

Erdős–Ko–Rado Theorem for a restricted universe

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Abstract

A family \mathcal{F} of k -element subsets of the n -element set $[n]$ is called *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. In 1961 Erdős, Ko and Rado showed that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ if $n \geq 2k$. Since then a large number of results providing best possible upper bounds on $|\mathcal{F}|$ under further restraints were proved. The paper of Li et al. is one of them. We consider the restricted universe $\mathcal{W} = \left\{ F \in \binom{[n]}{k} : |F \cap [m]| \geq \ell \right\}$, $n \geq 2k$, $m \geq 2\ell$ and determine $\max |\mathcal{F}|$ for intersecting families $\mathcal{F} \subset \mathcal{W}$. Then we use this result to solve completely the problem considered by Li et al.

Mathematics Subject Classifications: 05D05

1 Introduction

Let n, k be positive integers, $n \geq 2k$. Let $[n] = \{1, 2, \dots, n\}$ be the standard n -element set and $\binom{[n]}{k}$ the collection of all its k -element subsets. Subsets \mathcal{F} of $\binom{[n]}{k}$ are called k -uniform families.

A family \mathcal{F} is called *intersecting* if $F \cap F' \neq \emptyset$ holds for all $F, F' \in \mathcal{F}$. The simplest example of a large intersecting family is the star:

$$\mathcal{S} = \left\{ F \in \binom{[n]}{k} : 1 \in F \right\}.$$

Obviously, $|\mathcal{S}| = \binom{n-1}{k-1}$. The classical Erdős–Ko–Rado Theorem [EKR] states that no k -uniform intersecting family can surpass $|\mathcal{S}|$.

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Hilton and Milner [HM] showed that for $n > 2k$ up to isomorphism the star is the unique maximal family. The Erdős–Ko–Rado Theorem was the origin of a lot of research, there are many papers strengthening, generalizing or extending it. We refer the interested reader to the recent monograph by Gerbner and Patkos [GP].

Motivated by a paper of Li, Chen, Huang and Li [LCHL] we consider the following problem.

Let (m, ℓ) be a pair of integers, $m \geq 2\ell > 0$. Let us consider

$$\mathcal{W} = \left\{ G \in \binom{[n]}{k} : |G \cap [m]| \geq \ell \right\}.$$

In order to guarantee that \mathcal{W} is not empty, we suppose that $k \geq \ell$. Since increasing m beyond n does not change \mathcal{W} , we suppose $n \geq m$ as well. Whenever we need to stress the parameters, we write $\mathcal{W}(n, k, m, \ell)$ instead of the short form \mathcal{W} . We call \mathcal{W} the *restricted universe*.

Definition 1. Let $g(n, k, m, \ell)$ denote the maximum of $|\mathcal{F}|$ where $\mathcal{F} \subset \mathcal{W}(n, k, m, \ell)$ is intersecting. Let us define the *restricted star* $\mathcal{R} = \mathcal{R}(n, k, m, \ell)$ by $\mathcal{R} = \mathcal{S} \cap \mathcal{W}$, that is,

$$\mathcal{R} = \left\{ R \in \binom{[n]}{k} : 1 \in R, |R \cap [m]| \geq \ell \right\}.$$

For the case $m \geq k + \ell$ we are going to show that

$$(1) \quad g(n, k, m, \ell) = |\mathcal{R}|.$$

In the case $m \leq k + \ell - 1$ one can add further sets to \mathcal{R} . Define $t = m + 1 - \ell$ and note $\ell < t \leq k$ (the first part follows from $m \geq 2\ell$). Define

$$\mathcal{P} = \{P \in \mathcal{W} : |P \cap [m]| \geq t\}, \quad \mathcal{P}_0 = \{P \in \mathcal{P} : 1 \notin P\}.$$

Then $|\mathcal{R} \cup \mathcal{P}| = |\mathcal{R}| + |\mathcal{P}_0|$ and $\mathcal{R} \cup \mathcal{P}$ is still intersecting. Indeed, if $R \in \mathcal{R}$, $P \in \mathcal{P}$, then $\ell + t > m$ implies $R \cap P \cap [m] \neq \emptyset$. For $P, P' \in \mathcal{P}$ the same follows from $t + t > \ell + t > m$.

Our main result is the following

Theorem 2. Let n, k, m, ℓ satisfy $n \geq 2k$, $m \geq 2\ell$, $n \geq m$, $k \geq \ell \geq 1$. Then either $m \geq k + \ell$ and (1) holds or $m < k + \ell$ and

$$(2) \quad g(n, k, m, \ell) = |\mathcal{R} \cup \mathcal{P}_0|.$$

We prove Theorem 2 in Section 2. In Section 3 we use it for the complete solution of the following problem that was raised in [LCHL].

