Perfect State Transfer on Cayley Graphs over Dihedral Groups: The Non-Normal Case

Xiwang Cao
School of Mathematical Sciences
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China

State Key Laboratory of Information Security
Institute of Information Engineering
Chinese Academy of Sciences, Beijing 100093, China

xwcao@nuaa.edu.cn

Bocong Chen
School of Mathematics
South China University of Technology
Guangzhou 510641, China

bocong_chen@yahoo.com

San Ling
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
Singapore 639798

lingsan@ntu.edu.sg

Submitted: Dec 1, 2019; Accepted: May 8, 2020; Published: May 29, 2020

Abstract

Recently, perfect state transfer (PST for short) on graphs has attracted great attention due to their applications in quantum information processing and quantum computations. Many constructions and results have been established through various graphs. However, most of the graphs previously investigated are abelian Cayley graphs. Necessary and sufficient conditions for Cayley graphs over dihedral groups having perfect state transfer were studied recently. The key idea in that paper is the assumption of the normality of the connection set. In those cases, viewed as an element in a group algebra, the connection set is in the center of the group algebra, which makes the situations just like in the abelian case. In this paper, we study the non-normal case. In this case, the discussion becomes more complicated. Using the representations of the dihedral group D_n, we show that $\mathrm{Cay}(D_n, S)$ cannot have PST if n is odd. For even integers n, it is proved that if $\mathrm{Cay}(D_n, S)$ has PST, then S is normal.

Mathematics Subject Classifications: 05C25 81P45 81Q35

1 Introduction

Let Γ be an undirected simple graph whose vertex set is denoted by $V(\Gamma)$. Let A be an adjacency matrix of Γ. For an real number t, the *transfer matrix* of Γ is defined as the following $n \times n$ matrix:

$$H(t) = H_\Gamma(t) = \exp(-\imath tA) = \sum_{s=0}^{+\infty} \frac{(-\imath tA)^s}{s!} = (H_{u,v}(t))_{u,v\in V(\Gamma)},$$

where $\imath = \sqrt{-1}$ and $n = |V(\Gamma)|$ is the number of vertices in Γ.

Definition 1. Let Γ be a graph. For two distinct vertices $u, v \in V(\Gamma)$, we say that Γ has a perfect state transfer (PST) from u to v at the time $t(> 0)$ if the (u, v)-entry of $H(t)$, denoted by $H(t)_{u,v}$, has absolute value 1. We say that Γ is periodic at u with period t if $H(t)_{u,u}$ has absolute value 1. If Γ is periodic with period t at every point, then Γ is said to be periodic.

The phenomenon of perfect state transfer (PST) in quantum communication networks was originally introduced by Bose in [14]. This work has attracted much research interest since many applications have been found in quantum information processing and cryptography (see [1, 2, 3, 5, 11, 15, 17, 18, 34, 38, 37] and the references therein).

For perfect state transfer and related questions such as the close relationship with algebraic combinatorial objects, we refer the reader to Godsil [23, 24, 25] and Coutinho [19] and the references therein.

Since the adjacency matrix A is symmetric, its eigenvalues are all real numbers. A graph Γ is called *integral* if all eigenvalues of the adjacency matrix A are integers [26]. For abelian Cayley graphs, a complete characterization of integral graphs over abelian groups was obtained by Bridges and Mena [12] in 1982.

Despite many results about PST on graphs, there are only a few known constructions of PST on non-abelian Cayley graphs in the literature [13]. In [13], the authors studied the existence of PST on Cayley graphs over dihedral groups. Some necessary and sufficient conditions for a normal Cayley graph $\mathrm{Cay}(D_n, S)$ (which means that S is conjugation-closed, namely, $g^{-1}Sg = S$ for each $g \in G$) having PST were presented. As an application, it was proved that $\mathrm{Cay}(D_n, S)$ is periodic if and only if it is integral. Meanwhile, it was showed that $\mathrm{Cay}(D_n, S)$ has PST for some connection set S and some even integers n.

In the present paper, we consider the general case. Actually, it is not an easy task to get an explicit expression for the spectra and eigenspaces of Cayley graphs over non-abelian groups when the connection set is non-normal. If the underlying group is a dihedral group, then we show that one can use the Fourier transform to obtain the spectra and eigenvectors of the Cayley graph. Based on this observation, it is proved that $\mathrm{Cay}(D_n, S)$ cannot have PST if n is odd. For even integers n, it is shown that $\mathrm{Cay}(D_n, S)$ has PST if and only if S is normal.

The rest of the paper is organised as follows: In Sect. 2, we recall some basic facts about finite group representations and the Fourier transform on finite groups. In Sect. 3, we provide the spectra and the corresponding eigenspace of Cayley graphs over dihedral groups. In Sect. 4, we present our main results. In Sect. 6, we make some concluding remarks.

2 The representation of groups and The Fourier Transform

Let G be a finite group. A *representation* of G is a homomorphism $\rho : G \to GL(U)$, where U is a (finite-dimensional) non-zero vector space over the field of complex numbers $\mathbb{C}$. The dimension of U is called the *degree* of ρ. Two representations $\rho : G \to GL(U)$ and $\varrho : G \to GL(W)$ are *equivalent*, denoted by $\rho \sim \varrho$, if there exists an isomorphism $T : U \to W$ such that $\rho_g = T\varrho_g T^{-1}$ for all $g \in G$. For a representation $\rho : G \to GL(U)$ of G, the *character* of ρ is defined by:

$$\chi_\rho : G \to \mathbb{C}, \chi_\rho(g) = \mathrm{tr}(\rho(g)) \text{ for all } g \in G,$$

where $\mathrm{tr}(\rho(g))$ is the trace of the matrix $\rho(g)$ with respect to a basis of U. A subspace W of U is said to be G*-invariant* if $\rho(g)\omega \in W$ for every $g \in G$ and $\omega \in W$. Obviously, $\{0\}$ and U are G-invariant subspaces, which are called trivial subspaces. If U has no non-trivial G-invariant subspaces, then ρ is called an *irreducible representation* of G and χ_ρ is called an *irreducible character* of G.

Let S be a subset of G with $|S| = d \geqslant 1$. The Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is defined by

$$\begin{aligned} V(\Gamma) &= G, \text{ the set of vertices,} \\ E(\Gamma) &= \{(u, v) : u, v \in G, uv^{-1} \in S\}, \text{ the set of edges.} \end{aligned}$$

We assume that the identity element of G is not belonged to S (denoted by $1_G \notin S$), $S = S^{-1} = \{s^{-1} : s \in S\}$ (which means that Γ is a simple graph) and $G = \langle S \rangle$ (G is generated by S which means that Γ is connected). The adjacency matrix of Γ is defined by $A = A(\Gamma) = (a_{g,h})_{g,h\in G}$ where

$$a_{g,h} = \begin{cases} 1, & \text{if } gh^{-1} \in E(\Gamma), \\ 0, & \text{otherwise.} \end{cases}$$

Let $L(G) = \{f : G \to \mathbb{C}\}$ be the set of complex-valued functions from G to $\mathbb{C}$. For $f_1, f_2 \in L(G)$, we define

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(gh^{-1})f_2(h).$$

Then $L(G)$ is a $\mathbb{C}$-algebra. Moreover, we have a scalar product $\langle \cdot, \cdot \rangle_{L(G)}$ on $L(G)$ by setting

$$\langle f_1, f_2 \rangle_{L(G)} = \sum_{g\in G} f_1(g)\overline{f_2(g)},$$

where $\bar{\cdot}$ is the conjugate. For every $g \in G$, we define a mapping $\delta_g \in L(G)$ by

$$\delta_g(x) = \begin{cases} 1, & x = g, \\ 0, & x \neq g. \end{cases}$$

Then it is easy to see that $\{\delta_g : g \in G\}$ forms a basis of $L(G)$ and thus $\dim(L(G)) = |G|$. Define $f_S \in \mathrm{Hom}(L(G), L(G))$ by

$$f_S : L(G) \to L(G),\ \sum_{g \in G} c_g \delta_g \mapsto \sum_{s \in S} \sum_{g \in G} c_g \delta_{sg}.$$

The matrix of f_S under the basis $\{\delta_g : g \in G\}$ is identical to the adjacent matrix of the Cayley graph $\mathrm{Cay}(G, S)$.

