

# On absolute points of correlations in $\text{PG}(2, q^n)$

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## Abstract

Let  $V$  be a  $(d+1)$ -dimensional vector space over a field  $\mathbb{F}$ . Sesquilinear forms over  $V$  have been largely studied when they are reflexive and hence give rise to a (possibly degenerate) polarity of the  $d$ -dimensional projective space  $\text{PG}(V)$ . Everything is known in this case for both degenerate and non-degenerate reflexive forms if  $\mathbb{F}$  is either  $\mathbb{R}$ ,  $\mathbb{C}$  or a finite field  $\mathbb{F}_q$ . In this paper we consider degenerate, non-reflexive sesquilinear forms of  $V = \mathbb{F}_{q^n}^3$ . We will see that these forms give rise to degenerate correlations of  $\text{PG}(2, q^n)$  whose set of absolute points are, besides cones, the (possibly degenerate)  $C_F^m$ -sets studied by Donati and Durante in 2014. In the final section we collect some results from the huge work of B.C. Kestenband regarding what is known for the set of the absolute points of correlations in  $\text{PG}(2, q^n)$  induced by a non-degenerate, non-reflexive sesquilinear form of  $V = \mathbb{F}_{q^n}^3$ .

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## 1 Introduction and definitions

Let  $V$  and  $W$  be two vector spaces over the same field  $\mathbb{F}$ . A map  $f : V \rightarrow W$  is called  $\sigma$ -semilinear if there exists an automorphism  $\sigma$  of  $\mathbb{F}$  such that

$$f(v + v') = f(v) + f(v') \quad \text{and} \quad f(av) = a^\sigma f(v)$$

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for all vectors  $v \in V$  and all scalars  $a \in \mathbb{F}$ .

If  $\sigma$  is the identity map, then  $f$  is a *linear map*.

Let  $V$  be an  $\mathbb{F}$ -vector space with dimension  $d + 1$  and let  $\sigma$  be an automorphism of  $\mathbb{F}$ . A map

$$\langle \cdot, \cdot \rangle : (v, v') \in V \times V \longrightarrow \langle v, v' \rangle \in \mathbb{F}$$

is a  $\sigma$ -*sesquilinear form* or a  $\sigma$ -*semibilinear form* on  $V$  if it is a linear map on the first argument and it is a  $\sigma$ -semilinear map on the second argument, that is:

$$\begin{aligned} \langle v + v', v'' \rangle &= \langle v, v'' \rangle + \langle v', v'' \rangle \\ \langle v, v' + v'' \rangle &= \langle v, v' \rangle + \langle v, v'' \rangle \\ \langle av, v' \rangle &= a \langle v, v' \rangle, \quad \langle v, av' \rangle = a^\sigma \langle v, v' \rangle \end{aligned}$$

for all  $v, v', v'' \in V$ ,  $a \in \mathbb{F}$ . If  $\sigma$  is the identity map, then  $\langle \cdot, \cdot \rangle$  is a *bilinear form*. If  $\mathcal{B} = (e_1, e_2, \dots, e_{d+1})$  is an ordered basis of  $V$ , then for  $x, y \in V$  we have  $\langle x, y \rangle = X_t A Y^\sigma$ . Here  $A = (a_{ij}) = (\langle e_i, e_j \rangle)$  is the *associated matrix* to the  $\sigma$ -sesquilinear form in the ordered basis  $\mathcal{B}$ , and  $X, Y$  are the columns of coordinates of  $x, y$  w.r.t.  $\mathcal{B}$ . In what follows we will denote by  $A_t$  the transpose of a matrix  $A$ . The term *sesqui* comes from the Latin and it means one and a half. For every subset  $S$  of  $V$  put

$$\begin{aligned} S^\perp &:= \{x \in V : \langle x, y \rangle = 0 \ \forall y \in S\} \\ S^\top &:= \{y \in V : \langle x, y \rangle = 0 \ \forall x \in S\} \end{aligned}$$

Both  $S^\perp$  and  $S^\top$  are subspaces of  $V$ . The subspaces  $V^\perp$  and  $V^\top$  are called the *left* and the *right radical* of  $\langle \cdot, \cdot \rangle$ , respectively.

**Proposition 1.** *The right and left radical of a sesquilinear form of a vector space  $V$  have the same dimension.*

Let  $d+1$  be the dimension of  $V$ . Then  $\dim V^\perp = d+1 - \text{rk}(A)$  and  $\dim V^\top = d+1 - \text{rk}(B)$  where  $B = (a_{ij}^{\sigma^{-1}})$ . The assertion follows from  $\text{rk}(B) = \text{rk}(A)$ .

A sesquilinear form  $\langle \cdot, \cdot \rangle$  is called *non-degenerate* if  $V^\perp = V^\top = \{\underline{0}\}$ . It is called *reflexive* if  $\langle x, y \rangle = 0$  implies  $\langle y, x \rangle = 0$  for all  $x, y \in V$ .

A *duality* or *correlation* of  $\text{PG}(d, \mathbb{F})$  is a bijective map preserving incidence between points and hyperplanes of  $\text{PG}(d, \mathbb{F})$ . It can be seen as a collineation of  $\text{PG}(d, \mathbb{F})$  into its dual space  $\text{PG}(d, \mathbb{F})^*$ . We will call *degenerate duality* or *degenerate correlation* of  $\text{PG}(d, \mathbb{F})$  a non-bijective map preserving incidence between points and hyperplanes of  $\text{PG}(d, \mathbb{F})$ . If  $V$  is equipped with a sesquilinear form  $\langle \cdot, \cdot \rangle$ , then there are two *induced* (possibly degenerate) dualities of  $\text{PG}(d, \mathbb{F})$ :

$$Y \mapsto X_t A Y^\sigma = 0 \quad \text{and} \quad Y \mapsto Y_t A X^\sigma = 0.$$

The following holds.

