript.*

[6] B. Ergemlidze and A. Methuku "Triangles in C_5-free graphs and Hypergraphs of Girth Six." arXiv:1811.11873 (2018).

[7] Z. Füredi, A. Kostochka and R. Luo. "Avoiding long Berge cycles." arXiv:1805.04195 (2018).

[8] Z. Füredi, A. Kostochka and R. Luo. "Avoiding long Berge cycles II, exact bounds for all n." arXiv:1807.06119 (2018).

[9] Z. Füredi and L. Özkahya. "On 3-uniform hypergraphs without a cycle of a given length." *Discrete Applied Mathematics*, 216 (2017), 582–588.

[10] D. Gerbner, A. Methuku and C. Palmer. "General lemmas for Berge-Turán hypergraph problems." *European Journal of Combinatorics* 86 (2020): 103082.

[11] D. Gerbner, A. Methuku and M. Vizer. "Asymptotics for the Turán number of Berge-$K_{2,t}$." *Journal of Combinatorial Theory, Series B* (2019).

[12] D. Gerbner and C. Palmer. "Extremal results for Berge-hypergraphs." *SIAM Journal on Discrete Mathematics*, 31.4 (2017), 2314–2327.

[13] D. Grósz, A. Methuku and C. Tompkins. "Uniformity thresholds for the asymptotic size of extremal Berge-F-free hypergraphs." *European Journal of Combinatorics* (2020): 103109.

[14] A. Gyárfás. "The Turán number of Berge-K_4 in triple systems." *SIAM Journal on Discrete Mathematics* 33.1 (2019): 383–392.

[15] E. Győri. "Triangle-Free Hypergraphs." *Combinatorics, Probability and Computing*, 15 (1-2) (2006), 185–191.

[16] E. Győri and N. Lemons. "3-uniform hypergraphs avoiding a given odd cycle." *Combinatorica* 32.2 (2012), 187–203.

[17] E. Győri and N. Lemons. "Hypergraphs with no cycle of a given length." *Combinatorics, Probability and Computing*, 21 (2012), 193–201.

[18] E. Győri, N. Lemons, N. Salia and O. Zamora. "The Structure of Hypergraphs without long Berge cycles." arXiv:1812.10737 (2018).

[19] T. Jiang and J. Ma. "Cycles of given lengths in hypergraphs." *Journal of Combinatorial Theory, Series B*, 133 (2018): 54–77.

[20] A. Kostochka, and R. Luo. "On r-uniform hypergraphs with circumference less than r." arXiv:1807.04683 (2018).

[21] F. Lazebnik and J. Verstraëte. "On hypergraphs of girth five." The Electronic Journal of Combinatorics 10 (2003): #R25.

[22] C. Palmer, M. Tait, C. Timmons and A. Z. Wagner. "Turán numbers for Berge-hypergraphs and related extremal problems." *Discrete Mathematics* 342.6 (2019): 1553–1563.

[23] I. Ruzsa and E. Szemerédi. "Triple systems with no six points carrying three triangles." in *Combinatorics, Keszthely, Colloq. Math. Soc. J. Bolyai* 18, Vol II (1976), 939–945.

[24] C. Timmons. "On r-uniform linear hypergraphs with no Berge-$K_{2,t}$." *The Electronic Journal of Combinatorics* 24, no. 4 (2017): #P4.34.