From now on, we focus on the unitary representations of G. Let $\hat{G}$ be a complete set of pairwise inequivalent irreducible unitary representations of G. Let $\mathcal{A}(G)$ be defined as

$$\mathcal{A}(G) = \bigoplus_{\rho \in \hat{G}} \mathrm{Hom}(W_\rho, W_\rho),$$

where W_ρ is the representation space corresponding to the unitary irreducible representation ρ. Suppose that $\{v_1^\rho, \cdots, v_{d_\rho}^\rho\}$ is an orthonormal basis of W_ρ for any $\rho \in \hat{G}$. For $\rho, \sigma \in \hat{G}$ and $1 \leqslant i, j \leqslant d_\rho$, define

$$T_{i,j}^\rho \omega = \delta_{\rho,\sigma} \langle \omega, v_j^\rho \rangle_{W_\rho} v_i^\rho,\ \omega \in W_\sigma,$$

where $\delta_{\rho,\sigma} = 1$ if $\rho \sim \sigma$ and 0 otherwise. Then $T_{i,j}^\rho \in \mathcal{A}(G)$ and $\{T_{i,j}^\rho : \rho \in \hat{G}, 1 \leqslant i, j \leqslant d_\rho\}$ form a basis of $\mathcal{A}(G)$. In correspondence to the basis $\{v_1^\rho, \cdots, v_{d_\rho}^\rho\}$ of W_ρ, we define

$$\varphi_{i,j}^\rho : G \to \mathbb{C},\ g \mapsto \langle \rho(g) v_j^\rho, v_i^\rho \rangle_{W_\rho}.$$

Then $\varphi_{i,j}^\rho \in L(G)$ and it is known that $\{\varphi_{i,j}^\rho : \rho \in \hat{G}, 1 \leqslant i, j \leqslant d_\rho\}$ forms an orthonormal basis of $L(G)$. For every $f \in L(G)$, the Fourier transform of f is defined by

$$\mathcal{F}(f) = \bigoplus_{\rho \in \hat{G}} \rho(f),$$

where $\rho(f)$ is defined as

$$\rho(f) : W_\rho \to W_\rho,\ \omega \mapsto \sum_{g \in G} f(g) \rho(g)(\omega).$$

In other words, $\rho(f) = \sum_{g \in G} f(g) \rho(g)$ and thus $\rho(f) \in \mathrm{Hom}(W_\rho, W_\rho)$. It is known that the Fourier transform $\mathcal{F}$ is an algebra isomorphism from $L(G)$ to $\mathcal{A}(G)$ and

$$\mathcal{F}\overline{\varphi_{i,j}^\rho} = \frac{|G|}{d_\rho} T_{i,j}^\rho.$$

See [35] for details. Thus we have the following commutative diagram:

$$\begin{array}{ccc} L(G) & \xrightarrow{f_S} & L(G) \\ \uparrow {\scriptstyle \mathcal{F}^{-1}} & & \downarrow {\scriptstyle \mathcal{F}} \\ \mathcal{A}(G) & \xrightarrow{\mathfrak{f}} & \mathcal{A}(G). \end{array}$$

Moreover,

$$\mathfrak{f}(T^{\rho}_{i,j}) = \frac{d\rho}{|G|}\mathcal{F} f_S \overline{\varphi^{\rho}_{i,j}}.$$

Now,

$$\begin{aligned} f_S\overline{\varphi^{\rho}_{i,j}}(g) &= \sum_{s\in S} \overline{\varphi^{\rho}_{i,j}}(s^{-1}g) \\ &= \sum_{s\in S} \overline{\langle \rho(s^{-1}g)v^{\rho}_j, v^{\rho}_i\rangle} \\ &= \sum_{s\in S} \overline{\langle \rho(g)v^{\rho}_j, \rho(s)v^{\rho}_i\rangle} \\ &= \sum_{s\in S} \overline{\Big\langle \rho(g)v^{\rho}_j, \sum_{k=1}^{d_\rho} \langle \rho(s)v^{\rho}_i, v^{\rho}_k\rangle v^{\rho}_k \Big\rangle} \\ &= \sum_{s\in S}\sum_{k=1}^{d_\rho} \langle \rho(s)v^{\rho}_i, v^{\rho}_k\rangle \overline{\langle \rho(g)v^{\rho}_j, v^{\rho}_k\rangle} \\ &= \sum_{k=1}^{d_\rho} \langle \sum_{s\in S} \rho(s)v^{\rho}_i, v^{\rho}_k\rangle \overline{\langle \rho(g)v^{\rho}_j, v^{\rho}_k\rangle} \\ &= \sum_{k=1}^{d_\rho} \langle \sum_{s\in S} \rho(s)v^{\rho}_i, v^{\rho}_k\rangle \overline{\varphi^{\rho}_{k,j}(g)}. \end{aligned}$$

Therefore,

$$\mathfrak{f}(T^{\rho}_{i,j}) = \frac{d_\rho}{|G|}\mathcal{F} f_S \overline{\varphi^{\rho}_{i,j}} = \frac{d_\rho}{|G|}\sum_{k=1}^{d_\rho}\langle \sum_{s\in S}\rho(s)v^{\rho}_i, v^{\rho}_k\rangle \mathcal{F}\overline{\varphi^{\rho}_{k,j}}.$$

That is

$$\mathfrak{f}(T^{\rho}_{i,j}) = \sum_{k=1}^{d_\rho}\langle \sum_{s\in S}\rho(s)v^{\rho}_i, v^{\rho}_k\rangle T^{\rho}_{k,j}, \rho \in \hat{G}, 1 \leqslant i,j \leqslant d_\rho. \tag{1}$$

If S is conjugation-closed, namely $gSg^{-1} = S$ for all $g \in S$, then by the Schur orthogonality relations ([36, Theorem 4.2.8]) we have the following result.

Lemma 2. *[36, pp. 69-70] Let $G = \{g_1, \cdots, g_n\}$ be a finite group of order n and let $\rho^{(1)}, \cdots, \rho^{(s)}$ be a complete set of unitary representatives of the equivalent classes of irreducible representations of G. Let χ_i be the character of $\rho^{(i)}$ and let d_i be the degree of χ_i. Let S be a subset of G with $1_G \notin S$, $S = S^{-1}$ and $gSg^{-1} = S$ for all $g \in G$. Then the eigenvalues of the adjacency matrix A of the Cayley graph $\mathrm{Cay}(G, S)$ with respect to S are given by*

$$\lambda_k = \frac{1}{d_k} \sum_{g \in S} \chi_k(g), 1 \leqslant k \leqslant s,$$

where each λ_k has multiplicity d_k^2. Moreover, the vectors

$$v_{ij}^{(k)} = \frac{\sqrt{d_k}}{|G|} \left(\rho_{ij}^{(k)}(g_1), \cdots, \rho_{ij}^{(k)}(g_n) \right)^T, 1 \leqslant i, j \leqslant d_k$$

form an orthonormal basis for the eigenspace V_{λ_k}.