**Theorem 2.** *Any duality of  $\text{PG}(d, \mathbb{F})$ ,  $d \geq 2$ , is induced by a non-degenerate sesquilinear form on  $V = \mathbb{F}^{d+1}$ .*

A duality applied twice gives a collineation of  $\text{PG}(d, \mathbb{F})$ . If  $\langle , \rangle$  is given by  $\langle x, y \rangle = X_t A Y^\sigma$ , then the associated collineation is the following map:

$$f_{A_t^{-1} A^\sigma, \sigma^2} : X \mapsto A_t^{-1} A^\sigma X^{\sigma^2} \quad (\text{see [14]}).$$

The most important and studied types of dualities are those whose square is the identity, namely the *polarities*. Note that in this case  $A_t^{-1} A^\sigma = \rho I$  and  $\sigma^2 = 1$ .

**Proposition 3.** [1, 2] *A duality is a polarity if and only if the sesquilinear form defining it is reflexive.*

Non-degenerate, reflexive forms have been classified long ago (not only on finite fields but also on  $\mathbb{R}$  or  $\mathbb{C}$ ). A reflexive  $\sigma$ -sesquilinear form  $\langle , \rangle$  is:

- *$\sigma$ -Hermitian* if  $\langle y, x \rangle = \langle x, y \rangle^\sigma$ , for all  $x, y \in V$ . If  $\sigma = 1$  such a form is called *symmetric*.
- *Alternating* if  $\langle y, x \rangle = -\langle x, y \rangle$ .

**Theorem 4.** *A non-degenerate, reflexive  $\sigma$ -sesquilinear form is either alternating, or a scalar multiple of a  $\sigma$ -Hermitian form. In the latter case, if  $\sigma = 1$  (i.e. the form is symmetric), then the scalar can be taken to be 1.*

For a proof of the previous theorem we refer to [1] or [2]. Note that in case of reflexive  $\sigma$ -sesquilinear forms, left- and right-radicals coincide. The set  $\Gamma : X_t A X^\sigma = 0$  of absolute points of a polarity is one of the well known objects in the projective space  $\text{PG}(d, \mathbb{F})$ : subspaces, quadrics and Hermitian varieties.

We recall that the classification theorem for reflexive  $\sigma$ -sesquilinear forms also holds if the form is degenerate.

*Remark 5.* Let  $\Gamma$  be the set of absolute points of a degenerate polarity in the projective space  $\text{PG}(d, \mathbb{F})$ , then  $\Gamma$  is one of the following: subspaces, degenerate quadrics and degenerate Hermitian varieties. All of these sets are cones with vertex a subspace  $\mathcal{V}$  of dimension  $h$  (corresponding to  $V^\perp$ ) and base a non-degenerate object of the same type in a subspace of dimension  $d - 1 - h$  skew with  $\mathcal{V}$ .

So everything is known for reflexive sesquilinear forms.

Up to our knowledge (at least on finite fields) almost nothing is known on non-reflexive sesquilinear forms of  $V = \mathbb{F}_q^{d+1}$ . Given a sesquilinear form of  $V$  we may consider the induced correlation in  $\text{PG}(d, q^n)$  and the set  $\Gamma$  of its *absolute* points, that is the points  $X$  such that  $X \in X^\perp$  (or equivalently  $X \in X^\top$ ). If  $\langle x, y \rangle = X_t A Y^\sigma$ , then  $\Gamma : X_t A X^\sigma = 0$ . We will determine in the next sections the set  $\Gamma$  in  $\text{PG}(1, q^n)$  and in  $\text{PG}(2, q^n)$ .

Let  $V = \mathbb{F}_q^{d+1}$ ,  $d \in \{1, 2\}$  and let  $\Gamma : X_t A X^\sigma = 0$ ,  $\sigma \neq 1$ , be the set of absolute points of a (possibly degenerate) correlation of  $V$ . We start with the following definition.

**Definition.** Let  $\alpha$  be a  $\sigma$ -sesquilinear form of  $\mathbb{F}_q^{d+1}$ . A  *$\sigma$ -quadric* ( *$\sigma$ -conic* if  $d = 2$ ) of  $\text{PG}(d, q^n)$ ,  $d \in \{1, 2\}$  is the set of absolute points of the induced correlation of  $\alpha$  in  $\text{PG}(d, q^n)$ .

See [10] for the definition and the properties of  $\sigma$ -quadrics of  $\text{PG}(d, q^n)$  for every integer  $d \geq 1$ . It is an easy exercise (that we include in this paper) to determine the set  $\Gamma$  in  $\text{PG}(1, q^n)$ . For  $d = 2$  the set  $\Gamma$  in  $\text{PG}(2, q^n)$  has been studied by B.C. Kestenband in case  $\langle , \rangle$  is non-degenerate and the form is non-reflexive. The results are contained in 10 different papers from 2000 to 2014. We will summarize some of his results in the last section.

Contrary to the reflexive case, we will see that in general the knowledge of the set  $\Gamma$  of absolute points of a correlation induced by a non-degenerate, non-reflexive sesquilinear form will not help to determine the set  $\Gamma$  in the degenerate case.

## 2 Absolute points of correlations of $\text{PG}(1, q^n)$

In the sequel, we will determine the set of absolute points in  $\text{PG}(2, q^n)$  for a degenerate correlation induced by a degenerate  $\sigma$ -sesquilinear form with associated automorphism  $\sigma : x \mapsto x^{q^m}$ ,  $(m, n) = 1$ , of  $V = \mathbb{F}_{q^n}^3$ .

First assume that  $V = \mathbb{F}_{q^n}^2$  and consider the set of absolute points of a correlation induced by a  $\sigma$ -sesquilinear form, that is the set of points in  $\text{PG}(1, q^n)$  given by  $\Gamma : X_t A X^\sigma = 0$ , where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a  $2 \times 2$ -matrix over  $\mathbb{F}_{q^n}$ . Here again, let  $\sigma$  be the automorphism  $\sigma : x \mapsto x^{q^m}$ ,  $(m, n) = 1$ , of  $V = \mathbb{F}_{q^n}^2$ .