Definition 3. Let n, m, k be positive integers, $n \geq 2k > m \geq k$. A family $\mathcal{F} \subset \binom{[n]}{k}$ is called m -complete if $\binom{[m]}{k} \subset \mathcal{F}$. Let $h(n, m, k)$ denote $\max\{|\mathcal{F}| : \mathcal{F} \subset \binom{[n]}{k}, \mathcal{F} \text{ is intersecting and } m\text{-complete}\}$. For $m = k$ and $k + 1$ the Erdős–Ko–Rado Theorem and the Hilton–Milner Theorem, respectively, give the answer:

$$h(n, k, k) = \binom{n-1}{k-1}, \quad h(n, k+1, k) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Theorem 4. *If $n \geq 2k > m \geq k$, then*

$$(3) \quad h(n, m, k) = \binom{m}{k} + \sum_{m-k+1 \leq i \leq k-1} \binom{m-1}{i-1} \binom{n-m}{k-i}.$$

We should mention that Li, Chen, Huang and Li [LCHL] determined $h(n, m, k)$ for $n > n_0(k)$ and also for $m = 2k - 1, 2k - 2, 2k - 3$. Their construction corresponding to (3) is

$$\mathcal{H}(n, m, k) = \binom{[m]}{k} \cup \left\{ H \in \binom{[n]}{k} : 1 \in H, m - k + 1 \leq |H \cap [m]| \leq k - 1 \right\}.$$

2 The proof of Theorem 2

We are going to apply induction on n . The base of the induction is the case $n = 2k$.

In the non-restricted universe, $\binom{[2k]}{k}$ the Erdős–Ko–Rado Theorem gives the upper bound $\binom{2k-1}{k-1} = \binom{2k}{k} / 2$. The proof is very easy. Partition $\binom{[2k]}{k}$ into $\binom{2k-1}{k-1}$ complementary pairs (U, V) , that is $|U| = |V| = k$ and $U = [2k] \setminus V$. Note that at most one of the two sets can be a member of an intersecting family.

Since the upper bound $\binom{2k-1}{k-1}$ is still valid in the restricted universe, we just have to show that our constructions match this bound. That is, for every complementary pair (U, V) one of the sets is in our family. By symmetry assume $1 \in U$. If $m \geq k + \ell$, then $U \cup V = [2k] \supset [m]$ and $|V| = k$ imply

$$|U \cap [m]| \geq m - k \geq \ell, \quad \text{i.e., } U \in \mathcal{R}.$$

Let $m \leq k + \ell - 1$. If $|U \cap [m]| \geq \ell$, then $U \in \mathcal{R}$. If $|U \cap [m]| \leq \ell - 1$, then $[m] \subset U \cup V$ implies $|V \cap [m]| \geq m - \ell + 1 = t$, i.e., $V \in \mathcal{P}$. This concludes the proof of (1) and (2) for the case $n = 2k$.

In the induction step we are going to prove (1) and (2) for the pair $(n+1, k)$ assuming its validity for the pairs (n, k) and $(n, k-1)$. Formally there could be a problem in the case $k = \ell$, because $k-1$ is no longer greater or equal to ℓ . However, if $k = \ell$, then our restricted universe is simply $\binom{[m]}{k}$ and (1) directly follows from the Erdős–Ko–Rado Theorem. Thus we assume $k > \ell$ in the sequel.

The inductive step is fairly simple, however it relies on shifting, an operation on families that was introduced by Erdős, Ko and Rado [EKR]. To keep this paper short let us refer to [F87] for the details concerning shifting. The point is that this operation does not destroy the intersecting property and it maintains the property $|F \cap [m]| \geq \ell$ as well. Applying it repeatedly eventually produces a family \mathcal{F} with the following property.

$$(4) \quad \begin{aligned} &\text{Whenever } F \in \mathcal{F}, 1 \leq i < j \leq n \text{ and } F \cap \{i, j\} = \{j\}, \\ &\text{then the set } (F \setminus \{j\}) \cup \{i\} \text{ is also in } \mathcal{F}. \end{aligned}$$

Families satisfying (4) are called *shifted*. In view of the above discussion, in proving Theorem 2 we may assume that the intersecting family $\mathcal{F} \subset \mathcal{W}$ is shifted. Shifted families have many useful properties but we only need one, namely Claim 5 below.

Recall the standard definitions

$$\mathcal{F}(i) = \{F \setminus \{i\} : i \in F \in \mathcal{F}\}, \quad \mathcal{F}(\bar{i}) = \{F \in \mathcal{F} : i \notin F\}.$$

The equality

$$|\mathcal{F}| = |\mathcal{F}(i)| + |\mathcal{F}(\bar{i})|$$

should be obvious. Since $\mathcal{F}(\bar{i}) \subset \mathcal{F}$, $\mathcal{F}(\bar{i})$ is intersecting.

Claim 5 ([EKR]). *If $\mathcal{F} \subset \binom{[n+1]}{k}$ is shifted and intersecting, $n+1 \geq 2k$, then $\mathcal{F}(n+1)$ is intersecting as well.*

Proof. Let $G, G' \in \mathcal{F}(n+1)$. Since $|G| + |G'| = 2(k-1) < n$, we can choose $i \in [n]$ such that $i \notin G \cup G'$. Set $F = G \cup \{n+1\}$, $F' = G' \cup \{i\}$. Now $F \in \mathcal{F}$ by definition and $F' \in \mathcal{F}$ by (4). Noting $F \cap F' = G \cap G'$, the statement follows. \square

Now we are ready to prove the induction step.