Note that a proof of Lemma 2 can also be found in [31, Theorem 9]. Babai [7] studied the spectra of Cayley color graphs, which generalizes the notion of Cayley graphs; however, [7] does not consider how to find the corresponding eigenvectors of the eigenvalues. If S is not conjugation-closed, to the best of our knowledge, no method in literature tells how one can find the eigenvalues and the corresponding eigenvectors of the Cayley graphs. In the next section, we will use (1) to find the eigenvalues and eigenvectors of f_S.

3 Spectra and the corresponding eigenspaces of Cayley graphs over dihedral groups

For the dihedral group $D_n = \langle a, b \,|\, a^n = b^2 = 1, bab = a^{-1} \rangle$, its representations and characters are well known as given below.

Lemma 3. *[33, pp. 36-37] (1) The irreducible representations of the dihedral group $D_n (n \geqslant 2)$ are listed in Table 1 for even n and in Table 2 for odd n.*

Table 1. Representations of D_n (n even)

	$a^k (0 \leqslant k \leqslant n-1)$	$ba^k (0 \leqslant k \leqslant n-1)$
ψ_1	1	1
ψ_2	1	-1
ψ_3	$(-1)^k$	$(-1)^k$
ψ_4	$(-1)^k$	$(-1)^{k+1}$
ρ_h $_{1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor}$	$\begin{pmatrix} \omega^{hk} & 0 \\ 0 & \omega^{-hk} \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^{-hk} \\ \omega^{hk} & 0 \end{pmatrix}$

Table 2. Representations of D_n (n odd)

	$a^k(0 \leqslant k \leqslant n-1)$	$ba^k(0 \leqslant k \leqslant n-1)$
ψ_1	1	1
ψ_2	1	-1
ρ_h $1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor$	$\begin{pmatrix} \omega^{hk} & 0 \\ 0 & \omega^{-hk} \end{pmatrix}$	$\begin{pmatrix} 0 & \omega^{-hk} \\ \omega^{hk} & 0 \end{pmatrix}$

(2) The character table of D_n is listed in Table 3 for even n and in Table 4 for odd n.

Table 3. Character table of D_n (n even)

	$a^k(0 \leqslant k \leqslant n-1)$	$ba^k(0 \leqslant k \leqslant n-1)$
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$
χ_h $1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor$	$2\cos(\frac{2hk\pi}{n})$	0

Table 4. Character table of D_n (n odd)

	$a^k(0 \leqslant k \leqslant n-1)$	$ba^k(0 \leqslant k \leqslant n-1)$
χ_1	1	1
χ_2	1	-1
χ_h $1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor$	$2\cos(\frac{2hk\pi}{n})$	0

where $\omega = \exp(\frac{2\pi\imath}{n})$ is a primitive n-th root of unity.

Let $\mathrm{Cay}(D_n, S)$ be a Cayley graph with $1_G \notin S$, $S = S^{-1}$ and $G = D_n = \langle S \rangle$. Let $S_1 = S \cap \langle a \rangle$ and $bS_2 = S \cap b\langle a \rangle$. We assume that $s_1 = |S_1|, s_2 = |S_2|$ and $d = s_1 + s_2 = |S|$.

3.1 n is odd

We first consider the one-dimensional representations, and we obtain the following simple result.

Lemma 4. *For the one-dimensional representations of D_n, one has*

(1) The eigenvalue corresponding to the trivial representation is $\lambda_1 = d = |S|$, and the associated eigenvector is

$$p_1 = \frac{1}{\sqrt{2n}} \varphi_{11}^{\psi_1} = \frac{1}{\sqrt{2n}} (1, 1, \cdots, 1)^t.$$

(2) The eigenvalue corresponding to the one-dimensional representation ψ_2 is $\lambda_2 = s_1 - s_2$, and the associated eigenvector is

$$p_2 = \frac{1}{\sqrt{2n}} \varphi_{11}^{\psi_2} = \frac{1}{\sqrt{2n}} (\underbrace{1, 1, \cdots, 1}_{n \text{ times}}, \underbrace{-1, -1, \cdots, -1}_{n \text{ times}})^t.$$

Proof. Now, suppose that $\rho = \psi_1$ is the trivial representation. By (1), we have

$$\mathfrak{f}(T_{i,j}^{\psi_1}) = \sum_{k=1}^{d_{\psi_1}} \langle \sum_{s\in S} \rho(s) v_i^{\psi_1}, v_k^{\psi_1} \rangle T_{k,j}^{\psi_1} = dT_{i,j}^{\psi_1},\ 1 \leqslant i,j \leqslant d_{\psi_1} = 1.$$

This implies that

$$f_S \mathcal{F}^{-1} T_{i,j}^{\psi_1} = \sum_{k=1}^{d_{\psi_1}} \langle \sum_{s\in S} \rho(s) v_i^{\psi_1}, v_k^{\psi_1} \rangle \mathcal{F}^{-1} T_{k,j}^{\psi_1} = d\mathcal{F}^{-1} T_{i,j}^{\psi_1},\ 1 \leqslant i,j \leqslant d_{\psi_1} = 1.$$

We have shown that d is the eigenvalue corresponding to the trivial representation and the associated eigenvector is $p_1 = \frac{1}{\sqrt{2n}} \varphi_{11}^{\psi_1} = \frac{1}{\sqrt{2n}}(1,1,\cdots,1)^t$.

Next we assume that $\rho = \psi_2$. Then

$$\mathfrak{f}(T_{i,j}^{\psi_2}) = \sum_{k=1}^{d_{\psi_2}} \langle \sum_{s\in S} \rho(s) v_i^{\psi_2}, v_k^{\psi_2} \rangle T_{k,j}^{\psi_2} = (s_1 - s_2) T_{i,j}^{\psi_2},\ 1 \leqslant i,j \leqslant d_{\psi_2} = 1.$$

It follows that the eigenvalue corresponding to ψ_2 is $s_1 - s_2$ and the associated eigenvector is

$$\frac{1}{\sqrt{2n}}(\underbrace{1,1,\cdots,1}_{n \text{ times}}, \underbrace{-1,-1,\cdots,-1}_{n \text{ times}})^t. \qquad \square$$

Suppose now that ρ_h is a two-dimensional representation for $1 \leqslant h \leqslant \lfloor \frac{n-1}{2} \rfloor$. We then have

$$\begin{cases} \rho_h(a^k) v_1^{\rho_h} = & \omega^{hk} v_1^{\rho_h}, \\ \rho_h(a^k) v_2^{\rho_h} = & \omega^{-hk} v_2^{\rho_h}, \\ \rho_h(b) v_1^{\rho_h} = & v_2^{\rho_h}, \\ \rho_h(b) v_2^{\rho_h} = & v_1^{\rho_h}. \end{cases} \tag{2}$$

Let

$$\eta_h(S_1) = \sum_{a^k \in S_1} \omega^{hk}, \eta_h(S_2) = \sum_{a^k \in S_2} \omega^{hk}.$$

Assume that, corresponding to the two-dimensional representation ρ_h, μ_h is an eigenvalue of f_S associated with an eigenvector v_{ρ_h}.