The points of  $\Gamma$  satisfy

$$(ax_1 + cx_2)x_1^\sigma + (bx_1 + dx_2)x_2^\sigma = 0.$$

- If  $a = 1, b = c = 0, d \neq 0$ , then  $\Gamma : x_1^{\sigma+1} + dx_2^{\sigma+1} = 0$ . If  $q$  is odd and  $-d$  is a non-square of  $\mathbb{F}_{q^n}$ , then  $\Gamma = \emptyset$ . If  $q$  is even,  $r = (q^n - 1, q^m + 1)$  and  $d$  is an element of  $\mathbb{F}_{q^n}$  such that  $d^{\frac{q^n-1}{r}} \neq 1$ , again  $\Gamma = \emptyset$ .
- If  $a = 1, b = c = d = 0$ , then  $\Gamma : x_1^{\sigma+1} = 0$  is the point  $(0, 1)$ .
- If  $b = 1, a = c = d = 0$ , then  $\Gamma : x_1 x_2^\sigma = 0$  is the union of the points  $(1, 0), (0, 1)$ .
- From the examples above, we see that  $\Gamma$  can be the empty set,  $\Gamma$  can be a point or the union of two points. Hence we may assume that  $|\Gamma| > 2$  and so we may suppose that  $(1, 0), (0, 1), (1, 1) \in \Gamma$ , therefore  $a = d = 0, b = -c$ . So

$$\Gamma : x_1 x_2 (x_1^{\sigma-1} - x_2^{\sigma-1}) = 0.$$

Hence  $\Gamma$  is given by the union of  $\{(1, 0), (0, 1)\}$  and the set of points  $(x_1, x_2)$  in  $\text{PG}(1, q^n)$  such that  $(\frac{x_2}{x_1})^{\sigma-1} = 1$ . Hence,  $\Gamma$  is the set of points of an  $\mathbb{F}_q$ -subline  $\text{PG}(1, q)$  of  $\text{PG}(1, q^n)$ .

**Lemma 6.** *The set of the absolute points in  $\text{PG}(1, q^n)$  of a (possibly degenerate) correlation induced by a  $\sigma$ -sesquilinear form of  $\mathbb{F}_{q^n}^2$  is one of the following:*

- *the empty set, a point, two points;*
- *an  $\mathbb{F}_q$ -subline  $\text{PG}(1, q)$  of  $\text{PG}(1, q^n)$ .*

**Definition 7.** A  $\sigma$ -quadric  $\Gamma$  of  $\text{PG}(1, q^n)$  is *non-degenerate* if  $|\Gamma| \in \{0, 2, q+1\}$ , otherwise it is *degenerate*. A  $\sigma$ -conic  $\Gamma$  of  $\text{PG}(2, q^n)$  is *non-degenerate* if  $V^\perp \cap V^\top = \{\underline{0}\}$  and  $\Gamma$  does not contains lines. Otherwise, it is *degenerate*.

Note that if  $\Gamma$  is a degenerate  $\sigma$ -quadric of  $\text{PG}(1, q^n)$ , then there exists a degenerate sesquilinear form such that  $V^\perp = V^\top$  with  $\dim(V^\perp) = 1$ .

Next we determine the possible intersection configurations of a line and a  $\sigma$ -conic of  $\text{PG}(2, q^n)$ . So let  $\Gamma$  be the set of absolute points in  $\text{PG}(2, q^n)$  of a (possibly degenerate) correlation induced by a  $\sigma$ -sesquilinear form of  $\mathbb{F}_{q^n}^3$ ,  $\sigma : x \mapsto x^{q^m}$ ,  $(m, n) = 1$ . Hence  $\Gamma : X_t A X^\sigma = 0$ .

**Proposition 8.** *Every line intersects a  $\sigma$ -conic  $\Gamma$  of  $\text{PG}(2, q^n)$  in either 0 or 1 or 2 or  $q+1$  points (an  $\mathbb{F}_q$ -subline) or it is contained in  $\Gamma$ .*

*Proof.* Let  $Y, Z$  be two distinct points of  $\text{PG}(2, q^n)$  and let  $\ell : X = \lambda Y + \mu Z$ ,  $(\lambda, \mu) \in \mathbb{F}_{q^n}^2 \setminus \{(0, 0)\}$  be the line containing them. The number of points of  $\Gamma \cap \ell$  is given by the number of points of a  $\sigma$ -quadric on the line  $\ell$ . Indeed a point  $X = \lambda Y + \mu Z \in \ell \cap \Gamma$  if and only if  $a\lambda^{\sigma+1} + b\lambda\mu^\sigma + c\lambda^\sigma\mu + d\mu^{\sigma+1} = 0$ , where  $a = Y_t A Y^\sigma, b = Y_t A Z^\sigma, c = Z_t A Y^\sigma, d = Z_t A Z^\sigma$ . The assertion follows.  $\square$

### 3 Steiner's projective generation of conics and $C_F^m$ -sets

Before determining the sets of absolute points of correlations of  $\text{PG}(2, q^n)$  we will recall some subsets of  $\text{PG}(2, q^n)$ , called (possibly degenerate)  $C_F^m$ -sets, that have been introduced and studied in [6, 7, 8] generalizing the constructions of conics due to J. Steiner. We will see that  $\sigma$ -conics of  $\text{PG}(2, q^n)$  with  $|A| = 0$ , different from a cone with vertex a point and base a  $\sigma$ -quadric of  $\text{PG}(1, q^n)$  will be a (possibly degenerate)  $C_F^m$ -set. First we recall J. Steiner's construction of conics. Let  $\mathbb{F}$  be a field and let  $\text{PG}(2, \mathbb{F})$  be a Desarguesian projective plane. Let  $R$  and  $L$  be two distinct points of  $\text{PG}(2, \mathbb{F})$  and let  $\mathcal{P}_R$  and  $\mathcal{P}_L$  be the pencils of lines with centers  $R$  and  $L$ , respectively. In 1832 in [29] J. Steiner proves the following:

**Theorem 9.** *The set of  $\Gamma$  of points of intersection of corresponding lines under a projectivity  $\Phi : \mathcal{P}_R \rightarrow \mathcal{P}_L$  is one the following:*

- *if  $\Phi(RL) \neq RL$ , then  $\Gamma$  is a non-degenerate conic;*
- *if  $\Phi(RL) = RL$ , then  $\Gamma$  is a degenerate conic.*

Inspired by Steiner's projective generation of conics, in [6, 7, 8] the following definitions are given.