Lemma 6. *If $n \geq m$, then*

$$(5) \quad |\mathcal{F}| \leq g(n, k, m, \ell) + g(n, k-1, m, \ell).$$

Proof. Let $\mathcal{F} \subset \mathcal{W}(n+1, k, m, \ell)$ be intersecting. Because of $n \geq m$ both $\mathcal{F}(\overline{n+1}) \subset \mathcal{W}(n, k, m, \ell)$ and $\mathcal{F}(n+1) \subset \mathcal{W}(n, k-1, m, \ell)$ hold. In view of Claim 5 they are both intersecting. Consequently,

$$|\mathcal{F}(\overline{n+1})| \leq g(n, k, m, \ell), \quad |\mathcal{F}(n+1)| \leq g(n, k-1, m, \ell).$$

These two inequalities imply (5). \square

What about the case $n+1 = m$? Fortunately, in this case $k \geq \ell$ implies $\mathcal{W}(n+1, k, n+1, \ell) = \binom{[n+1]}{k}$ and the statement of Theorem 2 follows directly from the Erdős–Ko–Rado Theorem. Thus we may assume both $n > m$ and $k > \ell$. By Lemma 6, to conclude the proof in these cases the only thing that we have to show is that for the families $\mathcal{F} = \mathcal{R}(n+1, k, m, \ell)$ or $\mathcal{F} = \mathcal{R}(n+1, k, m, \ell) \cup \mathcal{P}_0(n+1, k, m, \ell)$

$$\begin{aligned} |\mathcal{F}(\overline{n+1})| &= g(n, k, m, \ell) \quad \text{and} \\ |\mathcal{F}(n+1)| &= g(n, k-1, m, \ell). \end{aligned}$$

However, both these facts are immediate from the definitions of \mathcal{R} and \mathcal{P}_0 . The only case that needs a slight verification is $\mathcal{P}_0 = \mathcal{P}_0(n+1, k, m, \ell)$. Recall that $t = m+1-\ell$ is independent of k and

$$\mathcal{P}_0 = \left\{ P \in \binom{[n+1]}{k} : 1 \notin P, |P \cap [m]| \geq t \right\}.$$

Consequently,

$$\begin{aligned} \mathcal{P}_0(\overline{n+1}) &= \left\{ P \in \binom{[n]}{k} : 1 \notin P, |P \cap [m]| \geq t \right\} = \mathcal{P}_0(n, k, m, \ell) \quad \text{and} \\ \mathcal{P}_0(n+1) &= \left\{ P \in \binom{[n]}{k-1} : 1 \notin P, |P \cap [m]| \geq t \right\} = \mathcal{P}_0(n, k-1, m, \ell), \end{aligned}$$

where in the second inequality we used $k-1 \geq \ell$ also.

The proof of Theorem 2 is complete. \square

3 The proof of Theorem 4

Let $\mathcal{F} \subset \binom{[n]}{k}$ be m -complete and intersecting. Set $\ell = m - k + 1$. Note that $\ell \leq k$ is equivalent to $m \leq 2k - 1$.

If $m = 2k - 1$, then $\binom{[2k-1]}{k} \subset \mathcal{F}$. Since no more k -set can be added to $\binom{[2k-1]}{k}$ without destroying the intersecting property, $|\mathcal{F}| = \binom{2k-1}{k}$ and $\mathcal{R}(n, 2k - 1, k) = \binom{2k-1}{k}$ follow. Thus we may assume that $m \leq 2k - 2$ whence $\ell = m - k + 1 \leq k - 1$. Adding ℓ we infer

$$2\ell \leq \ell + k - 1 = m.$$

Since \mathcal{F} is intersecting and $\binom{[m]}{k} \subset \mathcal{F}$ we infer $|F \cap [m]| \geq \ell$ for all $F \in \mathcal{F}$. That is, $\mathcal{F} \subset \mathcal{W}(n, k, m, \ell)$ and \mathcal{F} is intersecting. Consequently we may apply Theorem 2 to \mathcal{F} . Since $k + \ell = k + (m - k + 1) = m + 1$, we use (2) and conclude

$$(6) \quad |\mathcal{F}| \leq |\mathcal{R}(n, k, m, \ell)| + |\mathcal{P}_0(n, k, m, \ell)|.$$

Using the corresponding formulae (and $t = m + 1 - \ell = k$):

$$|\mathcal{R}(n, k, m, \ell)| = \sum_{\ell \leq i \leq k} \binom{m-1}{i-1} \binom{n-m}{k-i},$$

$$|\mathcal{P}_0(n, k, m, \ell)| = \binom{m-1}{k}.$$

These and (6) yield

$$|\mathcal{F}| \leq \binom{m}{k} + \sum_{\ell \leq i \leq k-1} \binom{m-1}{i-1} \binom{n-m}{k-i} \quad \text{as desired.} \quad \square$$

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