Lemma 5. *Let the notation be defined as above. The eigenvalues corresponding to the representation ρ_h are*

$$\mu_h = \eta_h(S_1) \pm \sqrt{\eta_h(S_2)\eta_h(S_2^{-1})}, (1 \leqslant h \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor), \tag{3}$$

each with multiplicity 2.

Proof. By (1) and (2), we have

$$\begin{cases} \mathfrak{f}(T_{11}^{\rho_h}) = \eta_h(S_1)T_{11}^{\rho_h} + \eta_h(S_2)T_{21}^{\rho_h}, \\ \mathfrak{f}(T_{12}^{\rho_h}) = \eta_h(S_1)T_{12}^{\rho_h} + \eta_h(S_2)T_{22}^{\rho_h}, \\ \mathfrak{f}(T_{21}^{\rho_h}) = \eta_h(S_2^{-1})T_{11}^{\rho_h} + \eta_h(S_1)T_{21}^{\rho_h}, \\ \mathfrak{f}(T_{22}^{\rho_h}) = \eta_h(S_2^{-1})T_{12}^{\rho_h} + \eta_h(S_1)T_{22}^{\rho_h}. \end{cases} \tag{4}$$

In matrix form, we have

$$\mathfrak{f}(T_{11}^{\rho_h}, T_{21}^{\rho_h}, T_{12}^{\rho_h}, T_{22}^{\rho_h}) = (T_{11}^{\rho_h}, T_{21}^{\rho_h}, T_{12}^{\rho_h}, T_{22}^{\rho_h}) \begin{pmatrix} \eta_h(S_1) & \eta_h(S_2^{-1}) & 0 & 0 \\ \eta_h(S_2) & \eta_h(S_1) & 0 & 0 \\ 0 & 0 & \eta_h(S_1) & \eta_h(S_2^{-1}) \\ 0 & 0 & \eta_h(S_2) & \eta_h(S_1) \end{pmatrix}. \tag{5}$$

Thus we get the desired result. □

By (1), we can assume that $v_{\rho_h} = a_{11}T_{11}^{\rho_h} + a_{12}T_{12}^{\rho_h} + a_{21}T_{21}^{\rho_h} + a_{22}T_{22}^{\rho_h}$ with $a_{ij} \in \mathbb{C}$. It follows that

$$\mathfrak{f}(a_{11}T_{11}^{\rho_h} + a_{12}T_{12}^{\rho_h} + a_{21}T_{21}^{\rho_h} + a_{22}T_{22}^{\rho_h}) = \mu_h(a_{11}T_{11}^{\rho_h} + a_{12}T_{12}^{\rho_h} + a_{21}T_{21}^{\rho_h} + a_{22}T_{22}^{\rho_h}).$$

This leads to

$$f_S(a_{11}\varphi_{11}^{\rho_h} + a_{12}\varphi_{12}^{\rho_h} + a_{21}\varphi_{21}^{\rho_h} + a_{22}\varphi_{22}^{\rho_h}) = \mu_h(a_{11}\varphi_{11}^{\rho_h} + a_{12}\varphi_{12}^{\rho_h} + a_{21}\varphi_{21}^{\rho_h} + a_{22}\varphi_{22}^{\rho_h}). \tag{6}$$

By solving this equation, we get the following result.

Lemma 6. *(1) If $\eta_h(S_2) = 0$, then the corresponding eigenvalue is $\mu_h = \eta_h(S_1)$ with multiplicity 4 and the associated eigenvectors are $\varphi_{11}^{\rho_h}, \varphi_{21}^{\rho_h}, \varphi_{12}^{\rho_h}, \varphi_{22}^{\rho_h}$.*

(2) If $\eta_h(S_2) \neq 0$, let $\ell_h = \sqrt{\frac{\eta_h(S_2^{-1})}{\eta_h(S_2)}}$ and $\iota_h = \ell_h\bar{\ell}_h n + n$. Then the eigenvectors associated with $\mu_h^{(1)} = \eta_h(S_1) + \sqrt{\eta_h(S_2)\eta_h(S_2^{-1})}$ are

$$p_h^{(1)} = \frac{1}{\sqrt{\iota_h}}(\bar{\ell}_h\varphi_{11}^{\rho_h} + \varphi_{21}^{\rho_h}),\ p_h^{(2)} = \frac{1}{\sqrt{\iota_h}}(\bar{\ell}_h\varphi_{12}^{\rho_h} + \varphi_{22}^{\rho_h}).$$

The eigenvectors associated with $\mu_h^{(2)} = \eta_h(S_1) - \sqrt{\eta_h(S_2)\eta_h(S_2^{-1})}$ are

$$p_h^{(3)} = \frac{1}{\sqrt{\iota_h}}(-\varphi_{11}^{\rho_h} + \ell_h\varphi_{21}^{\rho_h}),\ p_h^{(4)} = \frac{1}{\sqrt{\iota_h}}(-\varphi_{12}^{\rho_h} + \ell_h\varphi_{22}^{\rho_h}).$$

Proof. These facts are deduced from (5) and (6). □

3.2 $n = 2m$ is even

In this case, Lemmas 4-6 also can be applied in this case, but there are two additional one-dimensional representations, namely ψ_3, ψ_4, see Lemma 3. For these two representations, based on the above discussion, we have the following result.

Lemma 7. *Let*

$$S_{10} = \{a^k : 1 \leqslant k \leqslant n-1, a^k \in S_1,\ k \text{ even}\},\ S_{11} = \{a^k : 1 \leqslant k \leqslant n-1, a^k \in S_1,\ k \text{ odd}\}$$

and

$$S_{20} = \{a^k : 1 \leqslant k \leqslant n-1, a^k \in S_2,\ k \text{ even}\},\ S_{21} = \{a^k : 1 \leqslant k \leqslant n-1, a^k \in S_2,\ k \text{ odd}\}.$$

Then $S_1 = S_{10} \cup S_{11}, S_2 = S_{20} \cup S_{21}$. Let $d_{i0} = |S_{i0}|, d_{i1} = |S_{i1}|, i = 1, 2$. The eigenvalue corresponding to the representation ψ_3 is $\lambda_3 = d_{11} - d_{10} + d_{21} - d_{20}$, and the associated eigenvector is

$$p_3 = \frac{1}{\sqrt{2n}}((-1)^i : 0 \leqslant i \leqslant 2m-1, (-1)^i : 0 \leqslant i \leqslant 2m-1)^t.$$

The eigenvalue corresponding to the representation ψ_4 is $\lambda_4 = d_{11} - d_{10} + d_{20} - d_{21}$, and the associated eigenvector is

$$p_4 = \frac{1}{\sqrt{2n}}((-1)^i : 0 \leqslant i \leqslant 2m-1, (-1)^{i+1} : 0 \leqslant i \leqslant 2m-1)^t.$$