**Definition 10.** The set  $\Gamma \subset \text{PG}(2, q^n)$  of points of intersection of corresponding lines under a collineation  $\Phi : \mathcal{P}_R \rightarrow \mathcal{P}_L$  with accompanying automorphism  $x \mapsto x^{q^m}$  is one the following:

- if  $\Phi(RL) \neq RL$ , then  $\Gamma$  is called a  $C_F^m$ -set with *vertices*  $R$  and  $L$ ;
- if  $\Phi(RL) = RL$ , then  $\Gamma$  is called a *degenerate*  $C_F^m$ -set with *vertices*  $R$  and  $L$ .

Moreover in [8] the following is proved:

**Theorem 11.** A  $C_F$ -set  $\Gamma$  of  $\text{PG}(2, q^n)$  with vertices  $R = (1, 0, 0)$  and  $L = (0, 0, 1)$  has canonical equation  $x_1x_3^{q^m} - x_2^{q^m+1} = 0$ . Hence  $|\Gamma| = q^n + 1$  and it is a set of type  $(0, 1, 2, q + 1)$  w.r.t. lines. It is the union of  $\{R, L\}$  with  $q - 1$  pairwise disjoint scattered  $\mathbb{F}_q$ -linear sets of pseudoregulus type, each of which is isomorphic to the set of points and lines of a  $\text{PG}(n - 1, q)$ , that is called a component of  $\Gamma$ .

**Theorem 12.** A degenerate  $C_F^m$ -set of  $\text{PG}(2, q^n)$  with vertices  $R = (1, 0, 0)$  and  $L = (0, 1, 0)$  has canonical equation  $x_3(x_1x_3^{q^m-1} - x_2^{q^m}) = 0$ . Here  $|\Gamma| = 2q^n + 1$  and it is a set of type  $(1, 2, q + 1, q^n + 1)$  w.r.t. lines. It is the union of the line  $RL$  with a set  $\mathcal{A}$  of  $q^n$  affine points isomorphic to  $\text{AG}(n, q)$  whose directions on the line  $RL$  form a maximum scattered  $\mathbb{F}_q$ -linear set  $\mathcal{S}$  of pseudoregulus type with transversal points  $R$  and  $L$ . Moreover  $\mathcal{A} \cup \mathcal{S}$  is a maximum scattered  $\mathbb{F}_q$ -linear set of rank  $n + 1$ , that is a blocking set of Rédei type and vice versa.

We refer to [8] and to [28] for the definition and properties of the relevant objects in the previous two theorems such as maximum scattered  $\mathbb{F}_q$ -linear set and blocking set of Rédei type.

In the next section we will see that the set of absolute points of a degenerate correlation of  $\text{PG}(2, q^n)$  induced by a degenerate, non-reflexive  $\sigma$ -sesquilinear form of  $\mathbb{F}_{q^n}^3$  is either a  $C_F^m$ -set or a degenerate  $C_F^m$ -set or a cone with vertex a point  $R$  and base a  $\sigma$ -quadric of a line  $\ell$  not through  $R$ .

## 4 Absolute points of degenerate correlations of $\text{PG}(2, q^n)$

In this section we determine the possible structure for the set of absolute points in  $\text{PG}(2, q^n)$  of a degenerate correlation induced by a  $\sigma$ -sesquilinear form in  $\mathbb{F}_{q^n}^3$ . Let  $\Gamma : X_tAX^\sigma = 0$  be such a set of points. For the sequel we can assume  $\sigma \neq 1$ . Indeed if  $\sigma = 1$ , then  $\Gamma$  is always a (possibly degenerate) conic or the full pointset (a degenerate symplectic geometry). In what follows,  $\sigma$ , if not differently specified, will denote always the map  $x \mapsto x^{q^m}$ ,  $(m, n) = 1$ .

First assume  $\text{rank}(A) = 2$ , then  $V^\perp$  and  $V^\top$  are one-dimensional vector spaces of  $V$ , so points of  $\text{PG}(2, q^n)$ . If  $V^\perp \neq V^\top$ , then we may assume that the point  $R = (1, 0, 0)$  is the right-radical and the point  $L = (0, 0, 1)$  is the left-radical. It follows that

$$A = \begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\Gamma : X_t A X^\sigma = (ax_1 + cx_2)x_2^\sigma + (bx_1 + dx_2)x_3^\sigma = 0.$$

Consider the pencil of lines through the point  $R$ , that is

$$\mathcal{P}_R = \{\ell_{\alpha,\beta} : (\alpha, \beta) \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \setminus \{(0, 0)\}\}, \text{ where}$$

$$\ell_{\alpha,\beta} : \begin{cases} x_1 &= \lambda \\ x_2 &= \mu\alpha, (\lambda, \mu) \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \setminus \{(0, 0)\} \\ x_3 &= \mu\beta \end{cases}$$

and the pencil of lines through  $L$ , that is

$$\mathcal{P}_L = \{\ell'_{\alpha',\beta'} : (\alpha', \beta') \in \mathbb{F}_{q^n} \times \mathbb{F}_{q^n} \setminus \{(0, 0)\}\}, \text{ where } \ell'_{\alpha',\beta'} : \alpha'x_1 + \beta'x_2 = 0.$$

Let

$$\Phi : \mathcal{P}_R \longrightarrow \mathcal{P}_L$$

be the collineation between  $\mathcal{P}_R$  and  $\mathcal{P}_L$  given by  $\Phi(\ell_{\alpha,\beta}) = \ell'_{\alpha',\beta'}$ , where

$$(\alpha', \beta')_t = A'(\alpha, \beta)_t^\sigma.$$

where  $A'$  is the matrix

$$A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Note that  $|A'| \neq 0$  since  $\text{rank}(A) = 2$ . It is easy to see that  $\Gamma$  is the set of points of intersection of corresponding lines under the collineation  $\Phi$  and hence it is a (possibly degenerate)  $C_F^m$ -set (see [6, 7, 8]).