4 PST on Cay(D_n, S)

For a simple graph Γ, Spec(Γ) denotes the set of all eigenvalues of Γ. For any symmetric matrix A, assume that its eigenvalues are λ_i $(1 \leqslant i \leqslant n)$. There is a unitary matrix $P = (p_1, \cdots, p_n)$, where each p_i $(1 \leqslant i \leqslant n)$ is an eigenvector of λ_i $(1 \leqslant i \leqslant n)$, so that we have the following spectral decomposition of A

$$A = \lambda_1 E_1 + \cdots + \lambda_n E_n, \tag{7}$$

where $E_i = p_i p_i^*$ $(1 \leqslant i \leqslant n)$ satisfies

$$E_i E_j = \begin{cases} E_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Therefore, we have the decomposition of the transfer matrix

$$H(t) = \exp(-\imath\lambda_1 t)E_1 + \cdots + \exp(-\imath\lambda_n t)E_n. \tag{9}$$

We also need notation of the 2-adic exponential valuation of rational numbers which is a mapping defined by

$$v_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\},\ v_2(0) = \infty,\ v_2(2^\ell \frac{a}{b}) = \ell,\ \text{where } a, b, \ell \in \mathbb{Z} \text{ and } 2 \nmid ab.$$

We assume that $\infty + \infty = \infty + \ell = \infty$ and $\infty > \ell$ for any $\ell \in \mathbb{Z}$. Then v_2 has the following properties. For $\beta, \beta' \in \mathbb{Q}$,

(P1) $v_2(\beta\beta') = v_2(\beta) + v_2(\beta')$;

(P2) $v_2(\beta + \beta') \geqslant \min(v_2(\beta), v_2(\beta'))$ and the equality holds if $v_2(\beta) \neq v_2(\beta')$.

Finally, we label the elements of $D_n = \langle a, b \,|\, a^n = b^2 = 1, bab = a^{-1} \rangle$ as follows. For a number u, if $0 \leqslant u \leqslant n-1$, then u corresponds to element a^u, and if $n \leqslant u \leqslant 2n-1$, then u corresponds to ba^u.

4.1 n is odd

In this subsection, we show that if n is odd and S is a subset of D_n, then $\mathrm{Cay}(D_n, S)$ has no PST between any pair of distinct vertices.

Theorem 8. *Let $n = 2m+1$ and let S be a non-empty subset of D_n. Let $\Gamma = \mathrm{Cay}(D_n, S)$ be a connected Cayley graph with connection set S. Then Γ has no PST between two distinct vertices, and Γ is periodic if and only if it is integral and $S_2 = \emptyset$ or $\langle a \rangle$. The minimum period of the vertices is $\frac{2\pi}{M}$, where $M = \gcd(\lambda - \lambda_1 : \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$.*

Proof. Let $n = 2m+1$. By the above computation results, we have the following unitary matrix

$$P = (p_1, p_2, p_1^{(1)}, p_1^{(2)}, p_1^{(3)}, p_1^{(4)}, \cdots, p_m^{(1)}, p_m^{(2)}, p_m^{(3)}, p_m^{(4)}),$$

where

$$p_1 = \frac{1}{\sqrt{2n}}(1, 1, \cdots, 1)^t, p_2 = \frac{1}{\sqrt{2n}}(1, \cdots, 1, -1, \cdots, -1)^t. \tag{10}$$

The projective matrices E_i $(1 \leqslant i \leqslant 2)$ are

$$E_1 = \frac{1}{2n} J_{2n}, E_2 = \frac{1}{2n} \begin{pmatrix} J_n & -J_n \\ -J_n & J_n \end{pmatrix}, \tag{11}$$

where J_n is the all-one matrix of order n. For $1 \leqslant h \leqslant m$, if $\eta_h(S_2) = 0$, then the associated eigenvectors are

$$\begin{array}{ll} p_h^{(1)} = \frac{1}{\sqrt{n}}\varphi_{11}^{\rho_h} = \frac{1}{\sqrt{n}}(\{\omega^{hk}\}_{k=0}^{n-1}, 0)^t, & p_h^{(2)} = \frac{1}{\sqrt{n}}\varphi_{21}^{\rho_h} = \frac{1}{\sqrt{n}}(0, \{\omega^{-hk}\}_{k=0}^{n-1})^t, \\ p_h^{(3)} = \frac{1}{\sqrt{n}}\varphi_{12}^{\rho_h} = \frac{1}{\sqrt{n}}(0, \{\omega^{hk}\}_{k=0}^{n-1})^t, & p_h^{(4)} = \frac{1}{\sqrt{n}}\varphi_{22}^{\rho_h} = \frac{1}{\sqrt{n}}(\{\omega^{-hk}\}_{k=0}^{n-1}, 0)^t. \end{array} \tag{12}$$

The corresponding projective matrices are

$$\begin{array}{ll} E_h^{(1)} = \frac{1}{n}\begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix}, & E_h^{(2)} = \frac{1}{n}\begin{pmatrix} 0 & 0 \\ 0 & \bar{\Omega} \end{pmatrix}, \\ E_h^{(3)} = \frac{1}{n}\begin{pmatrix} 0 & 0 \\ 0 & \Omega \end{pmatrix}, & E_h^{(4)} = \frac{1}{n}\begin{pmatrix} \bar{\Omega} & 0 \\ 0 & 0 \end{pmatrix}, \end{array} \tag{13}$$

where $\Omega = C(1, \overline{\omega}^h, \cdots, \overline{\omega}^{(n-1)h})$ is the circulant matrix with first row $(1, \overline{\omega}^h, \cdots, \overline{\omega}^{(n-1)h})$. For $1 \leqslant h \leqslant m$, if $\eta_h(S_2) \neq 0$, then by Lemma 6, the associated eigenvectors are

$$p_h^{(1)} = \frac{1}{\sqrt{\iota_h}}(\bar{\ell}_h \varphi_{11}^{\rho_h} + \varphi_{21}^{\rho_h})$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{\iota_h}}(\bar{\ell}_h, \bar{\ell}_h\omega^h, \cdots, \bar{\ell}_h\omega^{(n-1)h}, 1, \bar{\omega}^h, \cdots, \bar{\omega}^{(n-1)h})^t,\\
p_h^{(2)} &= \frac{1}{\sqrt{\iota_h}}(\bar{\ell}_h\varphi_{12}^{\rho_h} + \varphi_{22}^{\rho_h})\\
&= \frac{1}{\sqrt{\iota_h}}(1, \bar{\omega}^h, \cdots, \bar{\omega}^{(n-1)h}, \bar{\ell}_h, \bar{\ell}_h\omega^h, \cdots, \bar{\ell}_h\omega^{(n-1)h})^t,\\
p_h^{(3)} &= \frac{1}{\sqrt{\iota_h}}(-\varphi_{11}^{\rho_h} + \ell_h\varphi_{21}^{\rho_h})\\
&= \frac{1}{\sqrt{\iota_h}}(-1, -\omega^h, \cdots, -\omega^{(n-1)h}, \ell_h, \ell_h\bar{\omega}^h, \cdots, \ell_h\bar{\omega}^{(n-1)h})^t,\\
p_h^{(4)} &= \frac{1}{\sqrt{\iota_h}}(-\varphi_{12}^{\rho_h} + \ell_h\varphi_{22}^{\rho_h})\\
&= \frac{1}{\sqrt{\iota_h}}(\ell_h, \ell_h\bar{\omega}^h, \cdots, \ell_h\bar{\omega}^{(n-1)h}, -1, , -\omega^h, \cdots, -\omega^{(n-1)h})^t,
\end{aligned}
$$

where $\iota_h = \ell_h\bar{\ell}_h n + n$ with $\ell_h = \sqrt{\frac{\eta_h(S_2^{-1})}{\eta_h(S_2)}}$. The corresponding projective matrices are

$$
\begin{aligned}
E_h^{(1)} = \frac{1}{\iota_h}\begin{pmatrix} \bar{\ell}_h\ell_h\Omega & \bar{\ell}_h Q \\ \ell_h\bar{Q} & \bar{\Omega} \end{pmatrix}, \qquad E_h^{(2)} = \frac{1}{\iota_h}\begin{pmatrix} \bar{\Omega} & \ell_h\bar{Q} \\ \bar{\ell}_h Q & \bar{\ell}_h\ell_h\Omega \end{pmatrix},\\
E_h^{(3)} = \frac{1}{\iota_h}\begin{pmatrix} \Omega & -\bar{\ell}_h Q \\ -\ell_h\bar{Q} & \bar{\ell}_h\ell_h\bar{\Omega} \end{pmatrix}, \qquad E_h^{(4)} = \frac{1}{\iota_h}\begin{pmatrix} \bar{\ell}_h\ell_h\bar{\Omega} & -\ell_h\bar{Q} \\ -\bar{\ell}_h Q & \Omega \end{pmatrix},
\end{aligned}
\tag{14}
$$

where

$$
Q = \begin{pmatrix} 1 & \omega^h & \cdots & \omega^{(n-1)h} \\ \omega^h & \omega^{2h} & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ \omega^{(n-1)h} & 1 & \cdots & \omega^{(n-2)h} \end{pmatrix}.
$$

Substituting (13)-(4.8) in (9), we see that the (u, v)-entry of the transfer matrix is divided into two cases:

(1) $0 \leqslant u, v \leqslant n-1$ or $n \leqslant u, v \leqslant 2n-1$. In this case,

$$
\begin{aligned}
(H(t))_{u,v} &= \frac{1}{2n}(\exp(-\imath\lambda_1 t) + \exp(-\imath\lambda_2 t))\\
&\quad + \frac{1}{n}\sum_{h=1,\eta_h(S_2)=0}^{m}(\omega^{(v-u)h}\exp(-\imath\mu_h t) + \omega^{(u-v)h}\exp(-\imath\mu_h t))\\
&\quad + \sum_{h=1,\eta_h(S_2)\neq 0}^{m}\frac{1}{\iota_h}(\ell_h\bar{\ell}_h\omega^{(v-u)h} + \omega^{(u-v)h})\exp(-\imath\mu_h^{(1)} t)\\
&\quad + \sum_{h=1,\eta_h(S_2)\neq 0}^{m}\frac{1}{\iota_h}(\ell_h\bar{\ell}_h\omega^{(v-u)h} + \omega^{(u-v)h})\exp(-\imath\mu_h^{(2)} t).
\end{aligned}
\tag{15}
$$

(2) $0 \leqslant u \leqslant n-1, n \leqslant v \leqslant 2n-1$ or $n \leqslant u \leqslant 2n-1, 0 \leqslant v \leqslant n-1$. In this case, one can show that $|H(t)|_{u,v} < 1$ and thus PST cannot occur in this case.

We only consider the case of $0 \leqslant u, v \leqslant n-1$ as the remaining cases are analogous. By (15), we have

$$\begin{aligned}|H(t)_{u,v}| &\leqslant \frac{1}{2n}(|\exp(-\imath\lambda_1 t)| + |\exp(-\imath\lambda_2 t))| \\ &\quad +\frac{1}{n}\sum_{h=1}^{m}(|\omega^{(v-u)h}\exp(-\imath\mu_h^{(1)}t)| + |\omega^{(u-v)h}\exp(-\imath\mu_h^{(2)}t)|) \\ &= \frac{2}{2n} + \frac{1}{n}\sum_{h=1}^{m} 2 \\ &= 1.\end{aligned}$$

Thus, $|H(t)_{u,v}| = 1$ if and only if for $1 \leqslant h \leqslant m$ with $\eta_h(S_2) \neq 0$,

$$\exp(-\imath\lambda_1 t) = \frac{1}{\iota_h}\omega^{(u-v)h}\exp(-\imath\mu_h^{(1)}t) = \frac{1}{\iota_h}\omega^{-(u-v)h}\exp(-\imath\mu_h^{(2)}t) = \frac{\bar{\ell}_h\ell_h}{\iota_h}\exp(-\imath\mu_h^{(1)}). \quad (16)$$

This implies that $\iota_h = \bar{\ell}_h\ell_h = 1$, a contradiction. Thus, we know that if there is an h such that $\eta_h(S_2) \neq 0$, then there is no PST in Γ. Therefore, we consider the case of $\eta_h(S_2) = 0$ for all $1 \leqslant h \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$. In this case, we know that $S_2 = \emptyset$ or $\langle a \rangle$ by the inverse Fourier transform of the group ring $\mathbb{C}[\mathbb{Z}_n]$. Putting $t = 2\pi T$, we get from (15) that

$$2(u-v) \equiv 0 \pmod{n}, \quad (17)$$

$$(\lambda_2 - \lambda_1)T \in \mathbb{Z}, \quad (18)$$

$$(\mu_h - \lambda_1)T \in \mathbb{Z}, 1 \leqslant h \leqslant m. \quad (19)$$

Since $0 = \mathrm{tr}(A) = \lambda_1 + \lambda_2 + 4\sum_{h=1}^{m}\mu_h$, from (18) and (19), we have $2nT \in \mathbb{Z}$, and thus $T \in \mathbb{Q}$, the field of rational numbers. Now $\lambda_1 = |S|$ is a positive integer, we know that all the eigenvalues are rational and thus integral (since they are algebraic numbers). Moreover, (17) means that $u = v$, this is to say that Γ cannot have PST between distinct vertices when n is odd.

When $u = v$, by (15), we know that $\Gamma = \mathrm{Cay}(D_n, S)$ is period at the vertex u if and only if

$$\exp(-\imath\lambda_1 t) = \exp(-\imath\lambda_2 t) = \exp(-\imath\mu_h^{(1)}t) = \exp(-\imath\mu_h^{(2)}t)$$

for all $1 \leqslant h \leqslant \left\lfloor \frac{n-1}{2} \right\rfloor$ and some $t > 0$. This is equivalent to saying that Γ is periodic at the vertex u if and only if $T(\lambda - \lambda_1) \in \mathbb{Z}$ for all $\lambda \in \mathrm{Spec}(\Gamma)$. Equivalently, $T\gcd(\lambda - \lambda_1 : \lambda \in \mathrm{Spec}(\Gamma)) \in \mathbb{Z}$. Thus, the minimum $T(>0)$ is obviously $1/M$, where $M = \gcd(\lambda - \lambda_1 : \lambda \in \mathrm{Spec}(\Gamma))$. □

5 Broglington Manifolds

This section describes background information about Broglington Manifolds.

Lemma 9. *Broglington manifolds are abundant.*

Proof. A proof is given here. □

Remark 10. We note that the conclusion of Theorem 8, which says that Γ cannot have PST when n is odd, may be deduced from Godsil [24, Lemma 7.1] directly. Since every Cayley graph is vertex-transitive, if Γ has PST, then $H(t)$ should be of the form ζP, where ζ is a unit-norm complex number and P is a permutation matrix with $P^2 = I$. Moreover, P corresponds to an element of order 2 which lies in the center of D_n. When n is odd, the center of D_n is 1. This is a contradiction. Here we just give a direct proof of the fact. Moreover using this proof, we show that if Γ has PST, then it should be normal, namely $g^{-1}Sg = S$ holds for all $g \in D_n$.