Let  $Y = (y_1, y_2, y_3)$  be a point of  $\Gamma$ , then the tangent line  $t_Y$  to  $\Gamma$  at the point  $Y$  is the line

$$t_Y : (ay_2^\sigma + by_3^\sigma)x_1 + (cy_2^\sigma + dy_3^\sigma)x_2 = 0.$$

So for every point  $Y$  of  $\Gamma$  the line  $t_Y$  contains the point  $L = (0, 0, 1)$ . The tangent line  $t_L$  to  $\Gamma$  at the point  $L$  is the line

$$t_L : bx_1 + dx_2 = 0.$$

First assume that  $t_L = RL : x_2 = 0$ , then  $b = 0$  and we can put  $d = 1$  obtaining

$$\Gamma : (ax_1 + cx_2)x_2^\sigma + x_2x_3^\sigma = 0,$$

with  $a \neq 0$ , so

$$\Gamma : x_2(ax_1x_2^{\sigma-1} + cx_2^\sigma + x_3^\sigma) = 0,$$

that is a degenerate  $C_F^m$ -set with vertices  $R(1, 0, 0)$  and  $L(0, 0, 1)$  since the collineation  $\Phi$  maps the line  $RL$  into itself (see [8]).

Next assume  $t_L$  is not the line  $RL$ , so we may suppose, w.l.o.g., that  $t_L : x_1 = 0$ , that is  $d = 0$  and  $b = 1$ . In this case

$$\Gamma : (ax_1 + cx_2)x_2^\sigma + x_1x_3^\sigma = 0,$$

with  $c \neq 0$ , is a non-degenerate  $C_F^m$ -set with vertices  $R(1, 0, 0)$  and  $L(0, 0, 1)$  since the collineation  $\Phi$  does not map the line  $RL$  into itself (see [8]).

Next assume that  $V^\perp = V^\top$  so they coincide as projective points of  $\text{PG}(2, q^n)$ . We may assume that  $R = L = (1, 0, 0)$  is both the left and right radical of  $\langle , \rangle$ . In this case the set of absolute points is the set  $\Gamma$  of points of  $\text{PG}(2, q^n)$  such that  $X_tAX^\sigma = 0$  with

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

so

$$\Gamma : (ax_2 + bx_3)x_2^\sigma + (cx_2 + dx_3)x_3^\sigma = 0.$$

It follows that  $\Gamma$  is the set of points of a cone with vertex the point  $R$ , the base of this cone is either the empty set, a point, two points or  $q + 1$  points of an  $\mathbb{F}_q$ -subline of a line not through the point  $R$ .

Note that  $\Gamma$  can be seen as the set of points of intersection of corresponding lines under the same collineation  $\Phi$  from  $\mathcal{P}_R$  to  $\mathcal{P}_L$ , with  $R = L$ , similar to the case  $R \neq L$ .

Next assume  $\text{rank}(A) = 1$ . In this case  $\dim V^\perp = \dim V^\top = 2$ , so in  $\text{PG}(2, q^n)$  the right- and left-radical are given by two lines  $r$  and  $\ell$ . First assume  $r \neq \ell$ , so we may put  $r : x_3 = 0$  and  $\ell : x_1 = 0$ , then  $\Gamma : x_1x_3^\sigma = 0$ , that is the union of the two lines  $r$  and  $\ell$ .

Finally assume that  $r = \ell$ , e.g.  $r = \ell : x_3 = 0$ , then  $\Gamma : (cx_1 + dx_2 + ex_3)x_3^\sigma = 0$  is again the union of two (possibly coincident) lines.

In the next proposition we summarize what has been proved with the previous arguments.

**Proposition 13.** *The set of absolute points in  $\text{PG}(2, q^n)$  of a degenerate correlation induced by a degenerate  $\sigma$ -sesquilinear form of  $\mathbb{F}_q^3$  is one of the following:*

- a cone with vertex a point  $R$  and base a  $\sigma$ -quadric of a line  $\ell$  not through  $R$  (i.e. just  $R$ , a line, two lines or an  $\mathbb{F}_q$ -subpencil of  $\mathcal{P}_R$ );
- a degenerate  $C_F^m$ -set (see [8] );
- a  $C_F^m$ -set (see [8]).

#### 4.1 Some applications for $C_F^m$ -sets of $\text{PG}(2, q^n)$

In recent years, (degenerate or not)  $C_F^m$ -sets of  $\text{PG}(2, q^n)$  have been used for several applications. We recall some of them here.

Let  $M_{m,n}(\mathbb{F}_q)$ , with  $m \leq n$ , be the vector space of all the  $m \times n$  matrices with entries in  $\mathbb{F}_q$ . The *distance* between two matrices is the rank of their difference. An  $(m \times n, q, s)$ -rank distance code is a subset  $\mathcal{X}$  of  $M_{m,n}(\mathbb{F}_q)$  such that the minimum distance between any two of its distinct elements is  $s$ . An  $\mathbb{F}_q$ -linear  $(m \times n, q, s)$ -rank distance code is a subspace of  $M_{m,n}(\mathbb{F}_q)$ .

It is known (see e.g. [4]) that the size of an  $(m \times n, q, s)$ -rank distance code  $\mathcal{X}$  satisfies the *Singleton-like bound*:

$$|\mathcal{X}| \leq q^{n(m-s+1)}.$$

When this bound is achieved,  $\mathcal{X}$  is called an  $(m \times n, q, s)$ -maximum rank distance code, or  $(m \times n, q, s)$ -MRD code, for short.

In finite geometry  $(m \times m, q, m)$ -MRD codes are known as *spread sets* (see e.g. [5]) and there are examples for both  $\mathbb{F}_q$ -linear and non-linear codes.

The first class of non-linear MRD codes, different from spread sets, are the non-linear  $(3 \times 3, q, 2)$ -MRD codes constructed in [3] by using  $C_F^1$ -sets of  $\text{PG}(2, q^3)$ .