5.1 n even

In this subsection, we assume that $n = 2m$ is even. From the discussion in the previous section, we know that in order to ensure that $\Gamma = \mathrm{Cay}(D_n, S)$ has PST, the set S_2 should be $\emptyset$ or $\langle a \rangle$. However, if $S_2 = \emptyset$, then Γ is not connected. As a consequence, Γ has PST only if $S = \langle a \rangle$ and thus is conjugation-closed. When $\Gamma = \mathrm{Cay}(D_n, S)$ is normal, a necessary and sufficient condition on which Γ has PST was provided in [13]. For completeness, we combine the results as follows.

Theorem 11. *Let $n = 2m$ and let S be a non-empty subset of D_n. Let $\Gamma = \mathrm{Cay}(D_n, S)$ be a connected Cayley graph with the connection set S. Then Γ cannot have PST between two distinct vertices if S is not conjugation-closed. Conversely, if S is conjugation-closed, then Γ has four eigenvalues (not necessarily dinstict) which correspond to the one-dimensional representations ψ_1 to ψ_4, respectively. One eigenvalue is $\lambda_1 = |S|$ and the other three eigenvalues are denoted by $\lambda_2, \lambda_3, \lambda_4$, and some multiple eigenvalues corresponding to the two-dimensional representations ρ_h, which are denoted by $\mu_h (1 \leqslant h \leqslant m-1)$. Moreover, Γ is periodic if and only if it is integral. The minimum period of the vertices is $\frac{2\pi}{M}$, where $M = \gcd(\lambda - \lambda_1 : \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$. Meanwhile,*

(i) when m is even, Γ has PST between two distinct vertices u and v if and only if

(i1) all eigenvalues of Γ are integers, namely, Γ is integral;

(i2) $v = u + m$;

(i3) there is a constant α such that $v_2(\mu_{2h'-1} - \lambda_1) = \alpha$ for every $1 \leqslant h' \leqslant m/2$ and for each eigenvalue $\lambda \neq \mu_{2h'-1} (1 \leqslant h' \leqslant m/2)$, we have that $v_2(\lambda - \lambda_1) > \alpha$.

(ii) when m is odd, Γ has PST between two distinct vertices u and v if and only if the following conditions hold:

(ii1) all the eigenvalues of Γ are integers;

(ii2) $v = u + m$;

(ii3) $v_2(\lambda_3 - \lambda_1), v_2(\lambda_4 - \lambda_1)$ and $v_2(\mu_{2h'-1} - \lambda)$ are the same for all $1 \leqslant h' \leqslant \frac{m-1}{2}$, say, β, and $v_2(\lambda_2 - \lambda_1), v_2(\mu_{2h'} - \lambda_1)$ are bigger than β for all $1 \leqslant h' \leqslant \frac{m-1}{2}$.

Furthermore, when the conditions hold, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where $M = \gcd(\lambda - \lambda_1 : \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$.

Proof. In this situation, we have a unitary matrix

$$P = (p_1, p_2, p_3, p_4, p_1^{(1)}, p_1^{(2)}, p_1^{(3)}, p_1^{(4)}, \cdots, p_{m-1}^{(1)}, p_{m-1}^{(2)}, p_{m-1}^{(3)}, p_{m-1}^{(4)}),$$

where $p_1, p_2, p_h^{(1)}, \cdots, p_h^{(4)}$ and $E_1, E_2, E_h^{(1)}, \cdots, E_h^{(4)}$, $1 \leqslant h \leqslant m-1$ are the same as in Subsection 4.1, and

$$p_3 = \frac{1}{\sqrt{2n}}(1,-1,1,-1,\cdots,1,-1)^t,\ p_4 = \frac{1}{\sqrt{2n}}(1,-1,\cdots,1,-1,-1,1,\cdots,-1,1)^t,$$
$$E_3 = \frac{1}{2n}((-1)^{u+v})_{2n\times 2n},\ E_4 = \frac{1}{2n}(e_4(u,v))_{2n\times 2n},$$

where

$$e_4(u,v) = \begin{cases} (-1)^{u+v} & 0 \leqslant u,v \leqslant n-1 \text{ or } n \leqslant u,v \leqslant 2n-1, \\ (-1)^{u+v+1} & \text{otherwise.} \end{cases}$$

As in the above section, for $1 \leqslant h \leqslant m-1$, if $\eta_h(S_2) = 0$, then the associated projective matrices are the same as in (13); if there is an h with $1 \leqslant h \leqslant m-1$ such that $\eta_h(S_2) \neq 0$, then by Lemma 6, the corresponding projective matrices are the same as in (14).

We only consider the case $0 \leqslant u,v \leqslant n-1$. The (u,v)-entry of the transfer matrix is

$$\begin{aligned}(H(t))_{u,v} \ &= \ \frac{1}{2n}(\exp(-\imath\lambda_1 t) + \exp(-\imath\lambda_2 t) + (-1)^{u+v}(\exp(-\imath\lambda_3 t) + \exp(-\imath\lambda_4 t))) \\ &+\frac{1}{n}\sum_{h=1,\eta_h(S_2)=0}^{m-1}(\omega^{(v-u)h}\exp(-\imath\mu_h t) + \omega^{(u-v)h}\exp(-\imath\mu_h t)) \\ &+\sum_{h=1,\eta_h(S_2)\neq 0}^{m-1}\frac{1}{\iota_h}(\ell_h\bar{\ell}_h\omega^{(v-u)h} + \omega^{(u-v)h})\exp(-\imath\mu_h^{(1)}t) \\ &+\sum_{h=1,\eta_h(S_2)\neq 0}^{m-1}\frac{1}{\iota_h}(\ell_h\bar{\ell}_h\omega^{(v-u)h} + \omega^{(u-v)h})\exp(-\imath\mu_h^{(2)}t). \end{aligned} \tag{20}$$

It is readily seen that if there is an h, $1 \leqslant h \leqslant m-1$, such that $\eta_h(S_2) \neq 0$, then there is no PST in Γ. Namely, Γ has PST only if $S_2 = \langle a\rangle$, which means that S is conjugation-closed. The rest of the theorem was proved in [13]. □

6 Concluding remarks

In this paper, we have shown that, if a Cayley graph $\Gamma = \mathrm{Cay}(D_n, S)$ over the dihedral group $D_n = \langle a, b | a^n = b^2 = 1, bab = a^{-1}\rangle$ has PST, then $S \cap b\langle a\rangle = b\langle a\rangle$. As a consequence, S is normal, namely, conjugation-closed. Since $S \cap b\langle a\rangle = b\langle a\rangle$, for $0 \leqslant i,j \leqslant n-1$, one has that $a^i(ba^j)^{-1} = a^{i-j}b = ba^{j-i} \in S$. It follows that the adjacency matrix of Γ has the form

$$A = \begin{pmatrix} C & J \\ J & C \end{pmatrix} \tag{21}$$

where J is the $n \times n$ all-one matrix and C is a circulant matrix. Note that some joined graphs have the adjacency matrix as in (21), see for example, [6, Theorem 1]. However, there are some Cayley graphs over dihedral groups having PST even when the conditions of [6, Theorem 1] fail, see [13]. We leave the following questions for further research:

Open Question 1: Determine whether there is a circulant graph $\mathrm{Cay}(\mathbb{Z}_{2n}, S)$ which is isomorphic to a Cayley graph over the dihedral group D_n having PST.

Open Question 2: Determine whether circulant join graphs [6] and products or covers of graphs [19] are isomorphic to some Cayley graphs.

Acknowlegemants

The authors would like to thank the referee for his/her very careful reading and many valuable comments that helped us improve this paper. Parts of this work were done when the first author X. Cao was visiting Nanyang Technological University where he would like to express his grateful thankfulness for the hospitality. X. Cao's work is supported by the National Natural Science Foundation of China (11771007, 61572027). The work of B. Chen is supported by the National Natural Science Foundation of China Grant Numbers 11871025 and 11971175, as well as by Science and Technology Program of Guangzhou Grant Number 201804010102. The research of S. Ling was partially supported by Nanyang Technological University Research Grant M4080456.

References

[1] O.L. Acevedo, and T. Gobron, Quantum walks on Cayley graphs, J. Phys. A: Math. Gen., 39 (2006), 585–599.

[2] D. Aharonov, A. Ambainis, J. Kempe and U. Vazirani, Quantum walks on graphs, ACM Press, P. 50–59, Dec. 2000.

[3] A. Ahmadi, R. Belk, C. Tamon, C. Wendler, On mixing in continuous-time quantum walks on some circulant graphs, Quantum Comput. Inf. 3 (2003), 611–618.

[4] O. Ahmadi, N. Alon, I. F. Blake and I. E. Shparlinski, Graphs with integral spectrum, Linear Algebra Appl., 430 (2009), 547–552.

[5] B. Ahmadi, M.M.S. Haghighi and A. Mokhtar, Perfect state transfer on the Johnson scheme, arXiv:1710.09096v1, 2017.

[6] R.J. Angeles-Canul, R. Norton, M. Opperman, C. Paribello, M. Russell, C. Tamon, Perfect state transfer, integral circulants and join of graphs, Quantum Comput. Inf. 10 (2010), 325–342.

[7] L. Babai, Spectra of Cayley graphs, Jour. Combin. Theory Ser. B, 27 (1979), 180–189.

[8] M. Bašić, Characterization of circulant graphs having perfect state transfer, Quantum Inf. Process, 12(2013), 345–364.

[9] M. Bašić, M.D. Petković, Some classes of integral circulant graphs either allowing or not allowing perfect state transfer, Appl. Math. Lett., 22(2009), 1609–1615.

[10] M. Bašić, M.D. Petković and D. Stevanovic, Perfect state transfer in integral circulant graphs, Appl. Math. Lett. 22(7)(2009), 1117–1121.

[11] A. Bernasconi, C. Godsil and S. Severini, Quantum networks on cubelike graphs, Phys. Rev. Lett. 91(20)(2003), 207901.

[12] W.G. Bridge and R. A. Mena, Rational G-matrices with rational eigenvalues, Jour. Combin. Theory Ser. A, 32 (1982), 264–280.

[13] X. Cao, K. Feng, Perfect state transfer on Cayley graphs over dihedral groups, to appear in the journal Linear and Multilinear Algebra.

[14] S. Bose, Quantum communication through a unmodulated spin chain. Phys. Rev. Lett., 91(20): 207901, 2003.

[15] A. Casaccino, S. Lloyd, S. Mancini, S. Severini, Quantum state transfer through a qubit network with energy shifts and fluctuations, Int. Jour. Quantum Inf., 7(2009), 1417–1427.

[16] W. Cheung and C. Godsil, Perfect state transfer in cubelike graphs, Linear Algebra Appl., 435(10)(2011), 2468–2474.

[17] M. Christandl, N. Datta, T. Dorlas, A. Ekert, A. Kay and A.J. Landahl, Perfect state transfer of arbitary state in quantum spin networks, Phys. Rev. A, 73(3)(2005), 032312.

[18] M. Christandl, N. Datta, A. Ekert and A.J. Landahl, Perfect state transfer in quantum spin networks, Phys. Rev. Lett. 92(18)(2004), 187902.

[19] G. Coutinho, Quantum State Transfer in Graphs. PhD dissertation. University of Waterloo, 2014.

[20] G. Coutinho, C. Godsil, Perfect state transfer in products and covers of graphs, Linear and Multilinear Algebra, 64(2)(2016), 235–246, https://doi.org/10.1080/03081087.2015.1033381.

[21] G. Coutinho, C. Godsil, Perfect state transfer is poly-time, arXiv:1606.02264v1 [math.CO], 2016, see also: G. Coutinho, C. Godsil, Perfect state transfer is poly-time, Quantum inf. Comput., 17(5 and 6), May 2017, 495–602.

[22] G. Coutinho, C. Godsil, K. Guo and F. Vanhova, Perfect state transfer on distance-regular graphs and association schemes, Linear Algebra Appl., 478(2015), 108–130.

[23] C. Godsil, Periodic graphs, Electron. J. Combin., 18(1)(2011), ♯23. arXiv:0806.2074.

[24] C. Godsil, State transfer on graphs, Disc. Math. 312(1)(2012) 129–147.

[25] C. Godsil, When can perfect state transfer occur? Electron. J. Linear Algebra, 23(2012), 877–890.

[26] F. Harary and A. J. Schwenk, Which graphs have integral spectra? Graphs and Combinatorics (R. Bari and F. Harary, eds.), Springer-Verlag, Berlin (1974), 45–51.

[27] N. Johnston, S. Kirkland, S. Plosker, R. Storey and X. Zhang, Perfect state transfer using Hadamard diagonalizable graphs, Linear Algebra Appl., 531(2017), 373–398.

[28] W. Klotz and T. Sander, Integral Cayley graphs over abelian groups, Electronic Jour. Combin. 17(1) (2010), R81.

[29] L. Lu, Q. Huang and X. Huang, Integral Cayley graphs over dihedral groups, Jour. Algebra. Comb., https://doi.org/10.1007/s10801-017-0787-x.

[30] P.J. McCarthy, Introduction to arithmetical functions, Universitext, Springer-Verlag, New York, 1986. https://doi.org/10.1007/978-1-4613-8620-9.

[31] M.R. Murty, Ramanujan graphs, J. Ramanujan Math. Sco. 18, No. 1(2003), 1-20.

[32] M.D. Petkovic and M. Bašić, Further results on the perfect state transfer in integral circulant graphs, Computers Math. Appl., 61(2)(2011), 300–312.

[33] J.P. Serre, Linear representations of finite groups, Springer, New York, Translated from the second French edition by L. Scott, Graduate Text in Mathematics, vol. 42, 1997.

[34] M. Štefaňák and S. Shoupý, Perfect state transfer by means of discrete-time quantum walk on complete bipartite graphs, Quantum Inf. Process, 16(3)(2017),72. https://doi.org/10.1007/s11128-017-1516-z.

[35] T.C. Silberstein, F. Scarabotti and F. Tolli, Representation Theory of the Symmetric Groups, Cambridge University Press, 2010, Cambridge studies in advanced mathematics, 121.

[36] B. Steinberg, Representation Theory of Finite Groups: An Introductory Approach. Universitext. https://doi.org/10.1007/978-1-4614-0776-8-7.

[37] Y. Tan, K. Feng, and X. Cao, Perfect state transfer on abelian Cayley graphs, Linear Algebra Appl. 563 (2019), 331–352, arXiv:1712.09260v1 [quant-ph] 26 Dec 2017.

[38] H. Zhan, An infinite family of circulant graphs with perfect state transfer in discrete quantum walks, arXiv:1707.06703v1.