Indeed starting with a  $C_F^1$ -set  $\mathcal{C}$  of  $\text{PG}(2, q^3)$  in [3], they construct a set  $\mathcal{E}$  of  $q^3 + 1$  points that is an *exterior* set to the component  $\mathcal{C}_1$  of  $\mathcal{C}$  (i.e. the lines joining any two distinct points of  $\mathcal{E}$  is external to  $\mathcal{C}_1$ ).

By using field reduction from  $\text{PG}(2, q^3)$  to  $\text{PG}(8, q) = \text{PG}(M_{3,3}(q))$ , the component  $\mathcal{C}_1 \cong \text{PG}(2, q)$  corresponds to the Segre variety  $\mathcal{S}_{2,2}$ . This is the set of matrices with rank 1. The set  $\mathcal{E}$  corresponds to an exterior set to the Segre variety  $\mathcal{S}_{2,2}$  of  $\text{PG}(8, q)$  and hence a set of  $q^6$  matrices such that the difference between any two has rank at least two, i.e. a  $(3 \times 3, q, 2)$ -MRD code.

In this section, it is shown that, starting from a  $C_F^m$ -set of  $\text{PG}(2, q^n)$ , infinite families of non-linear  $(3 \times n, q, 2)$ -MRD codes can be constructed. (See also [10]).

Let  $R = (1, 0, 0)$  and  $L = (0, 0, 1)$  be two points of  $\text{PG}(2, q^n)$  with  $n \geq 3$  and let  $\mathcal{C}$  be the  $C_F^m$ -set with vertices  $R$  and  $L$  given by

$$\mathcal{C} = \{P_t = (t^{q^m+1}, t, 1) : t \in \mathbb{F}_{q^n}\} \cup \{R\}.$$

It follows that

$$\mathcal{C} = \bigcup_{a \in \mathbb{F}_q^*} \mathcal{C}_a \cup \{R, L\},$$

with  $\mathcal{C}_a = \{P_t : t \in N_a\}$  and  $N_a = \{x \in \mathbb{F}_{q^n} : N(x) = a\}$ , where  $N(x) = x^{\frac{q^n-1}{q-1}}$  denotes the *norm* of an element  $x \in \mathbb{F}_{q^n}$  w.r.t.  $\mathbb{F}_q$ . For every  $a \in \mathbb{F}_q^*$ , consider the partition of the points of the line  $RL$ , different from  $R$  and  $L$ , into subsets  $J_a = \{(-t, 0, 1) : t \in N_a\}$  and let  $\pi'_1 \cong \text{PG}(2, q)$  be a subgeometry contained in  $\mathcal{C}_1$ .

**Theorem 14.** *For every subset  $T$  of  $\mathbb{F}_q^*$  containing 1, the set*

$$\mathcal{X} = (\mathcal{C} \setminus \bigcup_{a \in T} \mathcal{C}_a) \cup \bigcup_{a \in T} J_a$$

is an exterior set with respect to  $\pi'_1$ .

*Proof.* The lines  $RP$  and  $LP$ , with  $P \in \mathcal{C} \setminus (\bigcup_{a \in T} \mathcal{C}_a \cup \{R, L\})$ , meeting  $\mathcal{C}$  exactly in two points, are external lines w.r.t.  $\pi'_1$ .

Similarly, for every  $P \in \mathcal{C}_b$  and  $P' \in \mathcal{C}_{b'}$  with  $b, b' \in \mathbb{F}_q^* \setminus T$  the line  $PP'$  is external to  $\pi'_1$ .

Finally, for every point  $P \in \mathcal{C}_b$  with  $b \in \mathbb{F}_q^* \setminus T$  and  $P' \in J_a$  with  $a \in T$ , the line  $PP'$  is external to  $\pi'_1$ .

Indeed suppose, by way of contradiction, that the line  $PP'$  meets  $\pi'_1$  in a point  $S$  with coordinates  $(x^{q^{2m}}, x^{q^m}, x)$ . Let  $P = (\alpha^{q^m+1}, \alpha, 1)$  with  $N(\alpha) = b$  and let  $P' = (-t, 0, 1)$  with  $N(t) = a$ . By calculating the determinant of the matrix  $M$  whose rows are the coordinates of the points  $S, P, P'$ , we have that  $|M| = -tM_1 + \alpha M_1^{q^m}$ , where  $M_1$  is the cofactor of the element  $m_{3,1}$  of  $M$ . Since  $S, P$  are distinct points,  $M_1 \neq 0$ , hence  $|M| = 0$  if and only if  $t = \alpha M_1^{q^m-1}$ , that is a contradiction since  $N(\alpha) = b$  while  $N(t) = a$  with  $a \neq b$ .  $\square$

From the previous theorem we have the following result.

**Corollary 15.** *For all  $n \geq 3, q > 2$ , the vectors  $\rho v \in M_{3,n}(q)$ ,  $\rho \in \mathbb{F}_q^*$ , whose corresponding points are in  $\mathcal{X}$ , plus the zero vector, give a non-linear  $(3 \times n, q, 2)$ -MRD code.*

*Remark 16.* These codes have been generalized first, by using bilinear forms and the cyclic representation of  $\text{PG}(n-1, q^n)$  by N. Durante and A. Siciliano in [11], to non-linear  $(n \times n, q, n-1)$ -MRD codes. Later G. Donati and N. Durante generalized these codes in [9] by using  $\sigma$ -normal rational curves, which is a generalization of normal rational curves in  $\text{PG}(d, q^n)$ , to non-linear  $((d+1) \times n, q, d)$ -MRD codes, where  $d \leq n-1$ .

Other applications of (degenerate or not)  $C_F^m$ -sets of  $\text{PG}(2, q^n)$  are:

- *Conics of  $\text{PG}(2, q^n)$ ,  $q$  even.*

Indeed the affine part, plus the vertices, of a degenerate  $C_F^1$ -set of  $\text{PG}(2, q^n)$ ,  $q = 2$  form a regular hyperoval.

- *Translation hyperovals of  $\text{PG}(2, q^n)$ ,  $q$  is even.*

Indeed the affine part, plus the vertices of a  $C_F^m$ -set of  $\text{PG}(2, q^n)$ ,  $q = 2, m > 1$ , form a translation hyperoval.

- *Semifield flocks of a quadratic cone of  $\text{PG}(3, q^n)$ ,  $q$  odd (and hence also translation ovoids of the parabolic quadric  $Q(4, q^n)$ ,  $q$  odd).*

Indeed, every semifield flock  $\mathcal{F}$  of a quadratic cone of  $\text{PG}(3, q^n)$ ,  $q$  odd is associated with a scattered  $\mathbb{F}_q$ -linear set  $\mathcal{I}(\mathcal{F})$  of internal points w.r.t. a non-degenerate conic  $\mathcal{C}$  of a plane  $\text{PG}(2, q^n)$  of  $\text{PG}(3, q^n)$ . The known examples of semifield flocks of a quadratic cone of  $\text{PG}(3, q^n)$  are the linear flock, the Kantor-Knuth, the Cohen-Ganley ( $q = 3$ ) and the sporadic semifield flock ( $q = 3, n = 5$ ). These semifield flocks give as set  $\mathcal{I}(\mathcal{F})$  a point, a scattered  $\mathbb{F}_q$ -linear set of pseudoregulus type on a secant line to  $\mathcal{C}$ , a component of a  $C_F^1$  set of  $\text{PG}(2, 3^n)$  and a component of a  $C_F^2$ -set of  $\text{PG}(2, 3^5)$ , respectively.

See [10] for more details on these applications.

## 5 Non-degenerate $\sigma$ -sesquilinear forms in $\mathbb{F}_{q^n}^3$

In this section we will determine the possible structure for the set of absolute points in  $\text{PG}(2, q^n)$  of a correlation induced by a non-degenerate  $\sigma$ -sesquilinear form of  $\mathbb{F}_{q^n}^3$ . So consider the set of points  $\Gamma : X_t A X^\sigma = 0$ . These sets of points have been studied in several papers by B. Kestenband. For this reason the set of points of this section have been called *Kestenband  $\sigma$ -conics* in [10]. We will try to summarize what is known up to now. First recall some easy properties for the set  $\Gamma$ . For the sequel we can assume  $\sigma \neq 1$ , since if  $\sigma = 1$ , then  $\Gamma$  is always a (possibly degenerate) conic.

**Proposition 17.** *Every line of  $\text{PG}(2, q^n)$  meets  $\Gamma : X_t A X^\sigma = 0$  in either 0, 1, 2 or  $q + 1$  points of an  $\mathbb{F}_q$ -subline.*

*Proof.* From Proposition 8 we only have to show that  $\Gamma$  does not contain lines. Suppose, by way of contradiction, that  $\Gamma$  contains a line, say  $\ell$ , then we can assume w.l.o.g. that  $\ell : x_3 = 0$ . Then the matrix  $A$  assumes the following shape:

$$A = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & d & e \end{pmatrix}$$

hence  $\text{rank}(A) \leq 2$ , a contradiction. □

In the next two theorems let  $\sigma : x \mapsto x^{q^m}$  with  $(m, 2n + 1) = 1$ .

**Theorem 18.** *Let  $\Gamma : X_t A X^\sigma = 0$  be the set of absolute points of a correlation of  $\text{PG}(2, q^{2n+1})$ ,  $n > 0$ ,  $q$  odd. Then we find that  $|\Gamma| \in \{q^{2n+1} + \epsilon q^{n+1} + 1 \mid \epsilon \in \{0, 1, -1\}\}$  and the set  $\Gamma$  depends on the number of points of  $\Gamma$  and points outside  $\Gamma$  fixed by the induced collineation  $f_{A_t^{-1} A^\sigma, \sigma^2}$ . The following holds:*

- *If  $f_{A_t^{-1} A^\sigma, \sigma^2}$  fixes no points outside  $\Gamma$ , then  $|\Gamma| = q^{2n+1} + 1$ . Moreover  $f_{A_t^{-1} A^\sigma, \sigma^2}$  fixes one point of  $\Gamma$  and  $\Gamma$  is projectively equivalent to:*

$$x_3 x_1^\sigma + x_2^{\sigma+1} + (x_1 + \epsilon x_2) x_3^\sigma = 0,$$

where  $e^{q^{2n}} + e^{q^{2n-1}} + \dots + e^q + e \neq 0$ .

- *If  $f_{A_t^{-1} A^\sigma, \sigma^2}$  fixes more than one point outside  $\Gamma$ , then  $|\Gamma| = q^{2n+1} + 1$ . Moreover,  $\Gamma$  is projectively equivalent to:*

$$x_1^{\sigma+1} + x_2^{\sigma+1} + x_3^{\sigma+1} = 0,$$

*the collineation  $f_{A_t^{-1} A^\sigma, \sigma^2}$  fixes  $\text{PG}(2, q)$  pointwise and hence it fixes the  $q + 1$  points of  $\Gamma$  on the subconic  $x_1^2 + x_2^2 + x_3^2 = 0$  in  $\text{PG}(2, q)$  and  $q^2$  points of  $\text{PG}(2, q) \setminus \Gamma$ .*

- If  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes one point outside  $\Gamma$ , then  $\Gamma$  is projectively equivalent to:

$$ax_1^{\sigma+1} + (bx_1 + cx_2)x_2^\sigma + dx_3^{\sigma+1} = 0,$$

with  $a, b, c, d \neq 0$ . Moreover  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes either 0, 1, 2 or  $q + 1$  points of  $\Gamma$ . In particular

- If  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes either 0, 2 or  $q + 1$  points of  $\Gamma$ , then  $\epsilon = 0$ . In the last case the  $q + 1$  points of  $\Gamma$  fixed by  $f_{A_t^{-1}A^\sigma, \sigma^2}$  are collinear.
- If  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes 1 point of  $\Gamma$ , then  $\epsilon \in \{-1, 1\}$ .

**Theorem 19.** Let  $\Gamma : X_tAX^\sigma = 0$  be the set of absolute points of a correlation of  $\text{PG}(2, q^{2n+1})$ ,  $n > 0$ ,  $q$  even. Then we find that  $|\Gamma| \in \{q^{2n+1} + \epsilon q^{n+1} + 1 | \epsilon \in \{0, 1, -1\}\}$  and the set  $\Gamma$  depends on the number of points of  $\Gamma$  and points outside  $\Gamma$  fixed by the induced collineation  $f_{A_t^{-1}A^\sigma, \sigma^2}$ . The following holds:

- If  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes no points outside  $\Gamma$ , then  $\epsilon \in \{-1, +1\}$ . Moreover  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes one point of  $\Gamma$  and  $\Gamma$  is projectively equivalent to:

$$x_3x_1^\sigma + x_2^{\sigma+1} + (x_1 + ex_2)x_3^\sigma = 0,$$

where  $e^{q^{2n}} + e^{q^{2n-1}} + \dots + e^q + e \neq 0$ .

- If  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes at least one point outside  $\Gamma$ , then  $\epsilon = 0$ . Moreover,  $\Gamma$  is projectively equivalent to:

$$x_1^{\sigma+1} + \rho x_1x_2^\sigma + x_2^{\sigma+1} + x_3^{\sigma+1} = 0,$$

for some  $\rho \in \mathbb{F}_{q^{2n+1}}$ . The trinomial  $x^{q^m+1} + \rho x + 1$  can have one, two,  $q + 1$  or no zeros.

1. If it has one or  $q + 1$  zeros, then  $\rho = 0$  and the collineation  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes  $\text{PG}(2, q)$  pointwise and hence it fixes the  $q + 1$  points of  $\Gamma$  on the subline  $x_1 + x_2 + x_3 = 0$  in  $\text{PG}(2, q)$  and  $q^2$  points of  $\text{AG}(2, q) = \text{PG}(2, q) \setminus \Gamma$ .
2. If it has two zeros, then the collineation  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes 2 points of  $\Gamma$ .
3. If it has no zeros, then the collineation  $f_{A_t^{-1}A^\sigma, \sigma^2}$  fixes no point of  $\Gamma$ .

In the next theorems let  $\sigma : x \mapsto x^{q^m}$  with  $(m, 2n) = 1$ .

**Theorem 20.** Let  $\Gamma : X_tAX^\sigma = 0$ , with  $A$  a diagonal matrix, be the set of absolute points of a correlation of  $\text{PG}(2, q^{2n})$ ,  $n > 0$ . Then

$$|\Gamma| \in \{q^{2n} + 1 + (-q)^{n+1}(q - 1), q^{2n} + 1 + (-q)^n(q - 1), q^{2n} + 1 - 2(-q)^n\}.$$

Moreover,

- if  $|\Gamma| = q^{2n} + 1 + (-q)^{n+1}(q - 1)$ , then  $\Gamma$  is projectively equivalent to

$$x_1^{\sigma+1} + x_2^{\sigma+1} + x_3^{\sigma+1} = 0;$$

- if  $|\Gamma| = q^{2n} + 1 + (-q)^n(q - 1)$ , then  $\Gamma$  is projectively equivalent to

$$x_1^{\sigma+1} + x_2^{\sigma+1} + ax_3^{\sigma+1} = 0,$$

with  $a \notin \{x^{q+1} : x \in \mathbb{F}_{q^{2n}}\}$ ;

- if  $|\Gamma| = q^{2n} + 1 - 2(-q)^n$ , then  $\Gamma$  is projectively equivalent to

$$ax_1^{\sigma+1} + bx_2^{\sigma+1} + x_3^{\sigma+1} = 0,$$

with  $a, b, a/b \notin \{x^{q+1} : x \in \mathbb{F}_{q^{2n}}\}$ .

**Theorem 21.** Let  $\Gamma : X_tAX^\sigma = 0$ , with  $A$  a non-diagonal matrix, be the set of absolute points of a correlation of  $\text{PG}(2, q^{2n})$ ,  $n > 0, q$  even. Then  $|\Gamma| \in \{q^{2n} - (-q)^{n+1} + 1, q^{2n} - (-q)^n + 1, q^{2n} + 1\}$ .

**Theorem 22.** Let  $\Gamma : X_tAX^\sigma = 0$ , with  $A$  a non-diagonal matrix, be the set of absolute points of a correlation of  $\text{PG}(2, q^{2n})$ ,  $n > 0, q$  odd. Then we can distinguish two cases.

1.  $\Gamma$  is projectively equivalent to

$$rx_1^{\sigma+1} + \rho x_1 x_2^\sigma + sx_2^{\sigma+1} + x_3^{\sigma+1} = 0$$

for some  $r, \rho, s$ . In this case, the trinomial

$$rx^{q^{m+1}} + \rho x + s \tag{1}$$

can have 0, 1, 2 or  $q + 1$  roots. Moreover, the following holds

$$|\Gamma| \in \{q^{2n} - (-q)^{n+1} + 1, q^{2n} - (-q)^n + 1, q^{2n} + 1\}.$$

2.  $\Gamma$  is projectively equivalent to

$$x_1^{\sigma+1} + x_1 x_2^\sigma + rx_2^{\sigma+1} + \tau x_2 x_3^\sigma + sx_3^{\sigma+1} = 0$$

for some  $r, \tau, s$ . Moreover  $|\Gamma| \in \{q^{2n} \pm q^n + 1, q^{2n} + 1\}$ .

In the first case,

- if Equation (1) has either  $q + 1$  zeros or no zeros, then  $\Gamma$  is equivalent to a set, which is already studied in the diagonal case (Cor. 7, Prop. 29, 30 in [4]),
- if Equation (1) has one zero, then

$$|\Gamma| \in \{q^{2n} - q^{n+1} + 1, q^{2n} + q^n + 1\} \text{ if } n \text{ is odd}$$

$$|\Gamma| \in \{q^{2n} + q^{n+1} + 1, q^{2n} - q^n + 1\} \text{ if } n \text{ is even,}$$

- if Equation (1) has two zeros, then  $|\Gamma| = q^{2n} + 1$ .

For the results and the examples of this section we refer to the following papers by B.C. Kestenband [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

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